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Computability-theoretic complexity of effective Banach spaces

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Abstract

We investigate the geometry of effective Banach spaces, namely a sequence of approximation properties that lies in between a Banach space having a basis and the approximation property. We establish some upper bounds on such properties, as well as proving some arithmetical lower bounds. Unfortunately, the upper bounds obtained in some cases are far away from the lower bound. However, we will show that much tighter bounds will require genuinely new constructions, and resolve long-standing open problems in Banach space theory. We also investigate the effectivisations of certain classical theorems in Banach spaces.

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Contents

Chapter 1

Introduction

1.1 Banach spaces

Banach spaces are fundamental to the field of functional analysis, with diverse applications in areas such as differential and integral equations, analysis, optimisation theory and many others [Kal07, Bot14, XGLS14]. Despite the complex and rich theory that comes from Banach spaces, the notion of a Banach space is sufficiently simple to be understood by beginners. We remind the reader that a *normed vector space* is simply a vector space V over a field F (usually $\mathbb{R}$ or $\mathbb{C}$, we will exclusively work with $\mathbb{R}$ in this thesis) endowed with a norm operator $\|\cdot\| : V \times V \to \mathbb{R}$ such that it satisfies the following axioms for all $x, y \in V$, $\lambda \in \mathbb{R}$:

- $\|x + y\| \leq \|x\| + \|y\|$
- $\|\lambda x\| = |\lambda| \, \|x\|$
- $\|x\| \geq 0$, $\|x\| = 0 \iff x = 0$.

Normed vector spaces can be naturally viewed as metric spaces with the metric defined by $d(x, y) = \|x - y\|$. A Banach space is simply a normed vector space such that when viewed as a metric space, the space is complete. Some well known examples of Banach spaces include L^p spaces, which are important in analysis and statistics. Sobolev spaces, which are fundamental in the theory of differential equations. Numerous other important examples of Banach spaces can be found in standard references such as [JL01, AK06].

Although the definition of a Banach space is simple to understand, the theory of Banach spaces is undoubtedly an integral part of mathematics. With fundamental results such as the Hahn-Banach theorem, the open mapping theorem, the Banach-Steinhaus theorem, and the Riesz theory of compact operators. The aforementioned theorems are fundamental and powerful as they apply to *all* Banach spaces, resulting in fruitful applications to many areas of classical analysis. This naturally motivates the study of Banach spaces for its own sake, analysing what results and properties can be concluded from general abstract Banach spaces, "pure" Banach space theory if one will.

As remarked by Lindenstrauss [Lin70], the general theory of Banach spaces, like the case for finite groups, is itself interesting and important for an independent study. Much like the situation for finite groups, classification, or the structure theory of Banach spaces has been a major theme of research. Investigations in this area has led to many well known open problems, some of which are even still open today. As we will later see, an important example of this is the "basis problem", which asks if every separable Banach space has a basis[1]. This question was first posed by Banach, and was later solved by [Enf73] in the negative after remaining open for nearly 40 years. Our goal in this thesis is to carry out a study regarding the structure theory of Banach spaces, through the lenses of *computability theory*. This gives a finer analysis of the algorithmic content of some of the classical results, and might even be useful in questions where computability theory is not concerned at all.

1.2 Effective Banach spaces

Effective Banach space theory, as opposed to the general theory of Banach spaces, attempts to analyse the *algorithmic content* of Banach space theory by dealing with computable Banach spaces. After all, much of the real-life applications are carried out using computers, it is therefore natural to wonder which "processes" are computable, and which ones are not. In order to rigorously discuss computability on Banach spaces, we must first agree on some form of "computability structure" on Banach spaces. This is not entirely straightforward, as Banach spaces are generally uncountable in nature, so we cannot naively carry over the definitions used in classical computable algebra.

One of the possible definitions is known as the *axiomatic approach*, pioneered in [PER87]. To motivate this definition, recall that our motivating question was to classify the "processes" that arise in functional analysis in terms of their computability. Before we could even discuss computability, we need to first decide on what "processes" we are considering. In this setting, it is natural to take the "processes" to be linear operators between Banach spaces, as regular processes such as Fourier transform, Laplace transform, etc. are all forms of linear operators.

Once we have decided that we are primarily interested in the computability of operators, it is natural to proceed and axiomatise the notion of *computable points* in Banach spaces, as they act as the inputs to operators. However, this definition fails to take into account the *topology* of Banach spaces, which is fundamental to the theory of Banach spaces. This leads to the axiomatisation of *computable sequences of points* in Banach spaces, capturing the topology of the space, while also being consistent with previous works such as [Grz57]. Indeed, this is the essence of the axiomatic approach, the computability structure on a Banach space is determined by its computable sequences, and a point x is computable if $(x, x, \ldots)$ is computable as a sequence. It was under this framework that the study of effective Banach spaces took off, leading to many remarkable results. For example, the *First Main Theorem* in [PER87] proved a link between the continuity and computability of operators on separable Banach spaces.

Theorem 1.2.1 (First Main Theorem, [PER83]). *Let X, Y be computable Banach spaces, and $\{e_i\}_i$ be a computable sequence in X whose linear span is dense. Let $T : X \to Y$ be a closed[2] linear operator whose domain contains $\{e_i\}_i$ and such that the sequence $\{T(e_i)\}_i$ is computable in Y. Then T maps every computable element of its domain onto a computable element of Y if and only if T is bounded.*

This result, in some sense, gives a complete characterisation of the operators that preserves computability structures, leading to numerous applications. For example, this theorem shows that certain integrals are computable.

[1] Basis here refers to Schauder basis.
[2] As in the graph of T is closed.

Example 1.2.2. As an application of the First Main Theorem, we show that the indefinite integral of a computable function $f \in C[a,b]$ is computable. This argument is due to [PER83]. To prove this, take $X = Y = C[a,b]$ in the First Main Theorem, $\{e_i\}_i$ as the sequence of monomials $\{x^i\}_i$ and T as the indefinite integral operator. The boundedness of indefinite integrals imply that T is a closed operator, and the computability of $\{T(x^i)\}_i$ is easy to verify. Applying the First Main Theorem then shows that T maps computable functions to computable functions.

We also give an application of the First Main Theorem in the negative direction.

Example 1.2.3. In the negative direction, we give a description of a result in [PER83] that there exists computable functions in $C[a,b]$ which have continuous derivatives, but whose derivatives are not computable (This result was originally proved in [Myh71] by an explicit construction). Let $X = Y = C[a,b]$ and $\{e_i\}_i = \{x^i\}_i$ as in the other example, and denote $T = \frac{d}{dx}$ where the domain of T is $C^1[a,b]$. In this case, the operator T is unbounded, and an application of the First Main Theorem shows that T must map some computable $f \in C^1[a,b]$ to a non-computable element in $C[a,b]$.

Although much of the theory of effective Banach spaces was pioneered under the axiomatic framework, there are other alternate approaches. One of the more notable and conventional approach is known as the *representation-based approach*. [Hau78, Hau80] were some of the first works to study the notion of *representations* in its own right, building on works such as [Kla61]. This approach was eventually adopted and popularised by Weihrauch's school of computable analysis [Wei00], which also incorporates works on the computability of *functionals*, such as [Grz55, Kle59, Ko91]. In the context of Banach spaces, the representation based approach fixes a dense sequence for each separable Banach space, and treats each separable Banach space as a space of Cauchy sequences[3]. This provides us with a natural computability framework on the separable Banach spaces, where an element is computable if it has a Cauchy sequence that is computable as a sequence of naturals. Note that this approach only works for separable Banach spaces.

For this thesis, we will be solely interested in separable Banach spaces, in which case the two approaches described above are equivalent. In fact, despite the fact that one can technically impose a computability structure on non-separable Banach spaces via the axiomatic approach, there are "bad" spaces where no natural computability structure exists [PER87]. Furthermore, it was shown in [Bra01a, Proposition 15.3] that there is *no* representation of l_∞ (a non-separable space) such that both $\{\|\cdot\|, +\}$ are computable operations. In some sense, separable Banach spaces is the natural arena where computability can be discussed, and central results such as the First Main Theorem are also proved under this setting. This also highlights a difficulty in the realm of effective Banach spaces, being that most classical results rely on the dual space, which might not be separable. To add on to this problem, some of the canonical theorems used in classical Banach space theory also fail in the effective case. For example, it was shown in [MNS85] that the Hahn-Banach theorem is *not* effective.[4] However, as we will see, these problems can be circumvented in some cases.

[3] We technically use Cauchy sequences where the rate of convergence is known, this will be expanded in Chapter 2.

[4] Although it is effective for the finite-dimensional case.

1.3 Complexity of effective Banach spaces

Our goal in this thesis is to analyse the algorithmic content of the classical structure theory of Banach spaces. To achieve this, we will be utilising results from the theory of effective Banach spaces and classical computability theory. Such investigations are fairly common in the case for countable structures, but are not as common for analytical objects.

There have been previous works to classify the complexity of natural analytical constructs. For example, [MN13] analysed the complexity of (locally) compact metric spaces, [BMM20] analysed the complexity of Lebesgue spaces and [CR99] analysed the complexity of some natural index sets occurring in analysis. Results in the general area of computable analysis are much more abundant in comparison. [Bra01b] examined the computability of Baire's Category Theorem, [Bra08] generalises the classical graph theorem in computability theory, [LW07] proves a computable version of the Riesz representation theorem, and [WZ07] analysed the computability of Cauchy's problem, just to name a few of the results.

Much like the theory of effective Banach spaces, the interaction between descriptive set theory and the theory of Banach spaces also has a long and rich history. Arguably the fundamental result due to [Szl68] proving that no separable reflexive Banach space can be universal for all separable reflexive Banach spaces, through the use of coanalytic ranks known as Szlenk indices can be viewed as a result in this area. Other results include works such as [Bos02], which proved a variety of complexity results regarding Banach spaces, as well as showing that the isomorphism relation between separable Banach spaces is non-Borel. [FLR06], which later showed that in fact the isomorphism relation is analytic complete. [God10] provides a list of some of the open problems in this area.

There have also been previous works to classify the complexity of the structural properties through the perspective of descriptive set theory. In this setting, we classify the complexity of sets using the Borel Hierarchy , which consists of classes $\boldsymbol{\Sigma}^0_\alpha, \boldsymbol{\Pi}^0_\alpha, \boldsymbol{\Delta}^0_\alpha$ for every countable ordinal α. In this hierarchy, $\boldsymbol{\Sigma}^0_1$ corresponds to the open sets, $\boldsymbol{\Pi}^0_\alpha$ sets are the complements of the $\boldsymbol{\Sigma}^0_\alpha$ sets, $\boldsymbol{\Sigma}^0_\alpha$ consists of countable union of elements from $\boldsymbol{\Pi}^0_\beta$ for $\beta < \alpha$, and finally $\boldsymbol{\Delta}^0_\alpha$ is the intersection of $\boldsymbol{\Sigma}^0_\alpha$ and $\boldsymbol{\Pi}^0_\alpha$. This gives a classification of the Borel sets in a Polish space.

For example, [Gha19] showed that the bounded approximation property and the π-property[5] are both $\boldsymbol{\Sigma}^0_6$, and [CDDK21, Theorem 7.13] showed that local basis structure is $\boldsymbol{\Sigma}^0_4$ and local Π basis structure is $\boldsymbol{\Sigma}^0_6$. We are, however, to the best of our knowledge, the first to look at the complexity of such structural properties from the computability point of view . To be precise, we will be classifying the complexity of structural properties using the *arithmetical hierarchy*[6]. This is also known as the lightface hierarchy, as it corresponds to the effectivised version of boldface hierarchy. For example, a set is $\boldsymbol{\Sigma}^0_1$ if and only if it is open, whereas a set is Σ^0_1 if and only if it is *effectively open*. In fact, the lightface hierarchy gives a more refined analysis. For example, all continuous functions are $\boldsymbol{\Delta}^0_1$ functions but not all of them are Δ^0_1, which corresponds to being computable. In the lightface setting, we obtained improved upper bounds for the results mentioned in the descriptive set theory case. In particular, we have the following[7] as part of our results:

- Bounded approximation property and π-property are both in Σ^0_4.

[5] These will be defined in Chapter 3.
[6] To be defined later.
[7] See Section 3.9 for a complete overview.

- Local basis structure is in Σ^0_3.[8]
- Local Π basis structure is in Σ^0_4.

In addition to the classical structural properties of Banach spaces, we also look at the complexity of the corresponding effective analogues. For example, we show that having a computable basis is Σ^0_3-complete, and so is having a computable finite dimensional Schauder decomposition.

[8] This was implicit in [Puj71] and [Bos08].

Chapter 2

Preliminaries

This chapter will provide the necessary background knowledge on computable analysis that is needed for this thesis. We will mostly follow standard notations and definitions. For a more detailed treatment of the materials introduced in this chapter, we refer the readers to [Soa16] for classical computability, [Ped89] for classical analysis and [Wei00] for computable analysis.

2.1 Computability theory

2.1.1 Basics

Computability theory, as the name suggests, is the study of how "computable" an object is, such as a set of natural numbers. To make this formal, we must have a rigorous definition of what it means for something to be computable. One way to achieve this is through the use of Turing machines. We omit the technicalities of defining what a Turing machine is, and rather invoke the widely accepted *Church-Turing thesis*. Which states that "a function would naturally be regarded as computable if and only if it can be computed by a Turing machine". Intuitively, Turing machines can be viewed as programs written in some modern programming language. In fact, this will be the default model used when Turing machines are referred to in this thesis.

Thus, we now have a notion of what it means for a function to be computable. However there is a key difference between the "functions" that can be computed by Turing machines and the ordinary functions one might be used to. A function that can be computed by a Turing machine does not necessarily have to be defined on all inputs. For example, if we view the Turing machines as computer programs, it is possible for a program to get stuck in some infinite loop on some inputs. To handle this, we will need the notion of partial functions.

Definition 2.1.1. $f : \mathbb{N} \to \mathbb{N}$ is a partial function if it is a function with a domain $X \subseteq \mathbb{N}$.

For a partial function φ, we write $\varphi(x) \downarrow$ to mean that φ halts (i.e. is defined) on input x, and $\varphi(x) \uparrow$ to mean that it does not halt. For two partial functions φ, ψ, we write $\varphi = \psi$ to mean that the functions are equivalent whenever either one of them halts, i.e. $\forall x[(\varphi(x) \downarrow = \psi(x) \downarrow) \vee (\varphi(x) \uparrow \wedge \psi(x) \uparrow)]$.

We can now view the computation modeled by a Turing machine as a partial function from $\mathbb{N}$ to $\mathbb{N}$, and define φ to be a partial computable function if and only if it is computed by some Turing machine.

Definition 2.1.2. A partial function $\varphi : \mathbb{N} \to \mathbb{N}$ is partially computable if it corresponds the computation modeled by some Turing machine. It is computable if it is both partially computable and total (i.e. halts on all inputs, $\text{dom}(\varphi) = \mathbb{N}$).

From now onwards, Turing machines are also identified by the partial computable functions they compute.

There is also another bit of terminology that needs to be introduced. In computer programs, although a computation might get stuck in an infinite loop and never halt, one can always run the loop for a finite number of steps and "force" the program to halt afterwards. In light of this, we define $\varphi(x)[s]$ to mean that s-steps have been computed on input x. $\varphi(x)[s]$ might be defined, in which case the result $\varphi(x)$ was computed in no more than s-many steps. Or it might be undefined, in which case either it takes more than s-steps to compute the result, or $\varphi(x) \uparrow$. The importance of $\varphi(x)[s]$ is that it is always computable, and if $\varphi(x) \downarrow$, then $\varphi(x)$ must halt after some finite number of steps.

Note that each Turing machine has a finite description, so we can effectively list all the partial computable functions through some coding. For example, we can treat each computer program as a sequence of bits, and code this into the natural numbers. Such that for each index, we can obtain the corresponding partial computable function and vice versa. We fix some such listing as $\varphi_0, \ldots$. The listing being effective gives the following.

Theorem 2.1.3 (Universal Turing machine)**.** *There is a partial computable function $\varphi(n, x)$ such that $\varphi(n, x) = \varphi_n(x)$ for every $n, x \in \mathbb{N}$. Any Turing machine that computes such a function is called an universal Turing machine.*

Intuitively, this machine can be thought of as a compiler, a program that can simulate all other programs. We omit the proof to avoid unnecessary details for this thesis, but the main idea is as follows. On input (n, x), the Turing machine can decode n and obtain the corresponding Turing machine for φ_n, and then run it on the input x.

The computability of classes of functions will also be of importance; we give a definition here.

Definition 2.1.4. Given a family of functions $f_0, f_1, \ldots$ that are (partial) computable, we say that they are uniformly (partial) computable if there exists a (partial) computable function $g(x, y)$ such that for all $n, x \in \mathbb{N}$, $g(n, x) = f_n(x)$.

With the above definitions, we can now define the halting problem.

Definition 2.1.5. The halting problem $\emptyset'$, is defined as:

$$\emptyset' = \{e | \varphi_e(e) \downarrow\}$$

Intuitively, the halting problem represents the information needed to determine if an arbitrary Turing machine will halt on any given input. It is a fundamental result that the halting problem is not computable.

Theorem 2.1.6. *$\emptyset'$ is not computable.*

2.1.2 Relativisation

An important aspect of computability is relativisation, which is when computations are carried out relative to oracles. Oracles can be thought of as external information that can be queried a finite number of times during computations. Intuitively, when a computation is relativised to a set A, it just means that the information contained in A can be used during the computation. We can extend the notion of partial computable functions to partial computable functions relativised to A, meaning that the set A is used as an oracle. These functions are denoted as φ_e^A. Similarly, whenever the phrase "relativised to A" is used, it just means that A is used as an oracle. All results discussed within this thesis hold when relativised to oracles. With the above definitions, we define Turing reducibilities.

Definition 2.1.7 (Turing reducibility)**.** A partial function f is Turing computable in a set A if there is an index e such that $\varphi_e^A = f$. A set A is Turing reducible to a set B (also written as $A \leq_T B$) if the characteristic function of A is computable from B.

For sets A, B, $A \leq_T B$ intuitively means that A can be computed using B, or B contains more information than A. If $A \leq_T B$ and $B \leq_T A$, we say that the sets are Turing equivalent and write $A \equiv_T B$.

The Turing jump of sets is another important operator. It is defined as follows.

Definition 2.1.8. For a set A, the (Turing) jump of A is defined:

$$A' = \{e | \varphi_e^A(e) \downarrow\}$$

It is the halting problem relativised to A. The n-th jump of A, denoted as A^n, is obtained by iterating the jump n times starting with A.

We now give an example of what it means for a result to be relativised, and provide some intuition as to why any result in this thesis will still hold when relativised to an oracle.

Theorem 2.1.9 (Theorem 2.1.6 relativised)**.** *For any set $A \subseteq \mathbb{N}$. We have:*

$$A' \not\leq_T A$$

i.e. A' is not computable relative to A.

Theorem 2.1.6 says that $\emptyset' \not\leq_T \emptyset$. So when relativised to A, we simply replace $\emptyset$ by A, and obtain the claim above.

By definition of the Turing jump, we get that $A \leq_T A'$ for any set A. Theorem 2.1.9 shows that this inequality is strict, meaning $A <_T A'$. As the Turing jumps can be iterated, we actually obtain a hierarchy of the form $A <_T A' <_T A'' <_T \ldots$. This gives rise to the *Arithmetical hierarchy*, introduced in the next section.

2.1.3 Arithmetical hierarchy and Post's theorem

The domains of the partial computable functions are of particular importance, they are known as the computably enumerable sets.

Definition 2.1.10 (Computably enumerable sets)**.** A set $A \subseteq \mathbb{N}$ is a computably enumerable set (c.e. set) if it is the domain of some partial computable function.

As hinted by the name, computably enumerable sets are called so because a set is c.e. if and only if it is the range of some computable function. In other words, each computably enumerable set can be enumerated by some computable function.

Theorem 2.1.11. *A set A is c.e. if and only if $A = \emptyset$ or A is the range of a computable function f.*

So, each c.e. set A can be computably enumerated by approximating the range of some computable function. Therefore whenever we say that a set A is c.e., we implicitly assume that there is a computable enumeration $\{A_s\}_{s\in\omega}$ of it. $\{A_s\}_{s\in\omega}$ being a computable enumeration means that it is uniformly computable, $\bigcup_s A_s = A$. And for all s, $A_{s+1} \supseteq A_s$, A_s is finite.

The computably enumerable sets are part of what is called the arithmetical hierarchy. In fact, they are equivalent to the Σ^0_1 sets, which are defined below:

Definition 2.1.12. A Σ^0_1 set is a set S such that:

$$S = \{x | \exists y R(x, y)\}$$

where $R \subseteq \mathbb{N} \times \mathbb{N}$ is some computable relation.

