Lattices of ideals in graph algebras: refining the join operation

by

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Abstract

We consider the ideal structure of Leavitt path algebras of graphs that satisfy Condition (K) over commutative rings with identity. The approach we use takes advantage of the relationship between Leavitt path algebras and Steinberg algebras of boundary path groupoids. We establish a lattice $\mathcal{F}'_L$ consisting of particular maps $\tau : E^0 \to \mathcal{L}(R)$ and show that this lattice is isomorphic to the lattice of ideals of $L_R(E)$. The advantage to our approach over previous lattice isomorphisms, even in the case that R is a field, is that we obtain convenient join and meet operations in $\mathcal{F}'_L$. Lastly, we provide three concrete examples.

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Chapter 1

Introduction

Lattice theory is a useful tool for studying order in many areas, including algebra. A *lattice* consists of an ordered set in which every two elements have a supremum and an infimum contained in the set [10]. We refer to the supremum as the *join* and the infimum as the *meet*. We are particularly interested in maps between lattices. A lattice isomorphism is a bijective map between lattices that preserves the join and meet operations [10].

This thesis is concerned with a particular lattice; the lattice of ideals of a ring. When ordered via containment, every subset of ideals of a ring has both a join and a meet (see Proposition 2.0.7). Since an algebra is also a ring, we can investigate the ideals of an algebra through its lattice of ideals.

We consider the lattice of ideals of *Leavitt path algebras*. Leavitt path algebras, introduced independently by Abrams and Aranda Pino in [2] and by Ara, Moreno and Pardo in [4], are algebras associated to paths in a directed graph such that multiplication in the algebra corresponds to concatenation of paths [15, Lemma 3.3]. For a survey of results on ideals in Leavitt path algebras over a field K, see [13]. This area has a strong connection to graph C*-algebras in operator theory. Leavitt path algebras are constructed, in part, by the Cuntz-Krieger relations [1] – relations that were initially used to develop C*-algebras (known as Cuntz-Krieger C*-

algebras) [14].

The paper 'Ideals of Steinberg Algebras of Strongly Effective Groupoids, with respect to Leavitt path algebras' [6] identifies the ideals of certain Leavitt path algebras, namely those associated to *row-finite* graphs with no *sources* that satisfy Condition (K). The authors consider Leavitt path algebras over an arbitrary, commutative ring R with identity and show how ideals in R give rise to ideals in the Leavitt path algebra. The approach in [6] employs the relationship between Leavitt path algebras and another algebraic structure; *groupoids*. We use the same approach to characterise the ideals of certain Leavitt path algebras associated to arbitrary graphs that satisfy Condition (K) [7].

Groupoids cross the threshold between group theory and category theory, but we will focus on the algebraic perspective. A groupoid, as the name suggests, is like a group. The difference is that the binary operation is only partially defined. The groupoids we are most interested in are those generated by directed graphs, called boundary path groupoids. Boundary path groupoids have a natural topology generated by a basis of compact open sets [15].

The step from topological groupoids to algebras is made using the Steinberg algebra of a groupoid, whose elements are linear combinations of characteristic functions of compact open subsets of the groupoid [15]. The delightful aspect of Steinberg algebras is that the Steinberg algebra of a boundary path groupoid is isomorphic to the Leavitt path algebra associated to the same graph [15]. By approaching Leavitt path algebras via groupoids, we are able to expand the understanding of Leavitt path algebras; finding the ideals of Steinberg algebras of boundary path groupoids also gives us the ideals of Leavitt path algebras.

In Chapter 2, we cover relevant lattice theory, with a particular focus on lattice isomorphisms. Chapter 3 continues with more preliminaries, this time pertaining to Leavitt path algebras and their connection to graph groupoids. We state the main results from [6] in Chapter 4, particularly the description of the ideal lattice of Steinberg algebras of strongly effective groupoids (Theorem 4.0.2) and the description of the ideals in a Leavitt path algebra associated to a row-finite graph with no sources that satisfy Condition (K) (Theorem 4.0.3).

Our main results are in Chapter 5. Previous work on the graded ideal structure of various Leavitt path algebras has used the lattice $\mathcal{T}_E$ of pairs (H, S) where H is a saturated and hereditary subset of E^0 and $S \subseteq B_H$ [1, 13]. Our first main result (Theorem 5.0.1) also takes this approach; we use a lattice consisting of certain maps from $\mathcal{T}_E$ to $\mathcal{L}(R)$.

The join of $\mathcal{T}_E$ is complicated, as we see in [1, Proposition 2.5.6]. An even more complex graph lattice is required when describing the all the ideals of a Leavitt path algebra over a field [1, Proposition 2.8.8]. Thus, one of the aims of this project is to find a "nicer" lattice structure in a non-row-finite setting. Our main theorem (Theorem 5.1.2) describes a graph based lattice for Leavitt path algebras such that the join and meet operations are simpler. Theorem 5.1.2 also shows that the ideals of a Leavitt path algebra over a ring are governed by the ideals in that ring. We apply this theorem to three concrete examples of Leavitt path algebras, and follow with a corollary for the situation when R is a field.

Part of our aim is to provide a more general description of ideals in Leavitt path algebras over a ring than in [6]. We were not the only ones interested in such results; in a paper [16] published last December, Rigby and van den Hove give a complete description of the ideal lattice of Leavitt path algebras over arbitrary rings. Since Rigby and van den Hove focus on classifying the ideals of any Leavitt path algebra over an arbitrary ring, the lattice they use to do this is complicated and the join is not straightforward [16, Proposition 6.16.]. Our results provide a lattice whose join and meet operations replicate those of an ideal lattice, thus our lattice is simpler and has a more intuitive join.

Chapter 2

Lattices

In [6], Clark et al propose a topology on the lattice of ideals of a ring. Before we look at this topology, we will delve into the properties of lattices and give some standard results. The following definitions are from [10].

Definition 2.0.1. Let L be a partially ordered set. Let $x, y \in L$. Then the *join* of x and y is

$$x \vee y := \text{sup}(x, y)$$

and the *meet* is

$$x \wedge y := \text{inf}(x, y).$$

Equivalently, the join of a subset $S \subseteq L$ is

$$\vee S := \sup(S)$$

and the meet is

$$\wedge S := \inf(S).$$

By definition, the join and meet might not necessarily exist, or they could exist outside of L.

Definition 2.0.2. Let $\mathcal{L}$ be a non-empty set with partial ordering $\leq$. If $x \vee y$ and $x \wedge y$ exist and are in $\mathcal{L}$ for all $x, y \in \mathcal{L}$, then $\mathcal{L}$ is a *lattice*. We denote a lattice by $(\mathcal{L}, \leq)$ or $\mathcal{L}$.

As an example, it is straightforward to check that the integers with respect to the usual $\leq$ is a lattice. The following are standard lattice theory definitions and results and can be found in [10].

Lemma 2.0.3. *Let* $(\mathcal{L}_1, \leq_1)$ *and* $(\mathcal{L}_2, \leq_2)$ *be lattices. Let* $\Gamma : \mathcal{L}_1 \to \mathcal{L}_2$ *be a bijection such that* $\Gamma(x) \leq_2 \Gamma(y)$ *if and only if* $x \leq_1 y$. *Then*

$$\Gamma(x \vee_1 y) = \Gamma(x) \vee_2 \Gamma(y) \quad \textit{and} \quad \Gamma(x \wedge_1 y) = \Gamma(x) \wedge_2 \Gamma(y). \tag{2.1}$$

Proof. For all $x, y \in \mathcal{L}_1$, we have $x, y \leq_1 x \vee_1 y$. So $\Gamma(x) \leq_2 \Gamma(x \vee_1 y)$ and $\Gamma(y) \leq_2 \Gamma(x \vee_1 y)$. Thus, $\Gamma(x) \cup \Gamma(y) \subseteq \Gamma(x \vee y)$. By definition,

$$\Gamma(x), \Gamma(y) \leq_2 \Gamma(x) \vee_2 \Gamma(y) \leq_2 \Gamma(x \vee_1 y).$$

But, since Γ is bijective, there exists a $z \in \mathcal{L}_1$ such that $\Gamma(x) \vee_2 \Gamma(y) = \Gamma(z)$. If $\Gamma(x) \vee_2 \Gamma(y) <_2 \Gamma(x \vee_1 y)$, then $x, y \leq_1 z <_1 x \vee_1 y$, which is a contradiction. Thus, we must have $\Gamma(x \vee_1 y) = \Gamma(x) \vee_2 \Gamma(y)$. Likewise, we get a contradiction unless $\Gamma(x \wedge_1 y) = \Gamma(x) \wedge_2 \Gamma(y)$. □

Definition 2.0.4. Let $\mathcal{L}_1, \mathcal{L}_2$ be lattices. Suppose there exists a map $\Gamma : \mathcal{L}_1 \to \mathcal{L}_2$ that satisfies (2.1). Then Γ is a *lattice homomorphism*. If Γ is bijective, then Γ is a *lattice isomorphism*.

Lemma 2.0.3 tells us that an order-preserving bijective map between lattices, with an order preserving inverse, is a lattice isomorphism. That is, the join and meet operations are preserved.

Definition 2.0.5. A Lattice $\mathcal{L}$ is said to be *complete* if $\vee S$ and $\wedge S$ exist for all $S \subseteq \mathcal{L}$.

Since a lattice isomorphism preserves the join and meet operations, we can conclude that a lattice isomorphism will preserve completeness. We show the details in the following lemma. Let S^u denote the set of upper-bounds of a set $S \subseteq \mathcal{L}$.

Lemma 2.0.6. *Suppose* $(\mathcal{L}_1, \leq)$ *and* $(\mathcal{L}_2, \leq)$ *are lattices. Suppose* $\Gamma : \mathcal{L}_1 \to \mathcal{L}_2$ *is a lattice isomorphism. Then, for all subsets* $S \subseteq \mathcal{L}_1$*, if* $\vee S$ *and* $\wedge S$ *exist, then* $\Gamma(\vee S)$ *and* $\Gamma(\wedge S)$ *exist and*

$$\Gamma(\vee S) = \vee\Gamma(S), \textit{ and}$$

$$\Gamma(\wedge S) = \wedge\Gamma(S).$$

That is, if $\mathcal{L}_1$ *is complete, so is* $\mathcal{L}_2$*.*

Proof. Suppose $\vee S \in \mathcal{L}_1$ for all $S \subseteq \mathcal{L}_1$. Since $\vee S \geq s$ for all $s \in S$, we have $\Gamma(\vee S) \geq \Gamma(s)$ for all $s \in S$. So $\Gamma(\vee S) \in \Gamma(S)^u$. Suppose there is a $x \in S^u$ such that $x \leq \Gamma(\vee S)$. Since Γ is bijective, there exits an $a \in \mathcal{L}_1$ such that $x = \Gamma(a)$. Since $\Gamma(a) \leq \Gamma(\vee S)$, $a \leq \vee S$. But $\Gamma(a) \geq \Gamma(s)$ for all $s \in S$. So $a \geq s \forall s \in S$. This is a contradiction unless $a = \vee S$, and hence $\Gamma(a) = \Gamma(\vee S)$. Suppose $y \in S^u$ is also a lower upper bound of S. Then there is a $b \in \mathcal{L}_1$ such that $\Gamma(b) = y$. But then b is a least upper bound of S, which is again a contradiction. Hence $\Gamma(\vee S)$ is the least upper bound of $\Gamma(S)$. The dual proof follows the same method. $\square$

We are interested in the lattice of two-sided ideals of a ring R, denoted $\mathcal{L}(R)$. It is stated in [10] that $(\mathcal{L}(R), \subseteq)$ is a lattice. We show the details below.

Proposition 2.0.7. *Let* R *be a ring and* $\mathcal{L}(R)$ *be the set of ideals of* R*. Then* $\mathcal{L}(R)$*, with order relation* $\subseteq$*, forms a complete lattice with join and meet defined by*

$$\bigwedge_{i \in I} L_i = \bigcap_{i \in I} L_i$$

and

$$\bigvee_{i \in I} L_i = \bigcap\{B \in \mathcal{L}(R) \mid \bigcup_{i \in I} L_{i \in I} \subset B\}. \tag{2.2}$$

Proof. The set of ideals of a ring satisfies two conditions:

(a) for every non-empty family $\{L_i\}_{i\in I} \subset \mathcal{L}(R)$,

$$\bigcap_{i\in I} L_i \in \mathcal{L}(R)$$

(see Corollary 3.2.3 [12]); and

(b) $R \in \mathcal{L}(R)$, since R is always an ideal of itself.

