VICTORIA UNIVERSITY OF WELLINGTON

Te Whare Wananga o te Upoko o te Ika a Maui

School of Mathematics, Statistics
and Operations Research

Te Kura Mātai Tatauranga, Rangahau Pūnaha

PO Box 600
Wellington
New Zealand

School Tel: +64 4 463 5341
School Fax: +64 4 463 5045
School Email: office@msor.vuw.ac.nz

Modeling Consecutive Failures of Repairable Systems, with Applications in Warranty Cost Analysis

by

Sima Varnosafaderani

Supervisors: Dr Stefanka Chukova & Dr Richard Arnold

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ABSTRACT

Most engineered systems are inclined to fail sometime during their lifetime. Many of these systems are repairable and not necessarily discarded and replaced upon failure. Unlike replacements, where the failed system is replaced with a new and identical system, not all repairs have an equivalent effect on the working condition of the system. Describing the effect of repairs is a requirement in modeling consecutive failures of a repairable system– at the very least, it is assumed that a repair simply returns the failed system to an operational state without affecting its working condition (i.e. the repair is minimal). Although this assumption simplifies the modeling process, it is not the most accurate description of the effect of repair in real situations. Often, along with returning a failed system to an operational state, repairs can improve the working condition of the system, and thus, increase its reliability which impacts on the rate of future failures of the system.

Repair models provide a generalized framework for realistic modeling of consecutive failures of engineered systems, and have broad applications in fields such as system reliability and warranty cost analysis. The overall goal of this research is to advance the state of the art in modeling the effect of general repairs, and hence, failures of repairable systems. Two specific types of system are considered:

(i) a system whose working condition initially improves with time or usage, and whose lifetime is modeled as a univariate random variable with a non-monotonic failure rate function;

(ii) a system whose working condition deteriorates with age and usage, and whose lifetime is modeled as a bivariate random variable with an increasing failure rate function.

Most univariate lifetime distributions used to model system lifetimes are assumed to have increasing failure rate functions. In such cases, modeling the effect of general repairs is straightforward– the effect of a repair can be modeled as a possible decrease, proportional to the effectiveness of the repair, in the conditional intensity function of the associated failure process. For instance, a general repair can be viewed as the replacement of the failed system with an identical system at a younger age, so that the conditional failure intensity following the repair is lower than the conditional failure intensity prior to the failure. When the failure rate function is initially decreasing, specifically bathtub-shaped, general repair models suggested for systems with increasing failure rate functions can only be applied when initial repairs are assumed to be minimal. In this study, we propose a new approach to modeling

the effect of general repairs on systems with a bathtub-shaped failure rate function. The effect of a general repair is characterized as a modification in the conditional intensity function of the corresponding failure process, such that the system following a general repair is at least as reliable as a system that has not failed. We discuss applications of the results in the context of warranty cost analysis and provide numerical illustrations to demonstrate properties of the models.

Sometimes the failures of a system may be attributed to changes in more than one measure of its working condition– for instance, the age and some measure of the usage of the system (such as, mileage). Then, the system lifetime is modeled as a bivariate random variable. Most general repair models for systems with bivariate lifetime distributions involve reducing the failure process to a one-dimensional process by, for instance, assuming a relationship between age and usage or by defining a composite scale. Then, univariate repair models are used to describe the effect of repairs. In this study, we propose a new approach to model the effect of general repairs performed on a system whose lifetime is modeled as a bivariate random variable, where the distributions of the bivariate inter-failure lifetimes depend on the effect of all previous repairs and following a general repair, the system is at least as reliable as a system that has not failed. The lifetime of the original system is assumed to have an increasing failure rate (specifically, hazard gradient vector) function. We discuss applications of the associated failure process in the context of two-dimensional warranty cost analysis and provide simulation studies to illustrate the results.

This study is primarily theoretical, with most of the results being analytic. However, at times, due to the intractability of some of the mathematical expressions, simulation studies are used to illustrate the properties and applications of the proposed models and results.

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Part I

Introduction

Chapter 1

Research Overview

A system's ability to perform its intended functions changes over time (or with use). A failure of the system occurs when it is no longer able to execute one or more of its functions satisfactorily. Most engineered systems are inclined to fail sometime during their lifetime. Many of these systems are repairable and not necessarily discarded and replaced upon failure.

The rate of future failures depends on, among others, the type of system and the type of rectification action performed following previous failures of the system. Rectification actions include replacing the failed system with a new and identical system, and repairing the failed system. Not all rectification actions have an equivalent effect on the working condition of the system. For instance, it is well-known that a repair, along with returning a failed system to an operational state, often improves the working condition of the system to somewhere between its working condition immediately prior to failure and its working condition following a replacement. It is possible that a repair may render a system unusable or worse than it was prior to failure– in this study, however, we do not consider these types of repair.

We use the term 'general repair' to refer to repairs that return the failed system to an operational state, and at the very least, do not further improve the working condition of the system. General repairs, based on their effectiveness in improving the working condition of the system, are classified as: 'minimal repair', which is the least effective repair; 'perfect repair', which is the most effective repair (and is in general modeled as a replacement); and 'imperfect repair', whose effectiveness is between those of the minimal and perfect repairs.

In order to model the effect of repairs, it is necessary to first define a measure of the working condition of the system. In most of the reliability literature, the working condition of the system is modeled with probabilistic measures, such as: the 'failure rate function' (which represents the rate of future failures of the system); the 'conditional reliability function' (which represents the probability of the system surviving an interval of time conditional on its current age); and the 'mean residual lifetime function' (which represents how long a system of a certain age is expected to survive). These measures can be a function of more than one variable; for instance, when the system deteriorates with increased age and usage. These functions reflect the type of system; for example, the lifetime of a system that deteriorates over time can be modeled with a distribution having an increasing failure rate function. Most often, the failure rate function, which uniquely determines the distribution of the original system lifetime, is the chosen metric.

For a repairable system, the working condition of the system at any point may depend on the effectiveness of previous repairs. Then, the successive lifetimes (i.e. lifetimes between consecutive failures) may not have the same distribution as the original lifetime. It is reasonable to assume that the distribution of an inter-failure lifetime depends on the preceding failure points and the effectiveness of the corresponding repairs. Therefore, having a model that adequately describes the effect of general repairs is a requirement in modeling consecutive failures of a repairable system.

With most general repair models, the effect of a repair is described in terms of changes in the failure rate or conditional reliability functions of the successive inter-failure lifetimes. The distributions of these successive inter-failure lifetimes uniquely determine the 'failure process' (i.e. the process of consecutive failure points of the repairable system).

1.1 Research Objectives

The overall goal of this research project is *to advance the state of the art in modeling the effect of general repairs, and hence, consecutive failures of repairable systems, using stochastic point processes*. Two specific types of system are considered in this study:

(i) a system whose working condition initially improves with age (or usage), and whose lifetime is modeled as a univariate random variable with a bathtub-shaped failure rate function;

(ii) a system whose working condition deteriorates with age and usage, and whose lifetime is modeled as a bivariate random variable having a bivariate increasing failure rate property.

1.1.1 Modeling Repairs in One Dimension

Most univariate lifetime distributions used to model system lifetimes are assumed to have increasing failure rate functions. In such cases, modeling the effect of general repairs is straightforward– the effect of a repair can be modeled as a possible decrease, proportional to the effectiveness of the repair, in the conditional intensity function of the associated failure process (the conditional intensity function of the process uniquely determines the distributions of the associated inter-failure lifetimes). For instance, a general repair can be viewed as the replacement of the failed system with an identical system at a younger age, so that the conditional intensity following the repair is lower than the conditional intensity immediately prior to the failure. This translates to an increased system reliability.

In practical situations it has been observed that systems do not always exhibit behavior with a monotonically degrading pattern. The lifetime of many electrical and mechanical systems have been successfully modeled with lifetime distributions having non-monotonic, specifically, bathtub-shaped failure rate (BFR) functions. These lifetime distributions are characterized by an initial period of "improvement", where the failure rate function is decreasing (and hence, the conditional reliability and mean residual lifetime functions are both initially increasing). When the failure rate function is initially decreasing, general repair models suggested for systems with increasing failure rate functions can only be applied when initial repairs are assumed to be minimal. General repair models describing the effect of non-minimal repairs performed while the system is still improving have not yet been developed.

The first objective of this study is *to model the effect of general repairs on systems whose lifetime is modeled with a distribution having a bathtub-shaped failure rate function, such that, the system following a general repair is at least as reliable as a system that has not failed.* Then, a point process is developed to model the consecutive failures of the system.

Although, time/age is used as the variable of interest, it can be replaced by any continuous, non-negative variable, such as some measure of usage (e.g. mileage).

1.1.2 Modeling Repairs in Two Dimensions

Sometimes the failures of a system may be attributed to changes in more than one measure of its working condition– for instance, the working condition of the system can deteriorate with age and some measure of the usage of the system (such as, mileage). Then the system lifetime is modeled as a bivariate random variable.

For a system whose lifetime is modeled as a bivariate random variable, where the two variables are not independent, the correlation structure of the variables needs to be taken into account when modeling the consecutive failures. This complicates the modeling process. To simplify the problem, most general repair models for systems with bivariate lifetime distributions involve reducing the failure process in two dimensions to a one-dimensional process by, for instance, assuming a relationship between age and usage or by defining a composite scale. Then, univariate repair models are used to describe the effect of general repairs. Full bivariate repair models, where the variables are correlated but not functionally related, have been suggested for minimal repair and perfect repair (or replacement), but not for imperfect repair.

The second objective of this research is *to model the effect of general (imperfect) repairs performed on a system whose lifetime is modeled as a bivariate random variable, such that following a general repair, the system is at least as reliable as a system that has not failed.* Then, a point process in two dimensions is developed to model the consecutive failures of the system.

Simulation studies will be carried out to demonstrate the properties and applications of this model.

1.1.3 Applications in Warranty Cost Analysis.

Repair models provide a generalized framework for realistic modeling of consecutive failures of repairable engineered systems, and have broad applications in many fields, such as, system reliability modeling and warranty cost analysis. For instance, having accurate estimates of the warranty servicing costs is necessary in developing and choosing optimal (most cost-effective) warranty policies (from the point of view of manufacturers). This can be achieved by defining realistic models of the effect of repairs, so that consecutive failures of the system (which may result in warranty claims) can be successfully modeled.

A secondary objective of this study is *to demonstrate the applications of the proposed general repair models in the context of warranty cost analysis.*

1.2 Research Methodology

This research is primarily theoretical, with most of the derived results being analytic. However, at times, due to the intractability of some of the mathematical expressions, we carry out simulation studies to illustrate the properties and applications of the proposed repair models and to compute estimates of the quantities of interest.

This project involves identifying appropriate functional forms of the univariate conditional intensity and the bivariate conditional reliability that realistically describe the effect of general repairs on the working condition of the systems considered.

The failure (or general repair) processes, in one and two dimensions, are generalized to include, as special cases, previously-suggested failure processes, such as the renewal process and the minimal repair process.

1.3 Thesis Structure and Outline

We conclude this chapter with an outline of the structure and organization of the thesis.

This thesis is organized into four parts. Part I of the thesis contains an overview of the study and a review of fundamental concepts in warranty analysis. The main contributions of the thesis are presented in Parts II and III, and both parts follow a common structure: in each part, the first chapter is a review of literature and contains revisions of fundamental concepts that will be used in the succeeding chapters of that part; the proposed repair models and their properties are presented next; the part concludes with numerical illustrations of the proposed repair models and applications in warranty cost analysis. Part IV of the thesis concludes this study. A detailed outline of the chapters follows.

Part I - Introduction

Chapter 1: In this chapter, we provided an overview of this study.

Chapter 2: In this chapter, we discuss the concepts of system failure, rectification actions (e.g. general repairs), warranty servicing strategies and the associated warranty servicing costs.

Part II - Modeling Repairs in One Dimension, with Applications in Warranty Analysis

Chapter 3: In this chapter, we discuss fundamental concepts used in modeling consecutive failures of a repairable system in one dimension. We also provide a review of existing general repair models for systems whose lifetimes are modeled as univariate random variables.

Chapter 4: In this chapter, we propose a new approach to model the effect of a general repair performed on a system whose lifetime is modeled with a distribution having a bathtub-shaped failure rate function. We then define a failure process to model consecutive failures of the system, where each failure is rectified by general repair.

Chapter 5: In this chapter, we suggest warranty servicing strategies for a system whose lifetime is modeled with a distribution having a bathtub-shaped failure rate function. We apply the repair models suggested in the previous chapter to derive the expected servicing cost for each strategy.

Part III - Modeling Repairs in Two Dimensions, with Applications in Warranty Analysis

Chapter 6: In this chapter, we provide a brief review of fundamental concepts used in modeling consecutive failures of a repairable system whose lifetime is modeled as a bivariate random variable. We also review existing models of general repairs in two dimensions.

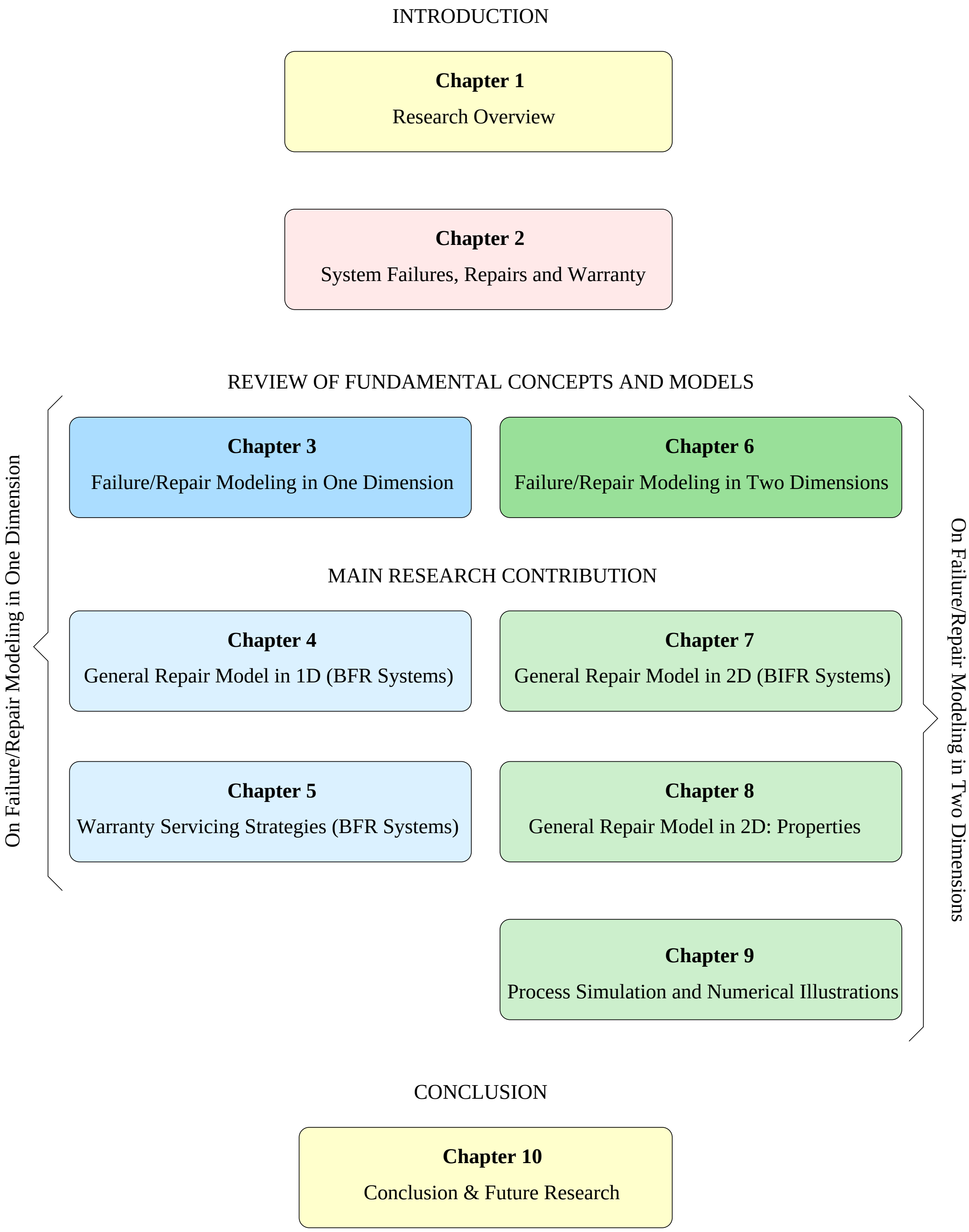
Chapter 7: In this chapter, we propose a new approach to model the effect of a general repair performed on a system deteriorating with age and usage, whose lifetime is modeled as a bivariate random variable. We then develop a failure (or general repair) process to model consecutive failures of the system, where each failure is rectified by general repair.

Chapter 8: In this chapter, we discuss various properties of the general repair process proposed in the previous chapter.

Chapter 9: In this chapter, we suggest a procedure for simulating the failure or general repair process in two dimensions. We illustrate the effect of general repairs through simulations of the failure process. We also illustrate applications of the repair model in the context of two-dimensional warranty cost analysis.

Part IV - Conclusion

Chapter 10: In this chapter, we conclude this study with a discussion and outline some possible directions for future research.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Chapter 2

System Failures, Repairs and Warranty

In this chapter, we review the concepts of system failure and rectification actions (e.g. replacements and general repairs). We provide a brief review of product warranty, warranty servicing strategies and servicing costs.

This chapter is arranged as follows. In Section 2.1, we review the concept of system failures. In Section 2.2, we discuss maintenance actions, in particular, general repairs. In Section 2.3, we discuss the various types of product warranty. In Section 2.4, we provide a brief review of warranty servicing strategies and discuss estimating warranty servicing costs. In Section 2.5, we conclude with a chapter summary.

2.1 System Failures

In the context of reliability engineering, a *system* is often defined as a collection of components (tools and equipments) that jointly perform a recognized set of functions. Each component of the system can itself be viewed as a single-component system. We equate products with systems.

