



Longevity risk and survivor derivative pricing

Paul Dawson

*Department of Economics, Management School, Lancaster University,
Lancaster, UK*

Hai Lin

*School of Economics and Finance, Victoria University of Wellington,
Wellington, New Zealand, and*

Yangshu Liu

Department of Financial Management, Xiamen University, Xiamen, China

Abstract

Purpose – Longevity risk, that is, the uncertainty of the demographic survival rate, is an important risk for insurance companies and pension funds, which have large, and long-term, exposures to survivorship. The purpose of this paper is to propose a new model to describe this demographic survival risk.

Design/methodology/approach – The model proposed in this paper satisfies all the desired properties of a survival rate and has an explicit distribution for both single years and accumulative years.

Findings – The results show that it is important to consider the expected shift and risk premium of life table uncertainty and the stochastic behaviour of survival rates when pricing the survivor derivatives.

Originality/value – This model can be applied to the rapidly growing market for survivor derivatives.

Keywords Insurance companies, Life insurance, Risk finance, Modelling, Survival rate, Survivor derivatives, Gamma distribution

Paper type Research paper

1. Introduction

Longevity risk, that is, the uncertainty of demographic survival rate, is very important for those who run pension schemes and provide annuities. Systematic underestimation of the demographic survival rate will result in the increase of payment by the benefit providers, putting at a huge loss and even causing bankruptcy. *The Economist* (2010) estimates that every additional year of life expectancy at age 65 is reckoned to bump up the present value of pension liabilities in British defined-benefit schemes by 3 per cent, or £30 billion (US\$48 billion). In Britain alone, the total exposure to longevity risk exceeds £2 trillion. The International Monetary Fund (2012) also reports a huge extra cost for pension funds if the individual life increases, that is, if people live three years longer than expected, an additional 50 per cent of GDP will need to be paid in advanced economies and 25 per cent of GDP in developing economies. The social security systems, especially pension funds, are vulnerable to the unexpected increase of demographic survival rate (the unexpected decrease of mortality rate).

JEL classification – G13

The authors thank Toby Daglish, Pai-Ta Shih, Chu Zhang and participants of the FMA Asian 2011 Conference for helpful comments.



Therefore, financial institutions need tools to hedge longevity risk and benefit from allocating money to longevity-related financial assets (Sherris and Wills, 2008; Coppola and D'Amato, 2012; Cocco and Gomes, 2012).

In recent years, survivor derivatives have developed quite rapidly to manage the longevity risk (*The Economist*, 2010). In 2009, Babcock International completed a longevity swap with Credit Suisse, and a group of bankers and insurers launched the Life and Longevity Markets Association (LLMA) on 1 February 2010 to spur the development of a liquid longevity market. For example, LLMA wants to speed up the survivor swaps transactions by standardizing documentation.

The recent development of the longevity market calls for corresponding pricing and hedging methods (Ballotta and Haberman, 2006; Dowd *et al.*, 2006; Bauer *et al.*, 2010; Wang and Yang, 2012; Cocco and Gomes, 2012). How to model the survival rate is one of the largest obstacles. The classical stochastic models for interest rate and equity derivatives are not applicable in describing the demographic survival rate. The modelling of the demographic survival rate (hereafter referred to as "SR") or demographic mortality rate (hereafter referred to as "MR") has two properties, including:

- (1) the value of SR should be always in the domain $[0,1]$; and
- (2) a slight disturbance on SR today will have a persistent effect on cumulative SR in the future.

This means that a small fitting error will induce a significant cumulative deviation in the long run. In addition, a good SR model should be able to capture the exhibited volatility pattern of historical SR. SR has historically been significantly less volatile than other stochastic variables, such as equity return or interest rates, and this creates significant challenges in curve fitting. The volatility of SR increases before the age of 80, reaches its peak around the 80s and decreases to zero at the age of 115.

The prior literature has done a lot of work to model SR. Janssen and Skiadas (2006), Dahl (2004) and Schrager (2006) relax some of the constraints mentioned above to construct stochastic SR (MR) models with mathematical convenience. For example, Janssen and Skiadas (2006) propose a first-passage time process to model individual mortality. Their model makes meaningful economic sense in describing individual deaths. Dahl (2004) introduces an affine stochastic mortality model analogous to the affine interest rate model framework, which is both analytically tractable and flexible. Schrager (2006) extends Dahl (2004) by allowing an age-varying parameter to fit the mortality of multiple cohorts. However, his model suffers considerable fitting errors. In particular, his model violates the requirement of non-negativity of MRs, which makes the practical implementation difficult. Some studies also attempt to explain the unexpected change of MR by adding jump components. For example, Chen *et al.* (2010) adopt the dynamic mortality models with jumps to capture the permanent effects caused by unexpected factors. Deng *et al.* (2012) introduce a double-exponential jump process to capture the asymmetric movement in MR. Lin *et al.* (2012) propose a multivariate jump diffusion process to capture the common mortality shock across countries.

On the other hand, some researchers focus on the effect of stochastic SR or MR on the price of derivatives. Dowd *et al.* (2006) use a transformed beta distribution to model longevity shocks. Their model is calibrated to match life tables and to capture historical SR volatility. Dawson *et al.* (2010) extend Dowd *et al.* (2006) by allowing age-varying parameters of the beta distribution. However, these two models neither

have a closed-form solution to cumulative SR nor do they consider the stochastic dynamics of interest rates. Ballotta and Haberman (2006) propose a stochastic MR model by introducing the effects of a stochastic reduction factor. They argue that the increase in volatility of MR will lead to the decrease of SR, and the price of guaranteed annuity options (hereafter referred to as “GAO”) becomes lower. Wang and Yang (2012) discuss the cohort mortality dependence under the Lee-Carter framework and apply their model on the pricing of survivor floor and survivor swaps.

The literature on MR keeps searching for a model either fitting the historical data well or pricing the derivatives properly, but few studies address the problems in both fields. As Cairns *et al.* (2011) state, a good MR model should satisfy the following condition:

[...] biological reasonableness; the plausibility of predicted levels of uncertainty in forecasts at different ages; and the robustness of the forecasts relative to the sample period used to fit the model.

In this paper, we attempt to fill the gap between fitting and pricing. We illustrate that our model can fit the historical data well and, at the same time, generate a MR volatile enough to describe the risk without violating any biological attribution. In particular, we use gamma distribution to model stochastic age-varying SR and take the impact of stochastic interest rates into account. We illustrate how to calibrate the model parameters from an SR curve and volatility structure using UK males’ life table data. We also apply our model to pricing survivor derivatives. The pricing results show that the prices of survivor derivatives are highly sensitive to the assumption about the expected life table shift in risk-neutral measure. It is important to consider the expected shift and risk premium of life table uncertainty and the stochastic behaviour of SRs when we price the survivor derivatives.