Since $\mathcal{L}(R)$ satisfies (a) and (b), [10, Corollary 2.32] tells us that $\mathcal{L}(R)$ is a lattice and that it is complete. □

The formula (2.2) says that the join of two ideals $I, J \in \mathcal{L}(R)$ is the ideal generated by $I \cup J$, hence $I \vee J = I + J$ (this is a standard ring theory result). We will use the following example of a lattice of ideals when looking at specific Leavitt path algebra examples later on. See Example 5.1.4.

Example 2.0.8. For $R = \mathbb{Z}$ every ideal is of the form $n\mathbb{Z}$, where $n \in \mathbb{Z}$. So $\mathcal{L}(R) = \{n\mathbb{Z} \mid n \in \mathbb{N}\}$. The lattice operations are defined by

$$n\mathbb{Z} \wedge m\mathbb{Z} = n\mathbb{Z} \cap m\mathbb{Z} = \text{lcm}(n, m)\mathbb{Z},$$

$$n\mathbb{Z} \vee m\mathbb{Z} = n\mathbb{Z} + m\mathbb{Z} = \text{gcd}(n, m)\mathbb{Z}.$$

To see this, fix $z \in n\mathbb{Z} \cap m\mathbb{Z}$. Since $z \in n\mathbb{Z} \cap m\mathbb{Z} \iff n|z$ and $m|z$, and $n|z$ and $m|z \iff \text{lcm}(n, m)|z$,

$$z \in n\mathbb{Z} \cap m\mathbb{Z} \iff z \in \text{lcm}(n, m)\mathbb{Z}.$$

For the join, we see that since $n\mathbb{Z} + m\mathbb{Z}$ is an ideal of $\mathbb{Z}$, there is a $d \in \mathbb{Z}$ such that $n\mathbb{Z} + m\mathbb{Z} = d\mathbb{Z}$. We then have $n\mathbb{Z} \subseteq d\mathbb{Z}$ and $m\mathbb{Z} \subseteq d\mathbb{Z}$, thus $d|m$ and $d|n$. So d is a common divisor of n, m. To see that it is the greatest common divisor, suppose $e \in \mathbb{Z}$ is a common divisor of m, n (i.e. $e|m$ and $e|n$). Then $n\mathbb{Z} \subseteq e\mathbb{Z}$ and $m\mathbb{Z} \subseteq e\mathbb{Z}$. Addition is closed in $e\mathbb{Z}$ since it is an ideal, so we have $n\mathbb{Z} + m\mathbb{Z} \subseteq e\mathbb{Z}$. Thus, $d\mathbb{Z} \subseteq e\mathbb{Z} \iff e|d$. Hence d is the greatest common divisor of m, n.

2.1 Topology on the lattice of ideals of a ring

The results in [6], as well as the results of this project, show that the elements of $\mathcal{L}(R)$ gives rise to ideals in graph algebras. So we are interested the properties of $\mathcal{L}(R)$. As such, we explore some questions about topological structures on $\mathcal{L}(R)$. The following topology on $\mathcal{L}(R)$ is introduced in [6, §5].

Proposition 2.1.1. *[6, p. 5474] Given a finite set $F \subset R$, define*

$$Z(F) := \{I \in \mathcal{L}(R) \mid F \subseteq I\}. \tag{2.3}$$

The collection of such $Z(F)$ forms a basis for a topology.

Before we prove this, we require a lemma. This result is stated in [6], but not proved.

Lemma 2.1.2. *For all finite subsets $F_1, F_2 \subset R$, we have $Z(F_1) \cap Z(F_2) = Z(F_1 \cup F_2)$.*

Proof. Let $L \in Z(F_1) \cap Z(F_2)$. Then $L \in \{I \in \mathcal{L}(R) \mid F_1 \subseteq I\}$ and $L \in \{I \in \mathcal{L}(R) \mid F_2 \subseteq I\}$. Thus, L contains both F_1 and F_2. That is, $L \in \{I \in \mathcal{L}(R) \mid F_1 \cup F_2 \subseteq I\} = Z(F_1 \cup F_2)$. Hence $Z(F_1) \cap Z(F_2) \subseteq Z(F_1 \cup F_2)$. It is easy to see that the converse holds. $\square$

Proof of Proposition 2.1.1. Let $\mathcal{B} = \{Z(F) \mid F \subset R, F \text{ is finite}\}$. This is a basis since

1. Fix $I \in \mathcal{L}(R)$. Let $F = \{0\}$. Then F is a finite subset contained in I. Since F is finite, there exists a $B \in \mathcal{B}$ such that $B = Z(F)$. Hence there exists a $b \in \mathcal{B}$ such that $I \in B$; and

2. Let $Z(F_1), Z(F_2) \in \mathcal{B}$ and $I \in Z(F_1) \cap Z(F_2)$. We have seen that $Z(F_1) \cap Z(F_2) = Z(F_1 \cup F_2)$, so $I \in Z(F_1) \cap Z(F_2) = Z(F_1 \cup F_2) \in \mathcal{B}$. $\square$

Since (2.3) is a basis for a topology on the lattice $\mathcal{L}(R)$, the obvious question arises: are the join and meet operations continuous with respect to this topology? As it turns out, they are. We show this in the next proposition.

Proposition 2.1.3. *When $\mathcal{L}(R) \times \mathcal{L}(R)$ is equipped with the product topology generated by the basis $\mathcal{B} \times \mathcal{B}$, the maps $\vee : \mathcal{L}(R) \times \mathcal{L}(R) \to \mathcal{L}(R)$ and $\wedge : \mathcal{L}(R) \times \mathcal{L}(R) \to \mathcal{L}(R)$ are continuous.*

Proof. By [8, Theorem 64.], it is sufficient to show that each inverse image of a basis element of $\mathcal{L}(R)$ is open in $\mathcal{L}(R) \times \mathcal{L}(R)$. Let $Z(F) \in \mathcal{B}$. Then

$$\wedge^{-1}(Z(F)) = \{(A, B) \in \mathcal{L}(R) \times \mathcal{L}(R) \mid A \cap B \in Z(F)\}.$$

Thus, $F \subseteq A \cap B$ for all $(A, B) \in \wedge^{-1}(Z(F))$, and hence $F \subseteq A$ and $F \subseteq B$. So $\wedge^{-1}(Z(F)) \subseteq Z(F) \times Z(F)$. Let $I, J \in Z(F)$. Then $I \cap J \in Z(F)$, so $(I, J) \in \wedge^{-1}(Z(F))$. That is, $Z(F) \times Z(F) \subseteq \wedge^{-1}(Z(F))$. Thus, $\wedge^{-1}(Z(F)) = Z(F) \times Z(F)$, which is, of course, open.

For the join function, we have

$$\vee^{-1}(Z(F)) = \{(A, B) \in \mathcal{L}(R) \times \mathcal{L}(R) \mid A + B \in Z(F)\}, \tag{2.4}$$

since $A \vee B = A + B$. That is, if $A \vee B \in Z(F)$, then for all $f \in F$, there is some $a \in A$ and $b \in B$ such that $f = a + b$. To show that the join is continuous, we must show that $\vee^{-1}(Z(F))$ is open. We will show this inductively, starting with the base case: when F is a singleton.

Fix $r \in R$. We then have

$$\vee^{-1}(Z(\{r\})) = \{(A, B) \in \mathcal{L}(R) \times \mathcal{L}(R) \mid A + B \in Z(\{r\})\}.$$

Fix $(A, B) \in \vee^{-1}(Z(\{r\}))$. Then there is some $a \in A$ and $b \in B$ such that $r = a+b$. The set $Z(a) \times Z(b)$ contains (A, B) and is open. We want to show that $Z(a) \times Z(b)$ is contained in $\vee^{-1}(Z(\{r\}))$. Let $(I, J) \in Z(a) \times Z(b)$. Then we have $a \in I$ and $b \in J$. So, by 2.4, $I + J \in \vee^{-1}(Z(\{r\}))$. Thus, $\vee^{-1}(Z(F))$ is open when F is a singleton.

Next, suppose $\vee^{-1}(Z(F))$ is open for all $F \subset R$ such that $|F| = n$. We will show that $\vee^{-1}(Z(G))$ is open for all $G \subset R$ such that $|G| = n+1$. Fix $G \subset R$ such that $|G| = n+1$ and fix $F \subset G$ such that $|F| = n$. Then $G = F \cup \{g\}$ where $g \in G/F$. By Lemma 2.1.2, we have $Z(G) = Z(F \cup \{g\}) = Z(F) \cap Z(g)$. Thus, $\vee^{-1}(Z(G)) = \vee^{-1}(Z(F)) \cap \vee^{-1}(Z(g))$. Since $|F| = n$, we know $\vee^{-1}(Z(F))$ is open, and we have already seen that $\vee^{-1}(Z(g))$ is open for singletons, hence $\vee^{-1}(Z(F)) \cap \vee^{-1}(Z(g))$ is open. Thus, $\vee^{-1}(Z(G))$ is open for all G such that $|G| = n+1$. By induction, we can conclude that $\vee^{-1}(Z(F))$ is open for all finite $F \subset R$, hence $\vee$ is continuous. $\square$

Chapter 3

Graph groupoids and graph algebras

In this chapter we define Leavitt path algebras associated to directed graphs (Definition 3.0.8), and boundary path groupoids associated to directed graphs (Definition 3.0.11). We then cover how Steinberg algebras connect groupoids to Leavitt path algebras in Section 3.1.

We begin by fixing some standard graph theory definitions and notation. Note that we follow the convention of [14] – our ranges and sources are inverted from the usual graph conventions. That is, where we take the range of an edge, the standard convention would take the source of the edge. See Definition 3.0.2.

Definition 3.0.1. [14, p. 5] A *directed graph* $E = (E^0, E^1, s, r)$ consists of two sets E^0, E^1 and functions $r, s : E^1 \to E^0$, where E^0 is the set of *vertices* and E^1 the set of *edges*. The function r maps an edge to its *range* and s maps an edge to its *source*.

While we have used the definition of directed graphs in [14], we have omitted that the sets E^0, E^1 be countable as we do not require it. We refer to directed graphs as graphs interchangeably.

A *row-finite* graph is one in which $r^{-1}(v)$ is finite for every $v \in E^0$. If

e is an edge with $s(e) = v$ and $r(e) = w$ then we say that v emits e and w receives e. A *source* is a vertex that receives no edges and an *infinite receiver* is a vertex that receives infinitely many edges. So row-finite means a graph has no infinite receivers. If $v \in E^0$ is either a source or an infinite receiver (that is, $r^{-1}(v)$ is either empty or infinite) then v is called *singular*. If v is neither a source nor an infinite receiver, then it is called *regular*. We denote the set of regular vertices by $\mathrm{reg}(E)$.

Definition 3.0.2. [14, p. 9] A *path of length* n in a directed graph E is a sequence $\mu = \mu_1\mu_2\cdots\mu_n$ of edges in E^1 such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \leq i \leq n-1$. The length of μ is denoted by $|\mu| := n$, and vertices are regarded as paths of length 0. We denote the set of finite paths in a directed graph by E^* and the infinite paths by E^∞. The set of paths of length n is denoted E^n. We extend the range and source maps to E^* by setting $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_{|\mu|})$ for $|\mu| > 1$, and $r(v) = v = s(v)$ for $v \in E^0$. If μ, ν are paths with $s(\mu) = r(\nu)$, we write $\mu\nu$ for the path $\mu_1\cdots\mu_{|\mu|}\nu_1\cdots\nu_{|\nu|}$.

Definition 3.0.3. [14, p. 35] A *cycle* in E is a path $\mu = \mu_1\ldots\mu_n$ with $n \geq 1$, $s(\mu_n) = r(\mu_1)$ and $s(\mu_i) \neq s(\mu_j)$ for $i \neq j$. We say that E satisfies *Condition (K)* if for every vertex v, either there is no cycle based at v, or there are two distinct paths μ, ν such that $s(\mu) = v = r(\mu)$, $s(\nu) = v = r(\nu)$, $r(\mu_i) \neq v$ for $i < |\mu|$, and $r(\nu_i) \neq v$ for $i \leq |\nu|$.