Most engineered systems are inclined to fail sometime during their lifetime. Formally, a *failure* of a system is defined as "the event when a function of the system is terminated" or its performance is outside acceptable bounds [1]. Failures of a system can often be classified as either sudden or gradual. A sudden failure is characterized by a sudden termination of a system function, whereas a gradual failure is characterized by a gradual decrease in the functional performance of the system; see Figure 2.1 for an illustration.

Most systems are *complex*, that is, they have more than a single function. In order to recognize and diagnose failures of a complex system, all of its functions, interrelationships

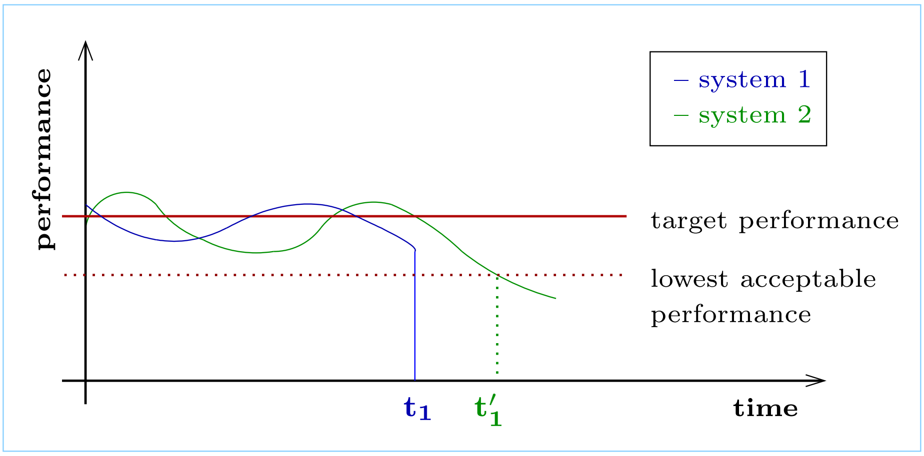


Figure 2.1: An illustration of the first failure of two systems, where System 1 has a sudden failure at time t_1 and System 2 has a gradual failure at time t_1'.

between functions, and related functional requirements must be known. It is not always possible to enumerate all manners in which the system may fail. For the purpose of this research, we will assume that all functions of the system are known and all failures are observable and a direct consequence of the system's lack of function [1].

For modeling purposes, it is necessary to distinguish between various causes of failure. Causes of failure include the following [1]:

(i) Faulty or lacking design.

(ii) Flawed manufacturing processes or divergence from standard processes.

(iii) Errors introduced while assembling the system.

(iv) Natural aging of the system (which may include age accumulation, usage accumulation, etc.).

(v) External stresses or shocks that the system may be subject to.

(vi) Weaknesses of the system, which could be a result of over-stated system capabilities.

(vii) Misuse or manhandling of the system.

Most probabilistic models of failures of a system are based on failures caused by the natural aging or the accumulated usage of the system or shocks delivered to the system.

2.2 System Repairs and Replacements

Failures of a system can be corrected or controlled through maintenance actions. These actions are usually classified as either *preventive* or *corrective* maintenance actions.

A *preventive maintenance action* refers to procedures performed in order to maintain the system in a desirable working condition by attempting to reduce the likelihood of future system failures. This type of maintenance is executed while the system is still in an operational state, although operation may have to cease when the system undergoes preventive maintenance. This cessation is managed and not a failure. For instance, a system can be serviced (maintained) periodically or when its performance falls below a preset limit [2].

A *corrective maintenance action* (also referred to as a *rectification action*) refers to procedures performed immediately after the occurrence of a failure in order to return the failed system to an operational state. Corrective maintenance actions include repair and replacement [2].

In terms of their repairability, systems are categorized as either *repairable* or *non-repairable*. Non-repairable systems or components (e.g. a fuse or a light bulb) are not designed to be repaired, and need to be replaced upon failure. Repairable systems or components, on the other hand, can be rectified upon failure without necessarily being replaced. However, repairable components that are beyond repair will need to be replaced. Multi-component systems can comprise both repairable and non-repairable components. In this study, we mainly consider corrective maintenance actions, repair in particular. Also, we assume that, unless stated otherwise, all systems are repairable.

Replacement. A *replacement* of the failed system involves replacing the failed system with a new and identical system. Therefore, the working condition of the system immediately following a failure is the same as that of a brand new system [2].

General repair. Repairs, along with restoring a system from a failed state to an operational state, are assumed to also affect the physical working condition of the repaired system. Not all repairs have the same effect on the working condition of a system. The working condition of a repaired system depends on the effectiveness of the repair. Based on their effect on the working condition of the system, repairs can be classified as one of the following three types:

(i) *minimal repair*, which simply restores the system to an operational state, leaving its working condition as it was immediately before the failure– therefore, the system following a minimal repair behaves as though it did not fail;

(ii) *perfect repair*, which results in the most improvement in the working condition of the system (when compared to a minimally-repaired system)– it is often defined as equivalent to a replacement and referred to as "as-good-as-new" repair;

(iii) *imperfect repair*, which restores the working condition of a failed system to a working condition between those following the two extremes: minimal repair and perfect repair.

Therefore, in terms of their effectiveness in improving the working condition of the repaired system, minimal repair is assumed to be least effective, perfect repair is assumed to be most effective, and imperfect repair is assumed to be more effective than a minimal repair but less effective than a perfect repair. These repairs are often collectively referred to as *general repair*.

Degree of repair. The effectiveness of a general repair is quantified by the *degree* of the repair. The degree of repair specifies the "degree to which the working condition of the system can be restored" [3]. Let δ denote the degree of repair. In most studies, δ (whether preassigned or random) is defined as a variable in the range $[0, 1]$, where the extreme $\delta = 0$ and $\delta = 1$ represent a minimal repair and a perfect repair respectively. Any general repair with degree in the range $(0, 1)$ represents an imperfect repair. The degree of an imperfect repair determines how far, in terms of effectiveness, it is from a perfect repair. As the degree of an imperfect repair increases from 0 to 1, the effectiveness of the imperfect repair is assumed to increase. Note that, depending on the model settings, minimal and perfect repairs can be viewed as special cases of imperfect repair.

This repair categorization is not exhaustive. A comprehensive classification of repairs includes repairs that can worsen the system and in extreme cases, render it useless, and also repairs that improve the system beyond its design, e.g. upgrades and improvements. In this research, we consider only what was defined above as general repair; refer to Pham [4] and Blischke & Murthy [2] for more on repairs.

When modeling consecutive failures of a repairable system, the following assumptions are made to simplify the modeling process:

(a) each failure of the system is followed immediately by a repair;

(b) the time to perform a repair is negligible in comparison to the operating time of the system, and is hence set to zero (i.e. repairs are instantaneous).

2.3 Warranty Policies

A *warranty* or *warranty policy* is a written contract that lists out the expected functions of a system and the manufacturers responsibility in the event that the warranted system breaks down or its performance is not satisfactory. Warranty policies also clearly specify the terms and conditions under which the policy holds. These terms include the proper usage and maintenance conditions of the warranted product [2, 3].

In the warranty literature, warranty policies are often categorized into one or more categories depending on the aspect of interest. Warranty servicing strategies and cost models are then specifically developed for policies in each of these categories.

The features that characterize a warranty policy are normally set out in the warranty agreement. Policies are usually distinguished based on two main characteristics: (i) the definition of the warranty coverage; and (ii) the nature of compensation under warranty.

2.3.1 Warranty Coverage

The warranty coverage is usually defined in terms of the variables that govern the failures of the warranted system, and the renewability of the warranty policy. These two characterizations of warranty policies are as follows.

2.3.1.1 Variables of the Warranty Policy

Under this categorization, warranty policies are distinguished based on the number of variables describing the warranty coverage. The two most common variables used in defining the warranty coverage are time (age) and some measure of usage (mileage or time spent in service).

One-dimensional warranties. When the failures of the system are governed by changes in a single variable– for instance, the age (or the usage) of the system– the warranty coverage can then be a one-dimensional interval, say $(0, w]$, starting immediately after the sale of the system. The warranty expires when the age (or usage) of the system exceeds the warranty limit w.

Two-dimensional warranties. When the failures of the system are governed by changes in two variables–for instance, both the age and the usage of the system– the warranty coverage can be a two-dimensional region with time and usage constituting the two dimensions [2]. For example, consider a rectangular region $(0, w_t] \times (0, w_u]$, where w_t and w_u denote the time and usage limits of the warranty, and the warranty expires as soon as one of these limits is exceeded.

In the two-dimensional case, the warranty limits determine the shape of the warranty region (coverage area); see Figure 2.2 for some examples (refer to Blischke & Murthy [3] for more).

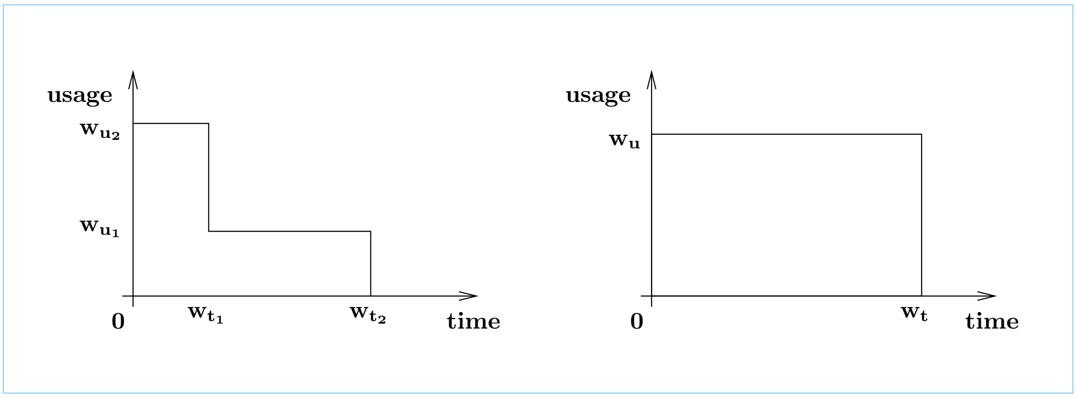


Figure 2.2: Some examples of two-dimensional warranty coverage.

In modeling warranty policies, the variables used are usually time in the one-dimensional case, and time and usage in the two-dimensional case. Both variables are defined on the positive real axis $\mathbb{R}_+$. Although discrete variables (such as, number of flights in warranty policies for aircraft) are sometimes used to define one of the dimensions, we only consider continuous variables.

The warranty coverage can have more than two dimensions, however, the one- and two-dimensional warranties are the ones commonly used. We restrict our study to one- and two-dimensional policies.

2.3.1.2 Renewability of the Warranty Policy

This categorization is based on the changes in the warranty coverage following the repair or replacement of the failed system under warranty. Based on this, warranty policies can be classified as follows.

Renewing warranty policies. A warranty is *renewing* if upon repair or replacement of the failed system, the warranty coverage is extended for a period/region equal to the original warranty period/region. Therefore, all repaired or replaced systems have the same warranty as the original system.

For example, consider a warranty period $(0, w]$. If the system fails before time w, and is either repaired or replaced, the repaired or new system is now warranted for a period of length w. In other words, when the $(n+1)$-th system failure at time T_{n+1} is within w units of T_n, i.e. $T_{n+1} \leq T_n + w$, the warranty limit is extended to $T_{n+1} + w$, for $n \in \mathbb{N}_+$, where $\mathbb{N}_+ = \{1, 2, \dots\}$ is the set of natural numbers. The warranty expires when the time between two successive failures is greater than w; see Figure 2.3.

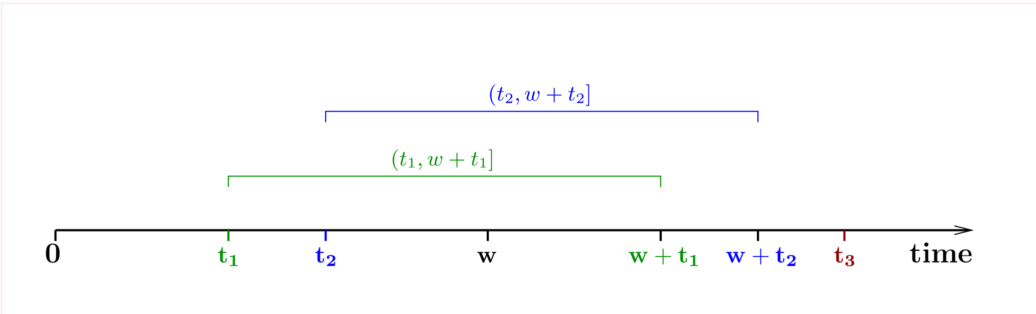


Figure 2.3: An example of a one-dimensional renewing warranty, where failures have occurred at times t_1, t_2 and t_3– the failure at time t_3 is not covered under warranty.

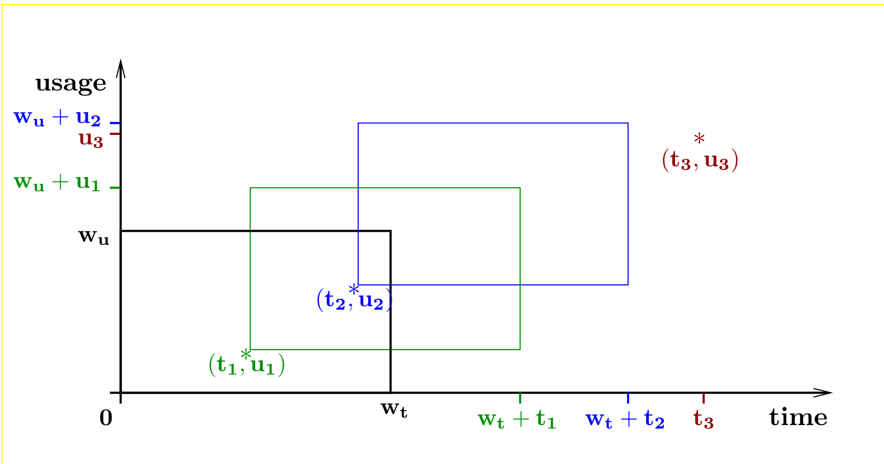


Figure 2.4: An example of a two-dimensional renewing warranty, where failures have occurred at points (t_1, u_1), (t_2, u_2) and (t_3, u_3)– the failure at point (t_3, u_3) is not covered under warranty.

Consider a warranted system with a two-dimensional rectangular warranty region, denoted by $(0, w_t] \times (0, w_u]$, where w_t and w_u denote the time and usage limits, respectively. When the $(n+1)$-th failure at the point (T_{n+1}, U_{n+1}) is within the warranty coverage, i.e. $T_{n+1} \leq T_n + w_t$ and $U_{n+1} \leq U_n + w_u$ or $(T_{n+1}, U_{n+1}) \in (T_n, T_n + w_t] \times (U_n, U_n + w_u]$, the system after the repair (replacement) has warranty coverage $(T_{n+1}, T_{n+1} + w_t] \times (U_{n+1}, U_{n+1} + w_u]$, for $n \in \mathbb{N}_+$. Here, the warranty expires when the time between failures exceeds w_t and/or the usage between failures exceeds w_u; see Figure 2.4.

Non-renewing warranty policies. A warranty is *nonrenewing* if the warranty coverage is fixed and does not change following a system repair or replacement. That is, the warranty coverage following a repair or replacement is the remaining warranty coverage of the system.

For instance, for a one-dimensional warranty coverage $(0, w]$, following the n-th failure of the system under warranty, i.e. $T_n \leq w$, the warranty coverage of the repaired or replaced system is the remaining warranty period $(T_n, w]$, for $n \in \mathbb{N}_+$. Therefore, the warranty expires when the age of the system exceeds w, which is equivalent to the warranty expiring immediately after the last failure in the warranty period $(0, w]$; see Figure 2.5.

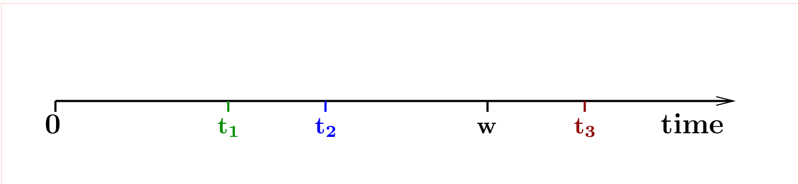


Figure 2.5: An example of a one-dimensional non-renewing warranty, where failures have occurred at times t_1, t_2 and t_3– the failure at time t_3 is not covered under warranty.

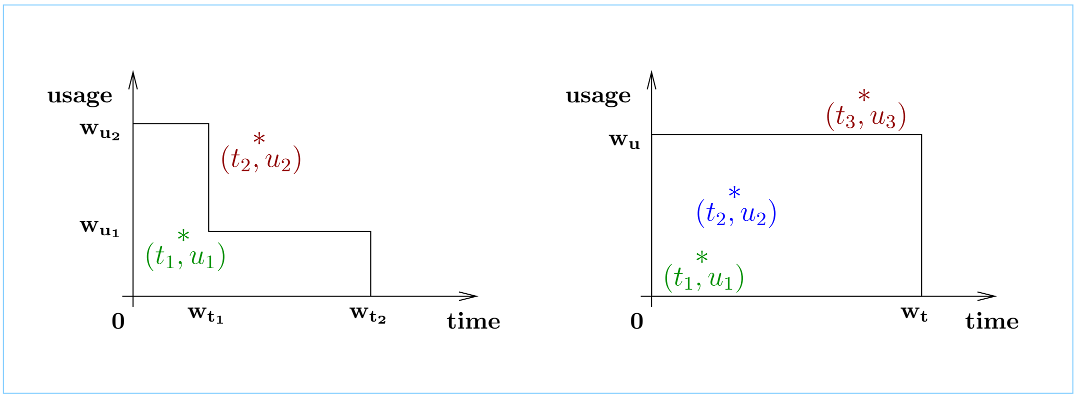


Figure 2.6: An example of a two-dimensional non-renewing warranty where failures have occurred at points (t_1, u_1), (t_2, u_2) and (t_3, u_3)– the failure at point (t_3, u_3) (right) and the failure at point (t_2, u_2) (left) are not covered under warranty.