Theorem 2.1.13. *A set S is computably enumerable if and only if it is Σ^0_1.*

Following on from the definition of Σ^0_1 sets, we will now define the arithmetical hierarchy.

Definition 2.1.14 (Arithmetical hierarchy)**.** Analogous to a Σ^0_1 set being characterised by an existential quantifier before a computable relation, a Σ^0_n set is a set with n alternating existential-universal quantifiers before a computable relation, with the first quantifier being an existential one (i.e. $\exists x_1 \forall x_2 \ldots \exists x_n R(x, x_1, \ldots, x_n)$). A Π^0_n set is the complement of a Σ^0_n set. A Δ^0_n set is a set that is both Σ^0_n and Π^0_n. The $\Sigma^0_n, \Pi^0_n, \Delta^0_n$ sets together form the arithmetical hierarchy.

It follows from the definition that the complement of a Σ^0_n set is a Π^0_n set, so Π^0_n sets are of the form $\{x | \forall x_1 \exists x_2 \ldots \forall x_n R(x, x_1, \ldots, x_n)\}$. The arithmetical hierarchy is probably best described by the following picture.

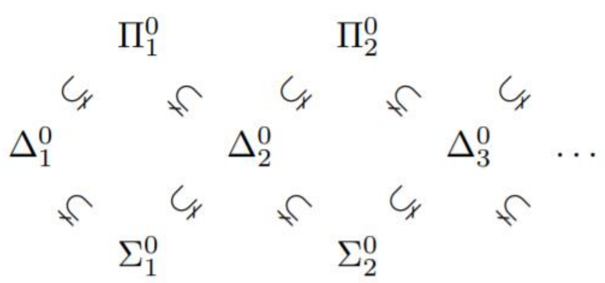

Figure 2.1: Arithmetical hierarchy

Associated to the arithmetical hierarchy is the notion of m-reducibility.

Definition 2.1.15 (m-reducibility)**.** Let A, B be subsets of $\mathbb{N}$, A is *many-one* reducible (m-reducible) to B (written $A \leq_m B$) if there is a computable function f such that

$$(\forall a \in \mathbb{N})(a \in A \iff f(a) \in B)$$

We also write $(\Sigma_n^0, \Pi_n^0) \leq_m (A, B)$ if for any $C \in \Sigma_n^0$, there exists a computable function f such that for all $c \in \mathbb{N}$

$$c \in C \iff f(c) \in A$$
$$c \notin C \iff f(c) \in B$$

Finally, a set A is Σ_n^0 (likewise for Π_n^0) complete if $(\Sigma_n^0, \Pi_n^0) \leq_m (A, A^C)$, where A^C denotes the complement.

The importance of $\leq_m$ is that it respects the arithmetical hierarchy. If $A \in \Sigma_n^0$ and $B \leq_m A$, then $B \in \Sigma_n^0$, which is not true for $\leq_T$ (E.g. $\emptyset' \in \Sigma_1^0$, $(\emptyset')^C \leq_T \emptyset'$, but $(\emptyset')^C \notin \Sigma_1^0$). Therefore, the type of reducibility used is quite important. Throughout this thesis, we will implicitly assume that the reduction is a m-reduction unless otherwise specified.

2.2 Analysis

We will be primarily working with Banach spaces, so let us define what they are. All results in this thesis hold for vector spaces over $\mathbb{C}$ as well, but we will only work with vector spaces over $\mathbb{R}$ for the sake of simplicity.

Definition 2.2.1 (Banach space)**.** A Banach space is a normed vector space that is complete in its norm.

More specifically, we will be working with separable Banach spaces.

Definition 2.2.2. A Banach space X is separable if there is a sequence $(e_i)_{i\in\omega} \in X^\omega$ that is dense in X.

Example 2.2.3. The following are some examples of the separability of some well-known Banach spaces.

- l_p spaces for $1 \leq p < \infty$ are indeed separable Banach spaces. By taking the dense set as $\{(r_0, \dots, r_n, 0, 0, \dots) | (r_0, \dots, r_n) \in \mathbb{Q}^{<\omega}\}$. c_0 is also separable via this dense set.
- $C[0,1]$, the Banach space of continuous functions on the unit interval under the sup norm, is also separable. One possible dense set is the set of polynomials with rational coefficients by the Stone-Weierstrass theorem.
- The Banach space l_∞ is not separable. To see this, suppose for the sake of contradiction that $(e_i)_{i\in\omega}$ is a dense sequence in l_∞, where $e_i = (e_{i,0}, e_{i,1}, \dots)$. We can then carry out a direct diagonalisation and construct an element $x = (x_0, \dots) \in l_\infty$ by setting $x_i = 0$ if $|e_{i,i}| > 1$ and $x_i = e_{i,i} + 1$ if $|e_{i,i}| \leq 1$. The resulting element satisfies $\|x - e_i\| \geq 1$ for all i, therefore a contradiction, and l_∞ is indeed not separable.

We remark here that there is the equally important notion of Hilbert spaces in functional analysis, which are Banach spaces where the norm is induced by an inner product. Not all separable Banach spaces are Hilbert spaces, and Hilbert spaces are very "nice" with respect to the properties that will be considered in this thesis, as they inherently contain the important notion of *angles*. For example, all separable Hilbert spaces will automatically have a basis, but as we will see later on, this is not always the case for Banach spaces.

Operators in this report will always be linear unless stated otherwise. Those operators that are bounded are of particular importance.

Definition 2.2.4. Let $T : X \to Y$ be an operator where X, Y are Banach spaces. The operator is bounded if there exists some $M \in \mathbb{R}$ such that

$$(\forall x \in X) \, \|T(x)\| \leq M \|x\|$$

Furthermore, if the operator is bounded, then the infimum of all such bounds is known as the norm of the operator.

Example 2.2.5. One of the more familiar examples of linear operators is that of matrices. Each matrix $M = (a_{ij})_{i \leq n, j \leq m}$ induces a linear operator from $\mathbb{R}^m$ to $\mathbb{R}^n$, where each basis vector $e_j \in \mathbb{R}^m$ is mapped to the vector $\sum_{i=1}^n a_{ij} e_i \in \mathbb{R}^n$. It is a classical result that such operators are always bounded.

It is in fact a fundamental result that linear operators are continuous if and only if they are bounded.

Lemma 2.2.6. *Let $T : X \to Y$ be a linear operator. Then T is bounded if and only if it is continuous.*

Proof. For the forward implication, we get that $\|Tx - Ty\| = \|T(x - y)\| \leq \|T\| \|x - y\|$, which implies continuity.

For the converse, as T is continuous, it is therefore continuous at 0, and there will be some $\delta > 0$ such that for all $v \in X$ with $\|v\| \leq \delta$, $\|T(v)\| = \|T(v) - T(0)\| \leq 1$. Now let $x \in X$ be some arbitrary element, we have

$$\|T(x)\| = \frac{\|x\|}{\delta} \left\| T\left(\frac{\delta x}{\|x\|}\right) \right\| \leq \frac{1}{\delta} \|x\|$$

So the norm of T is bounded by $\frac{1}{\delta}$. □

Operators that have a finite dimensional range are also of special importance, we will define them explicitly.

Definition 2.2.7. Let X, Y be normed vector spaces. An operator $T : X \to Y$ has *finite rank* if its range is finite-dimensional, in which case it is also called a *finite rank operator*.

The following is a central theorem that gives a criterion for open operators.[1]

Theorem 2.2.8 (Open mapping theorem)**.** *If X and Y are Banach spaces, and $T : X \to Y$ is a surjective continuous operator, then T is an open operator.*

An important corollary of the open mapping theorem is the following.

Corollary 2.2.9. *If X, Y are Banach spaces and T is a continuous linear operator that is bijective, then $T^{-1} : Y \to X$ is a continuous linear operator.*

Definition 2.2.10. A bounded linear operator $P : X \to X$ is a projection if $P^2 = P$. A subspace Y of X is complemented if there exists a projection $P : X \to Y, P(X) = Y$.

Alternatively, we can actually show that this is equivalent to the existence of some other subspace Z of X such that $X = Y \oplus Z$.

[1] An operator is open if it maps open sets to open sets.

Lemma 2.2.11. *A subspace Y of a Banach space X is complemented if and only if there exists a subspace Z of X such that $X = Y \oplus Z$. (i.e. Y is algebraically complemented)*

Definition 2.2.12. Let X, Y normed spaces. Then the set of all bounded linear operators from X to Y is denoted as $L(X, Y)$. It is well-known that if Y is a Banach space, then so is $L(X, Y)$. We will also denote the set of projections from X to Y as $P(X, Y)$. It is clear that $P(X, Y) \subseteq L(X, Y)$. Finally, $L_K(X, Y)$ denotes the set of all operators with norm bounded by K, and likewise for $P_K(X, Y)$.

The following results regarding finite dimensional spaces are also well known.

Theorem 2.2.13. *Let X be a normed space. Then the unit ball $B_X = \{x \in X : \|X\| \leq 1\}$ is compact if and only if X is finite dimensional.*

Theorem 2.2.14. *Let $S = \{x_0, \ldots, x_n\}$ be a set of linearly independent vectors in a normed space X. Then there is a constant C depending only on S such that for all $(\alpha_i)_{i \leq N} \in \mathbb{R}^N$, we have*

$$C \left\| \sum_{i=0}^{N} \alpha_i x_i \right\| \geq \sum_{i=0}^{N} |\alpha_i|$$

We will also need to introduce some technical terminologies.

Definition 2.2.15. Let $(X_i)_{i \in \omega}$ be a sequence of Banach spaces. Then their c_0 sum, denoted as

$$Y = (\oplus_i X_i)_{c_0}$$

is defined as the space of sequences $(x_i)_{i \in \omega} \in \prod_i X_i$ such that $(\|x_i\|)_i \to 0$. The norm of the element $\|(x_i)_{i \in \omega}\|$ is defined as

$$\|(x_i)_{i \in \omega}\| = \sup_i \|x_i\|$$

and the vector space operations are carried out term-wise. It can be verified that Y is indeed a Banach space as well.

Definition 2.2.16. Let X be a Banach space, and $(x_i)_{i \in \omega}$ be a sequence of elements in X. $[x_0, x_1, \ldots]$ is defined to be the closure of the finite linear span of $(x_i)_{i \in \omega}$.

2.3 Computable analysis

Computability theory studies the algorithmic content of mathematics, and has many sub areas. For example, the subfield of *computable algebra* studies the effective content of algebraic structures, such as groups. In this setting, we endow a computability structure on a group $(G, \cdot)$ by coding G as a computable subset of $\mathbb{N}$, and require the group operation to be computable. In contrast, the subfield of *computable analysis* studies the effective content of analytic objects, such as functions on $\mathbb{R}$, Banach spaces, etc. Distinct to the case for computable algebra, analytical objects are often uncountable in nature (e.g. $\mathbb{R}$, $\mathbb{C}$), and thus cannot be coded as a subset of $\mathbb{N}$.

There are many approaches to endow computability structures on analytical objects. In fact, Turing's original paper [Tur36] was actually motivated by trying to classify the *computable* *real numbers*, rather than computability on $\mathbb{N}$. Turing's original approach classified computable numbers as those that have a computable decimal expansion[2], Banach and Mazur had a notion of *Banach-Mazur* computability [Maz63] for functions $f : \mathbb{R}_c \to \mathbb{R}_c$[3], [Grz57] had a notion of computability for functions $f : [0,1] \to \mathbb{R}$, the "Russian school" of constructive mathematics led by Markov developed a notion of *Markov computable* for functions $f : \mathbb{R}_c \to \mathbb{R}_c$, and there are many other notions not mentioned here. Unsurprisingly, most of these notions are not equivalent, as shown by works such as [Her05, Abe84]. We refer the reader to [AB14] for a more comprehensive survey. In this thesis, we will be adopting the conventional and more modular approach, giving a general framework (not limited to $\mathbb{R}$) to work with sets of size continuum.

We will introduce computability on analytical objects through *representations*. This approach is known as "Type-two theory of effectivity" (TTE), which we introduce below. A more detailed treatment can be found in [Wei00].

2.3.1 Representation based approach

To motivate a representation based approach to computable analysis, we first re-examine the approach taken in computable algebra. Countable structures are treated as subsets of $\mathbb{N}$, formalised through the notion of *coding*. To be more precise, let S denote some countable structure. Then a *notation* for S is a surjective partial function $\mu : \mathbb{N} \to S$, where $n \in \mathrm{dom}(\mu)$ is said to be a *μ-name* for $\mu(n)$. The usual computability-theoretic operations are then carried out on $\mathrm{dom}(\mu)$, the names of S, instead of the actual elements of S. This approach has worked well for countable structures, but will clearly fail for larger structures such as $\mathbb{R}$ and $\mathbb{C}$, which all have a cardinality of the continuum, hence there cannot be a surjection from $\mathbb{N}$ to them.

To circumvent this problem, we will work with the theory of representations. As mentioned in the introduction, the theory of representations has its roots in [Kla61, Hau80, Hau78]. It was popularised by Weihrauch's school of computable analysis, originating in works such as [KW85, WK87]. As we will see, the theory of representations provides us with a powerful and modular framework to systematically deal with computability on sets that have a cardinality of the continuum.

Definition 2.3.1. Let S be a set, a *representation* of X is a surjective partial function $\nu : \mathbb{N}^{\mathbb{N}} \to S$. For any $s \in S$ and $n \in \mathbb{N}^{\mathbb{N}}$ such that $\nu(n) = s$, we say that n is a ν-name of x.

Example 2.3.2. The following are some examples of representations that the reader might be familiar with.

- The usual decimal representation for $[0,1]$ is in fact a representation in the sense defined. Where
$$\nu\left((x_0, x_1, \ldots)\right) = 0.x_0x_1x_2\ldots$$
- We could also easily obtain a similar representation for $\mathbb{R}$ by defining $\nu\left((x_0, x_1, \ldots)\right) = x_0.x_1x_2x_3\ldots$. Note that under this representation, the names are *not* unique. Since we have $\nu\left((1,0,0,\ldots)\right) = 1 = \nu\left((0,9,9,\ldots)\right)$.

[2]This notion was later corrected in [Tur37].
[3]R_c is the field of computable numbers.

- By using the standard notation from $\mathbb{N}$ to $\mathbb{Q}$, we can regard elements of $\mathbb{N}^{\mathbb{N}}$ as sequences of rationals. Then the representation $\nu((x_0, \ldots)) = \lim_{i\to\infty} x_i$ is a representation of $\mathbb{R}$ when defined on the set of convergent sequences.

We will now define a notion of computability on $\mathbb{N}^{\mathbb{N}}$.

Definition 2.3.3. A partial function $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is *computable* if there is an oracle Turing machine ψ such that $F(x) = y \iff (\psi^x(i))_{i\in\omega} = y$. As a result, we require ψ^x to not produce an infinite sequence for $x \in \mathbb{N}^{\mathbb{N}} \setminus \mathrm{dom}(F)$.

Example 2.3.4. Let $y \in \mathbb{N}^{\mathbb{N}}$, consider the constant function $I_y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined by $I_y(x) = y$ for all $x \in \mathbb{N}^{\mathbb{N}}$. Then I_y is computable if and only if y is computable as a function from $\mathbb{N}$ to $\mathbb{N}$.

Remark 2.3.5. We stress that in Definition 2.3.3, we are working with *oracle machines*, with its domain being $\mathbb{N}^{\mathbb{N}}$. (This can also be thought of as $\{f | f : \mathbb{N} \to \mathbb{N}\}$). Whereas classical computability works with regular Turing machines, with its domain being $\mathbb{N}$. This is why this approach is called "Type-two theory of Effectivity".

This leads to a natural notion of computability on sets with representations.

Definition 2.3.6. Let X, Y be sets and δ_X, δ_Y be representations for them. A partial function $f : X \to Y$ is (δ_X, δ_Y) computable if there is a computable (possibly partial) $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

$$\delta_Y \circ F(x) = f \circ \delta_X(x)$$

for all $x \in \mathrm{dom}(f \circ \delta_X)$. Note that this does not impose any requirement on $F(x)$ for $x \notin \mathrm{dom}(f \circ \delta_X)$.

Example 2.3.7. The following are some examples of Definition 2.3.6 in action.

- Let ρ denote the representation that treats $\mathbb{N}^{\mathbb{N}}$ as sequences of rationals and maps them to their limits (if they exist) in $\mathbb{R}$. One can check that the usual functions such as $\sin()$, $|\cdot|$, etc, are all (ρ, ρ) computable. But the ordering relation of $\mathbb{R}$ is not (ρ, ρ) computable. In fact, it is not even Σ^0_1.

- Let ρ_{10} denote the representation corresponding to the base-10 decimal expansion of $\mathbb{R}$. i.e.

  $$\rho_{10}((x_0, x_1, \ldots)) = x_0.x_1x_2x_3\ldots$$

  Under this representation, the innocent looking function $f(x) = 3x$ is in fact not (ρ_{10}, ρ_{10}) computable. To see this, suppose that there is some computable $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ that witnesses the computability of f as per Definition 2.3.6. Consider the value of $F((0, 3, 3, \ldots))$. The correct output should be $(1, 0, 0, \ldots)$ or $(0, 9, 9, \ldots)$ as $f(0.333\ldots) = 1$. However, as F is computable, it must have finite use for producing the first digit of its output. In other words, suppose that the output of $F((0, 3, 3, \ldots))$ is $(1, 0, \ldots)$. Then the "1" in the output $(1, 0, \ldots)$ must have been produced by only using a *finite* prefix of $(0, 3, \ldots)$. We can then find a string of the form $(0, 3, \ldots, 3, 0, \ldots)$ such that $F((0, 3, \ldots, 3, 0, \ldots))$ has 1 as the first digit of its output. But this is incorrect as $f(0.3\ldots30\ldots) < 1$. Using a similar argument for $F((0, 3, 3, \ldots)) = (0, 9, 9\ldots)$ completes the argument.

These definitions provide us with a general framework to define computability structures on sets such as $\mathbb{R}$, while only dependent on the representation chosen. This naturally leads to the question: "Which representation should we use?". As shown in the examples, representations such as ρ_{10} on $\mathbb{R}$ are certainly not desirable, since simple functions such as $f(x) = 3x$ should certainly be computable. As it turns out, there is in fact a natural representation to use if we require the regular operations we are familiar with to be computable ([Her99]).

Definition 2.3.8 (Cauchy representation of $\mathbb{R}$)**.** Fix $g : \mathbb{N} \to \mathbb{Q}$ as some coding of the rationals. Define the representation $\rho_{Cauchy} : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ by

$$\rho_{Cauchy}\left((x_0, \ldots)\right) = x \iff (\forall i)\left(|x - g(x_i)| < 2^{-i}\right)$$

As shown in [Her99], this is indeed the "only" representation for $\mathbb{R}$ if we require the usual operations to be computable. Thus, Cauchy representation has been regarded as the standard representation to use, and we will be adopting this convention in this thesis.

While TTE is very powerful and the theory of representation provides a modular approach for imposing computability structures on general sets, we will be using more direct definitions for the sake of simplicity on notations. Note that this is only a matter of preference as the definitions are in fact equivalent.

2.3.2 Computable reals

We will first introduce what it means for a real number to be computable. In the language of representations, a real $y \in \mathbb{R}$ is computable if and only if the constant function $f(x) = y$ is $(\rho_{Cauchy}, \rho_{Cauchy})$ computable.

Definition 2.3.9 ([Tur37])**.** A real number $\alpha \in \mathbb{R}$ is computable if there is a computable sequence of rationals $q_0 < q_1 < \ldots \to \alpha$ (converging to α) such that $|q_n - \alpha| < 2^{-n}$ for all n.

This means that the computable real numbers are the reals that have computable rational approximations and the rate at which the approximations converge is known. Almost all of the familiar mathematical constants fall under this category. For example, π, e and all rationals are computable. However, we note that there are only countably many computable numbers, so almost all real numbers are not computable.

In the definition for a computable real, the sequence of rationals converging to it is required to converge at a given rate. It is natural consider a relaxation of this condition, where there is not an *a priori* bound on the rate of convergence. This leads to the notion of c.e. reals.

Definition 2.3.10. A real number $\alpha \in \mathbb{R}$ is left-c.e if there is a computable sequence of rationals $q_0 < q_1 < \ldots \to \alpha$.

It is important to note that the above definition requires the sequence of rationals to be *increasing*. If we simply require $(q_i)_i \to \alpha$, then α is something known as a Δ^0_2 real, rather than being left-c.e. Intuitively, the left-c.e. reals are the reals that have computable approximations, but the rate of convergence is not known.