The results in [6] require a graph to be row-finite and with no sources, as well as satisfying Condition (K). In this thesis we seek to produce the same results, but for arbitrary graphs that satisfy Condition (K).

Definition 3.0.4. [14, p. 33] Let E be a directed graph. We define a *preorder* $\leq$ on E^0 given by:

$w \leq v$ if there is a path $\mu \in E^* \cup E^0$ such that $s(\mu) = v$ and $r(\mu) = w$.

As it turns out, this preorder on E^0 is very beneficial to us. In Theorem 5.1.2 it is used in the construction of a lattice that is isomorphic to the lattice of ideals of Leavitt path algebras.

Definition 3.0.5. [14, p. 34] We say $H \subseteq E^0$ is *hereditary* if whenever $v \in H$ and $w \in E^0$ for which $v \leq w$, then $w \in H$.

Definition 3.0.6. [14, p. 47] We say $H \subseteq E^0$ is *saturated* if whenever $v \in \text{Reg}(E)$ has the property that $\{s(e)|e \in E^1, r(e) = v\} \subseteq H$, then $v \in H$.

In [6, Theorem 6.1], Clark et al describe a lattice based on $\mathcal{H}_E$, the set of hereditary, saturated subsets of E^0. Since we aim to adapt [6, Theorem 6.1] to include singularities, $\mathcal{H}_E$ will no longer suffice.

Definition 3.0.7. [1, Definition 2.4.4.] Let H be a hereditary saturated subset of E^0. We say v is a *breaking vertex of H* if it belongs to the set

$$B_H := \{v \in E^0 \backslash H \mid v \text{ is an infinite receiver}, 0 < |r^{-1}(v) \cap s^{-1}(E^0 \backslash H)| \leq \infty\}.$$

That is, B_H is the set of infinite receivers not in H such that only a finite, non-zero number of edges being received have sources not in H.

[1, Definition 2.4.4.] has been included instead of that in [14] as it better suits our purposes. We have adjusted this definition to suit the path direction convention we use.

Let $\mathcal{T}_E$ be the set of elements (H, S) where H is a saturated and hereditary subset of E^0 and $S \subseteq B_H$. This set is a lattice [18, Definition 5.4] and is useful to us when adapting [6, Theorem 6.1] for non-row-finite graphs with sources. Now that we have established the necessary graph theory preliminaries, we define the graph algebra that we are concerned with finding the ideals of.

Definition 3.0.8 (Leavitt path algebras). [1, Definition 1.2.3] Let E be an arbitrary directed graph and R an arbitrary commutative ring with identity. We define a set $(E^1)^*$ consisting of symbols of the form $\{e^* \mid e \in E^1\}$. The *Leavitt Path Algebra of E with coefficients in R*, denoted $L_R(E)$, is the free associative R-algebra generated by the set $E^0 \cup E^1 \cup (E^1)^*$, subject to the following relations:

(V) $vv' = \delta_{v,v'}v$ for all $v, v' \in E^0$,

(E1) $r(e)e = es(e) = e$ for all $e \in E^1$,

(E2) $s(e)e^* = e^*r(e) = e^*$ for all $e \in E^1$,

CK1 $e^*e' = \delta_{e,e'}s(e)$ for all $e, e' \in E^1$, and

CK2 $v = \sum_{\{e \in E^1 | r(e)=v\}} ee^*$ for every regular vertex $v \in E^0$.

We think of each edge e^* as an edge with the reverse orientation of e. We call a path with reversed orientation a *ghost path* and denote the set of such paths $\widehat{E}$.

We have adjusted the definition of Leavitt path algebras to account for the direction of our paths and our purpose of finding ideals of Leavitt path algebras over commutative rings with identity. We interpret these relations – V, E1, E2, CK1, and CK2 – such that the vertices are represented by pairwise orthogonal idempotents $\{p_v \mid v \in E^0\}$ in an R-algebra A and the edges and ghost edges are represented by sets $\{s_e \mid e \in E^1\}$ and $\{s_{e^*} \mid e \in E^1\}$ in A. See [1, Remark 1.2.5] for more details. The relations CK1 and CK2, called the Cuntz-Krieger relations [14, p. 6-7], arose in operator theory for the purpose of developing graph C*-algebras. See [14] for a comprehensive survey of graph C*-algebras. [18, Corollary 3.2] gives an alternate definition of the Leavitt path algebra of E:

$$L_R(E) = \{\alpha\beta^* \mid \alpha, \beta \in E^* \text{ and } s(\alpha) = s(\beta)\}.$$

Next we cover groupoids and relevant groupoid theory. In essence, a groupoid is a a group with a partial function replacing the binary operation. For example, addition is a partial function on the set of all matrices.

Definition 3.0.9 (Groupoids). [17, Defintion 2.1.1.] A *groupoid* is a set G together with a distinguished subset $G^{(2)} \subseteq G \times G$, a multiplication map $(\alpha, \beta) \mapsto \alpha\beta$ from $G^{(2)}$ to G, and an inverse map $\gamma \mapsto \gamma^{-1}$ from G to G such that:

1. $(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in G$;
2. if (α, β) and (β, α) belong to $G^{(2)}$, then $(\alpha\beta, \gamma)$ and $(\alpha, \beta\gamma)$ belong to $G^{(2)}$, and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$; and
3. $(\gamma, \gamma^{-1}) \in G^{(2)}$ for all $\gamma \in G$, and for all $(\gamma, \eta) \in G^{(2)}$, we have $\gamma^{-1}(\gamma\eta) = \eta$ and $(\gamma\eta)\eta^{-1} = \gamma$.

Definition 3.0.10. [17, p. 5] Given a groupoid G we shall write

$$G^{(0)} := \{\gamma^{-1}\gamma \mid \gamma \in G\}$$

and refer to the elements of $G^{(0)}$ as *units* and $G^{(0)}$ itself as the *unit space* of G. Since $(\gamma^{-1})^{-1} = \gamma$ for all γ, we also have $G^{(0)} = \{\gamma\gamma^{-1} \mid \gamma \in G\}$. We define $r, s : G \to G^{(0)}$ by

$$r(\gamma) := \gamma\gamma^{-1} \quad \text{and} \quad s(\gamma) := \gamma^{-1}\gamma$$

for all $\gamma \in G$. We refer to r and s as the range and source maps, respectively.

Note that the source and range maps of a groupoid and the source and range maps of a directed graph share the same notation. Which maps we are referring to should be clear from context.

A groupoid G is a *topological* groupoid if there is a topology on G such that multiplication and inversion are continuous. Thus s and r are continuous with respect to the topology. If $U \subseteq G$ is an open set such that $s|_U$ and $r|_U$ are homeomorphisms onto open subsets of $G^{(0)}$, then U is called an *open bisection*. An *ample* groupoid is a topological groupoid with Hausdorff unit space and a base of compact open bisections.

Let R be a commutative ring with identity. A subset $U \subseteq G^{(0)}$ is said to be *invariant* if for all $g \in G$, $s(g) \in U$ implies $r(g) \in U$. Let X be a set. We say $\rho : G^{(0)} \to X$ is *G-invariant* if

$$\rho(s(\gamma)) = \rho(r(\gamma)) \quad \text{for all } \gamma \in G.$$

We say a groupoid is *effective* if $\mathrm{Int}(\mathrm{Iso}(G)) = G^{(0)}$, where Int denotes the interior and

$$\mathrm{Iso}(G) := \{\gamma \in G \mid s(\gamma) = r(\gamma)\}.$$

A groupoid is *strongly effective* if whenever F is a closed invariant subset of $G^{(0)}$, $G_F = \{\gamma \in G \mid s(\gamma) \in F\}$ is effective.

The groupoids which are of particular use to us in this thesis are boundary path groupoids, which are groupoids associated to paths in a directed graph. The *path space* of a directed graph E is $E^* \cup E^\infty$, the set of all finite and infinite paths. The *boundary path space* of E, denoted ∂E, is the set of all infinite paths or finite paths whose source is a singular vertex.

We use the definition of a graph groupoid that is in [9] as their path convention is the same as ours.

Definition 3.0.11 (Boundary path groupoids). [9, Example 2.1] The boundary path groupoid of a graph E is

$$G_E := \{(\alpha x, |\alpha - \beta|, \beta x) \mid \alpha, \beta \in E^*, x \in \partial E, s(\alpha) = s(\beta) = r(x)\}.$$

The unit space of G_E is $G_E^{(0)} = \{(x, 0, x) \mid x \in \partial E\}$, which we identify with ∂E. For $(x, k, y) \in G_E$, we have $r((x, k, y)) = x$ and $s((x, k, y)) = y$. The composition of morphisms and their inverses are defined by the formulae:

$$(x, k, y)(y, l, z) = (x, k + l, z),$$

$$(x, k, y)^{-1} = (y, -k, x).$$

We sometimes refer to the boundary path groupoid of E as the *graph groupoid* of E.

In [15, §2], Rigby verifies that there is a basis for a topology on the *path space* $E^* \cup E^\infty$. We have adjusted the following results to align with our path convention.

Definition 3.0.12. The *cylinder sets* of $E^* \cup E^\infty$ are the sets

$$Z(\alpha) := \{\alpha x | x \in E^* \cup E^\infty, s(\alpha) = r(x)\}.$$

The *generalised cylinder sets* of $E^* \cup E^\infty$ are the sets

$$Z(\alpha, F) = Z(\alpha) \backslash \cup_{e \in F} Z(\alpha e),$$

where $\alpha \in E^*$, $F \subseteq s(\alpha)E^1$.

The set of generalised cylinders sets forms a basis of open compact sets for a locally compact Hausdorff topology on $E^* \cup E^\infty$ (see [19, Theorem 2.1]). The topology on G_E by uses the same concept as the cylinder sets on $E^* \cup E^\infty$. For $\alpha, \beta \in E^* s(\alpha)$ and a finite set $F \subseteq s(\alpha)E^1$, we define

$$Z(\alpha, \beta) := \{(\alpha x, |\alpha| - |\beta|, \beta x) \mid x \in s(\alpha)\partial E\}, \text{ and}$$

$$Z(\alpha, \beta, F) := \mathbb{Z}(\alpha, \beta) \backslash \bigcup_{e \in F} Z(\alpha e, \beta e). \tag{3.1}$$

We then have:

Theorem 3.0.13. *[15, Theorem 2.17] Let E be a directed graph. Then G_E is an ample Hausdorff groupoid with the basis of open compact bisections given by (3.1).*

This result was known before it was included in [15], but Rigby notes that this is the first self-contained proof that we know of that applies to arbitrary directed graphs, and does not require the graph to be countable. The next section makes it clear why it is important that G_E be ample and Hausdorff.

3.1 Steinberg algebras

Steinberg algebras are algebras associated to groupoids. In this section, we define Steinberg algebras (Definition 3.1.3) and address how they relate to

Leavitt path algebras (Theorem 3.1.4). Lastly, we show that if two algebras A, B are isomorphic, then their ideal lattices are isomorphic (3.1.5). This last result is not new, but we have included it for completeness.

Definition 3.1.1. [12, §3, Definition 1.1] Let R be a ring. A *(left) R-module* is an additive abelian group A together with a function $\cdot : R \times A \to A$ such that, for all $r, s \in R$ and $a, b \in A$:

1. $r \cdot (a + b) = r \cdot a + r \cdot b$;
2. $(r + s) \cdot a = r \cdot a + s \cdot a$;
3. $(rs) \cdot a = r \cdot (s \cdot a)$.

If R has an identity element 1_R and

(iv) $1_R \cdot a = a$ for all $a \in A$,

then A is said to be a *unitary R-module*. If R is commutative, then left R-modules and right R-modules are the same thing and are called R-modules.

Let G be a group. Let R^G be the set of all functions $f : G \to R$. Then R^G has the structure of an R-module. The following definitions come from [15].