For a two-dimensional rectangular warranty coverage $(0, w_t] \times (0, w_u]$, the n-th repair or replacement of the system is covered by warranty if the point (T_n, U_n) is within the region $(0, w_t] \times (0, w_u]$, and the warranty coverage of the repaired or replaced system is the remaining coverage $(T_n, w_t] \times (U_n, w_u]$, for $n \in \mathbb{N}_+$. Analogous to the one-dimensional case, the warranty expires when either limit is exceeded or immediately after the last failure before time w_t and usage w_u; see Figure 2.6.

Note that while the coverage of the non-renewing warranty is fixed (constant), the coverage of the renewing warranty is stochastic [3].

2.3.2 Compensation Under Warranty

This categorization of warranty policies is based on the type of compensation provided to the consumer by the manufacturer in the event that the warranted system fails (and a claim is made under warranty).

Free-Replacement or Free-Repair Warranty. Under a *free-replacement warranty* (FRW)– sometimes referred to as a *free-repair warranty*– the manufacturer agrees to replace or repair a warranted system that fails in the warranty coverage region at no cost to the consumer. FRWs are commonly offered with both repairable and non-repairable products such as household appliances, automobile parts, electronics and other durable products.

Pro-rata Warranty. Under a *pro-rata warranty* (PrW), the manufacturer agrees to pay only a portion of the cost of the repair of a repairable system that fails under warranty, and the consumer is required to pay the remaining cost. The amount paid by the manufacturer is usually inversely proportional to the age (or age and usage in the two-dimensional case) of the system. That is, the refund is often a decreasing (or non-increasing) function of the variables classifying the warranty coverage. An example of this type of warranty policy is one that requires the consumer to pay an "excess" fee with each claim.

Full Rebate Warranty. Under a (full) *rebate warranty* (RW), the manufacture agrees to refund the full purchase price of the system, if the system fails before the warranty coverage ends. Under this type of warranty, the consumer is not obligated to buy a replacement product. RWs are not the same as FRWs. However, the consumer may decide to invest the refund provided by the manufacturer in a new and identical system.

Partial Rebate Warranty. Under a *partial rebate warranty* (PRW), the manufacturer agrees to refund a portion of the purchase price of the system, if it fails within the warranty coverage. The amount that is not refunded is a compensation for the use of the system (or service provided by the system) from the time of purchase until the system failure. PRWs are often offered with systems that deteriorate with age or usage, such as vehicle batteries.

Combination Warranty. The warranty policies defined so far are sometimes refereed to as "simple" warranty policies. Warranty combinations that are complex are often mixtures of these simple policies. A *combination warranty* (CW) is a mixture of two or more different types of simple warranty. These warranties are often characterized by portions of the warranty region being covered by different types of warranty– for instance, an initial period of FRW warranty followed by a period of PrW; see Figure 2.7. Combination warranty policies are sometimes used to cover multi-component systems where different groups or types of components are covered by different types of warranty.

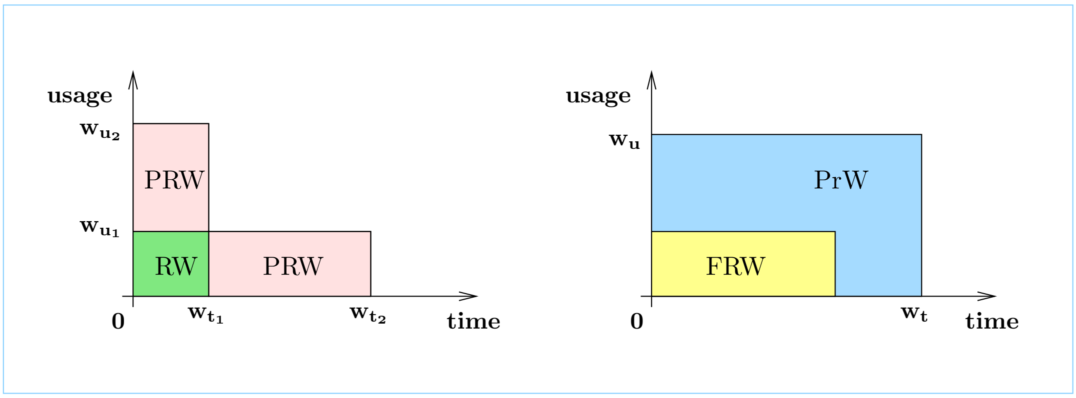


Figure 2.7: Some examples of combination warranty policies.

Types of warranty are not limited to the warranties mentioned above; see Blischke & Murthy [3] for more on types and examples of warranty policies.

2.4 Warranty Servicing Strategies and Costs

When a claim is made under warranty, it is first examined for its validity. Claims may be invalid for various reasons, such as: the claim being false (fraudulent); the claim being made outside the warranty coverage; the system failure leading to the warranty claim being due to the inappropriate use of the system; etc. [3]. For invalid claims, the manufacturer is not bound to provide the services outlined in the warranty policy. For claims that are valid, the manufacturer is required to resolve the claims based on the terms of the warranty policy– for instance, repairing the failed system or providing a replacement system. *Warranty servicing* refers to all actions taken towards processing and resolving a claim made under warranty.

The costs associated with servicing a warranty claim are referred to as *warranty servicing costs* [2]. The *total warranty servicing cost* for a single warranted system sold is the cost of servicing all claims made under its warranty.

The total warranty servicing cost for a system is a function of, among others, the warranty coverage, the nature of compensation under warranty, the number of warranty claims, and if rectification is involved, the type of rectification action (repair or replacement). The exact pricing of servicing warranties in most situations is not determinable and has to be estimated. Since the total warranty servicing cost depends on the number of claims, modeling the number of claims is important in estimating this cost. In most mathematical studies of warranty cost analysis, the process of modeling warranty costs therefore begins with modeling the process of warranty claims.

2.4.1 Consecutive Warranty Claims

In most studies on warranty cost modeling, four major assumptions are made when modeling consecutive warranty claims for a single warranted system:

(i) each failure of the warranted system is followed immediately by a claim;

(ii) all warranty claims are valid and the time taken to process a claim is negligible and is set equal to zero;

(iii) each claim is followed immediately by a repair or a replacement under warranty;

(iv) the time taken to repair a failed system is negligible and is set equal to zero.

With the above assumptions, the number of failures of a system during warranty coverage is equal to the number of claims under warranty, which is equal to the number of repairs (or replacements) covered by warranty. Therefore, modeling consecutive failures of the system during the warranty coverage is equivalent to modeling consecutive claims under warranty. In this study, by default the above assumptions hold, and the terms 'repair process', 'failure process' and 'warranty claim process' will be used interchangeably.

Since the total number of failures, and hence the total number of warranty claims is a random variable, the total warranty servicing cost per warranted system sold is also random. Finding a distribution function for the total warranty servicing cost is complex. Therefore, in most of the literature on warranty cost analysis the focus is on deriving an expression for the expected total warranty servicing cost.

2.4.2 Warranty Servicing Costs

Warranty servicing costs may include various costs, such as: administrative costs; costs of labor; costs of replacement components or parts; costs of servicing equipment; partial or full refunds; and so on. To simplify the process of modeling warranty servicing costs, often an aggregate of the associated costs is used as the cost of servicing a single warranty claim. When the servicing cost is modeled as a constant, the problem of estimating the total warranty servicing cost reduces to estimating the number of warranty claims.

Modeling the servicing cost as a constant simplifies the modeling process, but is not always reasonable. Other functional forms of this cost have been defined in the literature; see for instance, Yun & Kang [5] and Yun et al. [6]. Often, the suggested costs are a function of the age of the system, and for repairable systems, these costs are generally assumed to be proportional to the effectiveness of the repair, so that the cost of an imperfect repair is bounded between the cost of a minimal repair (from below) and the cost of a perfect repair (from above).

For warranty policies that involve rectification of the failed warranted system, it is often assumed that the majority of the servicing cost of a single claim is the cost of the rectification action. Then, the servicing cost is simply referred to as the *cost of repair* or *cost of replacement*. We will discuss some possible functional forms of these servicing costs in our numerical illustrations, in Chapters 5 and 9.

Refer to Blischke & Murthy for more on modeling warranty servicing costs [2, 3].

2.4.3 Warranty Servicing Strategies

Manufacturers attempt to reduce warranty servicing costs by various means, for instance, by improving the system reliability during the design and manufacturing phases (e.g. burning[I] in a system if it has an initially high rate of failure) or by performing scheduled preventive maintenance on the system within its warranty coverage. Both methods decrease the rate of future failures of the system. The costs of these actions are generally part of the sale price of the warranted system [6].

Another opportunity for reducing warranty servicing costs arises when the warranted system fails. When the system fails, the two most important factors in making the choice between repairing or replacing a failed system are the servicing cost and the working condi-

[I]Refer to Block & Savits [7] and Finkelstein & Cha [8] for a survey of the concept of burn-in and burn-in processes.

tion (reliability) of the system following the rectification action. There is usually a trade-off between the two, since often as the effectiveness of the rectification action increases (i.e. its reliability increases), its cost also increases– this may result in an increased total warranty servicing cost [6]. Therefore, to reduce the total warranty servicing cost, various strategies for resolving the warranty claims are considered.

One particular strategy is a *warranty servicing strategy* which refers to a planned series of actions (usually rectification actions) for servicing warranty claims. Most warranty servicing strategies found in the warranty literature, involve dividing the warranty period or region into disjoint sets, and performing different rectification actions when the system fails in each set. The sets and the rectification actions are chosen such that the total warranty servicing cost to the manufacturer is minimized.

Consider for instance the two-dimensional warranty coverage $(0, w_t] \times (0, w_u] =: \mathcal{W}$. The rectangular warranty region $\mathcal{W}$ is partitioned into n disjoint subregions, $\mathcal{W}_1, \ldots, \mathcal{W}_n$, for $n \in \mathbb{N}_+$, where

$$\mathcal{W}_i = (0, k_i] \times (0, l_i] \setminus (0, k_{i-1}] \times (0, l_{i-1}] \ , \qquad \forall i \in \{1, \ldots, n\} \ ; \tag{2.1}$$

$k_0 = l_0 = 0$; $k_n = w_t$ and $l_n = w_u$; see Figure 2.8. Then, a possible warranty servicing strategy is as follows: all repairs in the first and last subregions are minimal, and in the $n-2$ middle subregions, the first repair in each subregion is perfect (or imperfect) and all remaining repairs are minimal; see for instance Chukova et al. [9] and Iskandar et al. [10].

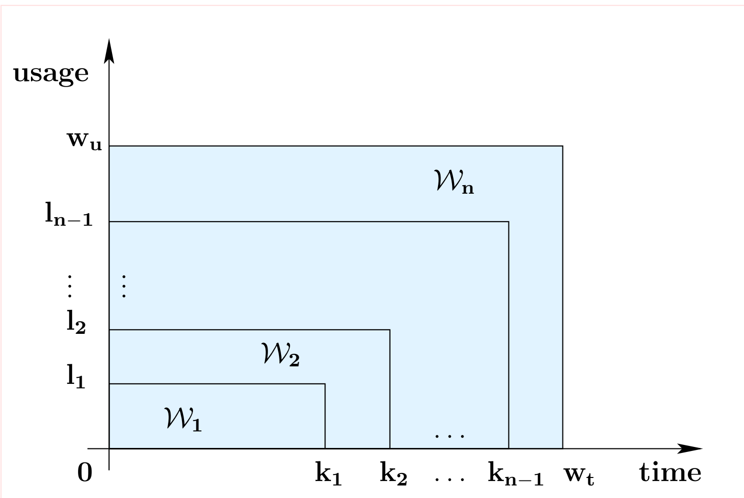


Figure 2.8: Illustration of the n subregions of the rectangular warranty region $\mathcal{W} = (0, w_t] \times (0, w_u]$.

The justification for having only minimal repairs at the start of the warranty coverage is that, for a deteriorating system, the rate of failures is low at the start of its lifetime, and therefore, repairs of higher degree may be unnecessary. Having only minimal repairs at the end of the warranty coverage is justified because performing expensive repairs (with higher degrees) may not be necessary as the warranty is near its end [10].

The number of subregions and the boundaries of the subregions are then chosen such that the expected total warranty servicing cost over the warranty region $\mathcal{W}$ is minimized.

An analogous one-dimensional warranty strategy defined for the warranty period $(0, w]$ is as follows. The warranty period is divided into n disjoint sub-intervals of the form $(k_{i-1}, k_i]$, for $i \in \mathbb{N}_+$, where $k_0 = 0$ and $k_n = w$. Then, the first failure in the $n-2$ middle sub-intervals is perfect (or imperfect) and all other repairs under warranty are minimal; see for instance Jack & Murthy [11] and Yun et al. [6].

Most warranty strategies are suggested for systems that deteriorate with age and/or usage. In Chapters 4 and 5, we discuss failure modeling and warranty servicing strategies for systems that initially improve with age/usage.

2.5 Chapter Summary

In this chapter, we reviewed the concepts of system failures and general repairs, and outlined the assumptions made in modeling successive failures of repairable systems.

We provided an overview of essential concepts and definitions of warranty policies, warranty servicing strategies and warranty servicing costs. We also outlined the assumptions made in modeling consecutive warranty claims.

Part II

Modeling Repairs in One Dimension with Applications in Warranty Analysis

Chapter 3

Failure Modeling in One Dimension

In this chapter, we discuss fundamental concepts used in modeling consecutive failures of a repairable system in one dimension and provide a review of existing general repair models for systems whose lifetimes are modeled as univariate random variables.

This chapter is organized as follows. In Section 3.1, we provide some background for failure modeling in one dimension. In Section 3.2, we provide a review of the literature on modeling general repairs in one dimension. In Section 3.3, we conclude with a brief chapter summary.

3.1 Fundamental Concepts

3.1.1 Stochastic Counting Processes

In the context of warranty analysis, failures of a system in one dimension are normally characterized by either the age of the system or the usage of the system. Here, for clarity, we will assume that the dimension of interest is time (age).

A common approach to model failures of a system is the probabilistic approach that assumes failures of the system occur randomly in time and attempts to model the distribution of the failure points. Then, the consecutive failures of the system, referred to as the *failure process*, is modeled as a stochastic point process in one dimension (here, time) [see Section 6.1.1].

Let the sequence $\{T_n; n \in \mathbb{N}_+\}$ denote the stochastic point process associated to the failure process, where $\mathbb{N}_+ = \{1, 2, \dots\}$, and T_n is the time of the n-th failure. The corresponding counting process, which counts the number of failures occurring in an interval

$(0, t]$, we denote by $\{N(t); t \in \mathbb{R}_+\}$, where $\mathbb{R}_+ = [0, \infty)$. Since the failure times are ordered, i.e. $0 < T_1 < T_2 < \ldots < T_n < \ldots$, the number of failures in any interval $(s, t] \subseteq \mathbb{R}_+$ is given by $N(t) - N(s)$, for $s < t$ [12]. Note that, we have used $N(t)$ to denote the number of failures in the set $(0, t]$, i.e. $N(t) \equiv N((0, t])$, and therefore, $N(t) - N(s) \equiv N((s, t])$.

Let the sequence $\{X_n; n \in \mathbb{N}_+\}$ denote the inter-failure lifetimes, where $T_1 = X_1$ and for $n \in \mathbb{N}_+$, $X_{n+1} = T_{n+1} - T_n$. Then, the processes $\{T_n; n \in \mathbb{N}_+\}$, $\{X_n; n \in \mathbb{N}_+\}$ and $\{N(t); t \in \mathbb{R}_+\}$ are equivalent, i.e. contain the same information, and are therefore exchangeable.

Assuming that at most one failure can occur in an infinitesimally small interval of time (hence, the strict ordering of the failure times), the corresponding counting process has the following properties:

(i) $N(0) = 0$;

(ii) $N(t) \in \mathbb{N}$ for all $t > 0$, where $\mathbb{N}$ is the set of natural numbers;

(iii) $N(t) = \inf_{s \in \mathbb{R}_+ \setminus [0,t]} N(s)$ for all $t > 0$, or $N(t) \leq N(s)$ for all $t < s$;

(iv) $\sup_{s \in [0,t)} N(s) \leq N(t) \leq \sup_{s \in [0,t)} N(s) + 1$ for all $t > 0$; in words, simultaneous failures do not occur in a small interval $(t - dt, t]$, $dt \to 0$ [13].

Independent increments. A counting process $\{N(t); t \in \mathbb{R}_+\}$ has *independent increments* if for all time points $0 < t_1 < t_2 < \cdots < t_n$, the distributions of the increments

$$N(t_1) \ \ , N(t_2) - N(t_1) \ \ , \ldots \ \ , N(t_n) - N(t_{n-1}) \tag{3.1}$$

are independent [13].

Stationary increments. A counting process $\{N(t); t \in \mathbb{R}_+\}$ has *stationary increments* if the increment $N(t + s) - N(t)$, for each $s \geq 0$, has the same distribution for all $t \geq 0$. In other words, the increments are stationary if the distribution of the change $N(t + s) - N(t)$ in the process value between any two points, t and $t + s$, depends only on the distance s between the two points [13].

For a counting process associated to the failure process, the distributions of the process increments depend on the type of rectification action performed on the system following each failure. We have assumed that the time taken to repair or replace a system following a failure is negligible (equal to zero), and also that a rectification action immediately follows a failure.

Since the failure times are strictly ordered, the number of failures in the interval $(0, t]$ can be expressed as

$$N(t) = \max\{n : T_n \leq t; \text{for } n \in \mathbb{N}_+\} = \sum_{n=1}^{\infty} \mathbb{I}_{\{T_n \leq t\}} \ , \tag{3.2}$$

i.e. $N(.)$ is right continuous. The indicator random variable $\mathbb{I}_{\{T_n \leq t\}}$ is defined as

$$\mathbb{I}_{\{T_n \leq t\}} = \begin{cases} 1 \ , & \text{if } T_n \leq t \\ 0 \ , & \text{if } T_n > t \ . \end{cases} \tag{3.3}$$

The distribution of the count $N(t)$ can be determined using the distributions of the failure times at time t, $t > 0$. Let $F_n(.)$ denote the distribution function of the n-th failure time T_n, i.e. $F_n(t) = \mathrm{P}\{T_n \leq t\}$. Then, the probability of n failures occurring in $(0, t]$ is given by

$$\begin{aligned} \mathrm{P}\{N(t) = n\} =& \mathrm{P}\{N(t) \geq n\} - \mathrm{P}\{N(t) \geq n+1\} \\ =& \mathrm{P}\{T_n \leq t\} - \mathrm{P}\{T_{n+1} \leq t\} \\ =& F_n(t) - F_{n+1}(t) \ . \end{aligned} \tag{3.4}$$

3.1.1.1 Cumulative Intensity Functions

The expected number of failures is often referred to as the *mean function* or the *cumulative intensity function* of the counting process $\{N(t); t \in \mathbb{R}_+\}$, and is denoted by $\Lambda(.)$.