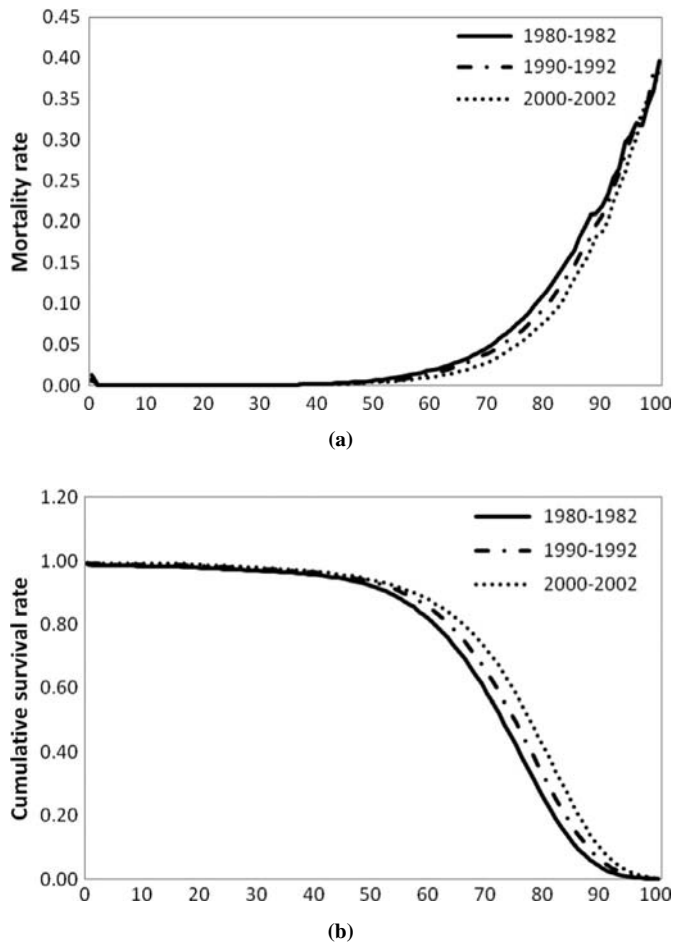
Our paper makes several contributions. First, our model satisfies all the desired properties of SR[1], and fits the historical statistical characteristics well. Second, though there are numerical solutions and approximations to get the value of longevity contingent (Cairns, 2011), our setting ensures that the distribution of cumulative SR is tractable, while another extraordinary value comes from the existence of closed-form solutions of the swap premium and the prices of survivor caps and floors. Moreover, compared with other distributions, modelling mortality with exponential gamma distribution allows the occurrence of extreme events, though with rare probability, such as war, an infectious disease epidemic or other catastrophes. We show that besides its use for the usual derivatives (survivor swap, survivor caps and survivor floors), it can also be a general framework in pricing contingent claims on either SR or interest rates or both, that is, GAOs, survivor caps and floors and survivor swaptions.

The paper is organized as follows. Section 2 introduces our SR model and calibrates it to the UK males’ life table data from 1980-1982 to 2004-2006. Section 3 tests the effects of stochastic interest rate. Section 4 uses our model to price survivor caps, survivor swaptions and GAOs. Section 5 concludes the paper.

2. Longevity risk model

In the last few decades, human MRs have decreased in a definite trend because of the improvement in life quality and medical techniques. Though slow, this trend has a significant impact on the management of insurance companies and pension funds which are exposed to long-horizon commitments. Thus, systematic longevity risk needs to be taken seriously.

To illustrate, we plot the MR (panel (a)) and cumulative SR (panel (B)) of UK males of different cohorts in Figure 1. MR is defined as the average probability that a person aged x exactly will die before reaching age $(x + 1)$, while cumulative SR is defined as the average probability that a person will survive until age x . Figure 1 plots three subperiods, 1980-1982, 1990-1992 and 2000-2002, to show the change of MR with time. The data are obtained from the web site of the UK Govenment actuary's department (historic interim life tables)[2]. MR and cumulative SR become steeper after the age of 60. We can also see a very gentle downward (upward) long-term trend of MR (cumulative SR)



Notes: Panel (a) plots the MR of UK males at different ages in the 1980-1982, 1990-1992 and 2000-2002 periods; panel (b) plots the cumulative SR of UK males at different ages in the 1980-1982, 1990-1992 and 2000-2002 periods; the data are obtained from the web site of the UK Govenment actuary's department (historic interim life tables)

Figure 1.
The MR and cumulative
SR of UK males at
different ages

at most ages, especially from age 50 to age 90. However, we will show later that a slight change of MR or SR will significantly affect the pricing of survivor derivatives.

To see the effects more specifically, we calculate the quantitative impact of a deterministic change of historical MR on the basic survivor derivative: a survivor swap. In a survivor swap, the pay-fixed party agrees to pay defined sums at defined intervals over the life of the contract and to receive in return payments predicated on the actual survivorship of the cohort referenced in the swap contract. The premium in a survivor swap can be written as (Dawson *et al.*, 2010)[3]:

$$\pi_{s,f} = \frac{\sum_{n=t+1}^f NY_n D(t,n) E_t^Q(S(t,n))}{\sum_{n=t+1}^f NY_n D(t,n) H(t,n)} - 1 \quad (1)$$

t : age at the time of the contract agreement.

s : age at the time of the first payment.

f : age at the time of the final payment.

n : age at the time of any given anniversary.

N : the size of the cohort at age t .

$E_t^Q(S(t,n))$ is expected cumulative SR from the age t to n at time t under the risk-neutral measure, Y_n is the payment per survivor due at age n ($= 0$ for $n < s$), $D(t,n)$ is the discount factor and $H(t,n)$ is the SR on the referenced life table at time t . As N , Y_n , $D(t,n)$ and $H(t,n)$ are known at time t , all we need to compute $\pi_{s,f}$ is $E_t^Q(S(t,n))$. For simplicity, we assume there is no risk premium for the life table uncertainty for the moment; that is, $E_t(S(t,n)) = E_t^Q(S(t,n))$, where $E_t(S(t,n))$ is expected cumulative SR from age t to n at time t under the physical measure[4]. We can write $E_t(S(t,n))$ as the following:

$$E_t(S(t,n)) = \prod_{i=t}^n (1 - \text{mortality}_i^u) \quad (2)$$

where u is an expected shift of $S(t,n)$ from that on life table at time t . When $u = 1$, the expected SR does not change; when $u > 1$, SR increases and otherwise SR decreases.

To capture the long-term trend of MR change, we relax the assumption of power term u in equation (2) and let it be age varying and linear with time, that is:

$$E_t(S(t,n)) = \prod_{i=t}^n \left(1 - \text{mortality}_i^{u_i(t)}\right) \quad (3)$$

where $u_i(t) = 1 + a_i(i - t)$. We then calibrate the parameters a_i to the annual life tables of UK males from 1980-1982 to 2004-2006[5].

Table I reports the calibration results. Panel (a) reports the summary statistics of a_i and panel (b) reports the calibration error of our models. The mean and median of a_i is positive, which implies that the SR increases in the long run. The max a_i happens at the age of 88, which is possibly due to the improvement of medical techniques since the 1980s. The minimum of a_i occurs at the age of 34, which probably reflects that increasing work pressures cause deteriorating health conditions at this age. In order to show whether this model calibrates the data well, we calculate the mean absolute

Panel (a) – summary statistics of α_i

α_i	Max	Min	Mean	Median	SD
Value	0.0086	− 0.0007	0.0044	0.0043	0.0029
Age	88	34			

Panel (b) – calibration errors of $u_i(t)$

MARE	Max	Min	Mean	Median	SD
Value	0.268	1.76×10^{-5}	4.66%	3.67%	0.0401
Age	21	20			

Notes: Table I reports the calibration results of the long-term trend model to the historical MR of UK males from 1980-1982 to 2004-2006; panel (a) reports the summary statistics of the parameters; panel (b) reports the calibration error, which is calculated by the MARE; the MARE is calculated as follows:

$$MARE_{i,t} = \frac{|\widehat{MR}_{i,t} - MR_{i,t}|}{MR_{i,t}}$$

where $MR_{i,t} = \text{mortality}_i^{1+a_i(i-t)}$

Table I.
Calibration results of the
long-term trend model

relative error (MARE) as follows:

$$MARE_{i,t} = \frac{|\widehat{MR}_{i,t} - MR_{i,t}|}{MR_{i,t}} \quad (4)$$

where $MR_{i,t} = \text{mortality}_i^{1+a_i(i-t)}$. Panel (b) reports the summary statistics of MARE. The results show that the calibration errors are limited. The mean and median of MARE are both smaller than 5 per cent.