Definition 3.1.2. A *characteristic function* of a subset U of G is $1_U : G \to R$ such that

$$1_U(g) = \begin{cases} 1 & \text{if } g \in U \\ 0 & \text{if } g \notin U. \end{cases}$$

Definition 3.1.3. Suppose G is an ample, Hausdorff groupoid and R is a commutative ring with identity. Let $A_R(G)$ be the R-submodule of R^G generated by the set

$$\{1_U | U \text{ is an open, compact, Hausdorff subset of } G\}.$$

The *convolution* of $f, g \in A_R(G)$ is defined as

$$f * g(x) = \sum_{\{y \in G | s(y)=s(x)\}} f(xy^{-1})g(y) = \sum_{\{(z,y) \in G^{(2)} | zy=x\}} f(z)g(y)$$

for all $x \in G$. The R-module $A_R(G)$, with the convolution, is called the *Steinberg Algebra* of G over R.

For all graphs E, the Steinberg algebra of G_E exists, since all boundary path groupoids are ample and Hausdorff (Theorem 3.0.13). This leads us to the next theorem.

Theorem 3.1.4. *[9, Example 3.2] Let E be an arbitrary directed graph, G_E a boundary path groupoid and R a commutative ring with identity. Then $L_R(E)$ and $A_R(G_E)$ are isomorphic as R-algebras.*

We provide a sketch of the proof. We represent the elements of $A_R(G_E)$ by the following indicator functions: for $v \in E^0$, $q_v := 1_{Z(v)}$, and for $e \in E^*$ $t_e := 1_{Z(e,s(e))}$ and $t_{e^*} := 1_{Z(s(e),e)}$. Using the universal property of Leavitt path algebras [1, Remark 1.2.5], we get a homomorphism $\theta : L_R(E) \to A_R(G_E)$ such that

$$\theta(p_v) = q_v, \theta(s_e) = t_e, \text{ and } \theta(s_{e^*}) = t_{e^*}. \tag{3.2}$$

In [9, Example 3.2], it is shown that θ (denoted π in [9]) is an isomorphism.

Theorem 3.1.4 says that every Leavitt path algebra is a Steinberg algebra. We are concerned with the ideals lattice of Leavitt path algebras, so we include the following lemma. Again, this result is not new, but we have included for lack of reference.

Lemma 3.1.5. *Let A, B be algebras. Suppose there is a map $\phi : A \to B$ such that ϕ is an isomorphism between algebras. Then the lattices of ideals $\mathcal{L}(A)$ and $\mathcal{L}(B)$ are lattice isomorphic.*

Proof. Define $\Phi : \mathcal{L}(A) \to \mathcal{L}(B)$ such that $\Phi(I_A) = \phi(I_A)$, for $I_A \in \mathcal{L}(A)$. Since A and B are isomorphic, it is easy to show that Φ is a well defined bijection. So we will show that Φ preserves the lattice structure of the ideals lattices. Both $\mathcal{L}(A)$ and $\mathcal{L}(B)$ have the same order relation; containment. Suppose $I_A \subseteq I'_A$. Then $\phi(I_A) \subseteq \phi(I'_A)$ since ϕ is isomorphic. Thus $\Phi(I_A) \subseteq \Phi(I'_A)$. Likewise, if $\Phi(I_A) \subseteq \Phi(I_A)$, then $\phi(I_A) \subseteq \phi(I'_A)$ and hence $I_A \subseteq I'_A$. So, by Lemma 2.0.3, Φ is a lattice isomorphism. □

It follows from Theorem 3.1.4 and Lemma 3.1.5 that the ideal lattice of the Leavitt path algebra of a graph E is lattice isomorphic to the ideal lattice of the Steinberg algebra of G_E. We use this isomorphism when describing the ideal lattice of Leavitt path algebras (see Theorems 5.0.1 and 5.1.2).

Chapter 4

Ideal lattices of Steinberg algebras

In [6], Clark et al consider the ideal lattices of certain Steinberg algebras and Leavitt path algebras. Our results expand upon and use those in [6], so we include the relevant results in this chapter, in particular [6, Theorem 5.4] and [6, Theorem 6.1]. [6, Theorem 5.4] (see Theorem 4.0.2) describes the ideals Steinberg algebras of strongly effective groupoids over a commutative ring R with identity. [6, Theorem 6.1] (Theorem 4.0.3) describes the ideals of the Leavitt path algebra over R of a countable row-finite graph E with no sources that satisfies Condition (K). In the next chapter, we re-do [6, Theorem 6.1] for more general graphs and use [6, Theorem 5.4] to refine the lattice structure we use to describe $\mathcal{L}(L_R(E))$.

In this chapter we introduce three lattices from [6]: two that are isomorphic to the ideal lattice of the Steinberg algebra over R of a strongly effective groupoid ((4.1) and (4.2)); and one that is isomorphic to the ideal lattice of certain Leavitt path algebras over R. We will use the lattices (4.1) and (4.2) in the next chapter when describing the ideal lattice of Leavitt path algebras of graphs that satisfy Condition (K).

First, we explain the method used in [6] to describe ideals of Steinberg algebras. For an ample Hausdorff groupoid G that is strongly effective and a commutative ring R with identity, the authors describe a lattice $\mathcal{F}_A$ consisting of maps from the open invariant subsets of $G^{(0)}$ to $\mathcal{L}(R)$ (see (4.1) below). It is shown in [6, Theorem 4.4] that there is a bijection between this lattice and the lattice of ideals of a Steinberg algebra $A_R(G)$. While the bijection between these two lattices is order preserving, the resulting join is is difficult to describe explicitly. Thus, in [6, §5] Clark et al define another lattice (see (4.2) below) to solve this issue. We will apply both of these lattices to Leavitt path algebras associated to arbitrary graphs.

Let $\mathcal{O}$ be the set of all open invariant nonempty subsets of $G^{(0)}$. In [6, Theorem 4.4], the authors describe a set

$$\mathcal{F}_{\mathcal{A}} := \{\pi : \mathcal{O} \to \mathcal{L}(R) \mid \pi \text{ satisfies } \pi\big(\bigcup_{U\in\mathcal{A}} U\big) = \bigcap_{U\in\mathcal{A}} \pi(U)\ \forall\, \mathcal{A} \subseteq \mathcal{O}\} \tag{4.1}$$

and show that $\Gamma_A : \mathcal{F}_{\mathcal{A}} \to A_R(G)$ such that

$$\Gamma_A(\pi) = \operatorname{span}_R \bigcup_{U\in\mathcal{O}} \{rf \mid r \in \pi(U), f \in A_R(G), \operatorname{supp}(f) \subseteq G_U\}$$

is a bijection. The sets G_U and $\operatorname{supp}(f)$ are defined as

$$\begin{aligned} G_U &:= \{\gamma \in G \mid s(\gamma), r(\gamma) \in U\}, \quad \text{and} \\ \operatorname{supp}(f) &:= \{x \in R \mid f(x) \neq 0\}. \end{aligned}$$

In [6] $\mathcal{F}_A$ is denoted by $\mathcal{F}$. We introduce a lattice $\mathcal{F}_L$ based on $\mathcal{F}_A$ in Theorem 5.0.1. The notation for both uses $\mathcal{F}$ to emphasise their connection.

The bijection Γ_A gives rise to a lattice structure $(\mathcal{F}_A, \preceq)$ where

$$\pi_1 \preceq \pi_2 \iff \Gamma(\pi_1) \subseteq \Gamma(\pi_2).$$

However, it is difficult to describe the element $\pi_1 \vee \pi_2 \in \mathcal{F}_A$ such that $\Gamma_A(\pi_1 \vee \pi_2) = \Gamma_A(\pi_1) + \Gamma_A(\pi_2)$ (see [6, Example 5.1]). To address this difficulty, Clark et al develop another lattice that is isomorphic to $\mathcal{F}_A$:

$$\mathcal{F}'_A := \{\rho : G^{(0)} \to \mathcal{L}(R) \mid \rho \text{ is continuous and } G\text{-invariant}\}, \tag{4.2}$$

where ρ is continuous with respect to the topology on $\mathcal{L}(R)$ generated by the basis defined in Proposition 2.1.1. The relationship between two lattices $\mathcal{F}_A$ and $\mathcal{F}'_A$ is described in the following lemma.

Lemma 4.0.1. *[6, Lemma 5.3] Let G be an ample Hausdorff Groupoid and let R be a commutative ring with identity.*

(a) For any function $\rho : G^{(0)} \to \mathcal{L}(R)$, the function $\pi_\rho : \mathcal{O} \to \mathcal{L}(R)$ given by

$$\pi_\rho(U) = \bigcap_{u \in U} \rho(u)$$

satisfies

$$\pi_\rho\Big(\bigcup_{U \in \mathcal{A}} U\Big) = \bigcap_{U \in \mathcal{A}} \pi_\rho(U) \; \forall \, \mathcal{A} \subseteq \mathcal{O}.$$

(b) For any $\pi \in \mathcal{F}_A$, the formula

$$\rho_\pi(u) = \bigcup_{U \text{ open}, u \in U} \pi([U])$$

defines a G-invariant continuous function $\rho_\pi : \mathcal{O} \to \mathcal{L}(R)$.

(c) We have $\pi_{\rho_\pi} = \pi$ for $\pi \in \mathcal{F}_A$ and $\rho_{\pi_\rho} = \rho$ for $\rho \in \mathcal{F}'_A$. In particular, $\rho \mapsto \pi_\rho$ is a bijection from $\mathcal{F}'_A$ to $\mathcal{F}_A$.

The next result from [6] describes the lattice isomorphism between $\mathcal{F}'_A$ and $\mathcal{L}(A_R(G))$. We use Theorem 4.0.2 in the next chapter when showing that a particular lattice based on the vertices of a graph E is isomorphic to $\mathcal{L}(L_R(E))$ when E satisfies Condition (K).

Theorem 4.0.2. *[6, Theorem 5.4] Let G be an ample Hausdorff groupoid which is strongly effective, and let R be a commutative ring with identity. Let $\mathcal{F}'_A$ be the set of continuous G-invariant functions $\rho : G(0) \to \mathcal{L}(R)$. There is a bijection $\Gamma'_A : \mathcal{F}'_A \to \mathcal{L}(A_R(G))$ such that*

$$\Gamma'_A(\rho) = \mathit{span}_R\{r1_B \mid B \text{ is a compact open bisection and } r \in \cap_{u \in [s(B)]} \rho(u)\}.$$

Then $\Gamma'_A(\rho) = \Gamma_A(\pi_\rho)$ *for all* $\rho \in \mathcal{F}'_A$*. Define a relation* $\preceq$ *on* $\mathcal{F}'_A$ *by*

$$\rho_1 \leq \rho_2 \text{ if and only if } \rho_1(u) \subseteq \rho_2(u)$$

for all $u \in G^{(0)}$*. Then* $(\mathcal{F}'_A, \preceq)$ *is a lattice with join and meet operations given by*

$$\begin{aligned} &\rho_1 \vee \rho_2(u) = \rho_1(u) + \rho_2(u) \text{ and} \\ &\rho_1 \wedge \rho_2(u) = \rho_1(u) \cap \rho_2(u), \end{aligned}$$

and

$$\Gamma'_A : (\mathcal{F}'_A, \preceq) \to (\mathcal{L}(A_R(G)), \subseteq)$$

is a lattice isomorphism.

In [6, §6], Clark et al state a theorem that describes the ideals of certain Leavitt path algebras over a commutative ring with identity. The lattice $\mathcal{F}$ is based on $\mathcal{F}_A$.

Theorem 4.0.3. *[6, Theorem 6.1] Let* E *be a countable row-finite directed graph with no sources and let* R *be a commutative ring with identity. Suppose that* E *satisfies Condition (K).*

(a) Suppose that I *is an ideal in* $L_R(E)$*. Then*

$$I = \mathit{span}_R\{r s_\lambda s_{\mu^*} \mid r p_{s(\mu)} \in I\}.$$

(b) Let $\mathcal{H}_E$ *be the set of all saturated hereditary subsets of* E^0*, and let* $\mathcal{F}$ *be the set of all functions* $\pi : \mathcal{H}_E \to \mathcal{L}(R)$ *such that*

$$\pi\Big(\bigvee_{H \in \mathcal{A}} H\Big) = \bigcap_{H \in \mathcal{A}} \pi(H) \quad \text{for all } \mathcal{A} \subseteq \mathcal{H}_E.$$

Then the map $\Gamma : \mathcal{F} \to \mathcal{L}(L_R(E))$ *given by*

$$\Gamma(\pi) = \mathit{span}_R\{r s_\mu s_{\nu^*} \mid \exists H \in \mathcal{H}_E \text{ such that } r \in \pi(H) \text{ and } s_\mu s_{\nu^*} \in I_H\}$$

is a bijection.