Given the distribution of $N(t)$ in (3.4), the expected number of failures in the interval $(0, t]$ can be derived as follows:

$$\begin{aligned} \Lambda(t) = E[N(t)] &= \sum_{k=0}^{\infty} k \, \mathrm{P}\{N(t) = k\} = \sum_{k=1}^{\infty} \sum_{n=1}^{k} \ \mathrm{P}\{N(t) = k\} \\ &= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \ \mathrm{P}\{N(t) = k\} = \sum_{n=1}^{\infty} \ \mathrm{P}\{N(t) \geq n\} \\ &= \sum_{n=1}^{\infty} \ \mathrm{P}\{T_n \leq t\} = \sum_{n=1}^{\infty} F_n(t) \ , \end{aligned} \tag{3.5}$$

where clearly $\Lambda(0) = 0$. Alternatively, using the definition of $N(t)$ given in (3.2), the expected number of failures can be derived as follows:

$$\Lambda(t) = \sum_{n=1}^{\infty} E[\mathbb{I}_{\{T_n \leq t\}}] = \sum_{n=1}^{\infty} P\{T_n \leq t\} = \sum_{n=1}^{\infty} F_n(t) \ . \tag{3.6}$$

3.1.1.2 Intensity Functions

Consider the infinitesimally small interval $(t, t+dt]$, in which at most one failure can occur. Then, the increment $N(t+dt) - N(t)$, for $dt \to 0$, is a binary random variable, such that

$$N(t+dt) - N(t) = \begin{cases} 1 \ , & \text{with approximate probability } \lambda(t)\ dt \\ 0 \ , & \text{with approximate probability } 1 - \lambda(t)\ dt \ . \end{cases} \tag{3.7}$$

where the function $\lambda(.)$ is referred to as the *rate of occurrence of failures* (ROCOF), and is defined as follows:

$$\lambda(t) = \lim_{dt \to 0} \frac{\mathrm{P}\{N(t+dt) - N(t) = 1\}}{dt} \ . \tag{3.8}$$

When the process is orderly[I], the ROCOF function is equal to the *intensity function* of the process, which is defined as the derivative of the cumulative intensity function $\Lambda(.)$.

The increment $N(t+dt) - N(t)$ can be expressed as a sum of indicator random variables (see previous section), and therefore, its expected value can be written as follows:

$$E[N(t+dt) - N(t)] = \sum_{n=1}^{\infty} E\left[\mathbb{I}_{\{N(t+dt)-N(t) \geq n\}}\right] = \sum_{n=1}^{\infty} \mathrm{P}\{N(t+dt) - N(t) \geq n\} \tag{3.9}$$

On dividing both sides of the above equation by dt and passing through the limit, we get

$$\lim_{dt \to 0} \frac{E[N(t+dt) - N(t)]}{dt} = \sum_{n=1}^{\infty} \lim_{dt \to 0} \frac{\mathrm{P}\{N(t+dt) - N(t) \geq n\}}{dt} \ , \tag{3.10}$$

where the LHS is the derivative of the cumulative intensity function at time t, i.e. $d\Lambda(t)/dt$. When the process is orderly, the probability that simultaneous failures occur in the small interval $(t, t+dt]$ is approximately zero. Therefore, (3.10) reduces to

$$\frac{d}{dt} \Lambda(t) = \lim_{dt \to 0} \frac{\mathrm{P}\{N(t+dt) - N(t) \geq 1\}}{dt} = \lim_{dt \to 0} \frac{\mathrm{P}\{N(t+dt) - N(t) = 1\}}{dt} = \lambda(t) \ . \tag{3.11}$$

Therefore, given the ROCOF function $\lambda(.)$, the cumulative intensity function can be derived as follows:

$$\Lambda(t) = \int_0^t \lambda(s) ds \ . \tag{3.12}$$

The intensity and ROCOF functions although useful, do not completely determine the probability structure of the associated counting process [14]. Since the failure process is

[I] A process is *orderly* if the probability of simultaneous events occurring in a small interval $(t, t+dt]$ $(dt \to 0)$ is of order dt or less, i.e. $\mathrm{P}\{N(t+dt) - N(t) \geq 2\} = o(dt)$.

assumed to be orderly, henceforth, the term intensity function will also be used to refer to the ROCOF defined in (3.8).

Let $\mathcal{H}_t$ denote the history of the process at time t, $t \geq 0$. Then, $\mathcal{H}_t$ contains all relevant data available at time t, including the process trajectory $\{N(s); 0 \leq s < t\}$ [12]. Then conditional on this history, the *conditional intensity function* of the process is defined as follows:

$$\tilde{\lambda}(t|\mathcal{H}_t) = \lim_{dt \to 0} \frac{\mathrm{P}\{N(t+dt) - N(t) = 1 \mid \mathcal{H}_t\}}{dt} \quad . \tag{3.13}$$

Note that, at any time t, the conditional intensity function $\tilde{\lambda}(t|\mathcal{H}_t)$, is stochastic (since the history of the process is stochastic), whereas the intensity function is deterministic [14, 15]. The intensity function at any point t can be viewed as the expected value with respect to the history $\mathcal{H}_t$ of the conditional intensity function at that point.

The conditional intensity process $\{\tilde{\lambda}(t|\mathcal{H}_t); t \in \mathbb{R}_+\}$ uniquely defines the stochastic properties of the counting process $\{N(t); t \in \mathbb{R}_+\}$ [16, 17]. Therefore, to model the failure process it is enough to model the effect of general repairs (following the failures) on the conditional intensity function of the process.

3.1.1.3 Baseline Intensity (or Failure Rate) Functions

To model the failure process, we begin with determining the *initial conditional intensity function* or the *baseline intensity function*, which is the conditional intensity function of the failure process before the first failure of the system. We denote the baseline intensity function by $\lambda_0(.)$. The effect of a general repair can be defined as changes in this baseline intensity function.

The baseline intensity function is the *instantaneous failure rate function* corresponding to the lifetime of the original system (i.e. the time T_1 $(= X_1)$ to first failure of the system), which is denoted by $r(.)$. This function is also referred to as the *hazard rate function* or simply the *failure rate function*. It is a deterministic function of time and is defined as follows:

$$\begin{aligned} r(t) &= \lim_{dt \to 0} \frac{\mathrm{P}\{N(t+dt) - N(t) = 1 \mid N(t) = 0\}}{dt} \\ &= \lim_{dt \to 0} \frac{\mathrm{P}\{T_1 \leq t+dt \mid T_1 > t\}}{dt} \\ &= \lim_{dt \to 0} \frac{\mathrm{P}\{t < T_1 \leq t+dt\}}{dt\, \mathrm{P}\{T_1 > t\}} = \frac{1}{\bar{F}(t)} \lim_{dt \to 0} \frac{F(t+dt) - F(t)}{dt} \quad , \end{aligned} \tag{3.14}$$

where $F(.)$ and $\bar{F}(.)$ denote the distribution and reliability functions of the original lifetime, and $F = 1 - \bar{F}$. When the density exists, it is denoted by $f(.)$, and the failure rate function can be expressed in terms of the density function as follows: $r(t) = f(t)/\bar{F}(t)$. The product $r(t)\ dt$ is the approximate probability of a failure occurring in the interval $(t, t + dt]$, given that the system is operating at (and has not failed prior to) time t.

The failure rate function in (3.14) can be written in terms of the reliability function $\bar{F}(.)$ alone:

$$\begin{aligned}
\Rightarrow \qquad r(t) &= \frac{1}{\bar{F}(t)} \lim_{dt \to 0} \frac{\bar{F}(t) - \bar{F}(t+dt)}{dt} \\
\Rightarrow \qquad r(t) &= \frac{-\frac{d}{dt}\bar{F}(t)}{\bar{F}(t)} \\
\Rightarrow \qquad -r(t) &= \frac{d}{dt}\{\ln\,[\bar{F}(t)]\} \\
\Rightarrow \qquad -\int_0^t r(s)\ ds &= \ln\,[\bar{F}(t)] \\
\Rightarrow \qquad e^{-\int_0^t r(s)\ ds} &= \bar{F}(t)\ .
\end{aligned} \tag{3.15}$$

This exponential representation is important, since the reliability, density and distribution functions of the original lifetime can be constructed using just the failure rate function. In terms of the failure rate function, the distribution and density functions are given by

$$F(t) = 1 - \bar{F}(t) = \ 1 - e^{-\int_0^t r(s)\ ds}\ ; \tag{3.16}$$

$$f(t) = \frac{d}{dt}F(t) = -\frac{d}{dt}\bar{F}(t) = r(t)\ e^{-\int_0^t r(s)\ ds}\ . \tag{3.17}$$

The failure rate function characterizes the first failure of the system, and all succeeding failures are characterized by the conditional intensity function, which takes into account the effect of the general repairs performed following the failures of the system [3].

3.1.2 Stochastic Aging Classification

In reliability studies, lifetime distributions are classified based on their aging properties. The *aging* (or *stochastic aging*), in some probabilistic sense, describes how the working condition (or performance) of the system changes with time. The working condition of a system can: (i) improve with time (negative aging); (ii) deteriorate with time (positive aging); or (iii) remain constant [18].

In this section, we define various classes of univariate lifetime distributions based on three indicators of the working condition of a system: the failure rate function, the conditional reliability function and the mean residual lifetime function.

3.1.2.1 Classes based on Failure Rate

Let X denote the original lifetime of the system, and let $r(.)$ denote the associated failure rate function.

IFR/DFR/CFR. When the working condition of the system is modeled in terms of its failure rate, we have the following classification of aging:

- the lifetime distribution is *increasing failure rate* (IFR), if $r(t)$ increases as $t \geq 0$ increases;
- the lifetime distribution is *decreasing failure rate* (DFR), if $r(t)$ decreases as $t \geq 0$ increases;
- the lifetime distribution is *constant failure rate* (CFR), if $r(t)$ remains constant for all $t \geq 0$ [3].

Classes of non-monotonic failure rate functions have been defined as combinations of these monotonic classes. Non-monotonic failure rate functions are distinguished by the presence of one or more *change-point(s)*, which are points where the monotonicity of the function changes [18, 19, 20]. A non-monotonic class, frequently appearing in the reliability literature, is the *bathtub-shaped failure rate* (BFR) class of failure rate function. These curves are characterized by two change-points, which we denote by a_1 and a_2, and are defined piece-wise as follows:

$$r(t) \text{ is } \begin{cases} \text{decreasing } (r'(t) < 0)\ , & \text{for } t \in [0, a_1] \\ \text{constant } (r'(t) = 0)\ , & \text{for } t \in [a_1, a_2] \\ \text{increasing } (r'(t) > 0)\ , & \text{for } t \in [a_2, \infty)\ , \end{cases} \tag{3.18}$$

where $r(a_1) = r(a_2)$ and $r'(t) = dr(t)/dt$ [21]– we will further discuss BFR functions in the next chapter.

Another example of the non-monotonic class is the *U-shaped failures rate* (UFR) class of failure rate function, which is characterized by a single change-point a, before which the failure rate is decreasing and after which the failure rate is increasing. UFR functions can be derived from BFR functions, by setting $a_1 = a_2$.

In Figure 3.1, we have plotted example functions from the different classes of failure rate function. In general, non-monotonic failure rate functions can be reduced to monotonic functions by selecting appropriate change-points, and can be viewed as generalizations of the monotonic functions [18].

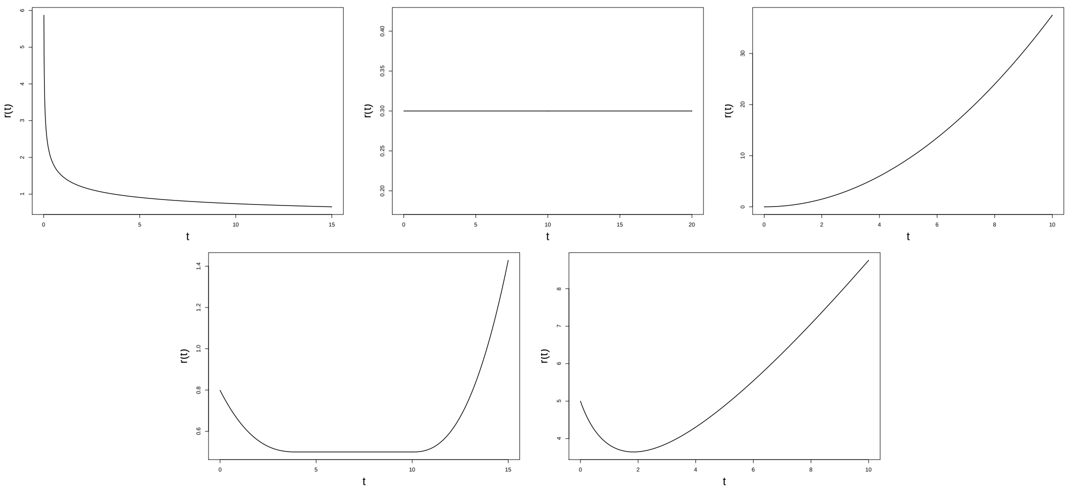


Figure 3.1: Examples of failure rate functions from various classes: DFR class (top,left); CFR class (top,middle); IFR class (top,right); BFR class (bottom, left); and UFR class (bottom right).

3.1.2.2 Classes based on Conditional Reliability

The conditional reliability of a system is the probability of the system surviving a finite interval of time, given that it has not failed prior to its current age. Let $\bar{F}_t(.)$ denote the conditional reliability function of a system of age t. Then, for $x > t$,

$$\bar{F}_t(x) = \mathrm{P}\{X > x | X > t\} = \frac{\mathrm{P}\{X > x\}}{\mathrm{P}\{X > t\}} = \frac{\bar{F}(x)}{\bar{F}(t)} \ , \tag{3.19}$$

where X denotes the lifetime of the system. Therefore, $\bar{F}_t(x)$ is the probability that a system of age t survives an additional $x - t$ units of time, given that it has not failed before t. When $t = 0$, (3.19) reduces to the reliability function at time x, i.e. $\bar{F}_0(x) = \bar{F}(x) = \mathrm{P}\{X > x\}$. The corresponding conditional distribution function, which we denote by $F_t(.)$, is given by

$$\begin{aligned} F_t(x) = \mathrm{P}\{X \leq x | X > t\} &= \frac{\mathrm{P}\{t < X \leq x\}}{\mathrm{P}\{X > t\}} = \frac{F(x) - F(t)}{\bar{F}(t)} \\ &= \frac{\bar{F}(t) - \bar{F}(x)}{\bar{F}(t)} = 1 - \frac{\bar{F}(x)}{\bar{F}(t)} = 1 - \bar{F}_t(x) \ . \end{aligned} \tag{3.20}$$

The monotone properties of the failure rate function $r(.)$ are the same as those of the conditional distribution function $F_t(.)$, since

$$r(t) = \lim_{dt \to 0} \frac{F(t + dt) - F(t)}{dt \ \bar{F}(t)} = \lim_{dt \to 0} \frac{F_t(t + dt)}{dt} \ . \tag{3.21}$$

When $F_t(t + dt)$ is increasing (decreasing) in $t \geq 0$, for all $dt \geq 0$, then $r(t)$ is increasing (decreasing) in $t \geq 0$ [18]. Since, $\bar{F}_t = 1 - F_t$, the monotone properties of the conditional reliability function are the opposite of those of the failure rate function.

IFR/DFR/CFR. When the working condition of the system is modeled in terms of its conditional reliability, the IFR/DFR classification can be equivalently expressed in terms of this function:

- the lifetime distribution is IFR, if $\bar{F}_t(t+s)$ decreases as $t \geq 0$ increases, for each $s \geq 0$;
- the lifetime distribution is DFR, if $\bar{F}_t(t+s)$ increases as $t \geq 0$ increases, for each $s \geq 0$;
- the lifetime distribution is CFR, if $\bar{F}_t(t+s)$ is constant for all $t \geq 0$, and for each $s \geq 0$ [18]; see Figure 3.2 for an illustration.

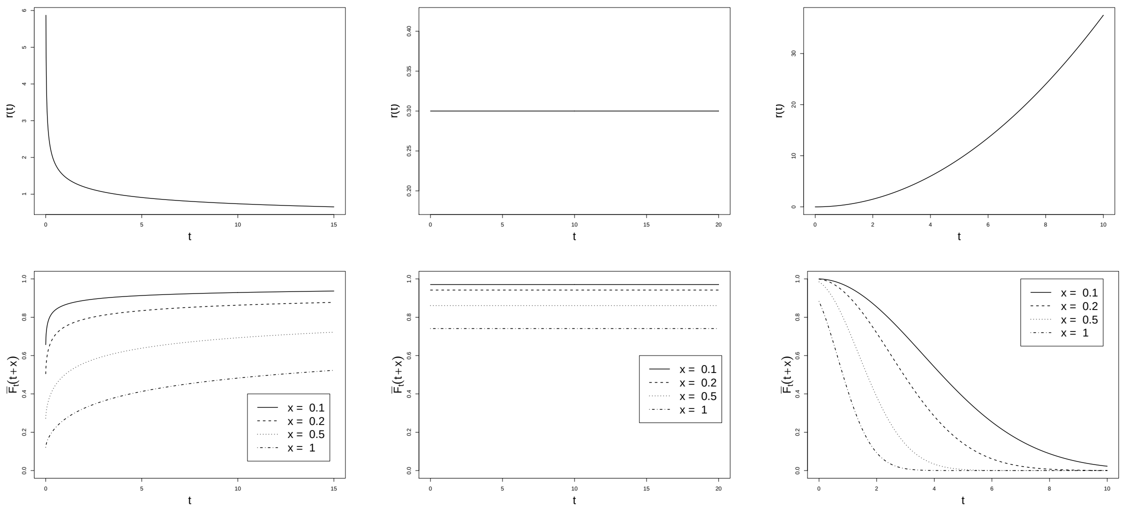


Figure 3.2: Examples of the conditional reliability function and the corresponding failure rate function: DFR class (left column); CFR class (middle column); and IFR class (right column).