To show the impact of MR modelling on the swap premium, we calculate the premium under a simple assumption that Y_n is constant after s and the interest rate is zero; thus, the swap premium can be written as:

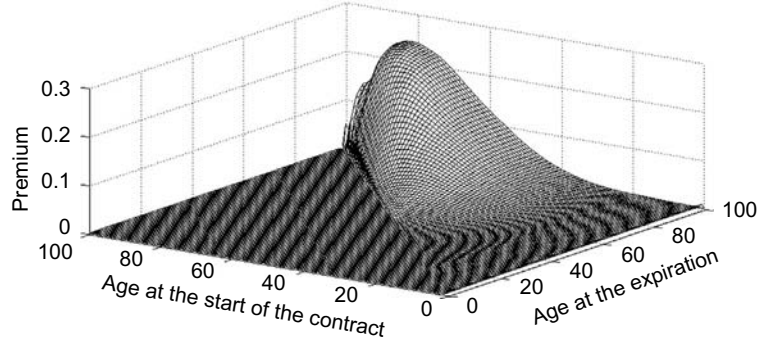
$$\pi_{s,f} = \frac{\sum_{n=s+1}^f E_t^Q(S(t, n))}{\sum_{n=s+1}^f H(t, n)} - 1 \quad (5)$$

Figure 2 plots the swap premium in equation (5) for different s and f . $H(t, n)$ is based on UK males' life tables in the 1980-1982 period, while $E_t^Q(S(t, n)) = E_t(S(t, n))$ and $E_t(S(t, n))$ are from the model calibration. The surface of the swap premium with respect to different start age and expiration age demonstrates that the swap premium is very sensitive at the 80s, when the SR decreases rapidly. This age-varying sensitivity places challenges on risk management, especially over this age period.

The realized future SR might deviate from its expectation. To model the uncertainty of future SR, we introduce a gamma-distributed stochastic variable in power terms:

$$S(t, n) = \prod_{i=t+1}^n p_i(t)^{\varepsilon_i(t)} \quad (6)$$

where $p_i(t)$ is the calibrated SR from age i to age $i + 1$ [6]. $\varepsilon_i(t) \sim \Gamma(k_{it}, -\theta_t / \ln p_i(t))$; that is, $\varepsilon_i(t)$ is assumed to be independently gamma distributed with two parameters. One is an age-varying shape parameter k_{it} and the other is a constant scale parameter θ_t . In order to make stochastic factor $\varepsilon_i(t)$ not affect the long trend of SR calibrated from the data, we choose these two parameters so that the expectation of $p_i(t)^{\varepsilon_i(t)}$ equals to $p_i(t)$.



Notes: This figure plots the survivor swap premium for different s and f under a simple assumption that Y_n is constant and the interest rate is zero, such that:

$$\pi_{s,f} = \frac{\sum_{n=s+1}^f E_t^Q(S(t, n))}{\sum_{n=s+1}^f H(t, n)} - 1,$$

Figure 2.
The survivor swap
premium

where $H(t, n)$ is based on the UK males' life table for the 1980-1982 period; $E_t^Q(S(t, n)) = E_t(S(t, n))$ and $E_t(S(t, n))$ are from the model calibration

In other words, we introduce a stochastic gamma variable to the SR which does not affect the expectation.

According to the property of gamma distribution:

$$y_i(t) = -\varepsilon_i(t) \ln p_i(t) \sim \Gamma(k_{it}, \theta_t)$$

we have:

$$p_i(t)^{\varepsilon_i(t)} = p_i(t)^{-\frac{y_i(t)}{\ln p_i(t)}} = e^{-y_i(t)}$$

Therefore:

$$E_t(p_i(t)^{\varepsilon_i(t)}) = E_t(e^{-y_i(t)}) \quad (7)$$

Since the introduction of the stochastic variable does not change the expectation, we also have:

$$E_t(e^{-y_i(t)}) = p_i(t)$$

The analytical solution of expectation and variance of $e^{-y_i(t)}$ can be computed as follows:

$$\begin{aligned} E_t(e^{-y_i(t)}) &= \int_0^\infty e^{-y_i(t)} g(y_i(t)) dy_i(t) = \int_0^\infty e^{-y_i(t)} y_i(t)^{(k_{it}-1)} \frac{e^{-(y_i(t))/\theta_t}}{\theta_t^{k_{it}} \Gamma(k_{it})} dy_i(t) \\ &= 1/(\theta_t^{k_{it}} \Gamma(k_{it})) \int_0^\infty e^{-(y_i(t))} [y_i(t)]^{(k_{it}-1)} e^{-(y_i(t))/\theta_t} dy_i(t) \end{aligned} \quad (8)$$

$$dy_i(t) = 1/(\theta_t^{k_{it}}) = (1 + \theta_t)^{-k_{it}}$$

$$\text{var}(e^{-y_i(t)}) = (1 + 2\theta_t)^{-k_{it}} - (1 + \theta_t)^{-2k_{it}} \quad (9)$$

Given the information on the expectation and volatility level of SR, we could take two steps to get the k_{it} and θ_t :

(1) Given the expectation, k_{it} is a function of θ_t . Thus, k_{it} can be written as:

$$k_{it} = -\frac{\ln(E_t(e^{-y_i(t)}))}{\ln(1 + \theta_t)} = -\frac{\ln p_i(t)}{\ln(1 + \theta_t)} \quad (10)$$

where $p_i(t) = 1 - \text{mortality}_i^{u_i(t)}$ and $u_i(t) = 1 + a_i(i - t)[7]$.

(2) Calibrate θ_t to volatility level of SR:

$$\begin{aligned} \text{var}(e^{-y_i(t)}) &= (1 + 2\theta_t)^{-k_{it}} - (1 + \theta_t)^{-2k_{it}} \\ &= (1 + 2\theta_t)^{\frac{\ln(p_i(t))}{\ln(1 + \theta_t)}} - (1 + \theta_t)^{\frac{2\ln(p_i(t))}{\ln(1 + \theta_t)}} \end{aligned} \quad (11)$$

The variance of SR can be estimated from historical MR as follows:

$$\text{var}(e^{-y_i(t)}) = \frac{1}{M} \sum_{t=1}^M (SR_{it} - E(SR_i))^2$$

Using the historical data of UK males' life tables from 1980-1982 to 2004-2006, we calibrate the parameter of θ_t equal to 0.0010, and k_{it} varies from 0.2257 to 486.69 for different ages.

These parameters are calibrated in the physical measure. Under regular conditions, the drift term should be adjusted by the risk premium when transforming the distribution from the physical measure to the risk-neutral measure so that survivor derivatives can be priced. Since we have no ideas about the risk premium of mortality risk, we simply assume two levels of risk premium. One is low risk premium and the impact on shift equals to $(1/3)a_i(i - t)$, that is, $u_i^Q(t) = 1 + (2/3)a_i(i - t)$, where Q means the risk-neutral measure. The other is high risk premium and $u_i^Q(t) = 1 + (1/3)a_i(i - t)$. Because the impact of the risk premium is mainly on the expected shift of the life table in the risk-neutral measure, this analysis could also show the importance of expected shift changes for survivor derivatives pricing.

3. Stochastic interest rates

In the field of equity derivatives, stochastic interest rates usually play a trivial role (Bakshi *et al.*, 1997). The main reason for this is that interest rates are generally less volatile than equity returns. However, in the insurance and pensions markets, the uncertainty of future interest rates could be significant, since cumulative small changes in interest rates might eventually give rise to a large shock in the present value of long-duration contracts. Moreover, contrary to stock returns, the volatility of MRs is lower than that of interest rates. Thus, it is useful to check the impact of stochastic interest rates on survivor swap premiums.