Example 2.3.11. Let A be any c.e set that is not computable, and $\{a_s\}_{s\in\omega}$ be some computable enumeration of it. Construct the sequence $(q_i)_{i\in\omega}$ by setting

$$q_i = \sum_{i=0}^{n} 2^{-a_i - 1}$$

By construction, $(q_i)_{i\in\omega}$ is monotonically increasing and bounded from above by 1. Therefore, it converges to some real number α, and since $\{a_s\}_s$ is a *computable* enumeration, the sequence $(q_i)_{i\in\omega}$ is computable as well, making α a left-c.e real. However, note that if α was computable, we could compute its binary expansion and therefore compute A. Since A was chosen to be non-computable, α must be non-computable as well. The sequence $(q_i)_{i\in\omega}$ is known as a *Specker sequence*, an important example that proves the existence of strictly left-c.e reals.

We say that a sequence of reals $(x_i)_{i\in\omega}$ is uniformly left-c.e if there is a computable function $f(n,i)$ such that for each $n \in \mathbb{N}$, $(f(n,i))_{i\in\omega}$ gives a computable rational approximation to x_n as per Definition 2.3.10. Likewise, a sequence of reals $(x_i)_{i\in\omega}$ is uniformly computable if there is a computable function $g(n,i)$ such that $(g(n,i))_{i\in\omega}$ is an approximation to x_n as per Definition 2.3.9. This is distinct to the notion of a sequence of computable reals $(x_i)_{i\in\omega}$, which only requires x_i to be computable for each i, but lacks the uniformity requirement.

The reader should beware that computability in the Type Two case is a bit different from the usual (also known as Type One) notion of computability on $\mathbb{N}$. In the Type Two case, we almost exclusively only deal with *approximations*, rather than exact relations. For example, the equality relation $a = b$ between computable numbers a, b is Π^0_1 rather than decidable, and we instead work with the value $|a - b|$, which can be accurately approximated. Another notable example of this is the computability of sequences, let $(r_i)_{i\in\omega}$ be a sequence of rationals. Then $(r_i)_{i\in\omega}$ being uniformly computable in the Type One sense means that there is a Turing machine that outputs the *exact* codes of the rationals in the sequence. In the Type Two case, however, $(r_i)_{i\in\omega}$ being uniformly computable only guarantees uniformly computable *approximates* to the rationals of arbitrary precision.

2.3.3 Computable Banach spaces

We will now define what it means for a Banach space to be computable. Conceptually, this is very similar to the computability structure on $\mathbb{R}$. We will enforce the existence of "rationals", and impose computability conditions on the "rationals".

This is essentially why separable spaces provide natural computability structures. Intuitively, a dense sequence allows us to "fix" the elements on which computability features will be enforced upon.

Example 2.3.12. $\mathbb{Q}$ is a dense sequence for $(\mathbb{R}, |\cdot|)$, where $|\cdot|$ is the usual absolute value function.

This leads to the definition of *computable normed spaces*. The following is equivalent to the usual computability structure obtained by using the Cauchy representation on normed vector spaces.

Definition 2.3.13 (Computable spaces)**.** Let $(X, +, \cdot, \|\cdot\|)$ be a normed vector space with $(e_i)_{i\in\omega}$ as a dense sequence. An element $x \in X$ is said to be computable with respect to $(e_i)_{i\in\omega}$ if there is a computable function $d : \mathbb{N} \to \mathbb{N}$ such that

$$(\forall i)\left(\left\|e_{d(i)} - x\right\| < 2^{-i}\right)$$

We then define:

- A dense sequence $(e_i)_{i\in\omega}$ is a *presentation* if the operators $\{+,\cdot\}$ are uniformly computable on it with respect to itself, and $\|\cdot\|$ is uniformly computable on $(e_i)_{i\in\omega}$ as a real-valued function.
- A computable normed space is a normed space with a presentation. A computable Banach space is a computable normed space that is complete.

Example 2.3.14. The set of all rational polynomials on $[0,1]$ is a presentation for $C[0,1]$ by Stone-Weierstrass.

Remark 2.3.15. As a result of the preceding definition, normed spaces that are not separable cannot be considered as computable spaces. An example of this is the space l_∞, which cannot be given a presentation as it is not separable.

The observation above again highlights one of the main difficulties when dealing with computable Banach spaces, which is that the dual of a computable Banach space need not be separable, and therefore might not be computable. This problem occurs even for standard spaces such as $C[0,1]$. As most classical results in Banach space theory uses the dual, this drawback causes the effectivisation of most classical theorems to be non-trivial. There have been attempts to endow a computability structure on a non-separable normed space. For example, [Bra01a] defined the notion of a *general computable space*, which does not assume the space to be separable. But as remarked in the introduction, there are non-separable spaces where no "natural" computability structure can be imposed, so it is hard find a general way to circumvent this problem.

Let us stress that as per the preceding definition, a computable space inherently assumes some presentation, and is dependent on the presentation chosen. There are in fact Banach spaces where different presentations yield different computable Banach spaces ([PER87]). This is again distinct from the case for Hilbert spaces, where every presentation will yield the same[4] computable space [Mel13].

Definition 2.3.16. Let X, Y be computable finite dimensional Banach spaces. $L(X,Y)$ will always be assumed to be presented as a set of matrices of dimensions $\dim(Y) \times \dim(X)$ with rational coefficients unless stated otherwise.

We will also be discussing operators from one space to another. In view of this, we will also need to define what it means for elements of computable spaces to be computable, much like the case in $\mathbb{R}$.

Definition 2.3.17 (Cauchy names)**.** Fix X to be any computable Banach space and denote its presentation as $S = (e_i)_{i\in\omega}$. A Cauchy name for an element $x \in X$ is some sequence $(\alpha_i)_{i\in\omega}$ where $\alpha_i \in S$ for all i, and furthermore

$$\|\alpha_i - x\| < 2^{-i}$$

We say that $x \in X$ is computable if it has a computable Cauchy name (The Cauchy name is viewed as a sequence of naturals here).

The above definition is a generalisation of Definition 2.3.9, and is indeed equivalent to Definition 2.3.9 when we take $\mathbb{R}$ to be a computable normed space with $\mathbb{Q}$ as its representation.

With the definitions above, we can finally define what it means for an operator to be computable.

[4]Up to a computable isometry.

Definition 2.3.18. Let X, Y be computable normed spaces. Let $T : X \to Y$ be some function. T is computable if there is some Turing functional ψ such that for all $x \in X$, for all Cauchy names c of X, $(\psi^c(i))_{i\in\omega}$ is a Cauchy name for $T(x)$.

Remark 2.3.19. It is well known that computable functions are necessarily continuous, Theorem 2.3.24 gives a proof of this fact, and proves the stronger statement that the modulus of continuity at computable points will also be computable. In the case where $T : X \to Y$ is a linear operator, the modulus of continuity at 0 guarantees the operator T to be bounded. This can be viewed as a weak version of the First Main Theorem in [PER83].

We will also need an effectivised version of compactness.

Definition 2.3.20. Let X be a computable normed space. A compact subset $K \subseteq X$ is *effectively compact* if there is an effective procedure and a computable function f both uniform in $n \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ the procedure

- Gives a computable enumeration of a finite set of points $\{x_0, \dots, x_s\} \subseteq K$ such that

$$\bigcup_{i\leq f(n)} B\left(x_i, 2^{-n}\right) \supseteq K$$

  where $B(a, r)$ is the open ball of radius r centered around a.

Example 2.3.21. An example of an effectively compact set is the unit interval $[0, 1]$ in the computable normed space $(\mathbb{R}, |\cdot|)$. For a given n, we can approximate $[0, 1]$ through a list of dyadic rationals.

We obtain that the notion of effective compactness is preserved by computable functions. This result is an easy consequence of Theorem 2.3.25.

Theorem 2.3.22. *Let X be a computable normed space, with $K \subseteq X$ an effectively compact subset. Further let Y be some computable normed space, and $f : X \to Y$ be some computable function. Then $f(K) \subseteq Y$ is also effectively compact.*

Definition 2.3.23 (Modulus of continuity)**.** Let X, Y be normed spaces, and $f : X \to Y$ be some continuous function. Let $x \in X$ be some arbitrary point, we say that $d : \mathbb{N} \to \mathbb{N}$ is a modulus of continuity at x if

$$(\forall n)\,(\forall y \in X)\left(\|x - y\| < 2^{-d(n)} \implies \|f(x) - f(y)\| < 2^{-n}\right)$$

Analogously, if $f : X \to Y$ is uniformly continuous, we say that $d : \mathbb{N} \to \mathbb{N}$ is a modulus of uniform continuity if

$$(\forall n)\,(\forall x, y \in X)\left(\|x - y\| < 2^{-d(n)} \implies \|f(x) - f(y)\| < 2^{-n}\right)$$

It is well-known that all computable functions are continuous, we can in fact say something stronger.

Theorem 2.3.24. *Let X, Y be a computable normed spaces with $S = (e_i)_{i\in\omega}$ as X's computable presentation, let $f : X \to Y$ be some computable function. The modulus of continuity of f is uniformly computable on points in S.*

Proof. We describe an effective procedure uniform in $x \in S$ and $n \in \mathbb{N}$. Since S consists of elements from the dense sequence, $q = (x)_{i \in \omega}$ is indeed a computable Cauchy name for x. Let ψ be the underlying Turing functional for f, then its use function[5] $\varphi^q(n)$ is computable. We can now declare $d(n)$ to be $\varphi^q(n+1)+1$. If $\|x - y\| < 2^{-\varphi^q(n+1)-1}$, $q \upharpoonright d(n)$ is a prefix of some Cauchy name for y. Furthermore, since this prefix was constructed to be long enough, we get that $\psi^{q \upharpoonright d(n)}(n+1) \downarrow$. Hence by definition of ψ, we obtain

$$\left\| \psi^{q \upharpoonright d(n)}(n+1) - f(x) \right\| < 2^{-n-1}$$

and

$$\left\| \psi^{q \upharpoonright d(n)}(n+1) - f(y) \right\| < 2^{-n-1}$$

therefore, $\|f(x) - f(y)\| < 2^{-n}$. Since the described procedure is effective, we are done. □

We will now show that computable functions on effectively compact subsets of its domain will have a computable modulus of uniform continuity on that set. This result can also be found in [Wei00, Theorem 6.2.7].

Theorem 2.3.25. *Let X, Y be computable normed spaces, and $f : X \to Y$ be some computable function. Let $K \subseteq X$ be an effectively compact set, then there is a computable $d : \mathbb{N} \to \mathbb{N}$ such that d is a modulus of uniform continuity for f over K.*

Proof sketch. The main idea is to just effectivise the classical proof of the result that continuous functions over compact sets are uniformly continuous. By Theorem 2.3.24, for each ϵ, we can effectively find a corresponding δ-ball centered around each point. Then utilising effective compactness gives us an effective finite sub-cover, and we can just proceed as per the classical argument. □

Finally, the upshot of the above results is the following theorem (see [Wei00] for a TTE version).

Theorem 2.3.26. *Let X be a computable normed space, further let K be some effectively compact subset of X. Let $T : X \to \mathbb{R}$ be any computable function. Then $\max_K(f)$ is computable. (Therefore $\min_K(f)$ is also computable)*

Proof. First note that it is sufficient to show the relations $\max_K(f) < r$ and $\max_K(f) > r$ are Σ_1^0 for all $r \in \mathbb{Q}$, as we can carry out a binary-search style algorithm to compute the maximum if this was true. Without loss of generality, it is in fact sufficient to show that $\max_K(f) < r$ is Σ_1^0 uniformly in r. Fix some $r \in \mathbb{Q}$, denote $\{K_i\}_{i \in \omega}$ as some computable approximation of finite covers to the effectively compact set K, further denote $d : \mathbb{N} \to \mathbb{N}$ as the computable modulus of uniform continuity obtained via Theorem 2.3.25. We claim that

$$\max_K(f) < r \iff (\exists n)\,(\exists i > d(n))\,(\forall x \in K_i)\,\big(f(x) < r - 2^{-n}\big)$$

The forward implication follows directly. For the converse, let $x \in K$ be some arbitrary element. Since points in K_i generate an open cover for K, there is some $y \in K_i$ such that $\|x - y\| < 2^{-d(n)}$. By definition of $d(n)$, this implies that $\|f(x) - f(y)\| < 2^{-n}$. And since $f(y) < r - 2^{-n}$, we get that $f(x) < r$, which completes the proof. □

[5] Number of bits of q that was used to compute the n-th output.

Example 2.3.27 (Effective independence)**.** As an application of Theorem 2.3.26, we prove the effective independence lemma ([PER87]) from it. The effective independence lemma essentially states that in a computable normed space, linear independence is a Σ^0_1 relation. To prove this, let $\{x_0, \dots, x_n\}$ be our input. We then have

$$\{x_0, \dots, x_n\} \text{ is independent } \iff \min_{(\alpha_0,\dots,\alpha_n)\in\mathbb{R}^{n+1}\setminus 0} \left\| \sum_{i=0}^{n} \alpha_i x_i \right\| > 0$$

It is sufficient to note that instead of quantifying $(\alpha_0, \dots, \alpha_n)$ over $\mathbb{R}^{n+1} \setminus 0$, we can normalise and only quantify over the set

$$S = \left\{ (\alpha_0, \dots, \alpha_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} |\alpha_i| = 1 \right\}$$

which is an effectively compact set. Therefore, the value $\min_{(\alpha_0,\dots,\alpha_n)\in S} \|y\|$ is computable, which implies that the relation $\min_{y\in S} \|\sum_{i=0}^{n} \alpha_i x_i\| > 0$ is Σ^0_1, and we are done.

Remark 2.3.28. The example above illustrates another point of difference between computable normed spaces and countable computable vector spaces. For countable vector spaces, the corresponding notion of independence has been studied using computability through the notion of a *dependence algorithm*. Where a computable vector space V is said to have the dependence algorithm if

$$\{(v_0, \dots, v_k) \text{ is linearly dependent } | k \in \mathbb{N}, v_i \in V\}$$

is computable as a set of natural numbers. It was proven in [MN77] that any effective vector space has a dependence algorithm if and only if it has a c.e basis.[6] Whereas as shown in the example above, deciding if a finite set of vectors is linearly dependent is always Π^0_1 in a computable normed space.

It is also known that the unit balls of finite dimensional metric spaces are compact, we show that the effectivised version of the statement is also true.

Theorem 2.3.29. *Let $X = [e_0, \dots, e_N]$ be a computable normed space. The set*

$$\{x \in X : \|x\| \leq 1\}$$

is effectively compact.

Proof. Without loss of generality, we may assume that $\|e_i\| = 1$ for all i. Then the claim follows from the fact that $\{(\lambda_0, \dots, \lambda_N) : \lambda_i \in \mathbb{R}, \sum_i |\lambda_i| \leq 1\}$ is effectively compact. □

[6] Note that this refers to a Hamel basis.

Chapter 3

Complexity of effective Banach spaces

We are primarily interested in the *algorithmic content* of the various classical mathematical constructs that arise in Banach space theory. In contrast to the situation for countable vector spaces, the case for Banach spaces is much more complex and mysterious. We begin this chapter with some results regarding bases in Banach spaces that highlights this distinction.

A central notion in the theory of finite, or even countable dimensional vector spaces is the notion of a Hamel basis. Where $(v_i)_i$ is a Hamel basis for a vector space X if every $x \in X$ can be written uniquely as a *finite* linear combination of $(v_i)_i$. (i.e. $x = \sum_{i=0}^{M} \alpha_i v_i$ for an unique sequence of scalars $\{\alpha_i\}_{i \leq M}$). Unfortunately, in the case of Banach spaces, it is a consequence of the Baire category theorem that all Hamel bases will be uncountable. This highlights the major drawback of using Hamel bases for Banach spaces, namely that Hamel bases are purely *algebraic*, and do not take into account the *topology* of the spaces.

The core idea in generalising Hamel basis to topological vector spaces such as Banach spaces is to consider *infinite sums*, as this inherently relies on the topology of the space. Applying this to Banach spaces leads to the fundamental notion of *Schauder basis*.

Definition 3.0.1 (Schauder)**.** Let X be a Banach space. A sequence $(x_i)_{i \in \omega} \in X^\omega$ is a Schauder basis of X if for all $x \in X$, there is an *unique* sequence of coefficients $(a_i)_{i \in \omega} \in \mathbb{R}^\omega$ such that

$$\sum_{i=1}^{\infty} a_i x_i = x$$

A sequence that is the Schauder basis of the closure of its linear span is called a basic sequence.

Example 3.0.2 ([Haa10])**.** Define a sequence of functions $\{x_n(t)\}_{n \geq 1}$ by $x_1(t) \equiv 1$[1] and, for $k = 0, 1, 2, \ldots, l = 1, 2, \ldots, 2^k$,

$$x_{2^k+l}(t) = \begin{cases} 1 & t \in [(2l-2)2^{-k-1}, (2l-1)2^{-k-1}] \\ -1 & t \in ((2l-1)2^{-k-1}, 2l \cdot 2^{-k-1}] \\ 0 & \text{otherwise} \end{cases}$$

this is known as the *Haar system*, which forms a basis of $L_p(0,1)$ for every $1 \leq p < \infty$.

Remark 3.0.3. Not only is the existence of Schauder bases important classically, it also places a role in the theory of effective Banach spaces. For example, [BD07] analysed the computable

[1] $(\forall t)(x_1(t) = 1)$.

analogues of certain classical theorems regarding compact operators, under the assumption that the space possesses a computable Schauder basis. Another example is [BGM$^+$21] where certain theorems relied on the existence of a computable basis.

Throughout this thesis, basis will always refer to Schauder basis unless otherwise stated. It can be observed from the definition that a Banach space with a basis will necessarily be separable, as the finite linear span of the basis forms a countable dense set. However, the converse was known as the basis problem, originally posed by Banach and remained open for nearly 40 years, until Per Enflo [Enf73] came up with a negative example.

There was a huge effort towards solving the basis problem. As a part of the effort, many important properties regarding the geometry of Banach spaces were identified. Several of them will be examined in this thesis, and some of them are consequences of having a basis. One them is called the *approximation property*[2], and Enflo's example in fact lacks the approximation property. We will later expand on this important property as it enables us to prove lower bounds on certain index sets.

We note that the situation is vastly different from the case for countable vector spaces. In the arena of countable vector spaces, we are always guaranteed the existence of a Hamel basis. Whereas the existence of Schauder basis is not implied by the separability of a Banach space.

Interestingly, in the effective setting, a countable computable vector space need not have computable non-trivial independent sets. A computable infinite dimensional vector space was constructed in [MN77], where all computable independent sets are necessarily finite. This result was further generalised in [FSS83], where a computable infinite dimensional vector space was constructed such that every infinite independent set computed the halting problem. On the other hand, computable Banach spaces always have an infinite computable independent set.[3]

We will now show that in any computable infinite dimensional Banach space, there is a computable basic sequence. Unsurprisingly, a basic sequence is computable if it is uniformly computable as a sequence of points.

Definition 3.0.4. Let X be a computable Banach space. $(x_i)_{i\in\omega}$ is a computable basic sequence if it is a basic sequence and is uniformly computable as a sequence of points. A computable basis is a computable basic sequence that is also a basis.

The following is a classical lemma going back to Banach.

Lemma 3.0.5 (Banach). *Let X be a Banach space and $(x_i)_{i\in\omega} \subseteq X$ a sequence of elements. Then $(x_i)_{i\in\omega}$ is a basis of X if and only if:*

1. *$x_i \neq 0$ for all $i \in \mathbb{N}$.*
2. *There is a constant $K \in \mathbb{R}$ such that for all $n, m \in \mathbb{N}$ with $m < n$, for all sequences of scalars $(a_i)_{i\in\mathbb{N}}$, we have*
$$\left\|\sum_{i=1}^{m} a_i x_i\right\| \leqslant K \left\|\sum_{i=1}^{n} a_i x_i\right\|$$
3. *The finite linear span of $(x_i)_{i\in\mathbb{N}}$ is dense in X.*

[2] To be defined in Section 3.1.

[3] Essentially a consequence of Example 2.3.27.

The proof presented here is due to [LT77, Proposition 1.a.3].

Proof. For the forward implication, note that if $(x_i)_{i\omega}$ is a basis, then it trivially satisfies (1) and (3). Now consider the sequence of natural projections $\{S_i\}_{i\in\omega}$ associated with the basis, defined by

$$S_k\left(\sum_{i=0}^{\infty}\alpha_i x_i\right) = \sum_{i=0}^{k}\alpha_i x_i$$

(2) is equivalent to requiring the value $\sup_i \|S_i\|$ to be finite. To show this, define the alternate norm $\|\cdot\|_b$ on X by

$$\left\|\sum_{i=0}^{\infty}\alpha_i x_i\right\|_b = \sup_n \left\|\sum_{i=0}^{n}\alpha_i x_i\right\|$$

Note that this is well-defined as $(\sum_{i=0}^{n}\alpha_i x_i)_n \to \sum_{i=0}^{\infty}\alpha_i x_i$, so $\|\cdot\|_b$ is finite on any $v \in X$. Furthermore, $\|\cdot\|_b$ is indeed a norm on X, and $\|v\| \leq \|v\|_b$ for all $v \in X$. In fact, it is not hard to show that $(X, \|\cdot\|_b)$ is complete as well. An application of the open mapping theorem then proves that the norms $\|\cdot\|, \|\cdot\|_b$ are equivalent, and we are done.