Using the exponential representation in (3.15), the conditional reliability function in terms of the failure rate function is, for $x > t$, given by

$$\bar{F}_t(x) = \frac{\bar{F}(x)}{\bar{F}(t)} = \frac{e^{-\int_0^x r(u)\, du}}{e^{-\int_0^t r(u)\, du}} = e^{-\int_t^x r(u)\, du} . \tag{3.22}$$

Therefore, for distributions where the failure rate function is defined piece-wise (for instance, BFR functions), the conditional reliability function can be derived from the failure rate function; see Figure 3.3 for an illustration. Notice that, the conditional reliability function for a BFR function is non-monotonic (initially increasing, then decreasing); we will further discuss this in the next chapter.

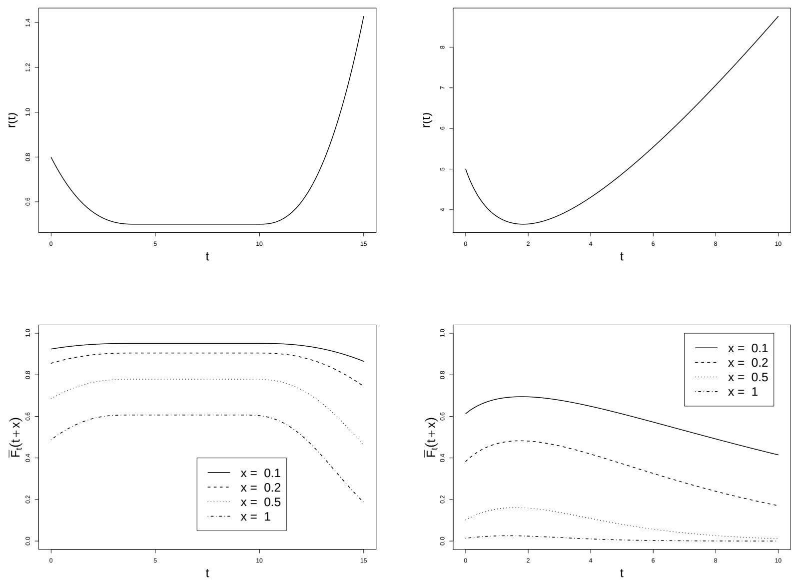


Figure 3.3: Examples of the conditional reliability function and the corresponding failure rate function: BFR class (left column); and UFR class (right column).

NBU/NWU. Another categorization of lifetime distributions defined based on conditional reliability, which is weaker (in terms of conditions) than the IFR/DFR classification, is as follows:

- the lifetime distribution is new-better-than-used (NBU), iff $\bar{F}_t(t+s) \leq \bar{F}(s)$, for all $t > 0$ and each $s \geq 0$;
- the lifetime distribution is new-worse-than-used (NWU), iff $\bar{F}_t(t+s) \geq \bar{F}(s)$, for all $t > 0$ and each $s \geq 0$ [18].

The CFR distribution can be a member of either class, if the definitions are not strict. According to this classification, the working condition of a used system is compared to that of a new system. When the probability of a system of age t not failing in the next s time units is less (greater) than that of a new system, then the new system is better (worse) than the used system.

When $\bar{F}_t(t+s)$ is increasing (decreasing) in $t \geq 0$, for each $s \geq 0$, then $\bar{F}(s) = \bar{F}_0(0+s) \leq (\geq)\bar{F}_t(t+s)$, for $t > 0$. Therefore, the IFR (DFR) class is contained in the NBU (NWU) class [18].

3.1.2.3 Classes based on Mean Residual Lifetime

The *residual lifetime* of a system is its remaining life conditional on the event that it has not failed prior to its current age. Let X_t denote the residual life of a system of age t. Then, $X_t = [X - t | X > t]$, where X is the original system lifetime (see Figure 3.4).

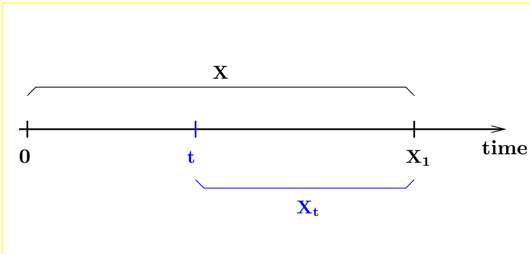


Figure 3.4: Illustration of the residual life of a system prior to its first failure at time $X \equiv X_1$.

The distribution function of the residual lifetime is the conditional distribution function in (3.20), i.e.

$$\mathrm{P}\{X_t \leq x - t\} = \mathrm{P}\{X \leq x | X > t\} = F_t(x) \ , \tag{3.23}$$

where $F_t(x)$ represents the probability of a failure occurring at or before time x given that the system has not failed prior to time t, for $x > t > 0$.

The aging properties of a system are often described in terms of its *mean residual lifetime* (MRL), which is given by

$$\begin{aligned}
\mu(t) :=& E[X_t] = E[X - t | X > t] = E\left[\int_t^{[X|X>t]} dx\right] \\
=& E\left[\int_t^{\infty} \mathbb{I}_{\{X>x|X>t\}} \, dx\right] = \int_t^{\infty} P\{X > x | X > t\} \, dx \\
=& \int_t^{\infty} \bar{F}_t(x) \, dx = \frac{1}{\bar{F}(t)} \int_t^{\infty} \bar{F}(x) \, dx \ .
\end{aligned} \tag{3.24}$$

The MRL at time $t = 0$ is the expected value of the original lifetime, i.e. $\mu(0) = E[X] := \mu$.

DMRL/IMRL. When the working condition of a system is modeled in terms of the MRL, we have the following classification of lifetime distributions:

- the lifetime distribution is *decreasing mean residual lifetime* (DMRL), iff $\mu(t)$ is decreasing in t, for $t \geq 0$;

- the lifetime distribution is *increasing mean residual lifetime* (IMRL), iff $\mu(t)$ is increasing in t, for $t \geq 0$ [18].

NBUE/NWUE. An alternate classification in terms of the MRL function is based on a comparison between a new system and a used system, where

- the lifetime distribution is new-better-than-used-in-expectation (NBUE), iff $\mu(t) \leq \mu$, for all $t \geq 0$;
- the lifetime distribution is new-worse-than-used-in-expectation (NWUE), iff $\mu(t) \geq \mu$, for all $t \geq 0$ [18].

When the MRL function $\mu(t)$ is decreasing (increasing) in t, then $\mu(t) \leq (\geq)\ \mu(0) = \mu$, for all $t > 0$. Therefore, the DMRL (IMRL) class is contained in the NBUE (NWUE) class.

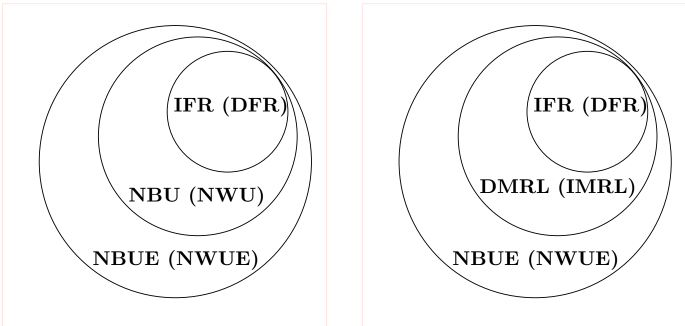


Figure 3.5: Illustration of the containment hierarchies of the aging classes.

Figure 3.5 depicts the two containment hierarchies of the aging classes. Let F denote a lifetime distribution. Then, the following chains of implications apply:

$$\begin{aligned} F \in\ \text{IFR (DFR)} \ &\Rightarrow\ F \in\ \text{NBU (NWU)} &&\Rightarrow\ F \in\ \text{NBUE (NWUE)}\ ; \\ F \in\ \text{IFR (DFR)} \ &\Rightarrow\ F \in\ \text{DMRL (IMRL)} &&\Rightarrow\ F \in\ \text{NBUE (NWUE)}\ . \end{aligned} \tag{3.25}$$

The proofs follow from the relationships between the failure rate, conditional reliability and MRL functions. The first chain of implications follows directly from the definitions of the three functions:

$$\begin{aligned} r(t) \text{ increasing (decreasing) in } t \ &\Leftrightarrow\ \bar{F}_t(t+s) \text{ decreasing (increasing) in } t, \text{ for each } s \geq 0 \\ &\Rightarrow\ \bar{F}_t(t+s) = e^{-\int\limits_t^{t+s} r(u)\, du} \leq (\geq)\ \bar{F}(s) = e^{-\int\limits_0^{s} r(u)\, du} \\ &\Rightarrow\ \mu(t) = \int\limits_0^{\infty} \bar{F}_t(t+s)\, ds\ \leq (\geq)\ \mu = \int\limits_0^{\infty} \bar{F}(s)\, ds\ . \end{aligned} \tag{3.26}$$

The second chain of implications follows from (3.26) and the definitions of classes based on MRL:

$$\begin{aligned} r(t) \text{ increasing (decreasing) in } t \ &\Rightarrow \ \mu(t) \text{ decreasing (increasing) in } t \\ &\Rightarrow \ \mu(t) \leq (\geq) \ \mu(0) = \mu \ . \end{aligned} \tag{3.27}$$

Figure 3.6 provides illustrations of the monotonic classes of failure rate function and the corresponding MRL functions. Notice that, the monotonicity of the failure rate function is the opposite of that of the MRL function for the monotonic classes.

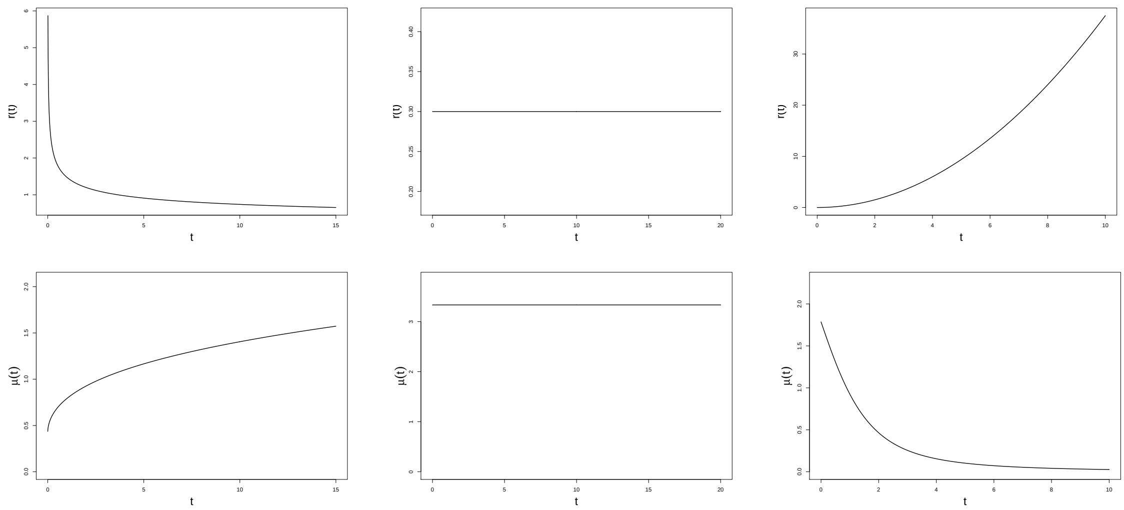


Figure 3.6: Examples of the failure rate function and the corresponding MRL function: DFR class (left column); CFR class (middle column); and IFR class (right column).

The MRL function can, using (3.22), be expressed in terms of the failure rate function as follows:

$$\mu(t) = \int_t^\infty \bar{F}_t(x) \, dx = \int_t^\infty e^{-\int_t^x r(u) \, du} \, dx \ . \tag{3.28}$$

Solving for the failure rate function, we get

$$r(t) = \frac{\frac{d}{dt} \, \mu(t) + 1}{\mu(t)} \ , \tag{3.29}$$

since

$$\begin{aligned} \frac{d}{dt}\mu(t) &= \int_t^\infty \frac{d}{dt} \, \bar{F}_t(x) \, dx - \bar{F}_t(t) = \int_t^\infty \bar{F}(x) \frac{d}{dt} \, \frac{1}{\bar{F}(t)} \, dx - 1 = \int_t^\infty \bar{F}(x) \frac{f(t)}{\bar{F}^2(t)} \, dx - 1 \\ &= \int_t^\infty \bar{F}(x) \frac{r(t)}{\bar{F}(t)} \, dx - 1 = r(t) \int_t^\infty \bar{F}_t(x) \, dx - 1 = r(t) \ \mu(t) - 1 \ . \end{aligned} \tag{3.30}$$

Therefore, the MRL can be constructed from the failure rate function. This is useful, for instance, when the failure rate function is defined piece-wise (e.g. the BFR function); see Figure 3.7 for an illustration. Notice that, the MRL function corresponding to a BFR function is non-monotonic (initially increasing, then decreasing) [21]; we will discuss this further in the following chapter.

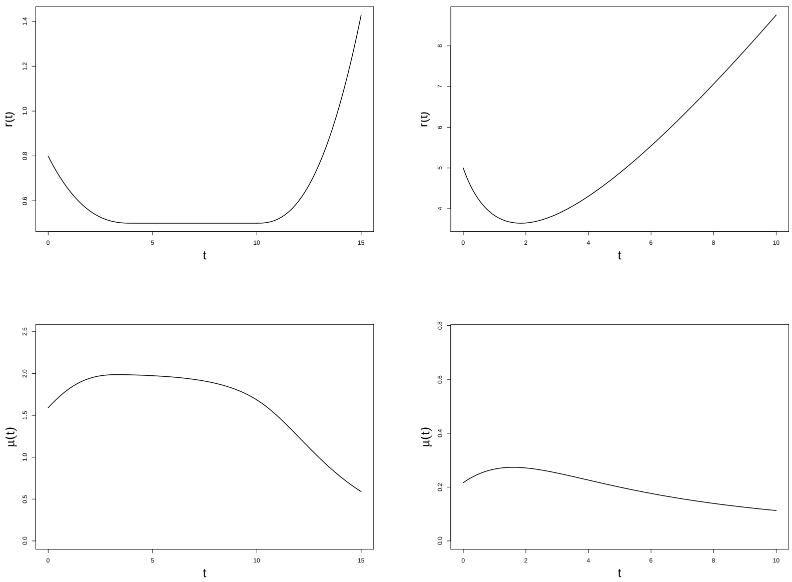


Figure 3.7: Examples of the failure rate function and the corresponding MRL function: BFR class (left column); and UFR class (right column).

As with the failure rate function, the MRL function completely determines the distribution of the system lifetime. The distribution function can be expressed solely in terms of the MRL function as follows:

$$\begin{aligned} F(t) &= 1 - e^{-\int_0^t r(u)\,du} = 1 - e^{-\int_0^t \frac{\frac{d}{du}\mu(u)+1}{\mu(u)}\,du} \\ &= 1 - e^{-\int_0^t \frac{\mu'(u)}{\mu(u)}\,du}\, e^{-\int_0^t \frac{1}{\mu(u)}\,du} = 1 - e^{-\left(\ln\mu(t)-\ln\mu(0)\right)}\, e^{-\int_0^t \frac{1}{\mu(u)}\,du} \\ &= 1 - e^{\ln\left(\frac{\mu(0)}{\mu(t)}\right)}\, e^{-\int_0^t \frac{1}{\mu(u)}\,du} = 1 - \frac{\mu(0)}{\mu(t)}\, e^{-\int_0^t \frac{1}{\mu(u)}\,du}\,. \end{aligned} \tag{3.31}$$

3.1.3 Partial Orderings of Distributions

While the aging classes describe (in some probabilistic sense) the aging properties of a system, partial orderings provide a comparison between lifetime variables in terms of some chosen aging property [18].

In this section, we provide a brief review of partial orderings of lifetime distributions in terms of failure rate, reliability and mean residual lifetime.

3.1.3.1 Partial Ordering based on Reliability

Let X and Y denote two lifetime variables with reliability functions denoted by $\bar{F}_X(.)$ and $\bar{F}_Y(.)$ respectively. Then, a partial ordering based on reliability is the *stochastic ordering* (abbreviated to 'ST'), which is defined as follows.

Definition 3.1. *X is greater than Y in stochastic ordering, i.e. $X \geq_{ST} Y$, iff*

$$\bar{F}_X(t) \geq \bar{F}_Y(t) \ , \qquad \forall \ t \geq 0 \ . \tag{3.32}$$

3.1.3.2 Partial Ordering based on Mean Residual Lifetime

Let $\mu_X(.)$ and $\mu_Y(.)$ denote the MRL functions of the random variables X and Y respectively. Then, a partial ordering based on MRL is the *mean residual ordering* (abbreviate to 'MR'), which is defined as follows.

Definition 3.2. *X is greater than Y in mean residual ordering, i.e. $X \geq_{MR} Y$, iff*

$$\mu_X(t) \geq \mu_Y(t) \ , \qquad \forall \ t \geq 0 \ . \tag{3.33}$$

It can be proved that this definition is equivalent to the ratio

$$\frac{\int_t^\infty \bar{F}_X(u) \ du}{\int_t^\infty \bar{F}_Y(u) \ du} \tag{3.34}$$

being increasing in $t \geq 0$ [18].

3.1.3.3 Partial Ordering based on Failure Rate

Let $r_X(.)$ and $r_Y(.)$ denote the failure rate functions of the two variables X and Y respectively. Then, the partial ordering based on failure rate is the *failure rate ordering* (abbreviated to 'FR', or sometimes, 'HR' for hazard rate), which is defined as follows.