The closed-form solution of $\pi_{s,f}$ can be solved under the assumption that interest rate and MR are independent. In this paper, we use the Cox, Ingersoll and Ross (hereafter referred to as "CIR") (Cox *et al.*, 1985) process to model the stochastic interest rate in the risk-neutral measure, that is:

$$dr_t = \kappa(b - r_t)dt + \sigma\sqrt{r_t}dW_t^Q$$

where dW_t^Q is a standard Brownian motion under the risk-neutral measure. The discount factor $D(t, n)$ can be represented analytically as:

$$D(t, n) = A(t, n)e^{-B(t, n)r_t}$$

where:

$$B(t, n) = \frac{2(e^{\gamma(n-t)} - 1)}{(\gamma + \kappa)(e^{\gamma(n-t)} - 1) + 2\gamma}$$

$$A(t, n) = \exp \left\{ \ln 2\gamma + \frac{1}{2}(\gamma + \kappa)(n - t) - \ln[(\gamma + \kappa)(e^{\gamma(n-t)} - 1) + 2\gamma] \right\}^{2\kappa b / \sigma^2}$$

$$\gamma = \sqrt{\kappa^2 + 2\sigma^2}$$

In order to show the impact of stochastic interest rates, we follow the CIR structure specified in Hull and White (1990), that is, $dr_t = 0.20(0.10 - r_t)dt + 0.06\sqrt{r_t}dW_t^Q$. The initial values r_0 are 0.08, 0.10 and 0.12 to account for the different shapes of term structure within the CIR framework[8]. We also calculate the survivor swap premium where the value of the constant interest rate is set to be equal to the long-term mean 0.10. We only report the result assuming no risk premium, that is, $u_i^Q(t) = 1 + a_i(i - t)$, for simplicity[9].

Table II reports the difference and relative difference of premiums between a constant interest rate and a CIR model when $u_i^Q(t) = 1 + a_i(i - t)$, where the parameters of a_i are calibrated to the annual life tables of UK males from 1980-1982 to 2004-2006. Panels (a)-(c) report the difference when $r_0 = 0.08, 0.10$ and 0.12 , respectively. The difference and relative difference are defined as:

$$\text{Difference} = \text{Premium}(\text{constant}) - \text{Premium}(\text{CIR})$$

$$\text{Relative difference} = \frac{\text{Premium}(\text{constant}) - \text{Premium}(\text{CIR})}{\text{Premium}(\text{constant})}$$

Both differences and relative differences increase with duration. When $r_0 = 0.08$, the largest relative difference is about -5.93 per cent of the premium of constant interest rate. When $r_0 = 0.10$, the largest relative difference is about -2.80 per cent of the premium of constant interest rate. When $r_0 = 0.12$, the largest relative difference is about 2.19 per cent of the premium of constant interest rate. Though the gap between constant and stochastic interest rate is small, its value in the relative measure is not negligible, especially for long duration. Figure 3 also plots the differences in premiums between a constant interest rate and a CIR model when $r_0 = 0.08, 0.10$ and 0.12 , respectively.

4. Pricing survivor derivatives

We first price survivor caps and floors. A survivor cap (floor) is a derivative in which the buyer receives payments at the end of each period in which the real (agreed) SR exceeds the agreed (real) level. Because the pricing of survivor caps and floors are related by put-call parity, we analyse only cap pricing in this section. The value of survivor caps can be expressed as:

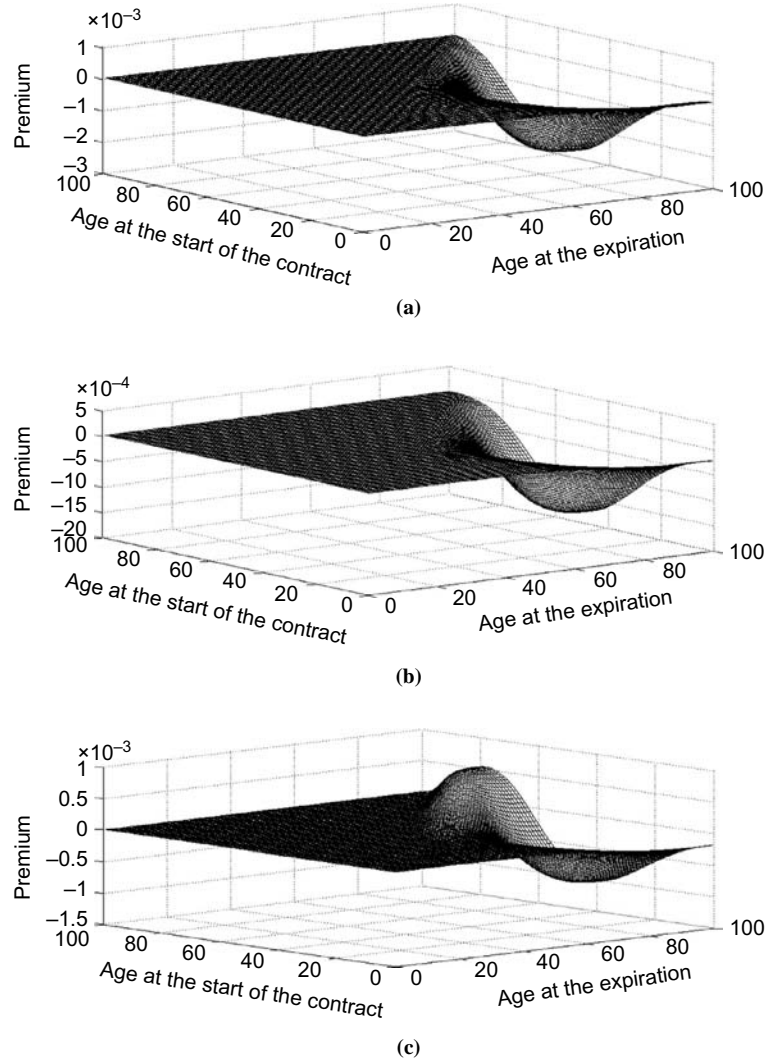
$$\text{caps} = N \sum_{n=t+1}^f D(t, n) E_t^Q [S(t, n) - S_{\text{strike}}(t, n)]^+ \quad (12)$$