(c) *Let* $\pi_1, \pi_2 \in \mathcal{F}$*. Then* $\Gamma(\pi_1) \subseteq \Gamma(\pi_2)$ *if and only if* $\pi_1(H) \subseteq \pi_2(H)$ *for all* $H \in \mathcal{H}_E$*.*

The set $\mathcal{F}$ is lattice isomorphic to $\mathcal{F}_{\mathcal{A}}$ since $\mathcal{O}$ is lattice isomorphic to $\mathcal{H}_E$ for row-finite graphs with no sources (the proof is analogous to that of [6, Lemma 6.4]). In the next chapter we re-do Theorem 4.0.3, but for arbitrary graphs satisfying Condition (K). When E is arbitrary, $\mathcal{O}$ is no longer lattice isomorphic to $\mathcal{H}_E$, so we must use a different graph lattice.

Chapter 5

Ideal lattices of Leavitt path algebras

This section contains our main results. First, in Theorem 5.0.1, we state a version of Theorem 4.0.3 for arbitrary graphs. To address singularities, we replace the set $\mathcal{H}_E$ of saturated hereditary subsets of E^0 with the set $\mathcal{T}_E$ of elements (H, S) where H is a saturated and hereditary subset and $S \subseteq B_H$.

Next, in Theorem 5.1.2 we use Theorem 4.0.2 to describe the ideals of the Leavitt path algebra of a graph E that satisfies Condition (K) over a commutative ring R with identity. The lattice we show to be isomorphic to $\mathcal{L}(L_R(E))$ has join and meet operations based on those of $\mathcal{L}(R)$, hence the join is nicely defined and relates well to the join of $\mathcal{L}(L_R(E))$. To illustrate Theorem 5.1.2, we apply it to three examples of Leavitt path algebras that each have singularities. This makes evident that the ideals of $L_R(E)$ are defined by the ideals in $\mathcal{L}(R)$. The final result of this section considers how R being a field affects Theorem 5.1.2.

Throughout this chapter, E refers to a directed graph and R is a commutative ring with identity. Since we have amassed a considerable amount of similar looking notation, we have included a table (Table 5) to aid the reader.

As we said above, in the following theorem we adapt Theorem 4.0.3

Table of notation	
Symbol	Definition
θ	The isomorphism $\theta : L_R(E) \to A_R(G_E)$, defined in (3.2), maps $p_v \mapsto q_v := 1_{Z(v)}$, $s \mapsto t_e := 1_{Z(e,s(e))}$, and $s_{e^*} \mapsto t_{e^*} := 1_{Z(s(e),e)}$.
Θ	The lattice isomorphism between ideal lattices of algebras that are isomorphic described in the proof of Lemma 3.1.5.
$\mathcal{T}_E$	The lattice consisting of elements (H,S) where H is a saturated and hereditary subset of E^0 and $S \subseteq B_H$.
$\mathcal{F}_L$	The set of maps $\zeta : \mathcal{T}_e \to \mathcal{L}(R)$ that satisfy (5.1).
Γ_L	A bijection from $\mathcal{F}_L$ to $\mathcal{L}(L_R(E))$
$\mathcal{O}$	The lattice of all open invariant nonempty subsets of $G^{(0)}$.
$\mathcal{F}_A$	A set consisting of maps $\pi : \mathcal{O} \to \mathcal{L}(R)$ (see (4.1)).
Γ_A	A bijection from $\mathcal{F}_A$ to $\mathcal{L}(A_R(G))$.
ϕ	The map $\phi : \mathcal{T}_E \to \mathcal{O}$ is given by $\phi((H,S)) = U_{H,S}$.
ρ	The inverse of ϕ.
Δ	A lattice isomorphism from $\mathcal{F}_L$ to $\mathcal{F}_A$.
$\mathcal{F}'_A$	A lattice of continuous G-invariant maps $\rho : \mathcal{O} \to \mathcal{L}(R)$. See (4.2).
Γ'_A	A lattice isomorphism from $\mathcal{F}'_A$ to $\mathcal{L}(A_R(G_E))$. See Theorem 4.0.2.
$\mathcal{F}'_L$	A lattice of maps $\tau : E^{(0)} \to \mathcal{L}(R)$ satisfying two conditions described in Lemma 5.1.1.
Γ'_L	A lattice isomorphism from $\mathcal{F}'_L$ to $\mathcal{L}(L_R(E))$.
Σ	A lattice isomorphism from $\mathcal{F}'_L$ to $\mathcal{F}'_A$ with inverse Υ.

Table 5.1: Since we have replicated the notation in [6], we have denoted the lattices and lattice isomorphisms that we have developed by subscript L. All mathematical objects with subscript A were proposed in [6]. The A refers to Steinberg algebras, and the L Leavitt path algebras.

for arbitrary graphs that satisfy Condition (K). We do this by replacing $\mathcal{H}_E$ with $\mathcal{T}_E$, the lattice of elements (H, S) where H is a saturated and hereditary subset and $S \subseteq B_H$. It is clear that the inclusion of singularities, infinite emitters in particular, makes for a more convoluted graph lattice – see [1, Proposition 2.5.6] for the lattice structure of $\mathcal{T}_E$.

Let H be a hereditary, saturated subset of E^0. For $v \in B_H$, we define an element v^H of $L_R(E)$ such that

$$v^H := v - \sum_{e \in r^{-1}(v) \cap s^{-1}(E^0 \setminus H)} ee^*.$$

We define $I_{(H,S)}$ as in [18, Definition 5.5]: if H is a saturated, hereditary subset of E^0 and $S \subseteq B_H$, let $I_{(H,S)}$ denote the ideal in $L_R(E)$ generated by $\{v \mid v \in H\} \cup \{v^H \mid v \in S\}$. In [18, Theorem 5.7] we see that $(H, S) \mapsto I_{(H,S)}$ is an isomorphism from the lattice $\mathcal{T}_E$ onto the graded ideals of $L_R(E)$.

Theorem 5.0.1. *Let E be a directed graph and let R be a commutative ring with identity. Suppose that E satisfies Condition (K).*

(a) Suppose that I is an ideal in $L_R(E)$. Then

$$I = \mathit{span}_R\{rs_\mu s_{\nu^*} \mid rp_{s(\mu)} \in I\}.$$

(b) Let $\mathcal{T}_E$ be the lattice consisting of elements (H, S) where H is a saturated and hereditary subset of E^0 and $S \subseteq B_H$, let $\mathcal{L}(R)$ be the set of ideals of R, and let $\mathcal{F}_L$ be the set of all functions $\zeta : \mathcal{T}_E \to \mathcal{L}(R)$ such that

$$\zeta\Big(\bigvee_{(H,S)\in\mathcal{A}} (H, S)\Big) = \bigcap_{(H,S)\in\mathcal{A}} \zeta((H, S)) \quad \text{for all } \mathcal{A} \subseteq \mathcal{T}_{\mathcal{E}}. \tag{5.1}$$

Then the map $\Gamma_L : \mathcal{F}_L \to \mathcal{L}(L_R(E))$ such that

$$\Gamma_L(\zeta) = \mathit{span}_R\{rs_\mu s_{\nu^*} \mid \exists (H, S) \in \mathcal{T}_E \text{ such that } r \in \zeta((H, S)), \\ s_\mu s_{\nu^*} \in I_{(H,S)}\}$$

is a bijection.

(c) *Let* $\zeta_1, \zeta_2 \in \mathcal{F}_L$*. Then* $\Gamma_L(\zeta_1) \subseteq \Gamma_L(\zeta_2)$ *if and only if* $\zeta_1((H,S)) \subseteq \zeta_2((H,S))$ *for all* $(H,S) \in \mathcal{T}_E$*.*

Proof. We require E to satisfy Condition (K) as then G_E is strongly effective [7, p. 8] and we can apply the Steinberg algebra results from [6]. To prove part (a) of Theorem 5.0.1 we apply [6, Proposition 4.1]. Suppose I is an ideal in $L_R(E)$. The Leavitt path algebra of E is isomorphic to the Steinberg algebra of G_E. Let $\theta : L_R(E) \to A_R(G_E)$ be the isomorphism from Theorem 3.1.4 such that $\theta(p_v) = 1_{Z(v)}$, $\theta(s) = 1_{Z(e,s(e))}$, and $\theta(s_{e^*}) = 1_{Z(s(e),e)}$. So $\theta(I)$ is an ideal in $A_R(G_E)$, and [6, Proposition 4.1] gives

$$\theta(I) = \text{span}_R\{r1_B \mid B \text{ is a compact open bisection and } r1_{s(B)} \in \theta(I)\}.$$

Fix $r1_B \in \theta(I)$. Then $B = \bigcup_{i=1}^n Z(\mu^i, \nu^i, F_i)$ where $\mu^i, \nu^i \in E^* s(\mu^i)$ for all $i \in \{1, \ldots, n\}$ and $F \subseteq s(\mu^i)E^1$ is finite. Fix $j, k \in \{1, \ldots, n\}$. By [15, Lemma 2.14], the intersection of two basis sets is $Z(\mu^j, \nu^j, F_j) \cap Z(\mu^k, \nu^k, F_k) =$

$$\begin{cases} Z(\mu^j, \nu^j, F_j \cup F_k) & \text{if } \mu^j = \mu^k, \nu^j = \nu^k \\ Z(\mu^j, \nu^j, F_j) & \text{if } \exists \kappa \in E^*, |\kappa| \geq 1, \mu^j = \mu^k\kappa, \nu^j = \nu^k\kappa, \kappa_1 \notin F_k \\ Z(\mu^k, \nu^k, F_k) & \text{if } \exists \kappa \in E^*, |\kappa| \geq 1, \mu^k = \mu^j\kappa, \nu^k = \nu^j\kappa, \kappa_1 \notin F_j \\ \emptyset & \text{otherwise.} \end{cases}$$

There are three cases where $Z(\mu^j, \nu^j, F_j) \cap Z(\mu^k, \nu^k, F_k) \neq \emptyset$. If

$$Z(\mu^j, \nu^j, F_j) \cap Z(\mu^k, \nu^k, F_k) = (Z(\mu^j, \nu^j, F_j),$$

then $Z(\mu^k, \nu^k, F_k) \subseteq Z(\mu^j, \nu^j, F_j)$. So we can remove $Z(\mu^k, \nu^k, F_k)$ from $\bigcup_{i=1}^n Z(\mu^i, \nu^i, F_i)$ without affecting B. We would then have

$$B = \bigcup_{i=1}^{k-1} Z(\mu^i, \nu^i, F_i) \bigcup_{i=k+1}^{n} Z(\mu^i, \nu^i, F_i).$$

Likewise, if

$$Z(\mu^j, \nu^j, F_j) \cap Z(\mu^k, \nu^k, F_k) = Z(\mu^k, \nu^k, F_k)$$

we can remove $Z(\mu^j, \nu^j, F_j)$. Lastly, if

$$Z(\mu^j, \nu^j, F_j) \cap Z(\mu^k, \nu^k, F_k) = Z(\mu^j, \nu^j, F_j \cup F_k)$$

then

$$Z(\mu^j, \nu^j, F_j) \cup Z(\mu^k, \nu^k, F_k) = Z(\mu^j, \nu^j, F_j \cap F_k).$$

So in this case we can replace $Z(\mu^j, \nu^j, F_j)$ and $Z(\mu^k, \nu^k, F_k)$ with

$$Z(\mu^j, \nu^j, F_j \cap F_k).$$

As a result, we can represent B as a disjoint union of basis sets. By the definition of the characteristic function, we then have $1_B = \sum_{i=1}^m 1_{Z(\mu^i,\nu^i,F_i)}$, so $r1_B$ is an element of

$J = \text{span}_R\{r1_{Z(\mu,\nu)} \mid Z(\mu, \nu)$ is a compact open bisection and $r1_{Z(\nu)} \in \theta(I)\})$.