We now deal with the converse. If $\sum_{i=0}^{\infty}\alpha_i x_i = 0$, then (1) and (2) forces $\alpha_i = 0$ for all i, so the representations are unique, and it remains to show that every $v \in X$ has an expansion of this form. In view of (3), it is sufficient to show that the space of elements of the form $\sum_{i=0}^{\infty}\alpha_i x_i$ is closed. Let $(\sum_{i=0}^{\infty}\alpha_{i,n}x_i)_n \to v$, (2) implies that each sequence $(\alpha_{i,n})_n$ will converge to some $\beta_i \in \mathbb{R}$. It then follows that $v = \sum_{i=0}^{\infty}\beta_i x_i$, and we are done. $\square$

The proof of Lemma 3.0.5 leads to the very important notion of *basis constants*, which will be central to this thesis.

Definition 3.0.6. Let X be a Banach space and $(x_i)_{i\in\omega}$ be a basis of X, and $\{S_i\}_{i\in\omega}$ as its associated sequence of projections. The *basis constant* of $(x_i)_{i\in\omega}$, denoted as $\mathrm{bc}((x_i)_{i\in\omega})$, is the value $\sup_i \|S_i\|$. Note that $\mathrm{bc}((x_i)_{i\in\omega})$ is equivalent to the infimum of all K that satisfies the requirements of Lemma 3.0.5. The basis constant of the space X, denoted $\mathrm{bc}(X)$, is the infimum of basis constants across all of its bases. Lemma 3.0.5 shows that this notion is well-defined. Finally, if $(x_i)_{i\in\omega}$ is not a basis we define $\mathrm{bc}((x_i)_{i\in\omega})$ to be $\mathrm{bc}((x_i)_{i\in\omega}) = \infty$.

Remark 3.0.7. In simpler spaces such as $\mathbb{R}^n$, or for Hilbert spaces, the natural bases in fact have basis constant 1. In the theory of bases for Banach spaces, bases with basis constant 1 are of special importance as well, and are called *monotone basis*. Example 3.0.2 gives monotone bases for $L_p(0,1)$ spaces. Motivated by the situation for $\mathbb{R}^n$ and the fact that all finite dimensional Banach spaces have a basis, one might wonder whether all finite dimensional Banach spaces have a monotone basis. Unfortunately, this is not the case. In fact, it was shown in [Sza83] that there *does not* exist any universal bound on the basis constant of all finite dimensional Banach spaces. (This was also known as the *finite dimensional basis problem*.)

We note that it is a direct consequence of effective compactness that basis constants of finite computable sequences are themselves computable as well. This lemma can also be found in [Bos08] under the TTE framework.

Lemma 3.0.8. *Let X be a computable Banach space, and $\{x_0, \ldots, x_n\}$ be a computable sequence of independent points. Then $bc(x_0, \ldots, x_n)$ is computable, uniform in $\{x_0, \ldots, x_n\}$.*

Proof. For the sake of simplicity, let $[x_0, \ldots, x_i]$ denote the space spanned by the points. Note that the basis constant is simply the maximum of the norms of the natural projection operators $\{P_i\}_{i\leq n}$, $P_i : [x_0, \ldots, x_n] \to [x_0, \ldots, x_i]$. Since the projections are uniformly computable, the norms will also be uniformly computable by Lemma 3.3.1, which implies the uniform computability of the basis constant. $\square$

For finite dimensional spaces, we can actually say that the basis constant of the overall space will be computable as well. This was shown to be right-c.e. in [Bos08].

Corollary 3.0.9. *Let X be a computable Banach space, and $\{x_0, \ldots, x_n\}$ be a computable sequence of linearly independent points. Then $bc([x_0, \ldots, x_n])$ is computable. Furthermore, this is uniform in $\{x_0, \ldots, x_n\}$.*

Proof. Denote $D = [x_0, \ldots, x_n]$, and let $(v_i)_{i\leq n}$ be an arbitrary sequence of elements in D. By definition, we may write $v_i = \sum_{j=0}^{n} \alpha_{i,j} x_j$, so the sequence $(v_i)_{i\leq n}$ is uniquely characterised by the sequences of coefficients

$$\alpha_{0,0}, \alpha_{0,1}, \ldots, \alpha_{0,n}, \alpha_{1,0}, \ldots, \alpha_{n,0}, \ldots, \alpha_{n,n}$$

Furthermore, as scalar scaling preserves the basis constant of $(v_i)_{i\leq n}$, we can assume without loss of generality that $\sum_{i=0}^{n} \sum_{j=0}^{n} |\alpha_{i,j}| = 1$. Consider the natural mapping $f : (\mathbb{R}^{n\times n}, \|\cdot\|_1) \to D^n$ given by

$$f\left((\alpha_{i,j})_{i,j\leq n}\right) = \left(\sum_{j=0}^{n} \alpha_{i,j} x_j\right)_i$$

under this mapping, we can naturally regard each basis of D as an element in the image. Therefore, the basis constant of D is equivalent to the minimum of basis constants on f's image. Now note that $\sum_{i=0}^{n} \sum_{j=0}^{n} |\alpha_{i,j}| = 1$ is an effectively compact subset of $(\mathbb{R}^{n\times n}, \|\cdot\|_1)$, f is a computable mapping, and basis constants are also computable on computable points. We can therefore conclude that the basis constant of D is computable by Theorem 2.3.26. Note that there is a subtlety here, as certain sequences in the image of f might not be linearly independent. But this is not hard to fix as linearly-dependent sequences have a basis constant of ∞, and we can just halt the computation once it exceeds $\mathrm{bc}(x_0, \ldots, x_n)$. $\square$

Question 3.0.10. Let X be a computable Banach space with basis. What is the complexity of $\mathrm{bc}(X)$? What if X has a computable basis? Corollary 3.0.9 shows that if X is a finite dimensional space with a computable basis, then $\mathrm{bc}(X)$ is computable.

We are now ready to prove the following theorem.

Theorem 3.0.11. *Let X be an infinite dimensional computable Banach space, then there is a computable basic sequence in X.*

Which is an effectivised version of a classical lemma already known by Banach.

Lemma 3.0.12 (Banach)**.** *Every infinite dimensional Banach space contains an infinite basic sequence.*

To prove Theorem 3.0.11, we will first need the following classical lemma, which is really the proof for Theorem 3.0.12.

Lemma 3.0.13 (Mazur)**.** *Let X be an infinite dimensional Banach space, $B \subset X$ be a finite-dimensional subspace, and $\epsilon > 0$. Then there is an $x \in X$ with $\|x\| = 1$ so that*

$$\|y\| \leq (1+\epsilon)\|y + \lambda x\|$$

for all $y \in B, \lambda \in \mathbb{R}$. In fact, x can be chosen so that this inequality is strict whenever $\|y\|, \lambda \neq 0$.

When working with separable Banach spaces, this lemma can be slightly strengthened so that we only have to deal with the dense elements.

Lemma 3.0.14. *In Lemma 3.0.13, further suppose that X is a separable Banach space with a dense sequence $(e_i)_{i\in\omega}$. We can require the desired $x \in X$ to be some element from $(e_i)_{i\in\omega}$.*

Proof. Let X be some separable Banach space with a dense sequence $(e_i)_{i\in\omega}$, $B \subset X$ be some finite-dimensional subspace and $\epsilon > 0$ be some pre-determined constant. Further denote $x \in X$ to be some element that satisfies the requirements as given by Lemma 3.0.13 with $\|x\| = 1$. Note that by homogeneity ($y \in B \iff \frac{y}{\lambda} \in B$) it is sufficient to find some $z \in (e_i)_{i\in\omega}$ which satisfies

$$\|y\| \leq (1+\epsilon)\|y+z\|$$

for all $y \in B$. As $x \notin B$, we have that $\delta_x = \min_{y\in B}\|x+y\|$ is both well-defined and positive. Let $z \in X$ be any element where $\|z\| = 1$, since $\|y+x\| \leq \|y+z\| + \|x-z\|$, we have

$$\delta_x = \min_{y\in B}\|y+x\| \leq \delta_z + \|x-z\|$$

From the inequality above, we can choose some z sufficiently close to x with $\|z\| = 1$ so that $\|x-z\| \leq \epsilon(1+\epsilon)^{-1}\delta_z$, we show that this choice works

$$\|y\| \leq (1+\epsilon)\|y+x\| = (1+\epsilon)\|y+x-z+z\|$$

$$\leq (1+\epsilon)(\|y+z\| + \|x-z\|) \leq (1+\epsilon)\left(\|y+z\| + \epsilon(1+\epsilon)^{-1}\delta_z\right)$$

And by definition of δ_z, we get that

$$(1+\epsilon)\left(\|y+z\| + \epsilon(1+\epsilon)^{-1}\delta_z\right) \leq (1+\epsilon)\|y+z\| + \epsilon\|y+z\|$$

$$= (1+2\epsilon)\|y+z\|$$

Since Lemma 3.0.13 works for all values of ϵ, the conclusion follows. In fact, the exact same argument shows that we can always choose the desired $x \in X$ to be some computable point when X is a computable Banach space. □

We are now ready to prove Theorem 3.0.11.

Proof of Theorem 3.0.11. In light of Lemmas 3.0.14 and 3.0.8, we can simply carry out the classical construction. Fix some sequence of computable reals $(\epsilon_i)_{i\in\omega}$ such that $\prod_{i=0}^{\infty}(1+\epsilon_i) < \infty$. We will construct a basic sequence $(u_i)_{i\in\omega}$ inductively. Having constructed $u_0, \ldots, u_n$, find some x in the effective dense sequence for X such that $\mathrm{bc}(u_0, \ldots, u_n, x) \leq \prod_{i=0}^{n+1}(1+\epsilon_i)$. The existence of such an element is guaranteed by Lemma 3.0.14. Furthermore, this process is computable as the basis constants are computable. □

The proof above is perfectly valid, but we also provide an alternate and perhaps more direct proof, which is more in the style of how the classical Effective Independence lemma was proved in [PER87]. This may be of interest for a more fine-grained analysis of this result (e.g. a complexity-theoretic analysis). To begin, we will define a notion of orthogonality.

Definition 3.0.15. For $x, y \in X$, we say that x is K-orthogonal to y if

$$\|y\| \leq K \|y + \lambda x\|$$

for all $\lambda \in \mathbb{R}$, and the inequality is strict unless $\|y\| = \lambda = 0$. For a subspace $Y \subseteq X$, we say that x is K-orthogonal to Y if it is K-orthogonal to all elements in Y.

We can now proceed to the proof of the effectivised statement. Similar to the proof for effective independence given in [PER87], we will first prove an orthogonality criterion.

Lemma 3.0.16 (Orthogonality Criterion)**.** *For $m, k \in \mathbb{N}$, denote $S_{m,k}$ as the set of all k-tuples $\{\beta_0, \beta_1, \ldots, \beta_k\}$ of rationals whose denominators are 2^m and satisfy*

$$1 \leq |\beta_0|^2 + |\beta_1|^2 + \ldots + |\beta_k|^2 \leq 4$$

Given a finite sequence of linearly independent elements $y_0, y_1, \ldots, y_n \in X$, we give a criterion for any $x \in X$ to be K-orthogonal to B. For the sake of simplicity, denote $y_{n+1} = x$. Further define a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$f(\vec{v}) = K \left\| \sum_{i=0}^{n+1} v_i y_i \right\| - \left\| \sum_{i=0}^{n} v_i y_i \right\|$$

Then x is K-orthogonal to B if and only if there is some m such that

$$\min \left\{ f(\vec{\beta}) : \vec{\beta} \in S_{m,n+1} \right\} > K 2^{-m} \left(\|y_{n+1}\| + 2 \sum_{i=0}^{n+1} \|y_i\| \right)$$

Proof. We first show the forward implication. Suppose that x is indeed K-orthogonal to B, this implies that f is always positive over the domain $D = \left\{ \vec{\beta} : 1 \leq \|\vec{\beta}\| \leq 2 \right\}$. As D is compact and f is continuous, f is therefore bounded from below by some positive value over D. Finally, since the right hand side of the inequality tends to 0 as m increases, any sufficiently large m will satisfy the property.

We prove the reverse by contrapositive. Suppose that x is not K-orthogonal to B, this implies that there is some non-zero vector $\vec{\gamma}$ such that $f(\vec{\gamma}) \leq 0$. Dividing through by a constant, we can assume that $\|\vec{\gamma}\| = \frac{3}{2}$, thus for each $m \in \mathbb{N}$ there is some $\vec{\beta} \in S_{m,n+1}$ such that $\|\beta_i - \gamma_i\| \leq 2^{-m}$ for all i. Now a direct computation yields

$$\begin{aligned}
f(\vec{\beta}) &= K \left\| \sum_{i=0}^{n+1} \beta_i y_i \right\| - \left\| \sum_{i=0}^{n} \beta_i y_i \right\| \\
&\leq K \left\| \sum_{i=0}^{n+1} \gamma_i y_i \right\| - \left\| \sum_{i=0}^{n} \beta_i y_i \right\| + K 2^{-m} \left(\sum_{i=0}^{n+1} \|y_i\| \right) \\
&\leq K \left\| \sum_{i=0}^{n+1} \gamma_i y_i \right\| - \left\| \sum_{i=0}^{n} \gamma_i y_i \right\| + K 2^{-m} \left(\sum_{i=0}^{n+1} \|y_i\| \right) + K 2^{-m} \left(\sum_{i=0}^{n} \|y_i\| \right)
\end{aligned}$$

$$= f(\vec{\gamma}) + K2^{-m}\left(\|y_{n+1}\| + 2\sum_{i=0}^{n+1}\|y_i\|\right)$$

$$\leq K2^{-m}\left(\|y_{n+1}\| + 2\sum_{i=0}^{n+1}\|y_i\|\right)$$

Therefore, for all m, $\min_{\vec{\beta}\in S_{m,n+1}}\{f(\vec{\beta})\}$ is no greater than the right hand side of the inequality, completing the proof. □

Proof of Theorem 3.0.11. In light of Lemma 3.0.16, we can simply carry out the classical construction. Fix some sequence of computable reals $(\epsilon_i)_{i\in\omega}$ such that $\prod_{i=0}^{\infty}(1+\epsilon_i) < \infty$. We will construct a basic sequence $(u_i)_{i\in\omega}$ inductively. Having constructed $u_0, \dots, u_n$, find some computable x in the effective dense sequence for X such that x is $(1+\epsilon_{n+1})$-orthogonal to $[u_0, \dots, u_n]$. The existence of such an element is guarenteed by Lemma 3.0.14. Furthermore, utilising the criterion from Lemma 3.0.16 gives an effective procedure to search for x, as at each stage m, the uniformly computable function f is only evaluated on $S_{m,n+1}$, a set consisting of finitely many vectors. Therefore, the construction is both valid and effective, and the proof is complete. □

Theorem 3.0.11 shows that every computable Banach space has a computable basic sequence, a natural generalisation is to ask for which Turing degrees does there exist corresponding basic sequences. We now show that all Turing degrees can be realised.

Theorem 3.0.17. *Let X be a computable Banach space of infinite dimension. For any Turing degree $\boldsymbol{a}$, there exists a basic sequence $(v_i)_{i\in\omega}$ in X such that $(v_i)_{i\in\omega} \equiv_T \boldsymbol{a}$.*

To prove this theorem, we will need the following classical lemma.

Lemma 3.0.18 ([KMR40])**.** *Let X be a Banach space, and $(x_i)_{i\in\omega}$ a normalised basic sequence with basis constant K. Then any sequence $(y_i)_{i\in\omega}$ such that $\sum_{i=0}^{\infty}\|x_i - y_i\| < \frac{1}{2K}$ is also a basic sequence.*[4]

Proof. Without loss of generality, we may assume that $(x_i)_{i\in\omega}$ is a basis for X. For $x = \sum_{i=0}^{\infty}\alpha_i x_i \in X$, define $T(x) = \sum_{i=0}^{\infty}\alpha_i y_i$. The latter series converges and

$$\|x - T(x)\| \leq \sum_{i=0}^{\infty}|\alpha_i|\,\|x_i - y_i\| \leq \max_i|\alpha_i|\sum_{i=0}^{\infty}\|x_i - y_i\|$$

$$\leq 2K\,\|x\|\sum_{i=0}^{\infty}\|x_i - y_i\|$$

under the assumptions stated in the lemma, the above calculation shows that $\|I - T\| < 1$, thus T is an automorphism of X and we are done. □

We are now ready to prove Theorem 3.0.17.

Proof of Theorem 3.0.17. By Theorem 3.0.11, we obtain a normalised computable basic sequence $(x_i)_{i\in\omega}$ in X. Let $U \in \mathbb{Q}$ be an upper bound on the basis constant of $(x_i)_{i\in\omega}$, and $A \in 2^{\omega}$ such that $A \equiv_T \boldsymbol{a}$. For each i, let $p(x_i) = \{y_{i,0}, y_{i,1}\}$ denote the first two distinct elements witnessed to have $\|x_i - y_{i,0/1}\| < \frac{2^{-i}}{4U}$ under some fixed computable enumeration of

[4]The original claim is actually stronger, these two sequences will in fact be equivalent.

$(e_i)_{i\in\omega}$, the presentation of X. Now consider the sequence $(y_{i,A(i)})_{i\in\omega}$, clearly $(y_{i,A(i)})_{i\in\omega} \leq_T \boldsymbol{a} \oplus (x_i)_{i\in\omega} \oplus U \equiv_T \boldsymbol{a}$. On the other hand, we also have $\boldsymbol{a} \leq_T (y_{i,A(i)})_{i\in\omega} \oplus (x_i)_{i\in\omega} \oplus U \equiv_T (y_{i,A(i)})_{i\in\omega}$. As any Cauchy name of $(y_{i,A(i)})_{i\in\omega}$ will allow us to determine which of $\{y_{i,0}, y_{i,1}\}$ is in the sequence uniformly for each i. Since the sequence $(y_{i,A(i)})_{i\in\omega}$ is basic by Lemma 3.0.18, the proof is complete. $\square$

Remark 3.0.19. Theorem 3.0.17 shows that in a computable Banach space with basis, the Turing degrees of bases will be closed upwards. We ask if anything can be said about the other direction.

Question 3.0.20. In a computable Banach space with basis, what can be said about the Turing degrees of its bases in addition to Theorem 3.0.17? Note that this is also related to the complexity of BASIS$_I$.[5]

As shown in Theorem 3.0.11, every infinite dimensional Banach space has a computable infinite basic sequence. In the proof, we utilised the fact that the basis constants (really the projection norms) of a computable finite sequence will be computable. A finer analysis of this technique shows that the basis constants of *infinite* computable bases are left-c.e, in contrast to the finite-dimensional case as mentioned in Corollary 3.0.9.

Lemma 3.0.21. *Let X be a computable Banach space and $(x_i)_{i\in\omega}$ a computable basis of it, then $bc\left((x_i)_{i\in\omega}\right)$ is a left-c.e. real.*

Proof. Let $\{S_i\}_{i\in\omega}$ be the sequence of projections associated to the basis. By Definition 3.0.6, the basis constant is $\sup_i \|S_i\|$, where $S_k\left(\sum_{i=0}^{\infty} \alpha_i x_i\right) = \sum_{i=0}^{k} \alpha_i x_i$. Now note that for each i, $\|S_i\|$ is a left-c.e. real as it can be approximated by S_i's norm limited to the spaces

$$\operatorname{span}\{x_0\}, \operatorname{span}\{x_0, x_1\}, \operatorname{span}\{x_0, x_1, x_2\}, \ldots$$

and clearly this is a left-c.e approximation for $\|S_i\|$, since the norm of operators with finite dimensional domain are computable as per Lemma 3.3.1. As the basis constant is then the supremum of a sequence of left-c.e reals, it is itself left-c.e as well. $\square$

It is therefore natural to ask if every left-c.e real can be realised as the basis constant of some basic sequence. We show that this is indeed the case in the following theorem.