Definition 3.3. *X is greater than Y in failure rate ordering, i.e. $X \geq_{FR} Y$, iff*

$$r_X(t) \leq r_Y(t) \ , \qquad \forall \ t \geq 0 \ . \tag{3.35}$$

Failure rate ordering is more stringent than both stochastic ordering and mean residual ordering, and it implies stochastic and mean residual orderings, i.e.

$$\begin{aligned} X \geq_{\text{FR}} Y \;&\Rightarrow\; X \geq_{\text{ST}} Y \;; \\ X \geq_{\text{FR}} Y \;&\Rightarrow\; X \geq_{\text{MR}} Y \;. \end{aligned} \tag{3.36}$$

The first implication follows from the relationship between the failure rate and reliability functions given in (3.15):

$$\begin{aligned} r_X(t) \leq r_Y(t) \;&\Rightarrow\; \int_0^t r_X(u)\,du \;\leq\; \int_0^t r_Y(u)\,du \\ &\Rightarrow\; e^{-\int_0^t r_X(u)\,du} = \bar{F}_X(t) \;\geq\; e^{-\int_0^t r_Y(u)\,du} = \bar{F}_Y(t) \;. \end{aligned} \tag{3.37}$$

The second implication follows from (3.21) and (3.24), where for $t > 0$,

$$\begin{aligned} r_X(t) \leq r_Y(t) \;&\Rightarrow\; \int_t^{t+s} r_X(u)\,du \;\leq\; \int_t^{t+s} r_Y(u)\,du \\ &\Rightarrow\; e^{-\int_t^{t+s} r_X(u)\,du} \;\geq\; e^{-\int_t^{t+s} r_Y(u)\,du} \\ &\Rightarrow\; \int_0^\infty e^{-\int_t^{t+s} r_X(u)\,du} ds = \mu_X(t) \;\geq\; \int_0^\infty e^{-\int_t^{t+s} r_Y(u)\,du} ds = \mu_Y(t) \;. \end{aligned} \tag{3.38}$$

3.2 Review of Repair Models in One Dimension

In this section, we provide a review of various general repair models for a system with a univariate lifetime. The associated failure processes are defined in one dimension, which is assumed to be time (or age).

Degree of a general repair. Recall that general repairs, based on their effectiveness in improving the working condition of the system, are categorized as one of the following types: (i) perfect repair; (ii) imperfect repair; (iii) minimal repair. Perfect repair is most effective, minimal repair least effective and imperfect repair has effectiveness between the two; see Section 2.2. The improvement in working condition following an imperfect repair can be anywhere between no improvement (minimal repair) or maximal improvement (perfect repair).

To model imperfect repairs, the effectiveness of repairs is translated to a variable, often referred to as the *degree of repair*, in the range $[0,1]$. The two extremes 0 and 1 correspond to a minimal repair and a perfect repair, respectively, and any repair with a degree between the two (i.e. in the range $(0,1)$) corresponds to an imperfect repair. The degrees of repair are ordered, such that, a general repair with a higher degree is more "effective" than one with a lower degree.

We have categorized the general repair models (processes) based on the three type of general repair. However, in most cases, the imperfect repair models include the minimal and perfect repair models.

3.2.1 Perfect Repair Process

The system following a perfect repair is assumed to be in an "as-good-as-new" working condition. This assumption is reasonable when the system deteriorates with time, and its working condition at the start of its lifetime is at its best (for instance the IFR class of lifetime distributions– we will discuss this further in the next chapter).

The sequence of consecutive failures of a system with a univariate lifetime, where failures are followed by immediate and instantaneous perfect repairs, is modeled as a *renewal process* in one dimension [3]. Since the repairs are immediate and instantaneous, the number of failures in any given interval is equal to the number of perfect repairs (or renewals).

A stochastic counting process $\{N(t); t \in \mathbb{R}_+\}$ is a renewal process if the sequence of non-negative inter-failure lifetimes $\{X_n; n \in \mathbb{N}_+\}$ are independent and identically distributed random variables [13]. This is equivalent to replacing a failed system with a new and identical system immediately following each failure.

Let $F(.)$ denote the distribution function of the original lifetime. Since, the inter-failure lifetimes are independent and identically distributed, $F(.)$ is the distribution function of all consecutive lifetimes $X_2, X_3, \ldots$. Also, let $\{T_n; n \in \mathbb{N}_+\}$ denote the sequence of failure points. Then, $N(t) = \max\{n : T_n \leq t\}$ counts the number of failures (or perfect repairs) performed in the interval $(0,t]$. Let $F_n(.)$ denote the distribution function of the n-th failure time T_n, for $n \in \mathbb{N}_+$. The distribution function of the first failure time, for $t \geq 0$, is given by

$$\begin{aligned} F_1(t) &= \mathrm{P}\{T_1 \leq t\} \\ &= \mathrm{P}\{X_1 \leq t\} = F(t) \ . \end{aligned} \tag{3.39}$$

Then, the distribution function of the second failure time T_2, is given by

$$\begin{aligned} F_2(t) &= \mathrm{P}\{T_2 \le t\} = \mathrm{P}\{T_1 + X_2 \le t\} = \mathrm{P}\{X_2 \le t - T_1\} \\ &= \int_0^t \mathrm{P}\{X_2 \le t - t_1 | T_1 = t_1\}\, dF_1(t_1) \\ &= \int_0^t F(t - t_1)\, dF_1(t_1) =: F^{**}F_1(t) \ , \end{aligned} \tag{3.40}$$

where $F^{**}F_1(.)$ denotes the convolution of F with $F_1(= F)$; and $dF_1(t) = f_1(t)\ dt$, when the corresponding density function $f_1(.)$ exists. In general, the distribution function of the $(n+1)$-th failure time T_{n+1}, for $n \in \mathbb{N}_+$, is given by

$$\begin{aligned} F_{n+1}(t) &= \mathrm{P}\{T_{n+1} \le t\} = \mathrm{P}\{T_n + X_{n+1} \le t\} = \mathrm{P}\{X_{n+1} \le t - T_n\} \\ &= \int_0^t \mathrm{P}\{X_{n+1} \le t - t_n | T_n = t_n\}\, dF_n(t_n) \\ &= \int_0^t F(t - t_n)\, dF_n(t_n) =: F^{**}F_n(t) \ . \end{aligned} \tag{3.41}$$

Therefore, the distribution function of the $(n+1)$-th failure time T_{n+1} is the convolution of F with F_n, which we denoted by $F^{**}F_n(.)$, for $n \in \mathbb{N}_+$; refer to Ross [22, 13]. Note that, when the times $\{t_1, \ldots, t_n\}$ of the previous n failures are given, the conditional distribution of the $(n+1)$-th failure time T_{n+1} depends only on the last failure time t_n before it, for $n \in \mathbb{N}_+$. It follows from the definition of a renewal process that, for all $n \in \mathbb{N}_+$ and $t > t_n$,

$$\begin{aligned} \mathrm{P}\{T_{n+1} \le t | T_1 = t_1, \ldots, T_n = t_n\} &= \mathrm{P}\{T_{n+1} \le t | T_n = t_n\} = \mathrm{P}\{X_{n+1} \le t - t_n | T_n = t_n\} \\ &= P\{X_1 \le t - t_n\} = F(t - t_n) \ . \end{aligned} \tag{3.42}$$

The distribution of the number $N(.)$ of failures and its expected value (or cumulative intensity function) $\Lambda(.) = E[N(.)]$ can be derived by substituting for the distribution functions of the failure times in equations (3.4) and (3.5), respectively. Note that, $\Lambda(t)$ represents the expected number of renewals (here, perfect repairs) in the interval $(0, t]$, for $t > 0$, and at $t = 0$, $\Lambda(0) = E[N(0)] = 0$.

The cumulative intensity function of the renewal process is sometimes referred to as the *renewal function*. The renewal function uniquely determines the distribution F [22].

By conditioning on the first failure at time $T_1 = X_1$, the renewal function can be expressed as the following *renewal equation*:

$$\begin{aligned}\Lambda(t) &= \int_0^t E\big[N(t)|T_1 = t_1\big]\, dF(t_1) = \int_0^t E\big[N(t) - N(t_1) + N(t_1)|T_1 = t_1\big]\, dF(t_1) \\ &= \int_0^t E\big[N(t_1)|T_1 = t_1\big]\, dF(t_1) + \int_0^t E\big[N(t) - N(t_1)|T_1 = t_1\big]\, dF(t_1) \\ &= \int_0^t dF(t_1) + \int_0^t \Lambda(t - t_1)\, dF(t_1) = F(t) + \int_0^t \Lambda(t - t_1)\, dF(t_1) \ ,\end{aligned} \tag{3.43}$$

since $E[N(t_1)|T_1 = t_1] = 1$, and

$$E[N(t) - N(t_1)|T_1 = t_1] = E[N(t - t_1)] = \Lambda(t - t_1) \ . \tag{3.44}$$

This follows from the definition of a renewal process: since a new and identical system is put into use after the first failure at t_1, the distribution of the number of renewals (or failures) at any point t, for $t > t_1$, is the same as the distribution of the number of renewals at the point $t - t_1$, i.e. $N(t) - N(t_1) \overset{\mathrm{d}}{=} N(t - t_1)$ [13].

Given a distribution function $F(.)$, the renewal equation can sometimes be analytically solved for the renewal function $\Lambda(.)$; see Ross [13] and Karlin & Samual [23]. Most often, however, numerical methods involving the Reimann-Stieltjes approximation of the integral equation are used to compute the renewal function; see for instance Xie [24].

The conditional intensity function of the renewal process at time $t \geq 0$ is given by

$$\tilde{\lambda}(t|\mathcal{H}_t) = r(t - (X_1 + \cdots + X_{N(t^-)})) = r(t - T_{N(t^-)}) \ , \tag{3.45}$$

where $T_{N(t^-)} = X_1 + \cdots + X_{N(t^-)}$ is the time of the last renewal before time t, and $r(.)$ is the failure rate function corresponding to the original lifetime [25].

3.2.2 Minimal Repair Process

By definition, a minimal repair restores the working condition of the system after the repair to its working condition immediately prior to the failure, i.e. the system behaves like a system that has not failed.

The sequence of consecutive failures of a system with a univariate lifetime, where failures are followed by immediate and instantaneous minimal repairs, is modeled as a *Poisson* *process* in one dimension. The Poisson process is unique in that the intensity function of the process is equal to the conditional intensity function of the process, i.e.

$$\lambda(t) = \tilde{\lambda}(t|\mathcal{H}_t) = r(t) \ , \tag{3.46}$$

where $r(.)$ is the failure rate function corresponding to the original system lifetime [17, 26]. The Poisson process can be used to model minimal repairs of systems with both monotonic (e.g. IFR or DFR) and non-monotonic (e.g. BFR) lifetime distributions– i.e. there is no restriction on the class of lifetime distribution.

A stochastic counting process $\{N(t); t \in \mathbb{R}_+\}$ is a Poisson process with intensity function $\lambda(.)$, if the following conditions hold:

(i) $\{N(t); t \in \mathbb{R}_+\}$ has independent increments;

(ii) $\mathrm{P}\{N(t+dt) - N(t) = 1\} = \lambda(t)\ dt + o(dt)$;

(iii) $\mathrm{P}\{N(t+dt) - N(t) \geq 2\} = o(dt)$; and

(iv) for $n \in \mathbb{N}$, the number of failures in the interval $(0, t]$, $t > 0$, has a Poisson distribution with parameter $\Lambda(t)$, i.e.

$$\mathrm{P}\{N(t) = n\} = \frac{[\Lambda(t)]^n\ e^{-\Lambda(t)}}{n!} \ , \tag{3.47}$$

where

$$\Lambda(t) = E[N(t)] = \int_0^t \lambda(s)\ ds \ . \tag{3.48}$$

The number of failures in any interval $(s, t]$, for $0 < s \leq t$, is given by $\Lambda(t) - \Lambda(s)$, and therefore, the distribution of the increment $N(t) - N(s)$ is given by

$$\mathrm{P}\{N(t) - N(s) = n\} = \frac{[\Lambda(t) - \Lambda(s)]^n\ e^{-\left\{\Lambda(t) - \Lambda(s)\right\}}}{n!} \ . \tag{3.49}$$

3.2.3 Imperfect Repair Processes

The stochastic properties of a failure process can be uniquely determined by the associated conditional intensity process, which we have denoted by $\{\tilde{\lambda}(t|\mathcal{H}_t); t \in \mathbb{R}_+\}$; see Section 3.1.1.2. Therefore, in order to define a failure process, where failures are followed by general repair, we can model the effect of the general repairs in terms of the conditional intensity function of the process.

In this section, we will review various approaches suggested for modeling the effect of imperfect repairs.

3.2.3.1 Age Modification Approach

This approach involves modeling the effect of general repairs as changes (reductions) in the age of the system, such that, after a general repair the system behaves like an identical system at a younger age. This approach assumes that the system has an IFR lifetime distribution (i.e. the system is deteriorating with time).

Let $A(t)$ denote the age of the system at real time t in terms of its working condition. This function is referred to as the *virtual age* or *effective age* function, and is different from the calendar age of the system. For systems with IFR functions, it is defined such that $A(t) \leq t$, for all $t > 0$ (at $t = 0$, $A(0) = 0$). When the system has not failed or if all failures of the system are rectified by minimal repair, then this virtual age is equal to the actual or calendar age of the system.

With virtual age models, the conditional intensity function of the failure process at any time $t \geq 0$, is defined as follows:

$$\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_0(A(t)) \ , \tag{3.50}$$

where $\lambda_0 = r$ is the baseline intensity (or failure rate) function of the original lifetime. The baseline intensity function remains unchanged, and the effect of a repair can be viewed as shifting or translating this baseline function along the time axis.

The virtual age function $A(.)$ at any point is defined as a function of the number of failures and the corresponding degrees of repair prior to that point. Usually the virtual age functions are of the form

$$A(t) = t - \xi_A(T_1, \ldots, T_{N(t^-)}; \delta_1, \ldots, \delta_{N(t^-)}) \ , \tag{3.51}$$

where $\xi_A(.)$ is some non-negative function of the failure times $\{T_n; n \in \mathbb{N}_+\}$ and the corresponding general repair degrees $\{\delta_n; n \in \mathbb{N}_+\}$, and $N(t^-)$ represents the number of failures prior to time t. This function is defined such that, when $\delta_i = 1$ for all $i \in \mathbb{N}_+$ (i.e. all repairs are perfect), then the virtual age reduces to $A(t) = t - T_{N(t^-)}$, for all $t \geq 0$, which is the virtual age corresponding to a renewal process in one dimension; see Section 3.2.1. Also, when $\delta_i = 0$ for all $i \in \mathbb{N}_+$ (i.e. all repairs are minimal), then the virtual age reduces to $A(t) = t$, for all $t \geq 0$, which is the virtual age corresponding to a Poisson process in one dimension; see Section 3.2.2.

The first virtual age models, suggested by Kijima [27], are defined in terms of a virtual age process, denoted by $\{V_n; n \in \mathbb{N}_+\}$, where V_n is the virtual age of the system immediately following the n-th repair, for $n \in \mathbb{N}_+$. That is, in terms of a virtual age function,

$$V_n = A(T_n^+) \ . \tag{3.52}$$

These virtual age models are characterized by the following property: the distribution of the $(n+1)$-th inter-failure lifetime X_{n+1}, $n \in \mathbb{N}_+$, given all previous failure points, is defined as

$$\mathrm{P}\{X_{n+1} > x | V_n = v_n\} = \mathrm{P}\{X_1 > v_n + x | X_1 > v_n\} = \frac{\bar{F}(v_n + x)}{\bar{F}(v_n)} \ , \tag{3.53}$$

where $\bar{F}(.)$ is the reliability function of the original lifetime X_1.

Kijima's Model I. Kijima's first virtual age model assumes a repair can at most undo damage accumulated by the system since the previous repair, and therefore,

$$V_n = V_{n-1} + D_n\, X_n = \sum_{i=1}^{n} D_i\, X_i \ , \tag{3.54}$$

where $\{D_n; n \in \mathbb{N}_+\}$ is a sequence of independent and identically distributed random variables in the range $[0, 1]$. Note that, here, 1 corresponds to a minimal repair and 0 corresponds to a perfect repair. Using our notations, this virtual age process is equivalent to having the following virtual age function:

$$A(t) = t - \sum_{i=1}^{N(t^-)} \delta_i \left[A(T_i) - A(T_{i-1}^+)\right] = t - \sum_{i=1}^{N(t^-)} \delta_i\, X_i \ , \tag{3.55}$$

since, when we set $\delta_i = 1 - D_i$, we get

$$\begin{aligned} A(T_n^+) &= T_n^+ - \sum_{i=1}^{n} \delta_i\, X_i = T_n^+ - \sum_{i=1}^{n} (1 - D_i)\, X_i \\ &= T_n^+ - \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} D_i\, X_i \\ &= T_n^+ - T_n + \sum_{i=1}^{n} D_i\, X_i \\ &= \sum_{i=1}^{n} D_i\, X_i = V_n \ . \end{aligned} \tag{3.56}$$

Kijima's Model II. Kijima's second virtual age model assumes a repair can undo all damage accumulated by the system since the start of its lifetime, and therefore,

$$V_n = D_n \left(V_{n-1} + X_n\right) = \sum_{i=1}^{n} \left(\prod_{j=i}^{n} D_j\right) X_i \ . \tag{3.57}$$

Using our notations, this virtual age process is equivalent to defining the virtual age function as follows:

$$A(t) = t - \sum_{i=1}^{N(t^-)} \delta_i \ A(T_i) \ . \tag{3.58}$$

The proof is straight-forward. When we set $1 - \delta_i = D_i$, for all $i \in \mathbb{N}_+$, we get

$$\begin{aligned} A(T_1^+) &= A(T_1) - \delta_1 \ A(T_1) = (1 - \delta_1) \ A(T_1) = (1 - \delta_1) \ T_1 \\ &= \sum_{i=1}^{1} \prod_{j=i}^{1} (1 - \delta_j) \ X_i = \sum_{i=1}^{1} \prod_{j=i}^{1} D_j \ X_i = V_1 \ . \end{aligned} \tag{3.59}$$

We assume that this is true for n, i.e.

$$A(T_n^+) = \sum_{i=1}^{n} \prod_{j=i}^{n} (1 - \delta_j) \ X_i = \sum_{i=1}^{n} \prod_{j=i}^{n} D_j \ X_i = V_n \ . \tag{3.60}$$

Then, for $n + 1$, we get

$$\begin{aligned} A(T_{n+1}^+) &= (1 - \delta_{n+1}) \ A(T_{n+1}) = (1 - \delta_{n+1}) \ \left[A(T_n^+) + X_{n+1}\right] \\ &= (1 - \delta_{n+1}) \left[\sum_{i=1}^{n} \prod_{j=i}^{n} (1 - \delta_j) \ X_i + X_{n+1}\right] \\ &= \sum_{i=1}^{n} \prod_{j=i}^{n+1} (1 - \delta_j) \ X_i + (1 - \delta_{n+1}) \ X_{n+1} \\ &= \sum_{i=1}^{n+1} \prod_{j=i}^{n+1} (1 - \delta_j) \ X_i = \sum_{i=1}^{n+1} \prod_{j=i}^{n+1} D_j \ X_i = V_{n+1} \ . \end{aligned} \tag{3.61}$$

Other functional forms of the virtual age function can be defined; see for example Doyen & Guadoin [28], Hollander [29], and Varnosafaderani & Chukova [30].