Thus $\theta(I) \subseteq J$. Since $J \subseteq \theta(I)$, we have $J = \theta(I)$. Finally, we have

$$I = \theta^{-1}(J) = \text{span}_R\{rs_\mu s_{\nu^*} \mid rp_{s(\mu)} \in I\}.$$

Our result for part (b) applies [6, Theroem 4.4]. Recall $\mathcal{F}_A$ is the set of maps $\pi : \mathcal{O} \to \mathcal{L}(R)$ such that for all $\mathcal{B} \subseteq \mathcal{O}$

$$\pi(\bigcup_{U\in\mathcal{B}} U) = \bigcap_{U\in\mathcal{B}} \pi(U).$$

Then [6, Theroem 4.4] tells us that there is a bijection $\Gamma_A : \mathcal{F}_A \to \mathcal{L}(A_R(G_E))$ such that

$$\Gamma_A(\pi) = \text{span}_R\big(\bigcup_{U\in\mathcal{O}} \{rf \mid r \in \pi(U), f \in A_R(G_E), \text{supp}(f) \subset G_U\big)\}.$$

By Lemma 3.1.5, there is a lattice isomorphism $\Phi : \mathcal{L}(A_R(G_E)) \to \mathcal{L}(L_R(E))$ such that $\Phi(I) = \theta^{-1}(I)$, for $I \in \mathcal{L}(A_R(G_E))$ and $\theta^{-1} : A_R(G_E) \to L_R(E)$ is the inverse of θ. We want to show that there is a bijection between $\mathcal{F}_A$ and $\mathcal{F}_L$, and hence a bijection between $\mathcal{F}_L$ and $\mathcal{L}(L_R(E))$. The following diagram should help with clarity. We will find Δ.

$$\begin{array}{ccc} \mathcal{F}_A & \xrightarrow{\Gamma_A} & A_R(G_E) \\ \Delta \uparrow & & \downarrow \Phi \\ \mathcal{F}_L & \xrightarrow[\Gamma_L]{} & L_R(E) \end{array}$$

It is shown in [5] that the lattices $\mathcal{O}$ and $\mathcal{T}_E$ are isomorphic. We define an element of $\mathcal{O}$ from an element in $\mathcal{T}_E$ as follows: for $(H, S) \in \mathcal{T}_E$

$$U_H := \{x \in G_E^{(0)} \mid s(x_n) \in H \text{ for some } n\}$$

$$U_S := \{\alpha \in E^* \mid s(\alpha) \in S\}$$

$$U_{H,S} := U_H \cup U_S.$$

On the other hand, given $U \in \mathcal{O}$, define

$$H_U := \{v \in E^0 \mid Z(v) \subseteq U\}$$

$$S_U := \{r(\alpha) \mid \alpha \in U \cap E^*, s(\alpha) \text{ is an infinite emitter and } s(\alpha) \notin H_U\}.$$

Note that [5] uses the standard path convention, so the source and range maps have been interchanged to match our convention. Now [5, Theorem 3.3] says that the map $\phi : \mathcal{T}_E \to \mathcal{O}$ given by $\phi((H, S)) = U_{H,S}$ is a lattice isomorphism and has inverse $\rho : \mathcal{O} \to \mathcal{T}_E$ given by $\rho(U) = (H_U, S_U)$. So we see:

$$\begin{array}{ccc} \mathcal{O} & & \\ \phi \uparrow & \overset{\pi}{\searrow} & \\ \mathcal{T}_E & \xrightarrow[\zeta]{} & \mathcal{L}(R) \end{array}$$

for $\pi \in \mathcal{F}_A$, $\zeta \in \mathcal{F}_L$. Let

$$W := \{\pi \circ \phi \mid \pi \in \mathcal{F}_A\}.$$

We claim that $W = \mathcal{F}_L$. Fix $\pi \circ \phi \in W$. We have $\pi \circ \phi((H,S)) = \pi(U_{H,S})$. Since ϕ is a lattice isomorphism, for all $\mathcal{A} \subset \mathcal{T}_E$

$$\begin{aligned}
\pi \circ \phi\Big(\bigvee_{(H,S)\in\mathcal{A}} (H,S) \Big) &= \pi\Big(\bigcup_{(H,S)\in\mathcal{A}} U_{H,S} \Big) \\
&= \bigcap_{(H,S)\in\mathcal{A}} \pi(U_{H,S}) \\
&= \bigcap_{(H,S)\in\mathcal{A}} \pi(\phi((H,S)).
\end{aligned}$$

Thus, $\pi \circ \phi \in \mathcal{F}_L$, and hence $W \subseteq \mathcal{F}_L$. Next, fix $\zeta \in \mathcal{F}_L$. Then

$$\zeta\Big(\bigvee_{(H,S)\in\mathcal{A}} (H,S) \Big) = \bigcap_{(H,S)\in\mathcal{A}} \zeta((H,S)).$$

We then employ ρ, the inverse of ϕ, to get

$$\zeta\Big(\rho(\bigcup_{(H,S)\in\mathcal{A}}) \Big) = \bigcap_{(H,S)\in\mathcal{A}} \zeta(\rho(U_{H,S})) \quad \text{for all } \mathcal{A} \subseteq \mathcal{T}_E.$$

Thus the composition map $\zeta \circ \rho$ is in $\mathcal{F}_A$. Hence there is some $\pi \in \mathcal{F}_A$ such that $\zeta \circ \rho = \pi$. Since ρ is the inverse of ϕ, we rearrange to get $\zeta = \pi \circ \phi \in W$. Therefore, we can conclude that $\mathcal{F}_L = W$.

Let $\Delta : \mathcal{F}_L \to \mathcal{F}_A$ be given by $\Delta(\pi \circ \phi) = \pi$. This is clearly surjective: for each $\pi \in \mathcal{F}_A$ there is a $\pi \circ \phi \in \mathcal{F}_L$ such that $\Delta(\pi \circ \phi) = \pi$. Suppose $\Delta(\pi \circ \phi) = \Delta(\pi' \circ \phi)$ for $\pi, \pi' \in \mathcal{F}_A$. Then $\pi = \pi'$, so $\pi \circ \phi = \pi' \circ \phi$. Thus, Δ is a bijection. Define $\tilde{\Gamma}_L := \Delta \circ \Gamma_A \circ \Theta$. Then $\Gamma_L = \tilde{\Gamma}_L$ is a bijection such that

$$\Gamma_L(\zeta) = \operatorname{span}_R\{r s_\mu s_{\nu^*} \mid \exists (H,S) \in \mathcal{T}_E \text{ such that } r \in \zeta((H,S)), s_\mu s_{\nu^*} \in I_{(H,S)}\}.$$

Part (c) follows from [6, Lemma 4.5]. $\square$

The bijection Γ_L induces a lattice structure $(\mathcal{F}_L, \preceq)$ via

$$\zeta_1 \preceq \zeta_2 \iff \Gamma_l(\zeta_1) \subseteq \Gamma_L(\zeta_2),$$

such that

$$\begin{aligned}\Gamma_L(\zeta_1 \vee \zeta_2) &= \Gamma_L(\zeta_1) + \Gamma_L(\zeta_2), \text{ and} \\ \Gamma_L(\zeta_1 \wedge \zeta_2) &= \Gamma_L(\zeta_1) \cap \Gamma(\zeta_2).\end{aligned}$$

So we do have a lattice structure on $\mathcal{F}_L$, but $\mathcal{F}_L$ inherits from $\mathcal{F}_A$ the same difficulty in describing $\zeta_1 \vee \zeta_2$ such that $\Gamma_L(\zeta_1 \vee \zeta_2) = \Gamma_L(\zeta) + \Gamma_L(\zeta)$. In [6, Example 5.1] we see that the map $g : \mathcal{O} \to \mathcal{L}(R)$ given by $g(U) = \pi_1(U) + \pi_2(U)$, which is the natural guess for the join, does not necessarily satisfy $\Gamma_A(\pi_1 \vee \pi_2) = \Gamma_A(\pi_1) + \Gamma_A(\pi_2)$, nor is g necessarily in $\mathcal{F}_A$. Considering that $g \circ \phi$ would not be in $\mathcal{F}_L$ if g is not in $\mathcal{F}_A$, $g \circ \phi$ cannot the join of $\mathcal{F}_L$. We also note that our description of the ideals of $L_R(E)$ is somewhat recursive as it relies on another description of the ideals. To remedy this, we provide another description of the ideals of $L_R(E)$ by applying Theorem 4.0.2.

Remark 5.0.2. While Theorem 4.0.3 requires that E be a countable graph, we found that this is not required when proving Theorem 5.0.1. Hence we have not only broadened the scope to include singularities, but graphs that are not countable as well. It is likely that this difference arose because we used Theorem 3.0.13, which does not require E to be countable.

5.1 Refining the join operation

In Theorem 4.0.2 we see that, for a groupoid G and an arbitrary commutative ring with identity R, the set $\mathcal{F}'_A$ of G-invariant continuous functions $\rho : G^{(0)} \to \mathcal{L}(R)$ is lattice isomorphic to the ideal lattice of the Steinberg algebra of G. We establish a lattice $\mathcal{F}'_L$ that is based on the vertices of a directed graph and show that it is lattice isomorphic to $\mathcal{F}'_A$, and hence is isomorphic to $\mathcal{L}(L_R(E))$. We say $v \in E^0$ *is on* $x \in \partial E$ if there is an i such that $v = s(x_i)$, $v = r(x_i)$ or both.

Lemma 5.1.1. *Let E be a directed graph satisfying Condition (K) and let R be a commutative ring with identity. Let $\mathcal{F}'_A$ be the set of continuous, G_E-invariant functions $\rho : G_E^{(0)} \to \mathcal{L}(R)$ and let $\pi_\rho : \mathcal{O} \to \mathcal{L}(R)$ be the map defined in of Lemma 4.0.1, a). Define $\mathcal{F}'_L$ be the set of functions $\tau : E^0 \to \mathcal{L}(R)$ satisfying:*

1. *If $v, w \in E^0$ such that $v \leq w$, then $\tau(v) \subseteq \tau(w)$; and*
2. *For $v \in E^0$*
$$\tau(v) = \bigcap_{x \in Z(v)} \Big(\bigcup_{w \text{ on } x} \tau(w) \Big). \tag{5.2}$$

Then there exists a bijection $\Upsilon : \mathcal{F}'_A \to \mathcal{F}'_L$ such that

$$\Upsilon(\rho) = \tilde{\rho}, \ \ \textit{where} \ \ \tilde{\rho}(v) = \pi_\rho(Z(v)).$$

Proof. By definition
$$\pi_\rho(U) = \bigcap_{u \in U} \rho(u).$$

So
$$\tilde{\rho}(v) = \bigcap_{x \in Z(v)} \rho(x).$$

where $Z(v)$ is a cylinder set in ∂E. Before we show that Υ is a bijection, we must show that $\tilde{\rho}$ is in $\mathcal{F}'_L$. That is, we must show that $\tilde{\rho}$ satisfies conditions 1 and 2 above. Fix $v, w \in E^0$. Suppose $v \leq w$. Then $Z(w) \subseteq Z(v)$. Hence

$$\tilde{\rho}(v) = \bigcap_{x \in Z(v)} \rho(x) \subseteq \bigcap_{y \in Z(w)} \rho(y) = \tilde{\rho}(w).$$

Thus, $\tilde{\rho}$ satisfies condition 1. Fix $v \in E^0$. We show that $\tilde{\rho}$ satisfies condition 2. That is, we show

$$\tilde{\rho}(v) = \bigcap_{x \in Z(v)} \Big(\bigcup_{w \text{ on } x} \tilde{\rho}(w) \Big),$$

which expands to

$$\bigcap_{z \in Z(v)} \rho(z) = \bigcap_{x \in Z(v)} \Big(\bigcup_{w \text{ on } x} \Big(\bigcap_{y \in Z(w)} \rho(y) \Big) \Big). \tag{5.3}$$

First, fix $a \in \bigcap_{z \in Z(v)} \rho(z)$ and fix $x \in Z(v)$. Let $w = r(x) = v$. Fix $y \in \mathbb{Z}(w) = Z(v)$. Then $a \in \rho(y)$, since $y \in Z(v)$. So we have containment one way. Next, fix

$$a \in \bigcap_{x \in Z(v)} \Big(\bigcup_{w \text{ on } x} \Big(\bigcap_{y \in Z(w)} \rho(y) \Big) \Big).$$

Fix $z \in Z(v)$. By definition, there is some w on z such that $a \in \rho(y)$ for every $y \in Z(w)$. We have $z = \mu z'$ where $s(\mu) = w$. Then $z' \in Z(w)$. Hence $a \in \rho(z')$. But, since ρ is an invariant map, we have $\rho(z') = \rho(z)$. So $a \in \rho(z)$ for some $z \in Z(v)$, and thus $a \in \bigcap_{z \in Z(v)} \rho(z)$.