Start age\end age	70	75	80	85	90	95	100
<i>Panel (a) – premium difference when $r_O = 0.08$</i>							
Absolute difference							
65	–0.0001	–0.0005	–0.0011	–0.0018	–0.0023	–0.0024	–0.0025
70	–	–0.0001	–0.0006	–0.0013	–0.0018	–0.0020	–0.0021
75	–	–	–0.0002	–0.0007	–0.0013	–0.0016	–0.0017
80	–	–	–	–0.0002	–0.0007	–0.0011	–0.0011
85	–	–	–	–	–0.0002	–0.0005	–0.0006
90	–	–	–	–	–	–0.0002	–0.0003
95	–	–	–	–	–	–	–0.0001
Relative difference							
65 (%)	–2.00	–3.20	–4.26	–5.13	–5.68	–5.89	–5.93
70 (%)	–	–2.04	–3.24	–4.26	–4.94	–5.22	–5.27
75 (%)	–	–	–2.08	–3.34	–4.23	–4.63	–4.72
80 (%)	–	–	–	–2.16	–3.38	–3.99	–4.15
85 (%)	–	–	–	–	–2.11	–3.19	–3.54
90 (%)	–	–	–	–	–	–2.37	–3.10
95 (%)	–	–	–	–	–	–	–1.88
<i>Panel (b) – premium difference when $r_O = 0.1$</i>							
Absolute difference							
65	–0	–0.0001	–0.0004	–0.0007	–0.0010	–0.0011	–0.0012
70	–	–0	–0.0001	–0.0004	–0.0007	–0.0008	–0.0008
75	–	–	–0	–0.0002	–0.0004	–0.0005	–0.0005
80	–	–	–	–0	–0.0001	–0.0003	–0.0003
85	–	–	–	–	–0	–0.0001	–0.0001
90	–	–	–	–	–	–0	–0
95	–	–	–	–	–	–	–0
Relative difference							
65 (%)	–0.21	–0.74	–1.45	–2.12	–2.58	–2.76	–2.80
70 (%)	–	–0.21	–0.72	–1.36	–1.86	–2.08	–2.13
75 (%)	–	–	–0.21	–0.71	–1.23	–1.50	–1.57
80 (%)	–	–	–	–0.21	–0.66	–0.99	–1.09
85 (%)	–	–	–	–	–0.19	–0.55	–0.72
90 (%)	–	–	–	–	–	–0.21	–0.43
95 (%)	–	–	–	–	–	–	–0.13
<i>Panel (c) – premium difference when $r_O = 0.12$</i>							
Absolute difference							
65	0.0001	0.0002	0.0003	0.0003	0.0002	0.0001	0.0001
70	–	0.0001	0.0003	0.0005	0.0004	0.0004	0.0004
75	–	–	0.0001	0.0004	0.0005	0.0005	0.0005
80	–	–	–	0.0002	0.0004	0.0005	0.0005
85	–	–	–	–	0.0002	0.0003	0.0004
90	–	–	–	–	–	0.0001	0.0002
95	–	–	–	–	–	–	0
Relative difference							
65 (%)	1.58	1.69	1.33	0.84	0.48	0.32	0.29
70 (%)	–	1.61	1.77	1.50	1.18	1.01	0.97
75 (%)	–	–	1.65	1.89	1.73	1.56	1.52
80 (%)	–	–	–	1.72	2.03	1.96	1.90
85 (%)	–	–	–	–	1.72	2.05	2.05
90 (%)	–	–	–	–	–	1.92	2.19
95 (%)	–	–	–	–	–	–	1.61

Notes: Table II reports the difference and relative difference between models using constant and stochastic interest rates with different start and end ages; the stochastic interest rates are assumed to follow the CIR structure specified in Hull and White (1990), that is, $dr_t = 0.20(0.10 - r_t)dt + 0.06\sqrt{r_t}dW_t^Q$; the value of the constant interest rate is set to be equal to the long-term mean 0.10; panels (a)-(c) report the difference when $r_O = 0.08, 0.10$ and 0.12 , respectively; the results, assuming a no-risk premium, that is, $u_i^Q(t) = 1 + a_i(i - t)$, are reported; the difference and relative difference are calculated as follows:

$$\text{Difference} = \text{Premium}(\text{constant}) - \text{Premium}(\text{CIR}) \quad \text{Relative difference} = \frac{\text{Premium}(\text{constant}) - \text{Premium}(\text{CIR})}{\text{Premium}(\text{constant})}$$

Table II.
Premium differences
between models using
constant and stochastic
interest rates



Notes: This figure plots the difference of swap premium between the constant interest rate and the stochastic interest rate for each start age and expiration age:

$$\text{Difference} = \text{Premium}(\text{constant}) - \text{Premium}(\text{CIR})$$

The risk premium of the life table is assumed to be zero, that is, $u_t^Q(t) = 1 + a_t(i - t)$; the stochastic interest rate is assumed to follow a CIR process with $dr_t = 0.20(0.10 - r_t)dt + 0.06\sqrt{r_t}dW_t^Q$; the value of the constant interest rate is set to be equal to the long-term mean 0.1; panel (a) plots the difference when the initial value $r_0 = 0.08$, panel (b) reports the difference when $r_0 = 0.1$ and panel (c) reports the difference when $r_0 = 0.12$

Figure 3.
Difference of swap
premium between
constant interest rate and
stochastic interest rate

First, we compute the value of caplet. Letting y :

$$(t+1, n) = \sum_{i=t+1}^n y_i$$

$$\sim \Gamma\left(\sum_{i=t+1}^n k_i, \theta\right)$$

we have:

$$\begin{aligned} E_t[S(t, n) - S_{strike}(t, n)]^+ &= E\left[\prod_{i=t+1}^n e^{-y_i} - S_{strike}(t, n)\right]^+ \\ &= E\left[e^{-\sum_{i=t+1}^n y_i} - e^{-(\ln 1)/(S_{strike}(t, n))}\right]^+ \\ &= \int_0^{(\ln 1)/(S_{strike}(t, n))} (e^{-y(t+1, n)} - (e^{-(\ln 1)/(S_{strike}(t, n))})^{\frac{y(t+1, n) \sum_{i=t+1}^n k_i - 1}{\theta \sum_{i=t+1}^n k_i}}) dy(t+1, n) \\ &= (1 + \theta)^{-\sum_{i=t+1}^n k_i \gamma\left(\frac{((1 + \theta)/\theta) \ln 1}{S_{strike}(t, n)} \cdot \sum_{i=t+1}^n k_i\right)} - S_{strike}(t, n) \gamma\left(\frac{(1/\theta) \ln 1}{S_{strike}(t, n)} \cdot \sum_{i=t+1}^n k_i\right) \\ &= E_t\left(\frac{S(t, n) \gamma(1 + \theta)}{\theta} \frac{\ln 1}{S_{strike}(t, n)} \cdot \sum_{i=t+1}^n k_i\right) - S_{strike}(t, n) \gamma\left(\frac{1}{\theta} \frac{\ln 1}{S_{strike}(t, n)}\right) \cdot \sum_{i=t+1}^n k_i \end{aligned} \quad (13)$$

where:

$$\gamma\left(\frac{((1 + \theta)/\theta) \ln 1}{S_{strike}(t, n)} \cdot \sum_{i=t+1}^n k_i\right) \text{ and } \gamma\left(\frac{(1/\theta) \ln 1}{S_{strike}(t, n)} \cdot \sum_{i=t+1}^n k_i\right)$$

are an incomplete gamma function in the form of:

$$\gamma(a, b) = \frac{1}{\Gamma(b)} \int_0^a e^{-x} x^{b-1} dx$$

We then calculate $E_t^Q[S(t, n) - S_{strike}(t, n)]^+$. We replace the $u_i(t)$ with $u_i^Q(t)$ and get k_i and θ under the risk-neutral measure following the similar calibration procedure using equations (10) and (11). The parameters of k_i and θ under the risk-neutral measure are then used to calculate $E_t^Q[S(t, n) - S_{strike}(t, n)]^+$ using equation (13). By substituting equation (13) into equation (12), we can get the value of caps. For instance, we compute a cap price by assuming interest rates to follow the CIR process and, assuming $S_{strike}(t, n)$ is given by u_{strike} , without losing any generality. Parameters are given as follows:

- Parameters of contracts: $N = 10,000$, $f = 100$, $t = 65$; $u_{strike} = 0.95, 0.98, 1.00, 1.02, 1.05$.
- Parameters of gamma distribution: $\theta = 0.0005, 0.0010, 0.0015$, k_i varies with SR.
- Parameters of CIR: $\kappa = 0.2$, $b = 0.10$, $\sigma = 0.06$, $r_0 = 0.10$.