3.2.3.2 Intensity Modification Approach

Another approach for modeling the effect of general repairs is the *intensity modification approach*. Here, the effect of a general repair is defined as a reduction in the baseline intensity function of the failure process, such that, the conditional intensity function following an imperfect repair is bounded between those following a minimal repair and a perfect repair [28]. These models also assume an IFR lifetime distribution.

Where the conditional intensity function for the age modification approach has the following form:

$$\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_0\Big(t - \xi_A(T_1, \ldots, T_{N(t^-)}; \delta_1, \ldots, \delta_{N(t^-)})\Big) \ , \tag{3.62}$$

the conditional intensity function for the intensity modification approach is of the form

$$\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_0(t) - \xi_{\lambda_0}(T_1, \ldots, T_{N(t^-)}; \delta_1, \ldots, \delta_{N(t^-)}) \ , \tag{3.63}$$

where, $\xi_{\lambda_0}(.)$ is some function of the baseline intensity function $\lambda_0(.)$ and also the times and degrees of the general repairs. This function is defined such that, when all repairs are minimal, the conditional intensity function reduces to $\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_0(t)$, for all $t \geq 0$, which is the conditional intensity function corresponding to the minimal repair process; and when all repairs are perfect, the conditional intensity function reduces to $\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_0(t - T_{N(t^-)})$, for all $t \geq 0$, which is the conditional intensity function of the perfect repair process; see Sections 3.2.1 and 3.2.2.

An example of these intensity modification models, analogous to Kijima's second virtual age model, given in (3.57), is

$$\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_0(t) - \sum_{i=1}^{N(t^-)} \delta_i \left[\tilde{\lambda}_i(t|\mathcal{H}_t) - \lambda_0(t - T_i)\right] \ , \tag{3.64}$$

where $\tilde{\lambda}_i(t|\mathcal{H}_t)$ represents the conditional intensity function after the i-th general repair, defined for $t \in (T_i, \infty)$ [31].

In Figure 3.8, we have plotted examples of this conditional intensity function, given the failure times $T_i = t_i$, $i \in \{1, 2, 3\}$, for minimal, imperfect and perfect repair processes.

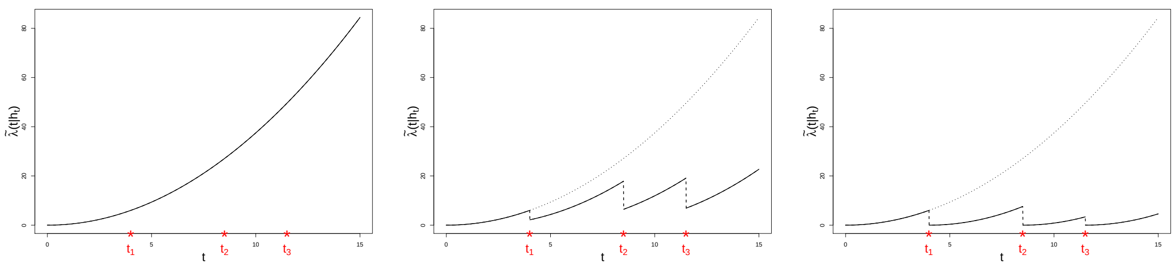


Figure 3.8: Illustrations of the conditional (failure) intensity, given the failure times t_1, t_2, and t_3: minimal repairs (left); imperfect repairs (middle); and perfect repairs (right).

Intensity modification models having forms other than (3.63) can be defined; see for example Doyen & Gaudoin [28], Lawless & Thiagarajah [15] and Lindqvist [32].

3.2.3.3 Monotone Stochastic Processes

The imperfect repair process can also be modeled as a monotone stochastic process, where the inter-failure lifetimes $\{X_n; n \in \mathbb{N}_+\}$ are monotone decreasing, i.e.

$$X_{n+1} \leq_{\mathrm{ST}} X_n \quad , \quad \forall n \in \mathbb{N}_+ \quad . \tag{3.65}$$

For example, consider the *quasi-renewal process*, a generalization of the ordinary renewal process, which is defined as follows. A point process is a quasi-renewal process with parameter $\alpha > 0$ and distribution F, iff

$$X_n \stackrel{\mathrm{d}}{=} \alpha^{n-1} \, Z_n \quad , \quad \forall n \in \mathbb{N}_+ \quad , \tag{3.66}$$

where the equality is in distribution, and $\{Z_n; n \in \mathbb{N}_+\}$ is a sequence of independent and identically distributed random variables with distribution function $F(.)$ (i.e. a renewal process) [33].

Let $G_n(.)$ denote the distribution function of the n-th inter-failure lifetime X_n of the quasi-renewal process, for $n \in \mathbb{N}_+$. Then, $G_1(x) = F(x)$, for $x \geq 0$; and the distribution function of the $(n+1)$-th inter-failure lifetime X_{n+1}, for $n \in \mathbb{N}_+$, is given by

$$\begin{aligned} G_n(x) &= \mathrm{P}\{X_{n+1} \leq x\} \\ &= \mathrm{P}\{\alpha^n \, Z_{n+1} \leq x\} = \mathrm{P}\{Z_{n+1} \leq \alpha^{-n} \, x\} = F\left(\frac{x}{\alpha^n}\right) \quad . \end{aligned} \tag{3.67}$$

This process with parameter $\alpha \in (0, 1]$ is used to model imperfect repairs, when the inter-failure lifetime following an imperfect repair is assumed to have the same distribution as the inter-failure lifetime leading to the repair scaled by α. When $\alpha = 1$, the process reduces to an ordinary renewal process. These processes are a generalization of the perfect repair process, but not the minimal repair process.

For more on quasi-renewal (and geometric) processes and their applications in imperfect maintenance see Lam [34], Pham & Wang [35], Park & Pham [36], Lam & Zhang [37], and Rehmert & Nachlas [38].

3.2.4 General Repair Processes and BFR Lifetime Distributions

The general repair models discussed above assume a monotonically increasing failure rate function, and are therefore unsuitable for modeling non-minimal general repairs when the system lifetime distribution has a non-monotonic failure rate function– in particular, BFR function. BFR functions are characterized by three lifetime phases: DFR ("burn-in" phase), CFR ("useful life" phase) and IFR ("wear-out" phase); see Figure 3.1.2.1.

As discussed earlier, when all repairs are minimal, the Poisson process can be used to model the consecutive failures of a system, regardless of the class of lifetime distribution. To model consecutive replacements of the system, the renewal process can be used, again regardless of the class of distribution. When perfect repairs are modeled as replacements however, there is the implicit assumption that the system is deteriorating with time, and therefore the system at the start of its lifetime is in a better working condition. For systems whose lifetimes are modeled with distributions having BFR functions, this is not the case; see Section 3.1.2 on aging properties.

Systems whose working condition initially improves with time are often subjected to a burn-in period in order to improve their reliability before they are released into the market. Most studies on burn-in strategies suggest only minimal repairs to rectify failures during the burn-in period. This is because existing general repair models are not appropriate when the failure rate function has a DFR phase.

Attempts have been made to adapt existing virtual age models to suit systems with lifetime distributions having a BFR function. In Dijoux [39] and Shafiee et al [40], a perfect repair in the non-decreasing phase returns the system to the start of the useful life period $[a_1, a_2]$ instead of the start of the system lifetime (note that a_1 and a_2 are the change-points of the BFR function; see Section 3.1.2.1). Therefore, virtual age models are applied only outside the DFR phase. In the DFR phase however, as with burn-in strategies, all repairs are modeled as minimal repairs.

Dijoux & Idée [41] introduce the concepts of "as-good-as-optimal repair" and "arithmetic increase in age" as a first attempt to model the effect of general repairs in the DFR phase. The effect of a repair, in both models, is expressed in terms of changes in the virtual age of the system. In both models, the effect of a non-minimal general repair in the DFR phase is reaching the useful life period faster than when a minimal repair is performed (i.e. aging the system). Once in the useful life period, the effect of a repair is modeled as a decrease in the virtual age, with a perfect repair returning the virtual age to a_1 (not 0).

Modeling the effect of a repair as an aging of the system, does not always preserve the order of the types of general repair in terms of their effectiveness (for instance, a perfect repair may not always be the most effective repair)– we will discuss this further in the next chapter, where we propose a new approach to model the effect of repairs while the system is still improving.

For more on imperfect repair modeling in general, refer to Wang & Pham [42], Lindqvist [32] and Blishcke & Murthy [2].

3.3 Chapter Summary

In this chapter, we provided a brief review of concepts (such as, stochastic counting processes, intensity processes and the failure rate function) necessary in modeling consecutive failures of systems in one dimension.

We discussed various aging classes and partial orderings of univariate distributions in terms of the failure rate, conditional reliability and mean residual lifetime functions.

We reviewed existing general repair models and the associated failure processes in one dimension.

Chapter 4

Modeling Repairs of a System with a Bathtub-Shaped Failure Rate Function

In this chapter, we propose a new approach to model the effect of a general repair performed on a system whose lifetime is modeled with a distribution having a bathtub-shaped failure rate function. We then define a failure process to model consecutive failures of the system, where failures are rectified by general repair.

This chapter is arranged as follows. In Section 4.1, we define the type of system considered. In Section 4.2, we discuss the definitions of the various types of general repair and their effect on the working condition of the system. In Section 4.3, we formulate the proposed general repair model and discuss its properties. In Section 4.4, we derive the distributions of the failure times and the inter-failure lifetimes of the proposed failure process. In Section 4.5, we conclude with a summary of the results.

4.1 Bathtub-Shaped Failure Rate Functions

We propose a general repair model for a system that is assumed to be from a homogeneous population having a lifetime distribution with a bathtub-shaped failure rate (BFR) function. (For brevity, we will say that a lifetime distribution is BFR, when the associated failure rate function is BFR.) We do not consider systems from a heterogeneous populations, where the BFR function is a result of a mixture of sub-populations of "weak" and "strong" systems; see Section 3.2.4. Instead, we consider systems that have a period of improvement following manufacture. During this *warm-up* or *burn-in* phase, the performance of the system improves. For instance, in a mechanical system, rough edges are smoothed, operating temperature increases, lubricant circulates and systems perform increasingly smoothly.

The BFR function is characterized by three lifetime phases: (i) a decreasing failure rate (DFR) phase or *wear-in period* (also, warm-up or burn-in period); (ii) a constant failure rate (CFR) phase or *useful life period*; and (iii) an increasing failure rate (IFR) phase or *wear-out period*. Let $r(.)$ denote a BFR failure rate function. Then, at time $t \geq 0$,

$$r(t) = \begin{cases} r_1(t) : r_1'(t) < 0 \ , & 0 \leq t \leq a_1 \\ r_2(t) : r_2'(t) = 0 \ , & a_1 \leq t \leq a_2 \\ r_3(t) : r_3'(t) > 0 \ , & t \geq a_2 \ , \end{cases} \tag{4.1}$$

where a_1 and a_2 denote the change-points, and $r_2(t) = r_1(a_1) = r_3(a_2)$, for all $t \in [a_1, a_2]$ [21]; see Figure 4.1.

During the initial, DFR phase, system failures are attributed to minor, undiscovered production defects whose effect on the working condition of the system diminishes over time. During the CFR phase, system failures are assumed to be random, due mostly to accidents (such as shocks, misuse, etc.) and not due to wear. During the IFR phase, system failures are due to accumulation of wear or damage and other negative effects of aging [43]; see Section 3.2.4.

Throughout this study, to illustrate concepts and models, and in the numerical examples, we use the following BFR function from Dijoux [39]:

$$r(t) = \begin{cases} \lambda + \alpha_1 \, \beta_1 \, (a_1 - t)^{\beta_1 - 1} \ , & t \leq a_1 \\ \lambda \ , & a_1 \leq t \leq a_2 \\ \lambda + \alpha_2 \, \beta_2 \, (t - a_2)^{\beta_2 - 1} \ , & t \geq a_2 \ , \end{cases} \tag{4.2}$$

where $\lambda > 0$, $\alpha_1, \alpha_2 > 0$, $\beta_1, \beta_2 > 1$, and $a_2 > a_1$, for $a_1, a_2 > 0$ [39]; see Figure 4.1. The derivative of this function is given by

$$r'(t) = \begin{cases} -\alpha_1 \, \beta_1 \, (\beta_1 - 1) \, (a_1 - t)^{\beta_1 - 2} \ , & t \leq a_1 \\ 0 \ , & a_1 \leq t \leq a_2 \\ \alpha_2 \, \beta_2 \, (\beta_2 - 1) \, (t - a_2)^{\beta_2 - 2} \ , & t \geq a_2 \ . \end{cases} \tag{4.3}$$

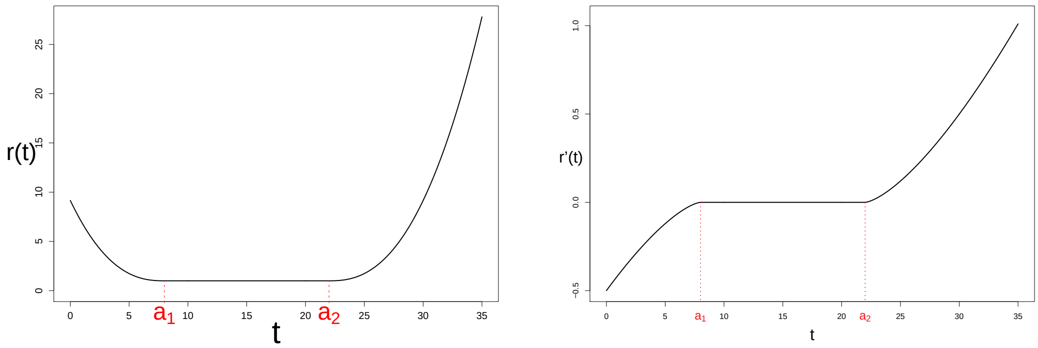


Figure 4.1: An example of the BFR function $r(.)$ and its derivative $r'(.)$, with parameters $\lambda = 1$, $\alpha_1 = \alpha_2 = 0.15$, $\beta_1 = \beta_2 = 3.45$, $a_1 = 8$, and $a_2 = 22$, plotted over the interval $[0, 35]$.

4.2 General Repairs

General repairs restore a system from a failed state to an operational state and are also assumed to affect the physical working condition of the repaired system. Based on their effectiveness, general repairs can be classified as one of the following three types: minimal, perfect and imperfect repairs, where minimal repairs are least effective, perfect repairs are most effective and imperfect repairs have effectiveness between the two. The effectiveness of a repair is represented by its degree, which is a variable in the range $[0, 1]$. The two extremes 0 and 1 correspond to minimal repair and perfect repair, respectively, and any repair with a degree between the two (i.e. in the range $(0, 1)$) corresponds to an imperfect repair. Thus, a general repair with a higher degree is more effective than one with a lower degree; see Section 3.2.

Throughout this text, we use the term *non-minimal repair* to refer to general repairs with degrees in the range $(0, 1]$, i.e. imperfect and perfect repairs.

4.2.1 System Condition Following General Repairs

A univariate lifetime distribution F can be characterized in terms of the following probabilistic properties of the residual lifetime: (i) the failure rate function $r(.)$; (ii) the conditional reliability function $\bar{F}_t(.)$; and (iii) the mean residual lifetime (MRL) function $\mu(.)$.

Let $X_t = [X - t | X > t]$ denote the residual lifetime of a system of age $t \geq 0$. Then, the relationships between the three functions are presented in the following equations:

$$\bar{F}_t(t+x) = \mathrm{P}\{X_t > x\} = \frac{\bar{F}(t+x)}{\bar{F}(t)} = e^{-\int\limits_t^{t+x} r(u)\, du} \ , \tag{4.4}$$

$$\mu(t) = E[X_t] = \int\limits_0^\infty \bar{F}_t(t+x)\, dx = \int\limits_t^\infty \bar{F}_t(u)\, du \ , \tag{4.5}$$

and

$$r(t)\, dt \approx \mathrm{P}\{X_t \leq dt\} = 1 - \bar{F}_t(t+dt) \ , \tag{4.6}$$

where $\bar{F}(.)$ denotes the reliability function of the original lifetime X; see Section 3.1.2. These metrics describe the aging properties of the system and are assumed to be a reflection of the physical working condition of the system.

We define the behavior of the system following a general repair in terms of the failure rate of the succeeding residual lifetime. Therefore, we need a more rigorous definition of the types of general repair based on this failure rate.