Theorem 3.0.22. *For any $\alpha \in \mathbb{R}$ that is left-c.e and $\alpha \geq 1$, there is Banach space X with basis $(e_i)_{i\in\omega}$ such that $bc\left((e_i)_{i\in\omega}\right) = \alpha$.*

Proof. In fact, we will show that it is sufficient to have $X = c_0$. Let $(e_i)_{i\in\omega}$ denote the standard basis, the idea is to replace blocks of $\{e_i, e_{i+1}\}$ by $\{e_i + e_{i+1}, e_i + \beta_i e_{i+1}\}$, where β_i is some parameter in $\mathbb{Q}$. Since $\mathrm{bc}(e_i + e_{i+1}, e_i + \beta_i e_{i+1})$ is simply a computable function continuous in β_i with range $[1, \infty)$, we can computably find β_i so that $0 < \alpha_i - \mathrm{bc}(e_i + e_{i+1}, e_i + \beta_i e_{i+1}) < 2^{-i}$. Finally, since the blocks are disjoint, the basis constants of the prefixes of the modified basis will form the sequence $\{\alpha_0 - \varepsilon_0, \alpha_1 - \varepsilon_1, \ldots\}$ where $0 < \varepsilon_i < 2^{-i}$, thus the supremum of the sequence clearly converges to α. $\square$

Since Theorem 3.0.22 shows that the basis constants can potentially be strictly left-c.e, it is natural to consider computability of the coordinate functionals. More formally, given a basis $(e_i)_i$ in a Banach space X, let $f(i, x)$ denote the i-th coefficient in the expansion of $x \in X$

[5] See Section 3.1.

under this chosen basis. Is this function always computable? Or can it be strictly left-c.e as well? As the following theorem shows, the coordinate functionals are in fact computable, although non-uniformly.

Theorem 3.0.23. *Let $(e_i)_{i\in\omega}$ be a fixed computable basis in a Banach space X. Let $f : \mathbb{N} \times X \to \mathbb{R}$ be its corresponding coordinate functional (i.e. $f(i,x)$ is the i-th coordinate of x in its expansion under the chosen basis), then f is computable, although non-uniformly.*

Proof. Let $U \in \mathbb{Q}$ be some upper bound on the basis constant of the chosen basis. We show that f is computable relative to U. Let z some arbitrary element in X, we will compute $\{f(i,z)\}_i$ inductively via a sequence of computable estimates $\{\vec{\beta}_0, \ldots\}$, $\vec{\beta}_k = (\beta_{k_0}, \beta_{k_1}, \ldots)$ such that $\left\|\vec{\beta}_{s+1} - \vec{\beta}_s\right\|_\infty \leq 2^{-s}$ and $\|z - \sum_i \beta_{s_i} e_i\| \leq \frac{4^{-s}}{U}$. Assume that $\{\vec{\beta}_0, \ldots, \vec{\beta}_s\}$ has been constructed, we will effectively search for some $(\alpha_i)_{i\leq M} \in \mathbb{Q}^{<\omega}$ such that the following hold

$$\left\|z - \sum_i^M \alpha_i e_i\right\| \leq \varepsilon_{s+1}$$

where ε_{s+1} is some value such that $\varepsilon_{s+1} \leq \frac{4^{-s-1}}{U}$ and

$$2U\left(\frac{4^{-s}}{U} + \varepsilon_{s+1}\right) \leq 2^{-s-1}$$

Assuming some such ε_{s+1} is chosen, then $(\alpha_i)_{i\leq M}$ clearly satistifes the requirements for β_{s+1}, as we have

- $\left\|z - \sum_i^M \alpha_i e_i\right\| \leq \frac{4^{-s-1}}{U}$.
- Let $\{P_i\}_i$ denote the corresponding projections for the basis, then

$$\left\|\vec{\beta}_{s+1} - \vec{\beta}_s\right\|_\infty \leq \max_k \left\|P_k(\sum_i \beta_{s_i} e_i) - P_k(\sum_i \alpha_i e_i)\right\|$$

$$\leq \max_k \|P_k\| \left(\frac{4^{-s}}{U} + \varepsilon_{s+1}\right) \leq 2^{-s-1}$$

  Where the final inequality follows from the fact that $\|P_k\| \leq 2U$ for all k.

Since z lies within the closure of the span of the chosen basis, ε_{s+1} can be made as small as possible, so such a value always exists for sufficiently large s. Furthermore, this process is effective. Thus, we can always find some desired $(\alpha_i)_{i\leq M}$ effectively, and the construction of the approximates $\{\vec{\beta}_0, \ldots\}$ is therefore effective by taking $\beta_{s+1} = (\alpha_i)_{i\leq M}$, and the proof is complete. □

Bases with a computable coordinate functional were introduced [BGM+21], where they were called *strongly computable*. Theorem 3.0.23 shows that this notion is in fact equivalent to the basis being computable.

3.1 Geometry of effective Banach spaces

As noted earlier, there are a number of important classical notions regarding the geometry of Banach spaces. We will now define the properties that we are interested in.

Definition 3.1.1 (Schauder decomposition)**.** Let X be a Banach space. A *Schauder decomposition* of X is an infinite sequence $(Z_i)_{i\in\omega}$ of closed subspaces of X such that for all $x \in X$, there exists an unique sequence $(z_i)_{i\in\omega}$, $z_i \in Z_i$ such that

$$x = \sum_{i=1}^{\infty} z_i$$

A Schauder decomposition where the spaces Z_i are all finite dimensional is called a *finite dimensional Schauder decomposition* (FDD).

One should view Schauder decompositions as a natural variation of Schauder bases. Intuitively, if a Banach space X has a basis $(e_i)_{i\in\omega}$, we can think of X being decomposed into one-dimensional spaces of the form $X = \operatorname{span}(e_0) \oplus \operatorname{span}(e_1) \oplus \ldots$. Schauder decompositions are then equivalent to requiring X to be decomposed into *closed subspaces* in the form $X = M_1 \oplus M_2 \oplus M_3 \oplus \ldots$, where the spaces M_i are no longer required to be one-dimensional. Finite dimensional Schauder decompositions simply enforces the spaces $\{M_i\}$ to be finite dimensional. As proven in [Sza87], these properties are indeed strictly weaker than having a basis.

Definition 3.1.2 (Local basis structure, [Puj71])**.** Let X be a Banach space. X is said to have the *local basis structure* if there is some universal constant $K \in \mathbb{R}$ such that for any finite dimensional subspace $B \subset X$, there exists a finite dimensional space $L \subset X$ such that $B \subseteq L$ and $\operatorname{bc}(L) \leq K$.

Remark 3.1.3. An interesting feature of LBS is that any computable Banach space has LBS if and only if it has the computable analogue of LBS. This further elaborated in Lemma 3.5.3.

Intuitively, a Banach space X having the local basis structure is one where it can be *approximated* by a sequence of finite dimensional subspaces, where each one of them have a "nice" basis of low basis constant. This should be thought of as the "local" version of having a Schauder basis.

It is not unreasonable to wonder if LBS in fact *equivalent* to having a basis. Since it might seem that we can always build a basis using LBS by inductively extending the current "basis elements" $\{b_0, \ldots, b_n\}$ to a bigger space $E \supseteq \operatorname{span}\{b_0, \ldots, b_n\}$ which still has a bounded basis constant. However, the problem with this line of reasoning is that while we are guaranteed $\operatorname{bc}(E) \leq K$ for some universal constant K, this only means that *some* basis of E has a low basis constant. It might be the case that no basis of E which extends the current "candidate basis" $\{b_0, \ldots, b_n\}$ has its basis constant bounded by K. As it turns out, this is indeed the case as shown by the original construction by Enflo in [Enf73], which has LBS yet lacks any basis.

The remaining properties are more technical in nature.

Definition 3.1.4 (π property)**.** Let X be a Banach space. X is said to have the *π property* if there is some universal constant K such that for any finite dimensional subspace $B \subset X$, there exists a projection $P : X \to L$ such that L is finite dimensional, $B \subseteq L$ and $\|P\| \leq \lambda$.

Definition 3.1.5 (Local Π basis structure, [Sza87])**.** Let X be a Banach space. X is said to have the *Local* Π *basis structure* if there is some universal constant λ such that for any finite dimensional subspace $B \subset X$, there exists a projection $P : X \to L$ such that $B \subseteq L$, $\mathrm{bc}(L) \leq \lambda$ and $\|P\| \leq \lambda$.

Remark 3.1.6. We note that the definition introduced in [Sza87] for LΠBS is a bit different from the definition we are using. The definition as stated is due to [Puj71], where such spaces are called B_ν spaces. However, as noted in [Sza87], both definitions are in fact equivalent, and stating it this way is simply a matter of preference.

Definition 3.1.7 (Approximation property)**.** Let X be a Banach space. X is said to have the *approximation property* if on all compact sets K, for all $\varepsilon > 0$, there is a finite rank operator T such that $(\forall x \in K)\,(\|Tx - x\| < \varepsilon)$.

Definition 3.1.8 (Bounded approximation property, [JRZ71])**.** Let X be a Banach space. X is said to have the *bounded approximation property* if there is a $\lambda \geq 1$ such that for every finite dimensional $E \subseteq X$ there is a finite rank operator $T : X \to M$. Where $E \subseteq M$, $T(e) = e$ for all $e \in E$, and $\|T\| \leq \lambda$.

Remark 3.1.9. The definition of BAP used here is due to [JRZ71, Proposition 1.1], which is of a different form to the original definition (e.g. [LT77, Definition 1.e.11]). But as shown in [JRZ71], the definitions are indeed equivalent.

To define the final property, we will first need the following notion.

Definition 3.1.10 ([JL01, Page 288])**.** Let X be a separable Banach space. An approximating sequence is a sequence $\{T_i\}_{i\in\omega}$ of finite rank operators on X converging strongly to the identity such that $T_m T_n = T_n$ for all $n < m$. Finally, the sequence is a λ-approximating sequence if $\sup_i \|T_i\| \leq \lambda$, and the sequence *commutes* if $T_m T_n = T_{\min(m,n)}$ for all m, n.

Which leads us to the following.

Definition 3.1.11 (Commuting bounded approximation property)**.** Let X be a separable Banach space. X has the *commuting bounded approximation property* (CBAP) if and only if it has a λ-approximating sequence for some λ such that the operators commute.

As we are interested in the complexities of these properties, we will introduce the following notations for the sake of simplicity.

Definition 3.1.12. In the following definitions, X is only quantified over all separable Banach spaces.

$$\mathrm{BASIS} = \{X : X \text{ has a Schauder basis}\}$$

$$\mathrm{SD} = \{X : X \text{ has a Schauder decomposition}\}$$

$$\mathrm{FDD} = \{X : X \text{ has a finite dimensional Schauder decomposition}\}$$

$$\mathrm{LBS} = \{X : X \text{ has the local basis structure}\}$$

$$\pi = \{X : X \text{ has the } \pi \text{ property}\}$$

$$\mathrm{L\Pi BS} = \{X : X \text{ has the local } \Pi \text{ basis structure}\}$$

$$\mathrm{AP} = \{X : X \text{ has the approximation property}\}$$

We use an overline to denote the complement within the space of Banach spaces. E.g.

$$\overline{\mathrm{AP}} = \{X : X \text{ does not have the approximation property}\}$$

As alluded to in the introduction, we are primarily interested in classifying the complexities of such properties. One way to rigorously define the complexity of such properties is through the use of *index sets*. To be precise, let P be some mathematical property. The index set P_I corresponding to P is defined as

$$P_I = \{e : M_e \text{ has property } P\}$$

where M_e is the e-th computable structure in some acceptable listing, and the complexity of P is defined to be the complexity of the index set P_I in terms of the arithmetical hierarchy. In view of this, we use a subscript of I to denote the corresponding index set for properties in Definition 3.1.12, and a subscript of C to denote the index set for the computable analogue

$$P_C = \{e : M_e \text{ has computable } P\}$$

where "computable P" is to be defined later.

The obvious implications of the classical properties are as follows (the arrows are directed by inclusion).

$$\begin{tikzcd}
 & & \mathrm{BASIS} \arrow[dll] \arrow[drr] & & \\
\mathrm{L\Pi BS} \arrow[d] \arrow[drr] & & & & \mathrm{FDD} \arrow[dll] \arrow[ddl] \arrow[d] \\
\mathrm{LBS} & & \pi \arrow[dd] & & \mathrm{SD} \\
 & & & \mathrm{CBAP} \arrow[dl] & \\
 & & \mathrm{BAP} \arrow[d] & & \\
 & & \mathrm{AP} & &
\end{tikzcd}$$

It has been shown through a cumulation of results that most of the reversals fail, we provide a summary here.

- FDD $\not\Rightarrow$ LBS: This was proven in [Sza87], where a Banach space with (unconditional) FDD but lacks local basis structure was constructed. This also shows that FDD $\not\Rightarrow$ Q where Q is any property stronger than LBS, and that P $\not\Rightarrow$ LBS where P is any property weaker than FDD.
- LBS $\not\Rightarrow$ AP: This was proven by the original construction due to Enflo [Enf73]. The constructed Banach space is seperable and has LBS, yet lacks the approximation property, which is weaker than having a basis. In fact, this also shows that LBS $\not\Rightarrow$ P where P is any property that implies AP. To the best of our knowledge, it is currently unknown as to whether LBS $\not\Rightarrow$ SD or not.

- AP $\not\Longrightarrow$ BAP: This was proven by [FJ73].
- CBAP $\not\Longrightarrow$ π: This was claimed to be proved in [Rea86], but the paper was never published.

To the best of our knowledge, the following implications are currently open.

- BAP $\Longrightarrow$ CBAP: This was noted to be an open problem in [JL01, Problem 4.2].
- π $\Longrightarrow$ CBAP: This was also noted to be open in [JL01, Problem 5.3].
- π $\Longrightarrow$ FDD: Finally, this was also noted to be open in [JL01, Problem 6.2].
- At last, we remark that SD $\Longrightarrow$ FDD also appears to be open, but we could not find this being explicitly stated in the literature.

3.2 Complexity of basis

As Enflo has shown, there exists separable Banach spaces with no basis at all. This naturally leads one to wonder what is the complexity of such constructions, and the complexity of Banach spaces that have a basis.

In this section, we first examine the complexity of computable bases. Bosserhoff [Bos08] was the first to show that there exists a computable Banach space with a basis, but does not have any computable basis. This construction was built on Davie's classical construction [Dav73], which Bosserhoff has shown to be computable. We utilise this construction to further show that the complexity of having a computable basis is exactly Σ^0_3. Carrying on, we then prove that the computability of Davie's space is strong enough to show a lower-bound of Π^0_3 on the computable Banach spaces that have a basis, and this is in some sense about the "best" one could do with the current constructions.

3.2.1 Complexity of computable basis

We will now show that having a computable basis is Σ^0_3 complete. The argument builds on a diagonalisation argument by [Bos08].

Theorem 3.2.1. *$BASIS_C$ is Σ^0_3 complete.*

We first introduce the construction used in [Bos08]. Let Z denote the Banach space constructed in [Dav73] that lacks the approximation property. It was proven in [Bos08] that this space is computable and also exhibits the local basis property.

Theorem 3.2.2 ([Bos08])**.** *There exists a computable Banach space without AP but has LBS.*

In particular, this implies that Z can be approximated by a sequence of "nice" subspaces.

Theorem 3.2.3 ([Bos08])**.** *There is a computable linearly independent sequence $(x_i)_{i\in\omega} \subseteq Z$, a computable increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ and an universal constant C such that $[x_0, \ldots] = Z$ and*

$$(\forall n \in \mathbb{N})(\mathrm{bc}([x_0, \ldots, x_\sigma(n)]) < C)$$

We first need the following definitions.

Definition 3.2.4 ([Bos08])**.** For any $n \in \mathbb{N}$, Z_n is defined as:

$$Z_n = [x_0, \ldots, x_{\sigma(n)}]$$

where $(x_i)_{i\in\omega}$ is given by Theorem 3.2.3. For any $\tau : \mathbb{N} \to \mathbb{N}$, the Banach space Y_τ is defined as:

$$Y_\tau = \left(\oplus_i Z_{\tau(i)}\right)_{c_0}$$

which is the sequence space where norms of elements within each sequence tends to 0, and the norm on the sequence is the supremum norm on the elements.

An important feature of this space is that it has a basis. Intuitively, as the columns have universally bounded basis constants, we can simply "join up" the bases of the columns in the larger space, and the resulting sequence will be a basis. This is also a consequence of Lemma 3.2.23.

Lemma 3.2.5 ([Bos08])**.** *The space Y_τ as defined in Definition 3.2.4 has a basis for any $\tau : \mathbb{N} \to \mathbb{N}$.*

The key idea is that Y_τ is a Banach space with basis, however each of its components can be made arbitrarily "large" such that no computable sequence can span it. For the sake of simplicity, also denote $Y = (\oplus_i Z)_{c_0}$. The following lemma is crucial.

Lemma 3.2.6 ([Bos08])**.** *For any basic sequence $(y_i)_{i\in\mathbb{N}} \in Y^\omega$ and $n \in \mathbb{N}$, we have*

$$emb^n(Z) \not\subseteq [y_0, y_1, \ldots]$$

Where $\mathrm{emb}^n : Z \to Y$ is the map defined by

$$\mathrm{emb}^n(x) = (0, \ldots, 0, x, 0, \ldots) \in Y$$

mapping $x \in Z$ to n-th position of a sequence that is otherwise entirely zero.

There is also a natural computability structure on the space Y_τ for certain classes of τ.

Definition 3.2.7. A function $\tau : \mathbb{N} \to \mathbb{N}$ is *lower semicomputable* if there is a c.e set $A \subseteq \mathbb{N}$ such that

$$\tau(n) = \sup\{k \in \mathbb{N} : \langle n, k\rangle \in A\}$$

for all $n \in \mathbb{N}$.

Lemma 3.2.8 ([Bos08])**.** *For any $\tau : \mathbb{N} \to \mathbb{N}$ that is lower semicomputable, the constructed space Y_τ equipped with the dense set $\{emb^j(x_i)\}_{i\leq\sigma(\tau(j)),j\in\mathbb{N}}$ is a computable Banach space.*

Finally, to construct a computable Banach space without any computable basis, it is sufficient to construct some lower semicomputable τ such that Y_τ does not contain any computable basis. Furthermore, by Lemma 3.2.6 and Theorem 3.2.3, we can construct τ by directly diagonalising against all computable basic sequences. The following is due to [Bos08], although presented in a slightly different fashion.

Lemma 3.2.9 ([Bos08])**.** *There is a lower semicomputable function $\psi : \mathbb{N}^3 \to \mathbb{N}$ such that for all $n, k, i \in \mathbb{N}$, if φ_n computes a basic sequence $(y_i)_{i\in\mathbb{N}} \in Y^\omega$ with basis constant smaller than k, we have*

$$emb^i(Z_{\psi(n,k,i)}) \not\subseteq [y_0, \ldots]$$

Corollary 3.2.10 ([Bos08]). *There exists a computable Banach space without computable basis.*

Proof. By Lemmas 3.2.8 and 3.2.9, define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(\langle n,k\rangle) = \psi(n,k,\langle n,k\rangle)$$

The resulting space Y_τ is a computable Banach space where $\tau(\langle n,k\rangle)$ is large enough so that $\mathrm{emb}^{\langle n,k\rangle}(Z_{\tau(\langle n,k\rangle)})$ is not spanned by φ_n (if it is a basic sequence with basis constant smaller than k). This implies that the space Y_τ cannot be spanned by any computable basic sequence[6], and therefore lacks basis. $\square$

It is worth noting that although the space constructed in Corollary 3.2.10 has no computable basis, it is unclear how uncomputable the bases are.

Question 3.2.11. Let Y_τ be the space used in the proof of Corollary 3.2.10 that was constructed by [Bos08]. What are the corresponding Turing degrees for the bases in this space?

We are now ready to prove Theorem 3.2.1, and begin with membership.

Proposition 3.2.12. $BASIS_C \in \Sigma^0_3$.

Proof. Let X_e be a computable Banach space. Then X_e has a computable basis if and only if there exists i such that:

$$\begin{aligned}
&\varphi_i \text{ computes some computable sequence } (x_n)_{n\in\mathbb{N}} \subseteq X \\
&\wedge (\forall n)(x_n \neq 0) \\
&\wedge (\exists K \in \mathbb{Q})\, (\forall m < n)\, (\forall (a_k)_{k\leq n} \subseteq \mathbb{Q}) \left\| \sum_{k=1}^{m} a_k x_k \right\| \leqslant K \left\| \sum_{k=1}^{n} a_k x_k \right\| \\
&\wedge \text{ the finite span of } (x_n)_{n\in\mathbb{N}} \text{ is dense in } X
\end{aligned}$$

The validity of this equivalence follows from Lemma 3.0.5. Furthermore, each individual condition is in Σ^0_3. Since X_e being a computable Banach space is also a Π^0_2 condition (see [BMM20]), the overall set S is therefore within Σ^0_3. $\square$

The following lemma is needed to show completeness.