Table III reports the price of caps under the parameter settings. Panels (a)-(c) report the results when $u_i^Q(t) = 1 + a_i(i - t)$ (no risk premium), $1 + (2/3)a_i(i - t)$ (low risk premium), $1 + (1/3)a_i(i - t)$ (high risk premium), respectively. There are

u_{strike}	0.95	0.98	1	1.02	1.05
<i>Panel (a) – $u_i^Q(t) = 1 + a_i(i - t)$</i>					
$\theta = 0.0005$	6,032.7	4,152.6	2,973.2	1,925.9	812.9
$\theta = 0.0010$	6,038.3	4,173.8	3,016.5	1,990.1	860.4
$\theta = 0.0015$	6,046.0	4,195.9	3,055.2	2,044.1	908.7
<i>Panel (b) – $u_i^Q(t) = 1 + (2/3) + a_i(i - t)$</i>					
$\theta = 0.0005$	5,060.8	3,182.6	2,017.2	1,038.3	222.3
$\theta = 0.0010$	5,066.7	3,207.5	2,071.3	1,116.3	261.8
$\theta = 0.0015$	5,075.2	3,233.6	2,118.8	1,181.4	305.7
<i>Panel (c) – $u_i^Q(t) = 1 + (1/3) + a_i(i - t)$</i>					
$\theta = 0.0005$	4,088.0	2,213.3	1,079.5	279.5	3.2
$\theta = 0.0010$	4,094.5	2,244.5	1,154.4	372.6	16.5
$\theta = 0.0015$	4,103.9	2,277.5	1,218.3	450.3	37.8

Notes: Table III reports the prices of caps under different θ and u_{strike} , which is calculated as follows:

$$\text{caps} = N \sum_{n=t+1}^f D(t, n) E_t^Q [S(t, n) - S_{\text{strike}}(t, n)]^+$$

The parameters are set as follows: parameters of contracts: $N = 10,000$, $f = 100$, $t = 65$; parameter of gamma distribution: $\theta = 0.0005, 0.0010, 0.0015$; k_i varies with SR; parameters of CIR: $\kappa = 0.20$, $b = 0.10$, $\sigma = 0.06$, $r_0 = 0.1$; the benchmark MR is based on the life table of UK males during the 1980-1982 period; panels (a)-(c) report the result when $u_i^Q(t) = 1 + a_i(i - t)$, $1 + (2/3)a_i(i - t)$ and $1 + (1/3)a_i(i - t)$, respectively

Table III.

Prices of caps
under different
values of θ and u_{strike}

significant differences among the results. The caps prices are higher when the risk premiums are low. For example, when $u_i^Q(t) = 1 + a_i(i - t)$, the price of the cap with $u_{\text{strike}} = 1.05$ and $\theta = 0.0005$ is 812.90. However, it is only 3.2 when $u_i^Q(t) = 1 + (1/3)a_i(i - t)$. This reflects the importance of considering the risk premium of SR when pricing survivor caps. On the other hand, the prices of caps decrease with the increase of strike price parameter u_{strike} and increase with the variance parameter θ .

We next price survivor swaptions. A survivor swaption gives the right, but not the obligation, to enter into a survivor swap contract at a specified rate of π_{strike} . Conditional on the future interest rate r_T and the assumed interest rate structure, the swaption payoff at maturity is (Dawson *et al.*, 2010):

$$C_T(r_T) = [\phi(\pi_{\text{expiry}} - \pi_{\text{strike}})]^+ N \sum_{n=0}^{f-(T+t)} D(T, T+n) Y_n H(t, n)$$

where:

$$\phi = \begin{cases} -1, & \text{receiver swaption} \\ 1, & \text{payer swaption} \end{cases}$$

The present value of swaption is given by:

$$\begin{aligned} C_t &= E_t^Q [D(t, T) C_T(r_T)] \\ &= D(t, T) E_t^Q [C_T(r_T)] \\ &= D(t, T) \int_R C_T(r_T) dF^Q(r_T) C \end{aligned} \tag{14}$$

Since the price of the payer and receiver swaptions can be connected by put-call parity, we analyse only the price of payer swaptions. Then:

$$C_T(r_T) = [(\pi_{\text{expiry}} - \pi_{\text{strike}})]^+ N \sum_{n=0}^{f-(T+t)} D(T, T+n) Y_n H(t, n)$$

Since this formula has no closed-form solution, we price it using 100,000 simulations. We choose the life table of UK males between 1980 and 1982 as the benchmark and use our calibrated expected shift model with gamma distribution to simulate the life table at time T . We also use a CIR model to simulate the interest rate r_T . Then $C_T(r_T)$ could be calculated in each simulation. The mean of 100,000 simulated $C_T(r_T)$ is discounted by $D(t, T)$ to get C_t . The simulated parameters are as follows:

- Parameters of contracts: $Y_n = 1$, $N = 10,000$, $T = 15$, $f = 100$, $t = 50$, $\pi_{\text{strike}} = -0.05, -0.02, 0.00, 0.02, 0.05[10]$.
- Parameter of gamma distribution: $\theta = 0.0005, 0.0010, 0.0015$, k_i varies with SR.
- Parameters of CIR: $\kappa = 0.20$, $b = 0.1$, $\sigma = 0.06$, $r_0 = 0.10$.

Table IV reports the prices of payer swaptions under the parameter settings. Panels (a)-(c) report the results when $u_i^Q(t) = 1 + a_i(i - t)$ (no risk premium), $1 + (2/3)a_i(i - t)$ (low risk premium) and $1 + (1/3)a_i(i - t)$ (high risk premium), respectively. The results are quite similar to those of caps. The prices of payer swaptions with no risk premiums are much higher than those under high risk premium. For example, when π_{strike} is -0.05 , the prices of payer swaptions if $u_i^Q(t) = 1 + a_i(i - t)$ are approximately two times those if $u_i^Q(t) = 1 + (1/3)a_i(i - t)$. It is about three times those when π_{strike} is 0.05 . The subtle change of assumption on u^Q makes a significant impact on the prices of survivor derivatives, which again reflects the importance of accounting for risk premium of life table uncertainty. The prices of payer swaptions decrease with the increase of π_{strike} , while the assumption of θ seems to have a limited impact on the prices of payer swaptions.

We finally turn to the pricing of GAOs. A GAO provides the holder of the contract the right to either receive at retirement a cash payment or receive an annuity which would be payable throughout his/her remaining lifetime and which is calculated at a guaranteed rate, depending on which has the greater value. This was a common feature of pension policies sold in the UK during the 1970s and 1980s. In a report by Bolton *et al.* (1997), GAOs could be applied to over 10 per cent of the long-term liabilities of the responding insurance companies. Ballotta and Haberman (2003) introduce a theoretical model to price the GAOs. However, they do not consider the longevity risk and derive SR directly from the life table. We extend their model by allowing SR to be a stochastic variable with respect to age. Our model accounts for uncertainty of the MR, which results in a larger risk exposure and different pricing result.

Following Ballotta and Haberman (2003), the present value of a GAO at maturity is:

$$V_i(T) = E_t^Q[D(t, T)C_T 1_{(\tau_i > T)}] \quad (15)$$

where:

π_{strike}	-0.050	-0.020	0	0.020	0.050
<i>Panel (a) - $u_i^Q(t) = 1 + a_i(i - t)$</i>					
$\theta = 0.0005$	4,026.9	3,571.6	3,319.9	3,076.2	2,727.6
$\theta = 0.0010$	4,020.2	3,564.9	3,313.2	3,069.6	2,721.3
$\theta = 0.0015$	4,012.6	3,557.4	3,305.8	3,062.3	2,714.1
<i>Panel (b) - $u_i^Q(t) = 1 + (2/3) + a_i(i - t)$</i>					
$\theta = 0.0005$	3,088.9	2,647.6	2,406.6	2,175.8	1,851.7
$\theta = 0.0010$	3,087.6	2,646.3	2,405.4	2,174.7	1,850.9
$\theta = 0.0015$	3,081.3	2,640.2	2,399.4	2,169.0	1,845.5
<i>Panel (c) - $u_i^Q(t) = 1 + (1/3) + a_i(i - t)$</i>					
$\theta = 0.0005$	2,072.8	1,657.9	1,439.1	1,237.7	969.6
$\theta = 0.0010$	2,067.9	1,653.6	1,435.2	1,234.3	966.8
$\theta = 0.0015$	2,067.2	1,653.2	1,435.2	1,234.6	967.4

Notes: Table IV reports the prices of payer swaption under different values of θ and π_{strike} using 100,000 Monte Carlo iterations; the benchmark MR is based on the life table of UK males during the 1980-1982 period; panels (a)-(c) report the result when $u_i^Q(t) = 1 + a_i(i - t)$, $1 + (2/3)a_i(i - t)$ and $1 + (1/3)a_i(i - t)$, respectively; the pricing equations are as follows:

$$\begin{aligned}
 C_t &= E_t^Q[D(t, T)C_T(r_T)] \\
 &= D(t, T)E_t^Q[C_T(r_T)] \\
 &= D(t, T) \int_R C_T(r_T) dF^Q(r_T)
 \end{aligned}$$

where:

$$C_T(r_T) = [(\pi_{expiry} - \pi_{strike})]^+ N \sum_{n=0}^{f-(T+t)} D(T, T+n) Y_n H(t, n).$$

Table IV.