We define a map in the inverse direction: define $\Sigma : \mathcal{F}'_L \to \mathcal{F}'_A$ such that, for $\tau \in \mathcal{F}'_L$,

$$\Sigma(\tau) = \tilde{\tau}, \quad \tilde{\tau}(x) = \bigcup_{w \text{ on } x} \tau(w).$$

Since there exists a path from $r(x)$ to each w on x, we have $\tau(r(x)) \subseteq \tau(w)$. Hence $\cup_{w \text{ on } x} \tau(w)$ is an ideal in R.

To see that Υ is a bijection, we show that $\Upsilon(\Sigma(\tau)) = \tau$ and $\Sigma(\Upsilon(\rho)) = \rho$. That is, we show that Σ is the inverse of Υ. Fix $\tau \in \mathcal{F}'_L$. Then

$$\Upsilon(\Sigma(\tau(v))) = \bigcap_{x \in [Z(v)]} \tilde{\tau}(x) = \bigcap_{x \in Z(v)} \Big(\bigcup_{w \text{ on } x} \tau(w) \Big)$$

for all $v \in E^0$. Notice that this gives us condition 2 of $\mathcal{F}'_L$, so $\Upsilon(\Sigma(\tau)) = \tau$.

From (5.3), we see that $\rho(x) = \bigcup_{w \text{ on } x} \bigcap_{y \in Z(w)} \rho(y)$. Hence

$$\tilde{\tilde{\rho}}(x) = \bigcup_{w \text{ on } x} \tilde{\rho}(w) = \bigcup_{w \text{ on } x} \Big(\bigcap_{y \in Z(w)} \rho(y) \Big) = \rho(x).$$

Thus, $\Sigma(\Upsilon(\rho)) = \rho$ and $\Upsilon(\Sigma(\tau)) = \tau$ for all $\tau \in \mathcal{F}'_L$ and $\rho \in \mathcal{F}'_A$. So Υ is bijective with bijective inverse Σ. □

The following theorem fully describes the lattice structure of ideals of Leavitt path algebras over R associated to graphs that satisfy Condition (K). In particular, the join operation is easy to define in $\mathcal{F}'_L$. Recall that $\Phi : \mathcal{L}(A_R(G_E)) \to \mathcal{L}(L_R(E))$ such that $\Phi(I) = \theta^{-1}(I)$ is a lattice isomorphism and θ is the isomorphism between a Leavitt path algebra and a Steinberg algebra defined in [9, Example 3.2].

Theorem 5.1.2. *Let R be a commutative ring with identity and let E be a directed graph. Let $\mathcal{F}'_L$ be the set defined in Proposition 5.1.1. Suppose E satisfies Condition (K). Define a map $\Gamma'_L : \mathcal{F}'_L \to \mathcal{L}_R(E)$ such that $\Gamma'_L := \Phi \circ \Gamma'_A \circ \Sigma$. Define a relation $\leq$ on $\mathcal{F}'_L$ such that*

$$\tau_1 \leq \tau_2 \text{ if and only if } \tau_1(v) \subseteq \tau_2(v) \text{ for all } v \in E^0.$$

Then $(\mathcal{F}'_L, \leq)$ is a lattice with join and meet operations given by

$$(\tau_1 \vee \tau_2)(v) = \tau_1(v) + \tau_2(v) \tag{5.4}$$

$$(\tau_1 \wedge \tau_2)(v) = \tau_1(v) \cap \tau_2(v), \tag{5.5}$$

and $\Gamma'_L : (\mathcal{F}'_L, \leq) \to (\mathcal{L}_R(E), \subseteq)$ is a lattice isomorphism with

$$\Gamma'_L(\tau) = \mathit{span}_R\{rs_\mu s_{\nu^*} \mid \mu, \nu \in E^* s(\nu), r \in \tau(s(\nu))\}.$$

Proof. Again, we require E to satisfy Condition (K) so that G_E is strongly effective and we can apply Theorem 4.0.2. To see that $(\mathcal{F}'_L, \leq)$ is a lattice, we want to check that $\leq$ is a partial order and that both $\tau_1 \vee \tau_2$ and $\tau_1 \wedge \tau_2$ are in $\mathcal{F}'_L$ for all $\tau_1, \tau_2 \in \mathcal{F}'_L$. It is straightforward to show that $\leq$ is a partial order: fix $\tau_1, \tau_2, \tau_3 \in \mathcal{F}'_L$. Then, for all $v \in E^0$, we have $\tau_1(v) \subseteq \tau_1(v)$ hence $\tau_1 \leq \tau_1$. Suppose $\tau_1 \leq \tau_2$ and $\tau_2 \leq \tau_1$. Then $\tau_1(v) \subseteq \tau_2(v)$ and $\tau_2(v) \subseteq \tau_1(v)$ for all $v \in E^0$. So $\tau_1(v) = \tau_2(v)$ for all $v \in E^0$. Thus $\tau_1 = \tau_2$. Lastly, we see that $\leq$ is transitive:

$$\begin{aligned}
\tau_1 \leq \tau_2 \text{ and } \tau_2 \leq \tau_3 &\iff \tau_1(v) \subseteq \tau_2(v) \text{ and } \tau_2(v) \subseteq \tau_3(v) \text{ for all } v \in E^0 \\
&\implies \tau_1(v) \subseteq \tau_3(v) \text{ for all } v \in E^0 \\
&\implies \tau_1 \leq \tau_3.
\end{aligned}$$

It is clear that $\tau_1 \vee \tau_2$ and $\tau_1 \wedge \tau_2$ both map to ideals of R, so we need to check that they satisfy the two conditions in Lemma 5.1.1. Fix $\tau_1, \tau_2 \in \mathcal{F}'_L$. Suppose $v \leq w$. Then $\tau_1(v) \subseteq \tau_1(w)$ and $\tau_2(v) \subseteq \tau_2(w)$. So

$$(\tau_1 \vee \tau_2)(v) = \tau_1(v) + \tau_2(v) \subseteq \tau_1(w) + \tau_2(w) = (\tau_1 \vee \tau_2)(w),$$

and

$$(\tau_1 \wedge \tau_2)(v) = \tau_1(v) \cap \tau_2(v) \subseteq \tau_1(w) \cap \tau_2(w) = (\tau_1 \wedge \tau_2)(w).$$

Hence $\tau_1 \vee \tau_2$ and $\tau_1 \wedge \tau_2$ satisfy the first condition of $\mathcal{F}'_L$. Next we check the second condition. Since τ_1 and τ_2 satisfy condition 2, we see that

$$\begin{aligned}
(\tau_1 \vee \tau_2)(v) &= \tau_1(v) + \tau_2(v) \\
&= \bigcap_{x \in Z(v)} \big[\bigcup_i \tau_1(r(x_i))\big] + \bigcap_{x \in Z(v)} \big[\bigcup_j \tau_2(r(x_j))\big] \\
&= \bigcap_{x \in Z(v)} \big[\bigcup_i \big(\tau_1(r(x_i)) + \tau_2(r(x_i))\big)\big] \\
&= \bigcap_{x \in Z(v)} \big[\bigcup_i (\tau_1 \vee \tau_2)(r(x_i))\big],
\end{aligned}$$

and

$$\begin{aligned}
(\tau_1 \wedge \tau_2)(v) &= \tau_1(v) \cap \tau_2(v) \\
&= \bigcap_{x \in Z(v)} \big[\bigcup_i \tau_1(r(x_i))\big] \cap \bigcap_{x \in Z(v)} \big[\bigcup_j \tau_2(r(x_j))\big] \\
&= \bigcap_{x \in Z(v)} \big[\bigcup_i \big(\tau_1(r(x_i)) \cap \tau_2(r(x_i))\big)\big] \\
&= \bigcap_{x \in Z(v)} \big[\bigcup_i (\tau_1 \wedge \tau_2)(r(x_i))\big].
\end{aligned}$$

So $(\mathcal{F}'_L, \leq)$ is indeed a lattice.

Recall that $\Sigma : \mathcal{F}'_L \to \mathcal{F}'_A$ maps $\Sigma(\tau) = \tilde{\tau}$ where $\tilde{\tau}(x) = \bigcup_{w \text{ on } x} \tau(w)$ for $x \in \partial E$. In Lemma 5.1.1 we saw that Σ is bijective. Fix $\tau_1, \tau_2 \in \mathcal{F}_L$. We show that

$$\tau_1 \leq \tau_2 \iff \Sigma(\tau_1) \leq \Sigma(\tau_2). \tag{5.6}$$

That is, we show that

$$\tau_1(v) \subseteq \tau_2(v) \iff \bigcup_{w \text{ on } x} \tau_1(w) \subseteq \bigcup_{w \text{ on } x} \tau_2(w)$$

for all $v \in E^0$ and all $x \in \partial E$. Suppose $\tau_1(v) \subseteq \tau_2(v)$ for all $v \in E^0$. Fix $x \in \partial E$ and fix $a \in \cup_{w \text{ on } x} \tau_1(w)$. Then there is some w on x such that

$a \in \tau_1(w)$. Since $\tau_1(v) \subseteq \tau_2(v)$ for all $v \in E^0$, $a \in \tau_2(w)$. Thus, for all $x \in \partial E$, we have

$$\bigcup_{w \text{ on } x} \tau_1(w) \subseteq \bigcup_{w \text{ on } x} \tau_2(w).$$

Next, suppose

$$\bigcup_{w \text{ on } x} \tau_1(w) \subseteq \bigcup_{w \text{ on } x} \tau_2(w) \quad \text{for all } x \in \partial E.$$

Fix $v \in E^0$. Then

$$\bigcap_{x \in Z(v)} \Big(\bigcup_{w \text{ on } x} \tau_1(w) \Big) \subseteq \bigcap_{x \in Z(v)} \Big(\bigcup_{w \text{ on } x} \tau_2(w) \Big).$$

So, by (5.2), $\tau_1(v) \subseteq \tau_2(v)$. Thus, Σ is bijective and satisfies (5.6), and hence Lemma 2.0.3 tells us that Σ is a lattice isomorphism.

The following diagram displays how $\Gamma'_L = \Sigma \circ \Gamma'_A \circ \Phi$ is constructed.

$$\begin{tikzcd}
\mathcal{F}'_A \arrow[r, "\Gamma'_A"] & A_R(G_E) \arrow[d, "\Phi"] \\
\mathcal{F}'_L \arrow[u, "\Sigma"] \arrow[r, "\Gamma'_L"'] & L_R(E)
\end{tikzcd}$$

In Theorem 4.0.2, Clark et al show that $\Gamma'_A : \mathcal{F}'_A \to \mathcal{L}(A_R(G))$ is a lattice isomorphism. So Γ'_L is composed of lattice isomorphisms, hence is a lattice isomorphism itself. Thus, the ideals of $L_R(E)$ are

$$\Gamma'(\tau) = \Phi(\Gamma'_A(\tilde{\tau})) = \text{span}_R\{rs_\mu s_{\nu^*} \mid \mu, \nu \in E^* s(\nu), r \in \bigcap_{x \in [s(Z(\mu,\nu))]} \tilde{\tau}(x)\}.$$

This can be simplified. Firstly, we have

$$\begin{aligned} s(Z(\mu, \nu)) &= \{s((\mu x, |\mu| - |\nu|, \nu x)) \mid x \in s(\nu)\partial E\} \\ &= \{(\nu x, 0, \nu x) \mid x \in s(\nu)\partial E\} \\ &= Z(\nu). \end{aligned}$$

Since $\tilde{\tau}$ is invariant,

$$\{\tilde{\tau}(r(\gamma)) : r(\gamma) \in Z(\nu)\} = \{\tilde{\tau}(s(\gamma)) : s(\gamma) \in Z(\nu)\}.$$

Thus,

$$\tilde{\tau}(Z(\nu)) = \{\tilde{\tau}(s(\gamma)) \mid s(\gamma) \in Z(\nu)\} = \{\tilde{\tau}(r(\gamma)) \mid s(\gamma) \in Z(\nu)\} = \tilde{\tau}([Z(\nu)].$$