Lemma 3.2.13. *Recall the construction carried out in Lemma 3.2.4. If τ is a computable function, then Y_τ contains a computable basis.*

Proof. As the basis constant of $Z_{\tau(i)}$ is uniformly bounded by some constant C, there is some basis $(a_{i,j})_{j\leq\sigma(\tau(i))}$ with basis constant smaller than C for each $Z_{\tau(i)}$. It was proved in [Bos08] that the natural embedding of these bases into Y_τ (i.e. $\{\mathrm{emb}^i(a_{i,j}) | i \in \mathbb{N}, j \leq \sigma(\tau(i))\}$) forms a basis for Y_τ. We will show that this is actually computable when τ is computable. If τ is computable, the sequence

$$x_0, x_1, \ldots, x_{\sigma(\tau(i))}$$

will be computable as well since $(x_i)_{i\in\mathbb{N}}$ and σ are both computable. Therefore, the rational span of the sequence will be computable as well. By continuity, we can therefore effectively find some basis that lies in the rational span of $(x_i)_{i\leq\sigma(\tau(i))}$ with basis constant smaller than C. As this procedure is uniform, it gives a computable basis in Y_τ. $\square$

[6] Note that any computable sequence in Y_τ is also a computable sequence in Y, so it is sufficient to diagonalise against computable sequences in Y.

We are now ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Due to Proposition 3.2.12, it remains to show that S is Σ^0_3 hard. It is a well known fact that for any set $A \in \Sigma^0_3$, there is a computable function $g : \mathbb{N}^2 \to \mathbb{N}$ such that

$$x \in A \iff (\exists y)(W_{g(x,y)} \text{ is infinite })$$

For all $x \in \mathbb{N}$, we construct a lower semicomputable function $h : \mathbb{N} \to \mathbb{N}$ in stages. Let $\{\psi_s\}$ be some computable enumeration of the function ψ constructed in Lemma 3.2.9. We also define the function $C : \mathbb{N} \to \mathbb{N}$, initially $C_0(n) = n$ for all $n \in \mathbb{N}$. $C(n)$ indicates the computable sequence that is diagonalised against at n. Initialise the construction by setting $h_s = 0$. At stage s, the following is carried out for each $n \leq s$.

- If $C(n) = -1$, do nothing. Otherwise:
- Enumerate $W_{g(x,C(n)),s}$. If a new element is enumerated, set $C(k)$ to $C(k-1)$ for all $k > n + |W_{g(x,C(n)),s}|$ and $C(n + |W_{g(x,C(n)),s}|)$ to -1.
- View $C(n)$ as a pair $\langle a, b\rangle$ and set $h_s(n)$ to $\max(h_{s-1}(n), \psi_s(a, b, n))$.

Finally we define h as $h = \lim_{s\to\infty} h_s$. This is the end of the construction, we now verify its validity.

Lemma 3.2.14. *The function h constructed is indeed a lower semicomputable function.*

Proof. The constructed sequence $\{h_s\}$ is clearly a computable enumeration of h. So it remains to verify that $\{h_s\}$ converges. For any $n \in \mathbb{N}$, we have $C(n) \leq n$. Therefore $h_s(n) \leq \max_{\langle a,b\rangle \leq n} \psi(a, b, n)$ for all s, and since $(h_s(n))_s$ is monotone, this implies convergence. □

We now show that the constructed h has the desired properties.

Lemma 3.2.15. *In addition to h being lower semicomputable, it also exihibit the following properties*

- *If $x \in A$, h is computable (although this might be non-uniform).*
- *If $x \notin A$, Y_h contains no computable basis.*

Proof. Suppose $x \in A$, thus there is some y such that $W_{g(x,C(y))}$ is infinite. By the construction, this means that

$$-1 = C(y + 1) = C(y + 2) = C(y + 3) = \dots$$

Therefore, to compute $h(k)$ for any $k > y$, we just have to run the computable construction for finitely many steps until $C(k) = -1$, in which case the current value of $h(k)$ will be its final value. And since there are only finite many values $h(k)$ for $k \leq y$, this can be computed non-uniformly. Hence, h is a computable function.

Now suppose $x \notin A$, in which case $W_{g(x,y)}$ is finite for all $y \in \mathbb{N}$. We will show that for all $\langle a, b\rangle \in \mathbb{N}$, there is some $n \in \mathbb{N}$ where $C(n) = \langle a, b\rangle$, implying that $h(n) \geq \psi(a, b, n)$ and therefore Y_h cannot contain any computable basis. At each stage s of the construction, there will be some index i_s where $C_s(i_s) = \langle a, b\rangle$. So it suffices to show that $(i_s)_s$ eventually stabilises. But by the construction, i_s can only increase when some new element has been enumerated in $W_{g(x,C(k))}$ for some $C(k) < \langle a, b\rangle$. And since $\{k : C(k) < \langle a, b\rangle\}$ is finite, and each set of the form $W_{g(x,y)}$ is finite as well, i_s can only increase for a finite number of steps until it eventually converges, and the proof is complete. □

Therefore, as the construction of h is uniform in x, we have established a reduction from an arbitrary Σ^0_3 set to BASIS_C, proving that BASIS_C is indeed Σ^0_3 hard. □

Remark 3.2.16. It was asked in [Bos08] as to whether there exists a computable Banach space with a $\{\text{monotone, unconditional}\}$[7] basis that does not have the corresponding computable analogue. In view of this, it would also be interesting to determine the complexity of the corresponding index sets.

Question 3.2.17. What is the complexity of MBASIS_C, the index set of Banach spaces that have a computable monotone basis?

Question 3.2.18. What is the complexity of UBASIS_C, the index set of Banach spaces that have a computable unconditional basis?

Finally, the construction in [Bos08] gives a computable Banach space without basis, but this is only for the chosen presentation. This naturally leads to the following questions.

Question 3.2.19. Is there a presentation for the space constructed in [Bos08] that has a computable basis?

Question 3.2.20. Is the existence of a computable basis dependent on the presentation chosen? That is, is there a computable Banach space with a computable basis, but does not have a computable basis in some other presentation?

Remark 3.2.21. As a specific case of the question above, it would be interesting to know if $C[0, 1]$ has a presentation that lacks any computable basis, since $C[0, 1]$ is known to be universal for separable Banach spaces.[8]

3.2.2 Complexity of BASIS_I

After showing that BASIS_C is Σ^0_3 complete, it is natural to ask whats the complexity of BASIS_I. By Lemma 3.0.5, it follows that $\mathrm{BASIS}_I \in \Sigma^1_1$, but it is unknown if this bound is tight. In this section, we establish the lower bound of Π^0_3. We first need the following.

Definition 3.2.22. Define the sets

$$\mathrm{BASIS}_k = \{X : X \text{ has a basis with basis constant no greater than } k\}$$

for all $k \geq 1$. Analogously, we obtain the corresponding definitions for BASIS_{I_k}, BASIS_{C_k}.

Before we proceed, we also need a technical lemma.

Lemma 3.2.23. *Let $(X_i)_{i\in\omega}$ be a sequence of Banach spaces that have uniformly bounded basis constant, then*

$$Y = (X_0 \oplus X_1 \oplus \ldots)_{c_0} \in BASIS$$

[7] An unconditional basis is one where the basis remains a basis under any permutation of its elements.
[8] Every separable Banach space can be isometrically embedded into it.

Proof. Denote M as the uniform bound on the basis constants, and let $(b_{k,i})_{i\in\omega}$ denote some basis in X_k that has a basis constant no greater than M. Fix some canonical enumeration of this collection of bases across all spaces (viewing them as elements of Y under the natural embedding), denote it as $\{\beta_i\}_{i\in\omega}$, we show that this sequence is a basis for Y. It is clear from the definition that if $y \in Y$ has an expansion $y = \sum_i \lambda_i\beta_i, \lambda_i \in \mathbb{R}$, then this expansion must be unique as each X_i is a complemented subspace of Y. Thus, it suffices to show that each $y \in Y$ has an expansion using $\{\beta_i\}_{i\in\omega}$. Let $(z_i)_{i\in\omega} = y \in Y$ be some arbitrary element in Y, and $(\lambda_i)_{i\in\omega}$ be the its natural coordinates by viewing β_i as bases in the spaces X_k and z_k as elements in X_k. Then we have the following partial sums

$$S_U = \left\| y - \sum_i^U \lambda_i\beta_i \right\|$$

As each β_i is an element of some space X_i, denote L_U as the maximum index on the spaces encountered for S_U. Then we have the following

$$S_U = \max\left(\max_{i\leq L_U}\left(\left\| z_i - \sum_j^{k_i} \lambda_j\hat{\beta}_j \right\| \right), \sup_{i>L_U} \|z_i\| \right)$$

where $\hat{\beta}_j$ denotes the natural projection onto the corresponding column. Finally, as the sequence $(\beta_i)_i$ was taken to a basis, we get that for all i:

$$\left\| z_i - \sum_j^{k_i} \lambda_j\hat{\beta}_j \right\| \leq \|z_i\| + \left\| \sum_j^{k_i} \lambda_j\hat{\beta}_j \right\| \leq \|z_i\| + M\|z_i\|$$

Finally, we get that

$$S_U \leq \max\left(\|z_i\| + M\|z_i\|, \sup_{i>L_U} \|z_i\| \right)$$

And since $\|z_i\| \to 0$ as $i \to \infty$, we get that $(S_U)_U \to 0$, as desired. □

Remark 3.2.24. In fact, the result above can be easily effectivised. If we have a computable sequence of Banach spaces $(X_i)_{i\in\omega}$ such that each of them has a computable basis and the basis constant is uniformly bounded, then the resulting space $Y = (\oplus_i X_i)_{c_0}$ is also a computable Banach space with a computable basis.

Lemma 3.2.25. *The following implication holds for all $n \in \mathbb{N}$, for all $k \in \mathbb{Q}$.*

$$\left(\Sigma_n^0, \Pi_n^0\right) \leq_m \left(BASIS_{I_k}, \overline{AP_I}\right) \implies \left(\Pi_{n+1}^0, \Sigma_{n+1}^0\right) \leq_m \left(BASIS_I, \overline{AP_I}\right)$$

Proof. We carry out a similar construction to [Bos08]. Suppose that we have $\left(\Sigma_n^0, \Pi_n^0\right) \leq_m \left(\mathrm{BASIS}_{I_k}, \overline{\mathrm{AP}_I}\right)$, denote g as some function that witnesses this reduction. Then define $f : \mathbb{N} \to \mathbb{N}$ such that

$$Y_{f(x)} = (X_{g(x,0)} \oplus X_{g(x,1)} \oplus \ldots)_{c_0}$$

We then have:

- $x \in \Pi_{n+1}^0 \implies (\forall i)\left((x,i) \in \Sigma_n^0\right) \implies (\forall i)(X_{g(x,i)} \in \mathrm{BASIS}_{I_k})$.
- $x \notin \Pi_{n+1}^0 \implies (\exists i)\left((x,i) \in \Pi_n^0\right) \implies (\exists i)(X_{g(x,i)} \in \overline{\mathrm{AP}_I})$.

In the case where $x \in \Pi^0_n$, the conclusion follows by Lemma 3.2.23. For the case where $x \notin \Pi^0_n$, we get that a complemented subspace of Y fails the to have the approximation property. And since approximation properties are preserved by complemented subspaces, this implies that $Y \in \overline{\mathrm{AP}_I}$, and the proof is complete. □

Finally, we have the following lower bound.

Theorem 3.2.26.

$$\left(\Pi^0_3, \Sigma^0_3\right) \leq \left(BASIS_I, \overline{AP_I}\right)$$

Proof. In view of Lemma 3.2.25, it is sufficient to show that $\left(\Sigma^0_2, \Pi^0_2\right) \leq \left(\mathrm{BASIS}_{I_k}, \overline{\mathrm{AP}_I}\right)$ for some $k \in \mathbb{R}$. We will utilise Davie's construction in [Dav73]. As Σ^0_2 is equivalent to the set of Σ^0_1 indices of finite sets, we can directly enumerate Davie's space. More formally, denote Z as Davie's space, it was shown that Z exhibits LBS in Theorem 3.2.3. Denote Z_n as was done in Definition 3.2.4. We will now reduce (FIN, INF) to $\left(\mathrm{BASIS}_{I_k}, \overline{\mathrm{AP}_I}\right)$, where k is the constant C in Theorem 3.2.3. For any $e \in \mathbb{N}$, we construct the space $Z_{|W_e|}$. Since W_e is c.e, this construction is effective and uniform in e. Furthermore, the constructed space will have basis constant no more than k if $e \in FIN$, and it will be Z if $e \in INF$, which lacks the approximation property by definition. □

As a corollary, we obtain that any property that is stronger than AP yet weaker than BASIS is also Π^0_3 hard. This will be discussed more in the sections corresponding to those properties.

3.3 π-space

It was proven in [Gha15] that the class of seperable Banach spaces with the π property is Borel, and it was implicit that this class is in fact Σ^0_6 (explicitly stated in [Gha19] in the Borel sense). We present this result here in a more modular fashion, and prove an improved bound of Σ^0_4 in the computable case. We first show that norms of finite rank operators are computable relative to its domain.

Lemma 3.3.1. *Given a computable finite rank operator T on a computable domain $D = [x_0, \ldots, x_n]$ to a computable Banach space Y, its norm is also computable.*

Proof. The norm of T is equivalent to the maximum of T over the unit ball of D. As D is finite dimensional, its unit ball is therefore compact. The claim then follows from Theorems 2.3.26 and 2.3.29. □

We will now prove a technical lemma that allows us to solely work in the subspaces generated by a given dense sequence. We first need the following result due to [JRZ71].

Lemma 3.3.2 ([JRZ71])**.** *Let T be an operator from a Banach space X onto an n-dimensional subspace $E \subset X$. Let $k \leq n$ and let F be a k-dimensional subspace of X such that $\|T|_F - I_F\| < \varepsilon < 1$, where $(1-\varepsilon)^{-1}\varepsilon k < 1$. Then*

1. *there is an operator S from X onto an n-dimensional subspace of X such that $S|_F = I|_F$, $\|S - T\| < (1-\varepsilon)^{-1}\varepsilon k \|T\|$.*

2. *If, in addition, T is a projection then S can be chosen to be a projection and* $\left\| S|_{T(X)} - I|_{T(X)} \right\| < (1-\varepsilon)^{-1}\varepsilon k$.

We can now show that projections "carry over" to sufficiently close spaces.

Lemma 3.3.3. *Let X be a separable Banach space with a dense sequence $(e_i)_{i\in\omega}$. Let L be a finite dimensional subspace of the form $L = [x_0, \dots, x_M]$. Further assume that there is a projection $T : X \to L$. Then for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $N = [y_0, \dots, y_M]$ where $\|y_i - x_i\| < \delta$ for all $i \leq M$, there exists a projection $S : X \to N$ where $\|S\| \leq \|T\| + \epsilon$.*

Proof. By Lemma 3.3.2, it is sufficient to show that for all projections $T : X \to L$, for all $\varepsilon > 0$, all spaces N_ε where the y_i's are chosen to be sufficiently close to x_i satisfies $\|T|_{N_\varepsilon} - I|_{N_\varepsilon}\| < \varepsilon$. Let $\sum_{i=0}^{M} \lambda_i y_i$ be some arbitrary element from N_ε, where the basis elements $\{y_i\}_{i\leq M}$ are to be determined later. We then have

$$\left\| T\left(\sum_{i=0}^{M} \lambda_i y_i\right) - \left(\sum_{i=0}^{M} \lambda_i y_i\right)\right\|$$

$$= \left\| \sum_{i=0}^{M} \lambda_i \left(T(y_i) - y_i\right) \right\|$$

$$\leq \sum_{i=0}^{M} |\lambda_i| \, \|T(x_i) - x_i + T(y_i - x_i) - (y_i - x_i)\|$$

$$\leq \sum_{i=0}^{M} |\lambda_i| \left(\|T\| + 1\right) \|x_i - y_i\|$$

As y_i could be chosen arbitrarily close to x_i for each i, we can without loss of generality assume that $\|x_i - y_i\| = \varepsilon_1$ for all i. Applying Theorem 2.2.14 to L gives a constant C *only dependent* on L such that $\sum_{i=0}^{M} |\lambda_i| \leq C \left\| \sum_{i=0}^{M} \lambda_i x_i \right\|$. Hence,

$$\sum_{i=0}^{M} |\lambda_i| \left(\|T\| + 1\right) \|x_i - y_i\|$$

$$\leq C\left(\|T\| + 1\right) \varepsilon_1 \left\| \sum_{i=0}^{M} \lambda_i x_i \right\|$$

Now note that $\left\| \sum_{i=0}^{M} \lambda_i (x_i - y_i) \right\| \leq \varepsilon_1 C \left\| \sum_{i=0}^{M} \lambda_i x_i \right\| \implies (1 - \varepsilon_1 C) \left\| \sum_{i=0}^{M} \lambda_i x_i \right\| \leq \left\| \sum_{i=0}^{M} \lambda_i y_i \right\|$. Chaining this together with the previous inequality gives

$$C\left(\|T\| + 1\right) \varepsilon_1 \left\| \sum_{i=0}^{M} \lambda_i x_i \right\| \leq \frac{\varepsilon_1 C\left(\|T\| + 1\right)}{1 - \varepsilon_1 C} \left\| \sum_{i=0}^{M} \lambda_i y_i \right\|$$

and since ε_1 can be made arbitrarily small, $\|T|_{N_\varepsilon} - I|_{N_\varepsilon}\|$ is smaller than ε for all sufficiently close $(y_i)_{i\leq M}$, and the proof is complete. $\square$

We immediately obtain the following corollary.

Corollary 3.3.4. *Let X be a separable Banach space with a dense sequence $(e_i)_{i\in\omega}$. Then X has the $\boldsymbol{\pi}$ property if and only if there exists λ such that for all $\vec{e} \in \{e_i : i \in \omega\}^{<\omega}$, there exists $\vec{v} \in \{e_i : i \in \omega\}^{<\omega}$ such that $span\{\vec{e}\} \subseteq span\{\vec{v}\}$ and there exists a projection $T : X \to span\{\vec{v}\}$ where $\|T\| \leq \lambda$.*

Proof. The forward implication follows from Lemma 3.3.3. For the converse, note that X has the $\boldsymbol{\pi}$ property if there exists a sequence of finite dimensional subspaces $\{E_i\}_{i\in\omega}$ directed under inclusion such that $X = \overline{\bigcup_i E_i}$ and for each i, there is an associated projection $T : X \to E_i$ where $\|T\| \leq \lambda$. Such a sequence can be built inductively using elements from the dense sequence, and we are done. □

Next, we need to show that projections between effective finite dimensional subspaces are contained in effectively compact sets.

Lemma 3.3.5. *Let X be a computable Banach space with $(e_i)_{i\in\omega}$ as its presentation. Let $M = [e_{n_0}, \ldots, e_{n_k}]$ be a finite dimensional subspace (with $(e_{n_i})_{i\leq k}$ being linearly independent), $E = [e_0, \ldots, e_i]$ be some arbitrary subspace such that $i > n_k$ (i.e. $M \subset E$), and $U \in \mathbb{Q}$ be some arbitrary constant. Consider $L(E, M)$ as a computable Banach space under its standard presentation as in Definition 2.3.16. Then there is an effectively compact set $K \subseteq L(E, M)$ such that $P_U(E, M) \subseteq K \subseteq P(E, M)$. Furthermore, such a set K can be produced uniformly in M, E and U.*

Proof. Intuitively, since matrices/operators in $P_U(E, M)$ have their norm bounded by U, this induces a bound on each of its entries. This shows that the coefficients lie in an effectively compact subset of $(\mathbb{R}^{n_k\times i}, \|\cdot\|_\infty)$ under its standard representation. Furthermore, there is a natural mapping from $(\mathbb{R}^{n_k\times i}, \|\cdot\|_\infty)$ to $L(E, M)$, and we will show that taking K to be the image of this natural mapping is sufficient.

Recall that an operator $T \in L(E, M)$ is uniquely determined by its values on $T(e_0), \ldots, T(e_i)$, which are in turn uniquely determined by their coefficients of the basis $e_{n_0}, \ldots, e_{n_k}$ in M. Let $\sigma_j^l \in \mathbb{R}$ denote the coefficient of e_{n_j} for $T(e_l)$, and V as twice the basis constant of $(e_{n_i})_{i\leq k}$, which is uniformly computable by Theorem 3.0.8. Let $l \leq i$ be some arbitrary index, we then have

$$V\,\|T(e_l)\| = V\left\|\sum_{j=0}^{k} \sigma_j^l e_{n_j}\right\|$$

Since V is a bound on the norms of the projection operators, this implies that

$$V\left\|\sum_{j=0}^{k} \sigma_j^l e_{n_j}\right\| \geq \left\|\sigma_j^l e_{n_j}\right\|$$

for each $j \in \{0, \ldots, k\}$. But recall that we are only interested in the case where $\|T\| \leq U$, so we have the following inequality.

$$\left\|\sigma_j^l e_{n_j}\right\| \leq UV\,\|e_l\| \implies |\sigma_j^l| \leq \left\|e_{n_j}\right\|^{-1} UV\,\|e_l\|$$

In conclusion, we get that

$$(\forall l \leq i)\,(\forall j \leq k)\left(|\sigma_j^l| \leq \left\|e_{n_j}\right\|^{-1} UV\,\|e_l\|\right)$$

Since $V, \|e_l\|, \|e_j\|^{-1}$ are all uniformly computable, we have shown that all entries σ_j^l have effective bounds uniform in M, E and U.