Prices of swaption under different values of θ and π_{strike}

Parameters are set as follows: parameters of contracts: $Y_n = 1$, $N = 10,000$, $T = 15$, $f = 100$, $t = 50$; parameter of gamma distribution: $\theta = 0.0005, 0.0010, 0.0015$; k_i varies with SR; parameters of CIR: $\kappa = 0.20$, $b = 0.10$, $\sigma = 0.06$, $r_0 = 0.1$

$$\begin{aligned}
 C_T &= \left\{ \left(g S_T \sum_{n=0}^{f-(T+t)} \Pr(\tau_{t+T} > n) D(T, T+n) \right) - S_T \right\}^+ \\
 &= g S_T \left\{ \left(\sum_{n=0}^{f-(T+t)} \Pr(\tau_{t+T} > n) D(T, T+n) \right) - \frac{1}{g} \right\}^+
 \end{aligned} \tag{16}$$

in which T is the option life time, S_T is the value of investment fund at time T , $\Pr(\tau_x > n)$ is the SR for people at age x to live n years, g is the guaranteed annuity rate, and f is the largest survival age. In Ballotta and Haberman (2003), C_T can be written as:

$$C_T = g S_T \sum_{n=0}^{f-(T+t)} \Pr(\tau_{t+T} > n) [D(T, T+n) - K_n]^+$$

where K_n is an “artificial” strike[11], which satisfies:

$$\left\{ \left(\sum_{n=0}^{f-(T+t)} \Pr(\tau_{t+T} > n) D(T, T+n) \right) - \frac{1}{g} \right\}^+ \\ = \sum_{n=0}^{f-(T-t)} \Pr(\tau_{t+T} > n) [D(T, T+n) - K_n]^+$$

To get K_n , it is necessary to find a critical value r_T^* as a function of $1/g$ and $\Pr(\tau_{t+T} > n)$, so that:

$$\sum_{n=0}^{f-(T+t)} \Pr(\tau_{t+T} > n) D_{r_T^*}(T, T+n) = \frac{1}{g}$$

However, when $\Pr(\tau_{t+T} > n)$ is a stochastic variable, r_T^* has many feasible values given g , which induces the inability to get a closed-form solution. Thus, we follow Ballotta and Haberman (2003) and model the interest rate under the Heath, Jarrow and Morton (hereafter referred to as ‘‘HJM’’) (Heath *et al.*, 1990) framework with constant volatility, derive prices of GAO by 100,000 Monte Carlo iterations and compare them with the price conditioned on a non-stochastic SR. The parameters are the same as Ballotta and Haberman (2003), that is:

$$\theta = 0.0010, \sigma_S = 0.2, \sigma_f = 0.001, t = 50, g = 0.111, S_0 = 100, T = 15, f = 100.$$

Table V reports the prices of GAOs under two models. Panels (a)-(c) report the results when $u_i^Q(t) = 1 + a_i(i - t)$ (no risk premium), $1 + (2/3)a_i(i - t)$ (low risk premium) and $1 + (1/3)a_i(i - t)$ (high risk premium), respectively. Similarly, the difference between the results in panels (a) and (c) reflects the importance of considering the risk premium of SR. The results assuming no risk premium with $u_i^Q(t) = 1 + a_i(i - t)$ are much higher than those assuming high risk premium ($u_i^Q(t) = 1 + (1/3) + a_i(i - t)$). The prices of GAOs with stochastic SR are lower than those with non-stochastic SR, although the expectation of stochastic SR is the same as the non-stochastic one[12]. These results are consistent with Ballotta and Haberman (2006), who explain that the increase of volatility of SR makes the mortality trend become more uncertain and the chance of surviving for another year deteriorates. The results demonstrate that the stochastic SR should be taken into account for the pricing of GAOs.

5. Conclusion

In this paper, we introduce the gamma distribution to describe the uncertainty of SR in a long horizon. Compared with prior models, our model has two main advantages. First, our method of setting the parameters of the gamma distribution make our model satisfy all the required properties of SR, including non-negativity, good curve-fitting capacity and low level of volatility structure. Second, in our framework, either the SR of a certain year or the cumulative SR for a long period follows gamma distribution and the latter’s shape parameter is expressed as the sum of the former, which is convenient for computing the prices of various survivor derivatives.

To make our model applicable, we illustrate how to calibrate the model parameters from an SR curve and volatility structure using UK males’ life table data. We then test the

Benchmark life table	1980-1982	1990-1992	2004-2006
<i>Panel (a) – $u_i^Q(t) = 1 + a_i(i - t)$</i>			
	$\rho = -1$		
Stochastic SR	20.18	28.44	45.74
Non-stochastic SR	23.88	32.12	49.82
	$\rho = 0$		
Stochastic SR	17.36	25.14	41.51
Non-stochastic SR	20.86	28.61	45.21
	$\rho = 1$		
Stochastic SR	14.66	21.97	37.49
Non-stochastic SR	18.05	25.59	40.97
<i>Panel (b) – $u_i^Q(t) = 1 + (2/3) a_i(i - t)$</i>			
	$\rho = -1$		
Stochastic SR	13.68	21.49	38.61
Non-stochastic SR	17.09	25.24	42.62
	$\rho = 0$		
Stochastic SR	11.30	18.62	34.79
Non-stochastic SR	14.64	21.99	38.41
	$\rho = 1$		
Stochastic SR	8.97	15.79	31.10
Non-stochastic SR	12.05	19.10	33.46
<i>Panel (c) – $u_i^Q(t) = 1 + (1/3) a_i(i - t)$</i>			
	$\rho = -1$		
Stochastic SR	7.39	14.51	31.06
Non-stochastic SR	10.53	17.84	34.92
	$\rho = 0$		
Stochastic SR	5.35	11.98	27.70
Non-stochastic SR	8.47	15.17	31.22
	$\rho = 1$		
Stochastic SR	3.42	9.55	24.46
Non-stochastic SR	6.17	12.75	27.57

Notes: Table V reports the price of the GAO under stochastic and non-stochastic SRs with different benchmark life tables and ρ ; ρ describes the correlation between value of future investment asset S_T and interest rate; panels (a)-(c) report the results when $u_i^Q(t) = 1 + a_i(i - t)$, $1 + (2/3)a_i(i - t)$ and $1 + (1/3) a_i(i - t)$, respectively; the interest rate term structure follows an HJM model with constant volatility:

$$V_t(T) = E_t^Q[D(t, T)C_T 1_{(\tau > T)}]$$

where:

$$C_T = gS_T \left(\sum_{n=0}^{f-(T+t)} \Pr(\tau_{t+T} > n) D(T, T+n) \right) - \frac{1}{g}^+.$$

Parameters are set as follows:

$$\theta = 0.0010, \sigma_S = 0.2, \sigma_f = 0.001, t = 50, g = 0.111, S_0 = 100, T = 15, f = 100.$$

effects of introducing stochastic interest rates. The results show that the impact of stochastic interest is non-negligible in some cases. Finally, our model is applied to pricing survivor caps, survivor swaptions and GAOs. We show that, in our setting, the price of survivor caps with stochastic interest rates has analytical solutions, while the swaptions and GAOs with stochastic interest rates could only be priced numerically. The pricing results show that the prices of survivor derivatives are highly sensitive to the assumption about the expected life table shift in the risk-neutral measure. It is very important to take

Table V.
Prices of GAO

the expected shift and risk premium of life table uncertainty and the stochastic properties of life tables into account when we price the survivor derivatives. The results of simulation also show that GAOs have a lower price when a stochastic SR is considered, which suggests that stochastic SR should be taken into account in the pricing of GAOs.