So we have

$$\bigcap_{x \in [Z(\nu)]} \tilde{\tau}(x) = \bigcap_{x \in Z(\nu)} \tilde{\tau}(x).$$

Fix $x \in Z(\nu)$. Then there is some $y \in Z(s(\nu))$ such that $x = \nu y$. By invariance we have $\tilde{\tau}(x) = \tilde{\tau}(s(\nu)y)$. Thus we get

$$\bigcap_{x \in Z(\nu)} \tilde{\tau}(x) = \bigcap_{y \in Z(s(\nu))} \tilde{\tau}(y).$$

This culminates to

$$\begin{aligned} \bigcap_{x \in [s(Z(\mu,\nu))]} \tilde{\tau}(x) &= \bigcap_{x \in Z(\nu)} \tilde{\tau}(x) \\ &= \bigcap_{y \in Z(s(\nu))} \tilde{\tau}(y) \\ &= \bigcap_{y \in Z(s(\nu))} \Bigg(\bigcup_{v \text{ on } y} \tau(v) \Bigg). \end{aligned}$$

We then apply the second condition of Lemma 5.1.1 to get

$$\bigcap_{y \in Z(s(\nu))} \bigcup_{v \text{ on } y} \tau(v) = \tau(s(\nu)).$$

So the ideals of $L_R(E)$ are of the form

$$\Gamma'_L(\tau) = \text{span}_R\{r s_\mu s_{\nu^*} \mid \mu, \nu \in E^* s(\nu), r \in \tau(s(\nu))\}. \qquad \square$$

Our motivation for Theorem 5.1.2 is not only to find all the ideals of $L_R(E)$, but to provide a convenient lattice structure on E such that join easily corresponds to that of $\mathcal{L}(L_R(E))$. We achieved this by finding a lattice of maps from E^0 to $\mathcal{L}(R)$ whose lattice structure was based on that of $\mathcal{L}(R)$. We see that

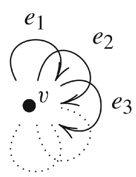
$$\begin{aligned}\Gamma(\tau_1) \vee \Gamma(\tau_2) &= \operatorname{span}_R\{rs_\mu s_{\nu^*} \mid \mu, \nu \in E^* s(\nu),\ r \in \tau_1(s(\nu)) + \tau_2(s(\nu))\} \\ &= \Gamma'_L(\tau_1 \vee \tau_2),\end{aligned}$$

which achieves our objective. Theorem 5.1.2 also makes clear that the ideals in $L_R(E)$ are dictated by those in $\mathcal{L}(R)$.

Remark 5.1.3. There is another way we could have approached finding the ideals of Leavitt path algebras of arbitrary graphs. *Drinen-Tomforde desingularisation* is a C^*-algebra technique used to extend results from algebras of row-finite graphs to algebras of countable graphs. If E is a graph, a desingularisation of E is a row-finite graph F such that the algebra of E is Morita equivalent to the algebra of F [11]. In [3, Theorem 5.6], Abrams and Pino show that this technique does indeed apply to Leavitt path algebras, so we could have tried to extend the results of [6] to arbitrary graphs via this method. The drawback of such a technique is that it is less direct and requires the graph to be countable.

We apply our results to three examples of Leavitt path algebras for illustrative purposes.

Example 5.1.4. Let E be the rose with one petal for each element of $\mathbb{N}$, pictured below (image from [1, Example 1.6.13]). We consider the ideals of $L_{\mathbb{Z}}(E)$.



Such a graph only has one vertex, v. Consequently, any map $\tau : E^0 \to \mathcal{L}(\mathbb{Z})$ trivially satisfies conditions 1 and 2 of Lemma 5.1.1. We can then surmise that

$$\mathcal{F}'_L = \{\tau_n : E^0 \to \mathcal{L}(\mathbb{Z}) \mid n \in \mathbb{N}, \tau_n(v) = n\mathbb{Z}\}.$$

The join and the meet of $\mathcal{F}_L$ are then

$$\begin{aligned}(\tau_{n_1} \vee \tau_{n_2})(v) &= n_1\mathbb{Z} + n_2\mathbb{Z} = \text{lcm}(n_1, n_2)\mathbb{Z}, \text{ and} \\ (\tau_{n_1} \vee \tau_{n_2})(v) &= n_1\mathbb{Z} \cap n_2\mathbb{Z} = \text{gcd}(n_1, n_2)\mathbb{Z}.\end{aligned}$$

The ideal corresponding to $\tau_n \in \mathcal{F}'_L$ is

$$\Gamma'_L(\tau_n) = \text{span}_{\mathbb{Z}}\{r s_\mu s_{\nu^*} \mid \mu, \nu \in E^*,\ n|r\} = \text{span}_{n\mathbb{Z}}\{s_\mu s_{\nu^*} \mid \mu, \nu \in E^*\}.$$

We have $\Gamma'_L(\tau_n) \subseteq \Gamma'_L(\tau_m)$ if and only if $m|n$. Theorem 5.1.2 shows that this describes all the ideals of $L_{\mathbb{Z}}(E)$. Note that this means that the ideals of $L_{\mathbb{Z}}(E)$ directly correspond to the ideals of $\mathbb{Z}$.

Example 5.1.4 makes clear an interesting outcome of Theorem 5.1.2: since r is in an ideal of R, we are not spanning over the whole of R, but that ideal.

Example 5.1.5. Let E be the directed graph with two sources pictured below.

$$\begin{tikzcd}[row sep=small] & w_1 \arrow[dl, "e_1"'] \\ v & \\ & w_2 \arrow[ul, "e_2"] \end{tikzcd}$$

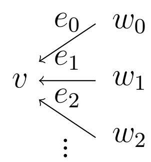
We consider the ideals of $L_{\mathbb{Z}}(E)$. We start by describing $\mathcal{F}_{\mathcal{L}}'$. Fix $\tau \in \mathcal{F}'_L$. Suppose $\tau(v) = n\mathbb{Z}$, $\tau(w_1) = m_1\mathbb{Z}$ and $\tau(w_2) = m_2\mathbb{Z}$ for $n, m_1, m_2 \in \mathbb{N}$. Then condition 1 of Lemma 5.1.1 says that $\tau(v) \subseteq \tau(w_1)$ and $\tau(v) \subseteq \tau(w_2)$. Condition 2 then becomes

$$\begin{aligned}\tau(v) &= \bigcap_{i\in\{1,2\}} \big(\tau(v) \cup \tau(w_i)\big) \\ &= \bigcap_{i\in\{1,2\}} \big(\tau(w_i)\big) \\ &= \tau(w_1) \cap \tau(w_2).\end{aligned}$$

Thus $n = \text{lcm}(m_1, m_2)$. The corresponding ideal is

$$\begin{aligned}\Gamma'_L(\tau) = \text{span}_{\mathbb{Z}}\{\, & rp_{w_1} \mid r \in m_1\mathbb{Z},\\ & rp_{w_2} \mid r \in m_2\mathbb{Z},\\ & rp_v \mid r \in \text{lcm}(m_1, m_2)\mathbb{Z},\\ & rs_{e_1} \mid r \in m_1\mathbb{Z},\\ & rs_{e_2} \mid r \in m_2\mathbb{Z},\\ & rs_{e_1}s_{e_1^*} \mid r \in m_1\mathbb{Z},\\ & rs_{e_2}s_{e_2^*} \mid r \in m_2\mathbb{Z},\\ & rs_{e_1^*} \mid r \in \text{lcm}(m_1, m_2)\mathbb{Z},\\ & rs_{e_2^*} \mid r \in \text{lcm}(m_1, m_2)\mathbb{Z}\}.\end{aligned}$$

Example 5.1.6. Let E be the graph with an infinite receiver v pictured below.



Suppose $\tau(v) = n\mathbb{Z}$ and $\tau(w_i) = m_i$ for $i \in \mathbb{N}$. As in Example 5.1.5, $\tau(v) \subseteq \tau(w_i)$ for all $i \in \mathbb{N}$. So condition 2 of Lemma 5.1.2 becomes

$$\tau(v) = \bigcap_{i \in \mathbb{N}} \tau(w_i).$$

Thus, $n = \text{lcm}(\{m_i \mid i \in \mathbb{N}\})$. Note, if one $\tau(m_i) = 0$, so does $\tau(v)$. The ideals of the Leavitt path algebra of E over $\mathbb{Z}$ follow the same format as in Example 5.1.5.

Corollary 5.1.7. *Let K be a field and E be a directed graph. Suppose E satisfies Condition (K). Then, for some $\tau \in \mathcal{F}'_L$, $\Gamma'_L(\tau)$ is either $\{0\}$ or*

$$\mathit{span}_K\{s_\mu s_{\nu^*} \mid \mu, \nu \in E^* s(\nu), \tau(s(\nu)) = K\}.$$

Proof. There are no non-trivial ideals in K since it is a field, so τ maps each vertex to either $\{0\}$ or K. If $\tau(s(\nu)) = \{0\}$, then $r = 0$. So the only non-zero elements of an ideal occur when $\tau(s(\nu)) = K$. $\square$

Future work will investigate how the lack of non-trivial ideals in K allows us to simplify the conditions of $\mathcal{F}'_L$.

Bibliography

[1] ABRAMS, G., ARA, P., AND MOLINA, M. S. *Leavitt Path Algebras*. Lecture Notes in Mathematics. Springer-Verlag, London, 2017.

[2] ABRAMS, G., AND PINO, G. A. The Leavitt path algebra of a graph. *J Algebra 293* (2005), 319–334.

[3] ABRAMS, G., AND PINO, G. A. The Leavitt path algebras of arbitrary graphs. *Houston Journal of Mathematics 34*, 2 (2008).

[4] ARA, P., MA, M., AND E, P. Nonstable k-theory for graph algebras. *Algebra Representation Theory 10*, 2 (2007), 157–178.

[5] CLARK, L. O., BARQUERO, D. M., GONZALEZ, C. M., AND MOLINA, M. S. Using Steinberg algebras to study decomposability of Leavitt path algebras. *Forum Mathematica 29*, 6 (2017), 1311–1324.

[6] CLARK, L. O., EDIE-MICHELL, C., HUEF, A. A., AND SIMS, A. Ideals of Steinberg algebras of strongly effective groupoids. *Transactions of the American Mathematical Society 371*, 8 (2019), 5461–5486.

[7] CLARK, L. O., AND HAZRAT, R. *Leavitt Path Algebras and Classical K-Theory*. Springer, 2020, ch. Etale groupoids and Steinberg algebras a concise introduction.

[8] CLARK, L. O., OCHOA, J., AND ANDERSON, S. Point-set topology. *Journal of Inquiry-Based Learning in Mathematics 41* (2016).

[9] CLARK, L. O., AND SIMS, A. Equivalent groupoids have Morita equivalent Steinberg algebras. *Journal of pure and applied algebra 219*, 8 (2015), 2062–2075.

[10] DAVEY, B., AND PRIESTLY, H. *Introduction to Lattices and Order*, 2 ed. Cambridge University Press, 2002.

[11] DRINEN, D., AND TOMFORDE, M. The C^*-algebras of arbitrary graphs. *The Rocky Mountain journal of mathematics 35*, 1 (2005), 105–135.

[12] HUNGERFORD, T. W. *Algebra*, 2 ed. Graduate Texts in Mathematics. Springer-Verlag, New York, 1974.

[13] KANUNI, M., AND SERT, S. A survey on the ideal structure of Leavitt path algebras. *Cornell University* (2018).

[14] RAEBURN, I. *Graph Algebras*. No. 103 in CBMS Regional Conference Series in Mathematics. The American Mathematical Society, 2005.

[15] RIGBY, S. W. The groupoid approach to Leavitt path algebras. Master's thesis, Ghent University, 2019.

[16] RIGBY, S. W., AND VAN DEN HOVE, T. A classification of ideals in Steinberg algebras and Leavitt path algebras over arbitrary rings. *Journal of Algebra 588* (2021), 200–249.

[17] SIMS, A. Hausdorff *é*tale groupoids and their C^*-algebras. *University of Wollongong* (2019).

[18] TOMFORDE, M. Uniqueness theorems and ideal structure for Leavitt path algebras. *J. Algebra 318*, 1 (2007), 270–299.

[19] WEBSTER, S. B. G. The path space of a directed graph. *Proc. Amer. Math. Soc. 142*, 1 (2014), 213–225.