It now follows that the coefficients $\{\sigma_j^l\}_{l,j}$ lie within an uniformly effectively compact subset of $(\mathbb{R}^{k\times i}, \|\cdot\|_\infty)$. Note that we have not limited ourselves to projections in the calculations above, rather the bounds calculated will hold for any operator in $L(E, M)$. However, limiting ourselves to $P(E, M)$ is just equivalent to requiring $\sigma_j^l = \delta_{ln_j}$ (where δ_{pq} is the Kronecker delta) for $l \in \{n_0, n_1, \ldots, n_k\}$, and this is still an uniformly effectively compact subset of $(\mathbb{R}^{k\times i}, \|\cdot\|_\infty)$, denote this subset as J.

It now remains to show that the natural mapping (the "identity" map that takes each matrix to its corresponding operator) $f : (\mathbb{R}^{k\times i}, \|\cdot\|_\infty) \to L(E, M)$ is indeed a computable mapping, as it follows by construction that $P_U(E, M) \subseteq f(J) \subseteq P(E, M)$. Furthermore, this function is in fact effectively uniformly continuous, and therefore it must be a computable mapping. This concludes the proof. □

We now show that the existence of projections of a bounded norm onto finite dimensional subspaces generated from the dense sequence in Π_1^0.

Lemma 3.3.6. *Let X be a computable Banach space with $(e_i)_{i\in\omega}$ as its computable presentation. Let M be a finite dimensional subspace of the form $M = [e_{n_0}, \ldots, e_{n_M}]$. Then the formula*

$$J_{M,K} = (\exists P : X \to M)(P \text{ is a projection} \wedge \|P\| \leq K)$$

is uniformly Π_1^0 for $K \in \mathbb{Q}$.

Proof. We first claim that

$$J_{M,K} \iff (\forall i > n_M)(\exists T : [e_0, \ldots, e_i] \to M)(T \text{ is a projection} \wedge \|T\| \leq K)$$

The forward implication clearly holds. For the converse, let $\{T_i\}_i$ denote a seqeunce of projections where $\|T_i\| \leq K$ and $T_i : [e_0, \ldots, e_i] \to M$, we will define a projection $T : X \to M$ from this sequence. Let $y \in (e_i)_{i\in\omega}$ be any element from the dense sequence, then the sequence $\{T_i(y)\}_i$ lies within $\{x \in M : \|x\| \leq K\|y\|\}$, which is a compact set as M is finite dimensional. Therefore, by a diagonal argument, we can define a linear operator T such that $T(y)$ is the limit of some subsequence of $\{T_i(y)\}_i$ for all $y \in (e_i)_{i\in\omega}$. It follows from the construction that this operator will be a projection with norm bounded by K. Furthermore, since the domain on which T is defined is dense in X, there is an unique extension of T onto X. Finally, the constructed operator T will be a projection from X to M as well, as it is a projection on the dense sequence.

We now know that $J_{M,K}$ is a Π_2^0 statement uniform in $K \in \mathbb{Q}$. However, note that the existential statement

$$(\exists T : [e_0, \ldots, e_i] \to M)(T \text{ is a projection} \wedge \|T\| \leq K)$$

is equivalent to evaluating the minimum value of the computable operator $f(T) = \|T\|$ on the effectively compact subset obtained in Lemma 3.3.5. Since Lemma 3.3.5 shows that this space is uniformly effectively compact, this statement is in fact also Π_1^0 by Theorem 2.3.26, and we are done. □

Chaining the previous results together, we obtain the following.

Theorem 3.3.7. *The set of computable Banach spaces with the π-property is Σ_4^0.*

Proof. This follows by combining Corollary 3.3.4 and Lemma 3.3.6. □

3.4 Local Π basis structure

After Enflo's example of a separable Banach space that lacks the approximation property, people naturally wondered if there is some separable Banach space without basis that *has* the approximation property (i.e. is the approximation property strictly weaker than having a basis?). This was first solved by [FJ73], where a space with the approximation property but lacked the bounded approximation property (therefore lacks a basis) was constructed. Extending the previous question, it is then natural to ask if there is some separable Banach space without a basis that *has* the bounded approximation property. This question was solved in [Sza87], where a separable Banach space without basis but exhibits the bounded approximation property was constructed. In [Sza87], LΠBS was introduced, and it has been open since as to whether LΠBS is equivalent to having a basis.

We are mostly interested in the complexity of effective Banach spaces that exhibit this property. For the Borel hierarchy, it has been proven in [CDDK21] that its complexity is $\mathbf{\Sigma^0_6}$. We show that similar to the π-property, the property of local Π basis structure is Σ^0_4 in the effective case.

The overall structure of the proof is very similar to the π-property case. We will first show that it is sufficient to consider solely the subspaces generated from a dense sequence.

Lemma 3.4.1. *Let X be a separable Banach space with $(e_i)_{i\in\omega}$ as a dense sequence. Then X has the LΠBS property if and only if there exists λ such that for all $\vec{e} \in \{e_i : i \in \omega\}^{<\omega}$, there exists $\vec{v} \in \{e_i : i \in \omega\}^{<\omega}$ such that span$\{\vec{e}\} \subseteq$ span$\{\vec{v}\}$ and a projection $T : X \to$ span$\{\vec{v}\}$ where $\|T\| \leq \lambda$ and bc(span$\{\vec{v}\}) \leq \lambda$.*

Proof. The results on the projection operators follow directly from Lemma 3.3.3, and it remains to prove that subspaces where with close basis elements have close basis constants. But this follows directly as the basis constant is continuous in its basis elements (see e.g. [Puj71]), and we are done. □

This yields the following result.

Theorem 3.4.2. *LΠBS$_I \in \Sigma^0_4$.*

Proof. Define $J_{M,\lambda}$ as the statement

$$J_{M,\lambda} = (\exists P : X \to M)(\text{P is a projection} \wedge \|P\| \leq \lambda)$$

By Lemma 3.4.1, we obtain the following characterisation of LΠBS$_I$ for a Banach space X with $(e_i)_{i\in\omega}$ as its presentation.

$$X \in \text{L}\Pi\text{BS}_I \iff (\exists \lambda \in \mathbb{Q})\,(\forall \vec{e})\,(\exists \vec{v})\left(J_{\text{span}\{\vec{v}\},\lambda} \wedge \text{bc}\left(\text{span}\{\vec{v}\}\right) \leq \lambda\right)$$

Where $\vec{e}, \vec{v}$ here refers to finite strings generated by elements from $(e_i)_{i\in\omega}$. Since the basis constant of a computable finite dimensional subspace is computable ([Bos08]), and $J_{\text{span}\{\vec{v}\},\lambda}$ is Π^0_1 by Lemma 3.3.6, it follows that the above characterisation is Σ^0_4. □

Although we do not have a completeness result, it follows from Theorem 3.2.26 that LΠBS$_I$ is at least Π^0_3 hard, so the upperbound is not too far off. Despite its simplicity, we also explicitly state the following corollary, which might be viewed as a possible reason as to why showing high lowerbounds for BASIS$_I$ is non-trivial.

Corollary 3.4.3. *Showing $\Pi^0_4 \leq BASIS_I$ would also show a separation between $BASIS_I$ and $L\Pi BS_I$, thereby answering the question posed in [Sza87].*

3.5 Local basis structure

For completeness, we note that it has been implicitly shown in both [Puj71] and [Bos08] that local basis structure is equivalent to its effectivised version, making it arithmetical and even Σ^0_3. We have the following.

Lemma 3.5.1 ([Puj71], [Bos08]). *A Banach space X has LBS if and only if there exists a constant $K \in \mathbb{R}$, a sequence $(x_i)_{i\in\omega} \in X^\omega$ and a function $\sigma : \mathbb{N} \to \mathbb{N}$ such that*

$$(\forall i) bc(span\{x_0, \ldots, x_{\sigma(i)}\}) < K$$

and the sequence $(x_i)_{i\in\omega}$ is dense in X.

This gives rise to a natural definition for computable LBS.

Definition 3.5.2. A computable Banach space X has computable LBS if there exists a computable sequence $(x_i)_{i\in\omega}$ and a computable $\sigma : \mathbb{N} \to \mathbb{N}$ that witnesses $X \in$ LBS.

The following was implicit in both [Puj71] and [Bos08].

Lemma 3.5.3 ([Puj71], [Bos08]). *A computable Banach space X has LBS if and only if it has computable LBS.*

Thus, we have the following

Corollary 3.5.4.

$$LBS_I = LBS_C \in \Sigma^0_3$$

LBS_I will also trivially have a lower bound of Π^0_2, as the set of all computable Banach spaces is Π^0_2 complete. Part of the difficulty in proving better lower bounds for LBS_I is due to the lack of *classical* constructions that lack LBS. In fact, to the best of our knowledge, the only space that achieves this is the construction in [Sza87], and it is not even clear if this construction can be made effective.

Question 3.5.5. Can space constructed in [Sza87] be made computable? In a similar vein, is there a computable Banach space without LBS?

Question 3.5.6. Determine the exact complexity of LBS_I.

3.6 Finite dimensional Schauder decomposition

The complexity of FDD_I by itself is quite mysterious, much similar to the case for BASIS_I. Theorem 3.2.26 does give us a Π^0_3 lower bound, and it is not hard to see that $\text{FDD}_I \leq \Sigma^1_1$. But proving anything much finer appears difficult. Much similar to the case for BASIS_I, it is also natural to consider the effectivised version of FDD_I. We show that under reasonable assumptions, the complexity of having a computable FDD is also Σ^0_3 complete. We begin by showing that FDD_I is indeed Σ^1_1.

Recall that a Banach space X has a FDD if there exists a sequence of finite dimensional subspaces $\{M_i\}_{i\in\omega}$ such that for all $x \in X$, there exists an unique sequence $\{y_i\}_{i\in\omega}$ such that $x = \sum_{i=0}^{\infty} y_i$ and $y_i \in M_i$ for all i. Like the case for BASIS_I, this can be shown to be Σ_1^1 via an characterisation using "decomposition constants".

Lemma 3.6.1 ([Mar69, Page 93]). *Let $\{M_i\}_{i\in\omega}$ be a sequence of closed subspaces of X, then $\{M_i\}_{i\in\omega}$ forms a Schauder decomposition if and only if:*

- *$\{M_i\}_{i\in\omega}$ is dense. i.e. The closure of span $(\bigcup_i M_i)$ is X.*
- *There exists a constant $K \in \mathbb{R}$ such that*

$$\left\| \sum_{i\le n} x_i \right\| \le K \left\| \sum_{i\le m} x_i \right\|$$

for all n, m with $n \le m$ and for all seqeunces $\{x_i\}_i$ with $x_i \in M_i$.

With the Lemma above, it is not hard to obtain the following corollary.

Corollary 3.6.2. *$FDD_I \in \Sigma_1^1$.*

We now turn to the question of effectivisation, but the "natural" definition is not immediately obvious. We will first work with the following definition of a computable FDD.

Definition 3.6.3 (Computable FDD). Let X be a computable Banach space, a sequence of finite dimensional subspaces $\{M_i\}_{i\in\omega}$ is a *computable FDD* for X if their basis elements and the sequence $\{\dim(M_i)\}_{i\in\omega}$ are uniformly computable.

Note that a basis $(x_i)_{i\in\omega}$ forms a one-dimensional FDD for the space, so any computable basis forms a computable FDD. We will now show that similar to Theorem 3.2.1, the index set for computable FDD is Σ_3^0 complete.

Theorem 3.6.4. *FDD_C is Σ_3^0 complete.*

Proof. Utilising Lemma 3.6.1, it follows that $\text{FDD}_C \le \Sigma_3^0$. We will now show hardness, in a similar fashion to the proof for hardness of Theorem 3.2.1. Recall the space Y_τ as constructed in Definition 3.2.4, defined as $Y_\tau = \left(\oplus_i Z_{\tau(i)}\right)_{c_0}$. Where $Z_n = [x_0, \ldots, x_{\sigma(n)}]$ are approximates to Davie's space such that $(\forall n)\ (\text{bc}(Z_n) \le K)$ for some constant K. For any lower semicomputable $\tau : \mathbb{N} \to \mathbb{N}$, the space Y_τ is a computable Banach space with a basis, thus with a FDD as well. We will now show that similar to Lemma 3.2.9, we can diagonalise against all computable FDD via a lower semicomputable function. Given a triple (φ_n, K, i), where $\{\text{span}(\varphi_n(i,0), \varphi_n(i,1), \ldots, \varphi_n(i, \dim(M_i)))\}_i$ is the computable FDD that we are trying to diagonalise, we carry out the following computation at stage s:

- If we witness $\varphi_{n,s}$ not outputting a Cauchy name for any of its basis elements at stage s, we terminate the computation.
- If we witness $\varphi_{n,s}$ outputting a sequence of finite dimensional spaces that have decomposition constant greater than k, we terminate the computation.
- Otherwise, search for a sufficiently large p such that $\text{emb}^i(Z_p)$ is not contained in the linear span of $\varphi_{n,s}$.

As defined, this construction will diagonalise against all potential computable FDDs. It therefore remains for us to show that this construction is actually valid, and can be carried out uniformly in stages $s = 0, 1, \ldots$. To see this, we have the following:

- Let $\alpha = \{\alpha_i\}_i$ be a sequence from the dense set, then saying α is *not* a Cauchy name is equivalent to the statement $(\exists k)(\|\alpha_k - \alpha_{k+1}\| > 2^{-k} + 2^{-k-1})$, which we can approximate in stages.
- Again, let $\{M_i\}_{i \le l}$ be a sequence of computable finite dimensional subspaces, to say that the decomposition constant is greater than K, it is equivalent to the statement

$$(\exists p, q)(\exists (y_i)_{i \le q})(y_i \in M_i) \left(\left\| \sum_{i=0}^{p} y_i \right\| > K \left\| \sum_{i=0}^{q} y_i \right\| \right)$$

  which is a Σ_1^0 statement, and we can therefore approximate to s stages.
- To show that the final requirement can be carried out effectively, it is sufficient to show that for all $i \in \mathbb{N}$, for all sequences $\{M_i\}_i$ that forms a FDD for *its span*, we have

$$\text{emb}^i(Z) \not\subseteq [M_0, M_1, \ldots]$$

  This follows for much of the same reason of Lemma 3.2.9, if $\text{emb}^i(Z) \subseteq [M_0, M_1, \ldots]$, then $\text{emb}^i(Z)$ is a complemented subspace of a space with FDD. But FDD implies the approximation property, and approximation property is preserved by complemented subspaces, so this implies that $\text{emb}^i(Z)$ has the approximation property, a contradiction.

Therefore, the construction is valid and we have proven an analogous result for Lemma 3.2.9, from which the same arguments in the proof of Theorem 3.2.1 follow, completing the proof. $\square$

Remark 3.6.5. In the proof above, we actually did not use the requirement that the dimensions of a computable FDD should also be uniformly computable. So if we were to only require the basis elements of a computable FDD to be given in a Σ_1^0 way, we would still obtain the corresponding Σ_3^0 completeness result.

Finally, we show that similar to the case for Theorem 3.0.23, the natural projections associated to a computable FDD is also computable.

Theorem 3.6.6. *Let $\{M_i\}_{i \in \omega}$ be a computable FDD for a computable Banach space X. Since each $x \in X$ can be* uniquely *expressed as a sequence $(y_i)_{i \in \omega}$ where $y_i \in M_i$, there is a natural sequence of associated projections $\{P_i\}_{i \in \omega}$ where $P_i : X \to M_i$, $P_i\left((y_k)_{k \in \omega}\right) = y_i$. This sequence of projections is in fact uniformly computable relative to an upper bound on the decomposition constant of $\{M_i\}_{i \in \omega}$.*

Proof. The proof of this is very similar to the proof for Theorem 3.0.23, so we avoid stating the explicit details. Let $U \in \mathbb{Q}$ denote some upper bound on the decomposition constant of $\{M_i\}_{i \in \omega}$. To compute P_k for some k on $y = \sum_{i=0}^{\infty} y_i$, we simply enumerate finite linear combinations of the form $\hat{y} = \sum_{i=0}^{l} \hat{y}_i$ until $\|y - \hat{y}\| < \varepsilon_1$, where $\varepsilon_1 > 0$ is to be chosen later. As $\|P_k\| \le 2U$, we get that

$$\|P_k(y) - P_k(\hat{y})\| \le \|P_k\| \, \|y - \hat{y}\| < 2\varepsilon_1 U$$

Since ε_1 can be made arbitrarily small, and $P_k(\hat{y})$ can be computed exactly, this shows that $P_k(y)$ can be computed uniformly, completing the proof. $\square$

Remark 3.6.7. Again, we did not need the fact that $\dim(M_i)$ is computable in the proof above either. So the same result still holds true if we only require the spaces $\{M_i\}_i$ to be given in a Σ_1^0 fashion.

3.7 Bounded approximation property

Recall the definition of the approximation property, which says that for any given compact sets, there is a sequence of finite rank operators that converges in norm over that compact set to the identity operator. However, there is no limitation on the norms of such operators. One possible generalisation is to require the existence of some universal constant such that the norms of the operators are bounded by it. This notion is known as the bounded approximation property. It was proven by [Gha19] that BAP is in $\mathbf{\Sigma_6^0}$ in the Borel hierarchy. We show in this section that like the π-property, BAP_I is in fact in Σ_4^0. Furthermore, as BAP implies AP, it follows that BAP_I is Π_3^0 hard. So despite the fact that we do not have a completeness result, the bounds are rather tight.

To start off, recall Definition 3.1.8 for BAP, we re-write it in the following form.

Definition 3.7.1. Let X be Banach space, then X has the bounded approximation property if and only if the hold.

$$(\exists \lambda)\ (\forall E \subset X \text{ that is finite dimensional})\ (\exists \text{ finite dimensional } M \subset X \wedge E \subseteq M)$$

$$(\exists T : X \to M)\ (\|T\| \leq \lambda \wedge T(x) = x \forall x \in E)$$

The plan of attack is now clear. Similar to the π-property, we will first argue that it is sufficient to quantify E, M over finite dimensional spaces generated from a dense sequence. And then argue that the existence of the operator T is in fact a Π_1^0 statement by compactness. We begin by giving the analog of Lemma 3.3.3.

Lemma 3.7.2. *Let X be a separable Banach space with a dense sequence $(e_i)_{i\in\omega}$. Let L be a finite dimensional subspace of the form $L = [x_0, \ldots, x_M]$, and let $I \subseteq \{0, \ldots, M\}$ be an index set such that $x_k \in (e_i)_{i\in\omega}$ for all $k \in I$. Further assume that there is an operator $T : X \to L$ such that $T(x_i) = x_i \forall i \in I$. Then for all $\epsilon > 0$, there is a space $N = [y_0, \ldots, y_M]$ where $y_i \in (e_i)_{i\in\omega}$, and an operator $S : X \to N$ where $\|S\| \leq \|T\| + \epsilon$ and $y_k = x_k$ for all $k \in I$.*

Proof. Choosing y_i sufficiently close to x_i induces an isomorphism $Q : M \to N$ such that $1 \leq \|Q\| \leq \epsilon$ by mapping each x_i to y_i, and this $\epsilon \geq 1$ can be made arbitrarily close to 1. Thus, the composition $QT : X \to N$ has $\|QT\| \leq \|Q\| \|T\| \leq \|T\| + \epsilon$ where $\epsilon > 0$ can be made arbitrarily small. Furthermore, it satisfies the property $QT(y_i) = QT(x_i) = Q(x_i) = x_i = y_i$ for all $i \in I$. $\square$

We arrive at the following analogue for Corollary 3.3.4.

Corollary 3.7.3. *Let X be a separable Banach space with a dense sequence $(e_i)_{i\in\omega}$. Then X has BAP if and only if there exists λ such that for all $\vec{e} \in \{e_i : i \in \omega\}^{<\omega}$, there exists $\vec{v} \in \{e_i : i \in \omega\}^{<\omega}$ and an operator $T : X \to span\{\vec{v}\}$ such that $span\{\vec{e}\} \subseteq span\{\vec{v}\}$, $\|T\| \leq \lambda$ and $T|_{span\{\vec{e}\}} = I|_{span\{\vec{e}\}}$.*

Proof. The forward direction follows by Lemma 3.7.2. For the converse, note that X has BAP if and only if there is a sequence $\{T_i\}_{i\in\omega}$ of finite rank operators with universally bounded norm that converges to the identity operator. And this can be built solely with subspaces from a given dense sequence. □

With the above results, we obtain a Σ_4^0 upperbound for BAP_I.

Theorem 3.7.4.

$$BAP_I \in \Sigma_4^0$$

Proof. The proof is almost identical to that of Theorem 3.3.7. By first noting that Lemma 3.3.6 carries over as the analogue of Lemma 3.3.5 still hold. This then induces a Σ_4^0 characterisation of BAP via Corollary 3.7.3. □

As remarked earlier, we also have $\Pi_3^0 \leq \text{BAP}_I$.