Notes

1. Which, as mentioned above, include (a) the value of SR should be always in the domain $[0,1]$, and (b) a slight disturbance on SR today will have a persistent effect on cumulative SR in the future.
2. www.gad.gov.uk/demography/%20data/life/%20tables/historic_interim_life_tables.html
3. In pricing survivor derivatives, we follow the standard no-arbitrage assumptions that justifies a risk-neutral pricing measure. Naturally, we recognize that, at present, no such market exists.
4. This assumption will be relaxed later by introducing the risk premium of life table uncertainty when we price the other survivor derivatives.
5. That is, we have 25 annual life tables data.
6. $p_i(t)$ also equals $1 - \text{mortality}_i^{1+a_i(i-t)}$.
7. A different value of $u_i^Q(t)$ will be used when we calibrate the parameters under the risk-neutral measure.
8. When $r_O = 0.08$, the initial value is less than the long-term mean 0.10 and the term structure tends to be upward sloping. On the other hand, it tends to be downward sloping when $r_O = 0.12$.
9. The results under low risk premium $u_i^Q(t) = 1 + (1/3) + a_i(i - t)$ and high risk premium $u_i^Q(t) = 1 + (2/3) + a_i(i - t)$ are available upon request.
10. These parameters of p_{strike} are approximately equal to $u_{strike} = 0.95, 0.98, 1.00, 1.02$ and 1.05 when we calculate the u_{strike} from p_{strike} .
11. K_n is computed inversely from equation (16) in Ballotta and Haberman (2003), thus it always satisfies equation (16).
12. Our further simulated results show that which one is higher depends on the assumption of volatility. If the volatility parameter θ is as small as what we use in this paper, the GAO prices of stochastic SR are lower than those of static SR. On the other hand, if the volatility parameter θ is large enough (θ equals to 0.06, for example), the GAO prices of stochastic SR will become larger than those of static SR. These results are available upon request.

References

- Bakshi, G., Cao, C. and Chen, Z. (1997), "Empirical performance of alternative option pricing models", *The Journal of Finance*, Vol. 52 No. 5, pp. 2003-49.
- Ballotta, L. and Haberman, S. (2003), "Valuation of guaranteed annuity conversion options", *Insurance: Mathematics and Economics*, Vol. 33 No. 1, pp. 87-108.
- Ballotta, L. and Haberman, S. (2006), "The fair valuation problem of guaranteed annuity options: the stochastic mortality environment case", *Insurance: Mathematics and Economics*, Vol. 38 No. 1, pp. 195-214.
- Bauer, D., Börger, M. and Ruß, J. (2010), "On the pricing of longevity-linked securities", *Insurance: Mathematics and Economics*, Vol. 46 No. 1, pp. 139-49.
- Bolton, M.J., Carr, D.H., Collis, P.A., George, C.M., Knowles, V.P. and Whitehouse, A.J. (1997), *Reserving for Annuity Guarantees – The Report of the Annuity Guarantees Working Party*, Institute of Actuaries, London.

- Cairns, A.J.G. (2011), "Modelling and management of longevity risk: approximations to survivor functions and dynamic hedging", *Insurance: Mathematics and Economics*, Vol. 49 No. 3, pp. 438-53.
- Cairns, A.J.G., Blake, D., Dowd, K., Coughlan, G.D., Epstein, E. and Khalaf-Allah, M. (2011), "Mortality density forecasts: an analysis of six stochastic mortality models", *Insurance: Mathematics and Economics*, Vol. 48 No. 3, pp. 355-67.
- Chen, B., Zhang, L. and Zhao, L. (2010), "On the robustness of longevity risk pricing", *Insurance: Mathematics and Economics*, Vol. 47 No. 3, pp. 358-73.
- Cocco, J.F. and Gomes, F.J. (2012), "Longevity risk, retirement savings, and financial innovation", *Journal of Financial Economics*, Vol. 103 No. 3, pp. 509-29.
- Coppola, M. and D'Amato, V. (2012), "Backtesting the solvency capital requirement for longevity risk", *The Journal of Risk Finance*, Vol. 13 No. 4, pp. 309-19.
- Cox, J.C., Ingersoll, J.E. and Ross, S.A. (1985), "A theory of the term structure of interest rates", *Econometrica*, Vol. 53 No. 2, pp. 385-407.
- Dahl, M. (2004), "Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts", *Insurance: Mathematics and Economics*, Vol. 35 No. 1, pp. 113-36.
- Dawson, P., Dowd, K., Cairns, A.J.G. and Blake, D. (2010), "Survivor derivatives: a consistent pricing framework", *Journal of Risk and Insurance*, Vol. 77 No. 3, pp. 57-596.
- Deng, Y., Brockett, P.L. and MacMinn, R.D. (2012), "Longevity/mortality risk modelling and securities pricing", *Journal of Risk and Insurance*, Vol. 79 No. 3, pp. 697-721.
- Dowd, K., Blake, D., Cairns, A.J.G. and Dawson, P. (2006), "Survivor swaps", *Journal of Risk and Insurance*, Vol. 73 No. 1, pp. 1-18.
- (The) Economist (2010), "Live long and prosper", *The Economist*, Vol. 394 No. 8668, p. 83.
- Heath, D., Jarrow, R. and Morton, A. (1990), "Bond pricing and the term structure of interest rates: a discrete time approximation", *Journal of Financial and Quantitative Analysis*, Vol. 25 No. 4, pp. 419-40.
- Hull, J. and White, A. (1990), "Pricing interest-rate-derivative securities", *Review of Financial Studies*, Vol. 3 No. 4, pp. 573-92.
- International Monetary Fund (2012), "The financial impact of longevity risk", *Global Financial Stability Report: The Quest for Lasting Stability*, International Monetary Fund, Washington, DC, pp. 123-51.
- Janssen, J. and Skiadas, C.H. (2006), "Dynamic modelling of life table data", *Applied Stochastic Models & Data Analysis*, Vol. 11 No. 1, pp. 35-49.
- Lin, Y., Liu, S. and Yu, J. (2012), "Pricing mortality securities with correlated mortality indexed", *The Journal of Risk and Insurance*, 20 July.
- Schrager, D.F. (2006), "Affine stochastic mortality", *Insurance: Mathematics and Economics*, Vol. 38 No. 1, pp. 81-97.
- Sherris, M. and Wills, S. (2008), "Financial innovation and the hedging of longevity risk", *Asia-Pacific Journal of Risk and Insurance*, Vol. 3 No. 1, pp. 52-64.
- Wang, C. and Yang, S. (2012), "Pricing survivor derivatives with cohort mortality dependence under the Lee-Carter framework", *The Journal of Risk and Insurance*, 11 September.

Corresponding author

Hai Lin can be contacted at: cfc1080@gmail.com

To purchase reprints of this article please e-mail: reprints@emeraldinsight.com
Or visit our web site for further details: www.emeraldinsight.com/reprints