We note here that there is some similarity in the properties BAP, π, FDD, CBAP, which might be of interest. All of these properties essentially involve the existence of some form of approximating sequence. We remind the reader that the notion of an approximating sequence was defined in Definition 3.1.10.

The following is a direct characterisation of BAP, π, FDD, CBAP via approximating sequences.

Theorem 3.7.5 ([JRZ71])**.** *Fix X as some separable Banach space.*

- *X has BAP if and only if it has a λ-approximating sequence for some λ.*
- *X has π if and only if it has a λ-approximating sequence for some λ such that the operators are all projections.*
- *X has CBAP if and only if it has a λ-approximating sequence for some λ such that the operators commute.*
- *X has FDD if and only if it has a λ-approximating sequence for some λ such that the operators are projections and they commute.*

As shown, the four properties BAP, π, FDD, CBAP are just combinations of the properties $\{\text{commuting}, \text{projections}\}$ on the approximating sequences. What is perhaps interesting is that arithmetical characterisations are known for precisely those properties where the operators are not required to commute. So perhaps the inherent difficulty is in the commuting aspect of the approximating sequences.

3.8 Approximation property

The natural definition for AP is quite complicated, as it quantifies over all compact subsets of a Banach space. We note here that $\text{AP}_I \in \Pi_1^1$ by invoking a classical lemma regarding the structure of compact subsets in Banach spaces.

Lemma 3.8.1. *A closed subset K of a Banach space X is compact if and only if there is a sequence $\{x_n\}_{n\in\omega}$ in X such that $\|x_n\| \to 0$ and $K \subseteq \overline{conv}\{x_n\}_{n\in\omega}$. Furthermore, in the case where $\|x_n\| \to 0$, $\overline{conv}\{x_n\}_{n\in\omega}$ can be written as*

$$\overline{conv}\{x_n\}_{n\in\omega} = \left\{ \sum_{n=1}^{\infty} \lambda_n x_n : \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n \leq 1 \right\}$$

In fact, we can show something slightly stronger. In separable Banach spaces, it is sufficient to only consider the convex hulls generated by a particular dense sequence.

Lemma 3.8.2. *Let X be a separable Banach space with $(e_i)_{i\in\omega}$ as a dense sequence. Then X has the approximation property if and only if:*

$$(\forall (e_{n_i})_{i\in\omega})\,(\forall \epsilon > 0)\,(\exists T)\,(T \text{ is a finite rank operator} \wedge (\forall x \in \overline{conv}(e_{n_i})_{i\in\omega}))(\|T(x) - x\| < \epsilon)$$

Proof. The forward direction follows directly via Lemma 3.8.1. The converse is a conseqeunce of the fact that $(T - I)(x), (T - I)(y)$ must be close in norm if x, y are close in norm. □

The characterisation given in Lemma 3.8.2 is almost Π^1_1, except for the existential quantifier on T. We now show that we only need to look at the behaviour of T on a sufficiently long prefix of a given sequence.

Lemma 3.8.3. *Let X be a Banach space and $(y_i)_{i\in\omega}$ a sequence in X such that $\|y_i\| \to 0$, denote $\overline{conv}((y_i)_{i\in\omega})$ as K. Then the condition*

$$(\forall \epsilon > 0)\,(\exists T)\,(\forall x \in K)\,(\|T(x) - x\| \leq \epsilon)$$

is equivalent to

$$(\forall \epsilon > 0)\,(\exists m \in \mathbb{N})\,(\exists T)\,((\forall x \in \overline{conv}((y_i)_{i\leq m}))\,(\|T(x) - x\| \leq \epsilon) \wedge (\forall i > m)\,((\|T\| + 1)\,\|y_i\| \leq \epsilon))$$

Proof. The forward implication follows from the fact that $((\|T\| + 1)\sup_{i>m} \|y_i\|)_m \to 0$ as $m \to \infty$.
For the converse, recall that any $x \in K$ can be written in the form $x = \sum_{i=0}^{\infty} \lambda_i y_i$ where $\lambda_i \geq 0, \sum_{i=0}^{\infty} \lambda_i \leq 1$. So for all $x \in K$, we can obtain the following bound for $\|T(x) - x\|$

$$\|T(x) - x\| = \left\| T\left(\sum_{i=0}^{\infty} \lambda_i y_i\right) - \sum_{i=0}^{\infty} \lambda_i y_i \right\|$$

$$\leq \left\| T\left(\sum_{i=0}^{m} \lambda_i y_i\right) - \sum_{i=0}^{m} \lambda_i y_i \right\| + \left\| T\left(\sum_{i=m+1}^{\infty} \lambda_i y_i\right) - \sum_{i=m+1}^{\infty} \lambda_i y_i \right\|$$

$$\leq \left\| T\left(\sum_{i=0}^{m} \lambda_i y_i\right) - \sum_{i=0}^{m} \lambda_i y_i \right\| + (\|T\| + 1)\sup_{i>m} \|y_i\|$$

Thus, if $\|T(x) - x\| \leq \epsilon$ for all $x \in \overline{\text{conv}}\,((y_i)_{i\leq m}))$ and $(\|T\| + 1)\,\|y_i\| \leq \epsilon$ for all $i > m$, we get that

$$\left\| T\left(\sum_{i=0}^{m} \lambda_i y_i\right) - \sum_{i=0}^{m} \lambda_i y_i \right\| + (\|T\| + 1)\sup_{i>m} \|y_i\| \leq 2\epsilon$$

Since this holds for all $\epsilon > 0$, we are done. □

Finally, since the desired property of the finite rank operator T only depends on a fixed finite dimensional subspace, we can apply the same trick as in Lemma 3.3.6, to only require finite rank operators of the form $T_i : E_i \to M$ where $\overline{\bigcup_i E_i} = X$. This gives a Π^1_1 characterisation of AP_I.

Theorem 3.8.4.

$$AP_I \leq \Pi^1_1$$

Proof. This follows by Lemmas 3.8.2 and 3.8.3, and the technique used in Lemma 3.3.6 which only requires a sequence of finite rank operators $T_i : E_i \to M$, where $\{E_i\}_i$ is a dense sequence of finite dimensional subspaces. $\square$

Question 3.8.5. There is a natural definition for AP_C by requiring computable operators and only quantifying over effectively compact sets, what is the complexity of it?

3.9 Summary of results on complexities

The following diagram gives a summary of the complexities of the various properties considered.

$$\begin{tikzcd}
 & \Pi^0_3 \leq \mathrm{BASIS}_I \leq \Sigma^1_1 \arrow[dl] \arrow[drr] & & \\
\Pi^0_3 \leq \mathrm{L\Pi BS}_I \leq \Sigma^0_4 \arrow[d] \arrow[dr] & & & \Pi^0_3 \leq \mathrm{FDD}_I \leq \Sigma^1_1 \arrow[dll] \arrow[ddl] \arrow[d] \\
\mathrm{LBS}_I \leq \Sigma^0_3 & \Pi^0_3 \leq \pi_I \leq \Sigma^0_4 \arrow[dd] & & \mathrm{SD}_I \leq \Sigma^1_1 \\
 & & \Pi^0_3 \leq \mathrm{CBAP}_I \leq \Sigma^1_1 \arrow[dl] & \\
 & \Pi^0_3 \leq \mathrm{BAP}_I \leq \Sigma^0_4 \arrow[d] & & \\
 & \Pi^0_3 \leq \mathrm{AP}_I \leq \Pi^1_1 & &
\end{tikzcd}$$

Remark 3.9.1. Figure 3.9 is mostly a summary of the previous results, other than the bounds $\mathrm{CBAP}_I \leq \Sigma^1_1$ and $\mathrm{SD}_I \leq \Sigma^1_1$. However, these upper bounds follow directly from Theorem 3.7.5 and Lemma 3.6.1, so we avoid giving explicit proofs.

As shown, all of the lower bounds follow from Theorem 3.2.26. In some cases, this lower bound is not too far away from the upper bound. But in cases such as $\mathrm{BASIS}_I, \mathrm{AP}_I$, the bounds are not very tight.

We note that in general, showing a lower bound of Π^0_4 or greater is going to be non-trivial due to the lack of "classical" constructions. For example, suppose that we would like to show $\mathrm{BASIS}_I \geq \Pi^0_4$. To the best of our knowledge, the only Banach spaces that lack Schauder basis lack one of $\{\pi_I, \mathrm{LBS}_I\}$, but both of them have an upper bound of Σ^0_4, so they cannot be used to show Π^0_4 hardness. Furthermore, this is further supported by the fact that $\mathrm{L\Pi BS}_I \leq \Sigma^0_4$, so showing $\Pi^0_4 \leq \mathrm{BASIS}_I$ would give a proof that $\mathrm{L\Pi BS} \neq \mathrm{BASIS}$, answering a problem posed by [Sza87]. Similar arguments also hold for FDD_I, as (to the best of our knowledge) it is open whether π implies FDD. The situation on AP_I is less clear, and we are not too sure if there exist similar arguments.

Finally, we remark that we did not obtain any completeness result. We strongly suspect that $\{\mathrm{L\Pi BS}_I, \pi_I, \mathrm{BAP}_I\}$ should all be Σ^0_4 complete. However the trick used to prove Π^0_3 hardness does not quite work for the Σ^0_4 case, and perhaps more complicated constructions are needed.

3.10 Computable implications

Recall Figure 3.1, which detailed the classical implications. We will now give natural effective analogues for some of these properties, and show that the obvious implications still hold in the effective case.

Recall that we have already defined the computable analogues of BASIS and FDD in Definitions 3.0.4 and 3.6.3. As a natural extension from Theorem 3.7.5, we will define computable versions of $\pi, \mathrm{BAP}, \mathrm{CBAP}$. The definitions below are just effectivisations of Theorem 3.7.5.

Definition 3.10.1 (Computable bounded approximation property)**.** A computable Banach space X has the *computable bounded approximation property* if it has a sequence of uniformly computable finite rank operators $\{T_i\}_i$ that forms a λ-approximating sequence for some $\lambda \in \mathbb{R}$.

Definition 3.10.2 (Computable π-property)**.** A computable Banach space X has the *computable π-property* if it has a sequence of uniformly computable finite rank projections $\{T_i\}_i$ that forms a λ-approximating sequence for some $\lambda \in \mathbb{R}$.

Definition 3.10.3 (Computable commuting bounded approximation property)**.** A computable Banach space X has the *computable commuting bounded approximation property* if it has a sequence of uniformly computable commuting finite rank operators $\{T_i\}_i$ that forms a λ-approximating sequence for some $\lambda \in \mathbb{R}$.

Remark 3.10.4. The reader might wonder why we did not define FDD_C in this way, where a computable Banach space X has computable FDD if it has an uniformly computable sequence of commuting projections of bounded norm. By Theorem 3.6.6, this notion is weaker than our current definition, and is in fact equivalent to requiring the basis elements of the FDD to be given in a Σ^0_1 fashion. We opted for the stronger definition as the weaker definition fails to capture the *finiteness* part of the decomposition, hence fails to differentiate between a FDD and a general SD. However, we do not know if these two notions are indeed different.

Question 3.10.5. Define a FDD to be *weakly computable* if the corresponding basis elements are given in a Σ^0_1 fashion. Is there a computable Banach space $X \notin \mathrm{FDD}_C$ that has a weakly computable FDD?

With these definitions, we obtain the following implication diagram for these computable properties.

$$\begin{tikzcd}
& \mathrm{BASIS}_C \arrow[dl] \arrow[d] & \\
\mathrm{LBS}_C & \mathrm{FDD}_C \arrow[dl] \arrow[dr] & \\
\pi_C \arrow[dr] & & \mathrm{CBAP}_C \arrow[dl] \\
& \mathrm{BAP}_C &
\end{tikzcd}$$

Most of the implications above follow directly from definitions, as justified below.

Theorem 3.10.6. *The implications in Figure 3.10 hold.*

Proof. As mentioned, this is mostly unraveling the definitions.

- $\mathrm{BASIS}_C \implies \mathrm{FDD}_C$ follows from Definitions 3.0.4 and 3.6.3.
- $\mathrm{BASIS}_C \implies \mathrm{LBS}_C$ also follows directly from definitions, as any prefix of a basis will have bounded basis constant.
- $\mathrm{FDD}_C \implies \pi_C$, $\mathrm{FDD}_C \implies \mathrm{CBAP}_C$ both follows from Theorem 3.6.6, as the associated projections have bounded norm and commute.
- $\pi_C \implies \mathrm{BAP}_C$, $\mathrm{CBAP}_C \implies \pi_C$ both follow directly from the definitions.

$\square$

The implications above are all fairly easy to show due to the way they are defined, and trivial in some sense. The perhaps more interesting question is whether boolean combinations of effective properties with classical properties give rise to stronger properties. For example, does having computable BAP and knowing that the space has π classically show that the space has computable π? We show that comp.FDD + BASIS $\not\Longrightarrow$ comp.BASIS on the assumption that a specific Banach space exists.

Definition 3.10.7. A Banach space X is said to have the strong local basis property (SLBS) if there exists a sequence $(x_i)_{i\in\omega} \in X^\omega$, a constant $K \in \mathbb{R}$ and a function $\sigma : \mathbb{N} \to \mathbb{N}$ such that:

$$(\forall i)\mathrm{bc}([x_0, \ldots, x_{\sigma(i)}]) < K$$

and

$$\Big([x_{\sigma(i)+1}, \ldots, x_{\sigma(i)}]\Big)_{i\in\omega}$$

forms a FDD for X.

Theorem 3.10.8 (comp.FDD + BASIS $\not\Longrightarrow$ comp.BASIS)**.** *Assuming that there exists a computable Banach space X with computable SLBS but does not have a basis, we have*

$$\textit{comp.FDD} + \textit{BASIS} \not\Longrightarrow \textit{comp.BASIS}$$

Proof. Let X be a computable Banach space without basis but has SLBS. We carry out the same diagonalisation as in Corollary 3.2.10 and obtain the space Y_τ. We now claim that this space witnesses the statement comp.FDD + BASIS $\not\Rightarrow$ comp.BASIS. By construction, Y_τ does indeed have a basis but not a computable one. Furthermore, as τ is lower semi-computable, it can be approximated in stages computably. In otherwords, for each i, $X_{\tau(i)} = [x_0, \ldots, x_{\sigma(\tau(i))}]$ can be approximated by the sequence of spaces:

$$[x_0, \ldots, x_{\sigma(\tau_0(i))}] \oplus [x_{\sigma(\tau_0(i))+1}, \ldots, x_{\sigma(\tau_1(i))}] \oplus \ldots$$

Where each space is computable. Furthermore, as X has SLBS, the union of this computable sequence of spaces across all $X_{\tau(i)}$ indeed form a computable FDD for Y_τ. Therefore, Y_τ has a computable FDD and a basis but lacks a computable basis, completing the proof. $\square$

We note here that this only partially answers one of the many such questions, and it would interesting to investigate the other questions.

Question 3.10.9. Completely determine the relationships in Figure 3.10, allowing combinations of classical properties.

3.11 Complexity of reflexivity

It was implicitly proven in [Bos02] that the set of reflexive Banach spaces is coanalytic complete. We note here that the reduction is in fact computable, thus implying that the set of reflexive Banach spaces is Π^1_1 hard. Together with a classical characterisation for reflexive Banach spaces, this shows that the set of reflexive Banach spaces is indeed Π^1_1 complete.

Theorem 3.11.1. *$S_r = \{e : X_e$ is a computable reflexive Banach space$\}$ is Π^1_1 complete.*

We use the following definition of reflexive Banach spaces.

Definition 3.11.2. Let X be a Banach space. A basic sequence $(x_i)_{i\in\omega}$ of X is *boundedly complete* if whenever $(a_i)_{i\in\omega}$ is a sequence of reals such that

$$\sup_n \sum_{i=0}^{n} a_i x_i < \infty$$

then the series $\sum_{i=0}^{\infty} a_i x_i$ converges.

Definition 3.11.3 ([Sin62]). Let X be a Banach space. Then X is reflexive if and only if every basic sequence in X is boundedly-complete.

Before introducing the reduction, we need to first introduce the universal Banach space for Schauder basis, this is a classical fact.

Theorem 3.11.4 ([Pe**ł**69]). *There exists a Banach space U with basis $(u_i)_{i\in\omega}$ such that for any basic sequence $(x_i)_{i\in\omega}$ in any Banach space, there is some complemented subsequence of $(u_i)_{i\in\omega}$ that is equivalent*[9] *to it. Furthermore, this space is unique up to isomorphism.*

[9]Two basic sequences $(x_i)_i, (y_i)_i$ are *equivalent* if for all $(a_i)_i$ scalars, $\sum_i a_i x_i$ converges if and only if $\sum_i a_i y_i$ converges.

For our purposes, we would need to further show that the universal space is computable with respect to its natural presentation. The following is an easy consequence from the classical constructions.

Lemma 3.11.5. *There is a universal space U such that the norm function is uniformly computable on the rational span of one of its basis.*

Proof. The classical construction shown in [LT77] works. Denote $T = \mathbb{N}^{<\mathbb{N}}$ as the complete tree. Let $(x_i)_{i\in\omega}$ be some dense sequence in $C[0,1]$, and V be the vector space spanned by the Hamel basis $(e_t)_{t\in T}$. The elements of V are simply vectors with finitely many non-zero entries, we define a norm on V as follows. Recall that $y \in V$ can be written as $y = \sum_{\sigma\in T} y_\sigma e_\sigma$ where the coefficients y_σ are all 0 except for finitely many exceptions.

$$\|y\| = \sup_{S\in\mathbb{N}^{<\mathbb{N}}} \Big\| \sum_{\sigma\preceq S} y_\sigma x_{\sigma'} \Big\|$$

where for $\sigma \in \mathbb{N}^{<\mathbb{N}}$, σ' is its final element. Denote U as the completion of V under this norm, the obvious vectors $(e_t)_{t\in T}$ form a computable basis for U. It was shown in [LT77] that this space is universal for Schauder basis. As each element of V has only finitely many non-zero entries, the norm introduced is computable on V. This implies that the norm is uniformly computable on the rational spans of $(e_t)_{t\in T}$, as desired. $\square$

We will now introduce the construction used in [Bos02], which is actually very similar to the previous construction.

Definition 3.11.6 ([Bos02])**.** Let T, V be as defined in the proof of Lemma 3.11.5, and $(u_i)_{i\in\omega}$ be the computable basis for U. An interval I is a set such that there exists $s, \tau \in \mathbb{N}^{<\mathbb{N}}$ and $I = \{w | s \preceq w \preceq \tau\}$. An admissible choice of intervals is a finite set $\{I_j; 0 \leq j \leq k\}$ of intervals such that each branch (i.e. element of $\mathbb{N}^{\mathbb{N}}$) intersects at most one of these intervals. The Banach space $U_2(T)$ is defined as the completion of V under the following norm

$$\|y\| = \sup\left(\left(\sum_{j=0}^{k}\left\|\sum_{s\in I_j} y_s u_{|s|}\right\|^2\right)^{1/2}\right)$$

and the supremum is taken over all $k \in \mathbb{N}$ and all admissible choice of intervals $\{I_j; 0 \leq j \leq k\}$. Again, as the elements of V only have finitely many non-zero entries, this norm is uniformly computable on its natural presentation. For any subtree A of T, $U_2(A)$ is defined as the closed subspace of $U_2(T)$ generated by $(e_t)_{t\in A}$.

Theorem 3.11.7 ([Bos02])**.** *The mapping U_2 satisfies the following properties on a tree A:*

- *If A is not well-founded, then $U_2(A)$ is isomorphic to U, the universal space, which is not reflexive.*
- *If A is well-founded, then $U_2(A)$ is reflexive.*

Corollary 3.11.8. *As the mapping U_2 in Definition 3.11.6 is computable, the properties listed in Theorem 3.11.7 show that U_2 is a computable reduction from the set of computable well-founded trees to the set of computable reflexive Banach spaces.*

Corollary 3.11.9. *S_r is Π^1_1 hard.*

$S_r \in \Pi^1_1$ then follows from Definition 3.11.3.

Corollary 3.11.10. $S_r \in \Pi^1_1$.

Proof. By Definition 3.11.3, we obtain the following:

$$e \in S_r \iff \text{Every basic sequence in } X_e \text{ is boundedly complete}$$

$$\iff (\forall ((a_i)_{i\in\omega} \in X_e^\omega)) ((a_i)_{i\in\omega} \text{ is basic } \implies (a_i)_{i\in\omega} \text{ is boundedly-complete})$$

It is clear that describing a sequence to be boundedly complete is Π^1_1, and describing a sequence to be basic is arithmetical via Lemma 3.0.5. Thus, the proof is complete □

This completes the proof of Theorem 3.11.1.

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