Let T_n and δ_n, for $n \in \mathbb{N}_+$, denote the n-th failure time and the degree of the corresponding general repair, respectively. Note that, the failure times are ordered: $0 < T_1 < T_2 < \ldots < T_n < \ldots$. Then, we define the conditional residual lifetime of the system at time t, given that n failures have occurred before t, as follows:

$$X_{t;h_{t;n}}^{(\delta_1,\ldots,\delta_n)} \stackrel{\mathrm{d}}{=} [T_{n+1} - t \mid T_{n+1} > t, N(t) = n, T_1 = t_1, \ldots, T_n = t_n; \delta_1, \ldots, \delta_n] \ , \tag{4.7}$$

defined for $t > t_n$, where $\mathcal{H}_t = h_{t;n}$ denotes the relevant history at time t, which in this case is the event $\{N(t) = n, T_1 = t_1, \ldots, T_n = t_n\}$, along with the degrees $\delta_1, \ldots, \delta_n$. When $t = t_n^+$, then this residual lifetime is equal in distribution to the $(n+1)$-th conditional inter-failure lifetime, i.e. for $n \in \mathbb{N}_+$, and $X_{n+1} = T_{n+1} - T_n$,

$$X_{t_n^+;h_{t_n^+;n}}^{(\delta_1,\ldots,\delta_n)} \stackrel{\mathrm{d}}{=} [X_{n+1} \mid N(t_n^+) = n, T_1 = t_1, \ldots, T_n = t_n; \delta_1, \ldots, \delta_n] \ . \tag{4.8}$$

Since a minimal repair has no effect on the working condition of the repaired system, when all n repairs before t are minimal, then the conditional residual lifetime in (4.7) is equal in distribution to the residual lifetime of the original system. That is, when $\delta_i = 0$, for all $i \in \{1, \ldots, n\}$, then

$$X_{t;h_{t;n}}^{(0,\ldots,0)} \stackrel{\mathrm{d}}{=} [X_1 - t \mid X_1 > t] = X_t \ , \tag{4.9}$$

where $X_1 = T_1$ is the original lifetime (or the time to first failure) of the system and X_t is the corresponding residual lifetime at time t.

From the definitions of the types of general repair, we know that, following a repair

(i) the working condition of the system is no worse than a system that has not failed (or equivalently, a minimally-repaired system); and

(ii) the working condition of the system improves as the degree of the repair increases, with a perfect repair resulting in the most improvement.

Since we use the failure rate function as a measure of the working condition of the system, we require the following ordering to hold:

$$X_{t;h_{t;n}}^{(\delta_1,\ldots,\delta_i=0,\ldots,\delta_n)} \leq_{FR} X_{t;h_{t;n}}^{(\delta_1,\ldots,\delta_i=\delta,\ldots,\delta_n)} \leq_{FR} X_{t;h_{t;n}}^{(\delta_1,\ldots,\delta_i=\delta',\ldots,\delta_n)} \leq_{FR} X_{t;h_{t;n}}^{(\delta_1,\ldots,\delta_i=1,\ldots,\delta_n)} , \tag{4.10}$$

for any $i \in \{1,\ldots,n\}$, $n \in \mathbb{N}_+$, and all $\delta \leq \delta'$, where $\delta, \delta' \in (0,1)$ denote the degrees of imperfect repairs. The term '*FR*' specifies that the partial ordering of the distributions is based on failure (or hazard) rate; we use the terminology introduced in Lai & Xie [18]. The ordering in (4.10) implies that, given the number n and times $t_1,\ldots,t_n$ of failures before time t, the residual lifetime at $t > t_n$ is stochastically ordered (based on failure rate ordering) by the value of any of the n degrees $\delta_1,\ldots,\delta_n$, given that the other $n-1$ degrees are fixed.

Let $r_{t;h_{t;n}}(x|\delta_1,\ldots,\delta_n)$ denote the failure rate function, at time $t+x$, of the $(n+1)$-th conditional residual lifetime defined in (4.7). This function is defined for $t > t_n$ and $x \geq 0$. The ordering in (4.10) is equivalent to the failure rate function $r_{t;h_{t;n}}(x|\delta_1,\ldots,\delta_i=\delta,\ldots,\delta_n)$ being decreasing in $\delta_i = \delta \in [0,1]$, for any $i \in \{1,\ldots,n\}$, and each $t > t_n$ and all $x \geq 0$, when all other parameters are fixed. That is, given the history $h_{t;n}$ of the failure process,

$$\begin{aligned} & r_{t;h_{t;n}}(x|\delta_1,\ldots,\delta_i=0,\ldots,\delta_n) \geq r_{t;h_{t;n}}(x|\delta_1,\ldots,\delta_i=\delta,\ldots,\delta_n) \\ \geq\ & r_{t;h_{t;n}}(x|\delta_1,\ldots,\delta_i=\delta',\ldots,\delta_n) \geq r_{t;h_{t;n}}(x|\delta_1,\ldots,\delta_i=1,\ldots,\delta_n) , \end{aligned} \tag{4.11}$$

for any $i \in \{1,\ldots,n\}$, $n \in \mathbb{N}_+$, and for each $t > t_n$ and all $x \geq 0$. Therefore, the failure rate following an imperfect repair is bounded between those following a minimal repair and a perfect repair.

The failure rate has a probabilistic interpretation which makes it a reasonable measure to use when defining the effect of repairs. In general, $r(t)dt$ is the probability of a system failing for the first time in an interval of length dt immediately following its current age t.

Then, $r_{t;h_{t;n}}(x|\delta_1,\ldots,\delta_n)\,dx$ can be interpreted as the probability that the $(n+1)$-th failure of the system is in the interval $(t+x, t+x+dx]$, given that its n-th failure was before time t, and given all n previous failure times and the corresponding repair degrees. In terms of the corresponding conditional reliability function, the failure rate satisfies

$$\begin{aligned} r_{t;h_{t;n}}(x|\delta_1,\ldots,\delta_n)\,dx &\approx \mathrm{P}\{T_{n+1} \le t+x+dx \mid T_{n+1} > t+x, T_1 = t_1,\ldots,T_n = t_n; \delta_1,\ldots,\delta_n\} \\ &= 1 - \mathrm{P}\{T_{n+1} > t+x+dx \mid T_{n+1} > t+x, T_1 = t_1,\ldots,T_n = t_n; \delta_1,\ldots,\delta_n\}\ , \end{aligned} \tag{4.12}$$

defined for all $t > t_n$ [cf. (4.6)]. Therefore, since the failure rate is decreasing in any of the n degrees of repair, the conditional reliability function is increasing in that degree. That is, a general repair of a higher degree at any of the n failure times results in greater improvement in system reliability when compared to a repair of a lower degree at that same time (when all other parameters are fixed). When $t = t_n^+$, then the failure rate function in (4.12) becomes the failure rate function of the $(n+1)$-th conditional inter-failure lifetime at $t+x$, and when all repairs are minimal, it reduces to the failure rate function of the original lifetime.

Note that, if (4.10) holds, then this order of residual lives also holds when the partial ordering is based on conditional reliability and mean residual life; see Lai & Xie [18] for more on partial orderings.

4.2.2 Replacement vs. Perfect Repair

Often perfect repair is defined as one that leaves the working condition of the system in an "as-good-as-new" state, which implies that it is equivalent to a replacement. Defining perfect repair such that it coincides, in effectiveness, with a replacement is consistent with the definition that perfect repair is the most effective general repair, only when:

(i) the repair model is defined such that all damage accumulated since the start of the system lifetime can be undone; and

(ii) the lifetime distribution is IFR (i.e. the system deteriorates over time).

Example 1. To illustrate point (i), consider the virtual age models introduced by Kijima [27]; see Section 3.2.3. The first virtual age model is defined such that a general repair can at most undo damage accumulated since the previous repair. Then, the definition of the repair model allows a perfect repair to coincide with a replacement as long as all previous repairs are perfect. Any perfect repair performed after the first general repair that is not perfect does not coincide with a replacement. The second virtual age model is defined such that a repair can undo all damage accumulated since the start of the system lifetime. Then, a perfect repair always coincides with a replacement. Presented in Figure 4.2 are examples of the two virtual age functions

$$\begin{aligned} \text{Model I:} \quad & V(t) = t - \sum_{i=1}^{N(t^-)} \delta_i \left[V(T_i) - V(T_{i-1}^+) \right] \; ; \\ \text{Model II:} \quad & V(t) = t - \sum_{i=1}^{N(t^-)} \delta_i \; V(T_i) \; , \end{aligned} \tag{4.13}$$

where $V(t)$ is the virtual age at time t, $N(t)$ is the number of failures in $(0, t]$, and $V(T_i^+)$ is the virtual age immediately following the i-th repair, for $i \in \mathbb{N}_+$.

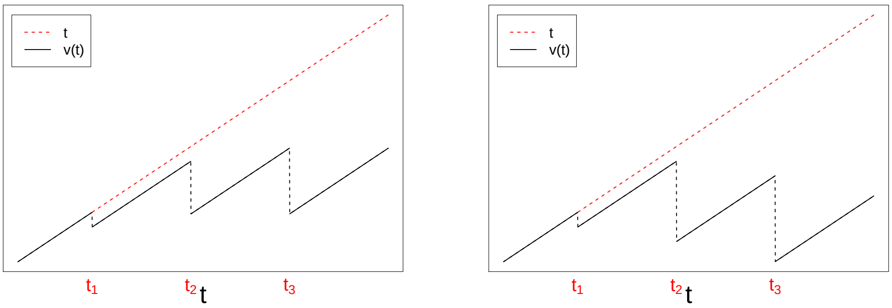


Figure 4.2: Example virtual age functions, denoted by $v(t)$, where a repair can undo damage accumulated since: (i) the previous repair (left) and (ii) the start of the system lifetime (right). In both plots, $\delta_1 = 0.3$, $\delta_2 = 0.8$ and $\delta_3 = 1$.

In Figure 4.2, notice that the effect of the perfect repair at time t_3 in both models is not the same: for Model I (left plot), the perfect repair at t_3 does not return the system to an as-good-as-new state, whereas for Model II (right plot), the perfect repair at t_3 returns the system to an as-good-as-new state. That is, the virtual ages immediately following the repairs are non-decreasing in Model I, whereas they are unrestricted in Model II. Yet, in each model, a perfect repair is the most effective according to the definition of the corresponding repair model given in (4.13). □

A system with an IFR function has a decreasing conditional reliability function and a decreasing mean residual life (MRL) function; see Section 3.1.2. This depicts a system that is deteriorating with time. Therefore, a repair that restores the working condition of the system to an as-good-as-new state is the most effective repair, since the system at the start of its lifetime is assumed to be in its best working condition. For a system with an IFR lifetime distribution, the ordering in (4.10) holds when perfect repair is defined as a replacement.

For a system with a BFR lifetime distribution, however, both the MRL function and the conditional reliability function are non-monotonic (initially increasing, and then decreasing); see Section 3.1.2. It has been shown that the point that maximizes the MRL for a BFR lifetime distribution is before the first change-point a_1 of the failure rate function and not necessarily at 0; see Mi [21]. When the system has a BFR lifetime distribution, the ordering in (4.10) does not always hold when perfect repair is defined as a replacement. That is, an as-good-as-new repair is not necessarily the most effective repair, since the working condition of the system is initially improving with time.

Example 2. To illustrate point (ii) on page 60, consider two systems: one with an IFR lifetime distribution and the other with a BFR distribution. Following the first failure at time $T_1 = t_1$, each system is replaced with a new and identical system. Figure 4.3 depicts the failure rate function of the original lifetime and the failure rate function of the second inter-failure lifetime conditional on $T_1 = t_1$, for the two systems.

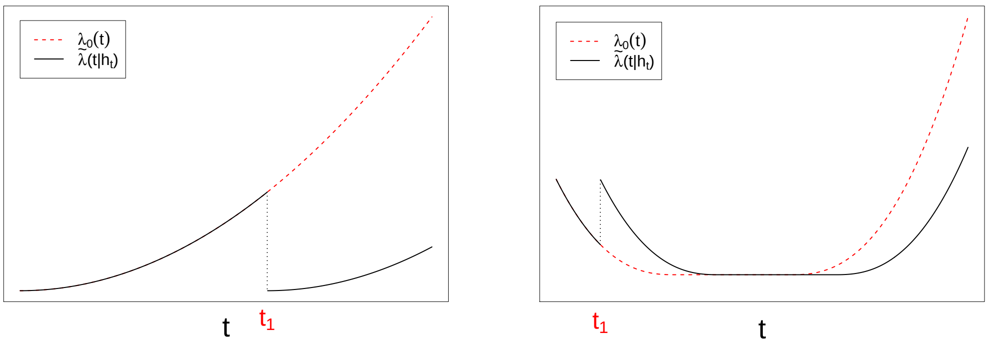


Figure 4.3: Conditional intensity function following a replacement at time $T_1 = t_1$: IFR distribution (left); and BFR distribution (right).

In Figure 4.3, notice that for the IFR case (left plot), for all $t > t_1$, the conditional intensity function following the repair is less than the original failure rate function. This is not the case for the BFR lifetime. This implies that following a replacement the system is not necessarily in a better condition than a system that has not failed (or equivalently, has been minimally repaired). In this case, perfect repair is not synonymous with as-good-as-new repair (or replacement). □

When dealing with lifetime distributions that are not IFR, we need to: (i) distinguish between as-good-as-new repair (or replacement) and perfect repair; and (ii) identify the most effective repair within the definition and constraints of the repair model, which will then be labeled perfect repair.

Here, we do not consider repairs that can worsen the system. Therefore, the working condition of the system following a repair either remains unchanged (following minimal repairs) or improves (following non-minimal repairs).

4.3 The General Repair Model

We model the effect of a general repair as a change in the conditional intensity function of the corresponding failure process, such that the system following the repair is at least as reliable as a system that has not failed. Although the proposed model is not a typical virtual age model, when the decreasing failure rate phase is removed, it reduces to the traditional virtual age models proposed by Kijima [27]; see Section 3.2.3.

Let $\{N(t); t \in \mathbb{R}_+\}$ denote the counting process associated with the failure process. Let $\tilde{\lambda}(t|\mathcal{H}_t)$ denote the conditional intensity (or failure intensity) of the process at time t, conditional on its history at this time. The history $\mathcal{H}_t$ contains all information available at time t, including the process trajectory up to this point [14]. Then, when the process is orderly, the conditional intensity function at time t is such that [cf. (3.7)]

$$\tilde{\lambda}(t|\mathcal{H}_t)\ dt \approx \mathrm{P}\{N(t+dt) - N(t) = 1 \mid \mathcal{H}_t\}\ . \tag{4.14}$$

The initial or baseline intensity function of the failure process, which we denote by $\lambda_0(.)$, is a deterministic function and equal to the failure rate function of the lifetime of the original system; see Section 3.1.1. Therefore, before the first failure, $\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_0(t) = r(t)$, where $r(.)$ is the failure rate function of the original lifetime, and

$$r(t)\ dt \approx \mathrm{P}\{N(t+dt) - N(t) = 1 \mid N(t) = 0\}\ . \tag{4.15}$$

Note that, here, the original lifetime distribution is BFR, i.e. the failure rate function $r(.)$ is bathtub-shaped and of the form defined in (4.1).

4.3.1 Model Description

We assume that a general, non-minimal repair, along with restoring the failed system to an operational state, improves the working condition of the system. This will be reflected in the reliability of the system following the repair.

The conditional intensity function of the failure process completely determines the conditional reliability of the system. Let $\bar{F}_t(t+x|\mathcal{H}_t)$ denote the conditional reliability of the system at time $t+x$, given that the system is operational at time t and conditional on the process history at t. In terms of the conditional intensity function, this conditional reliability function can be expressed as follows:

$$\bar{F}_t(t+x|\mathcal{H}_t) = \mathrm{P}\{N(t+x) - N(t) = 0 \mid \mathcal{H}_t\} = e^{-\int_t^{t+x} \tilde{\lambda}(s|\mathcal{H}_s)\, ds} , \tag{4.16}$$

where $\mathcal{H}_s = \mathcal{H}_t \cup \{N(t+s) - N(t) = 0\}$, for $s > t$. Therefore, we incorporate the effect of general repairs in the conditional intensity function of the failure process. Given (4.16), it is clear that this conditional reliability increases when the conditional intensity decreases.

For the general repair model, we make the following modeling assumptions:

(a) repairs are immediate and instantaneous (i.e. repair times are equal to zero);

(b) the system after a general repair can perform no worse than a minimally-repaired system; and

(c) the system can perform no better than it does during its useful life period.

We model the assumptions in (b) and (c) through the following bounds on the conditional intensity function. For all $t \geq 0$, and given the baseline intensity function $\lambda_0(.)$,

$$\min_{t' \in \mathbb{R}_+} \lambda_0(t') \quad \leq \quad \tilde{\lambda}(t|\mathcal{H}_t) \quad \leq \quad \lambda_0(t) \ . \tag{4.17}$$

We propose the following approach to model the effect of a general repair on the distribution of future failures of the system.

DFR phase: Early system failures are governed by initial defects whose impact on the working condition of the system reduces over time. Then a general repair following such failures removes some of the initial defects detected upon system failure, which results in improved system reliability. Therefore, in the DFR phase (i.e. when the conditional intensity function is decreasing), we model the effect of a general repair as a reduction in the conditional intensity along with an extension of the useful life period; see Figure 4.4 (left).

Both the drop in the conditional intensity and the extension of the useful life period following the general repair are proportional to the degree of the repair. Therefore, fol-

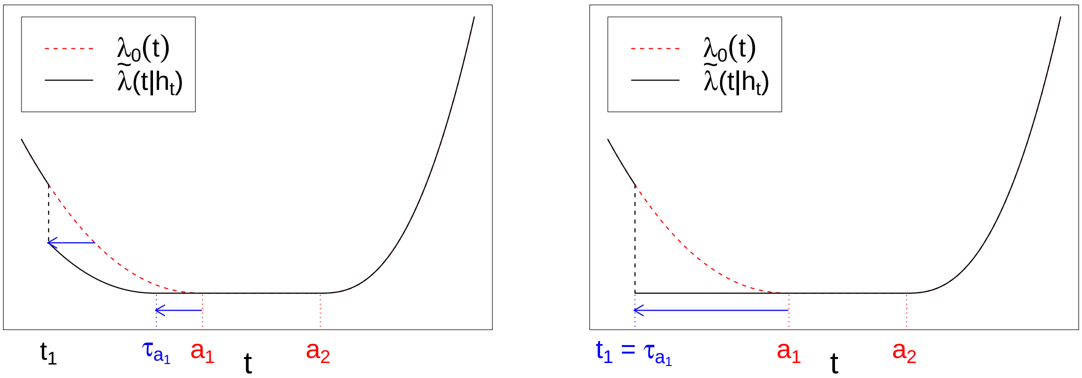


Figure 4.4: Example conditional intensity functions of a system whose failure at time t_1 is followed by a general repair: imperfect repair of degree $\delta_1 = 0.3$ (left), and perfect repair (right). Notice that, following the general repair, the useful life period is extended from $[a_1, a_2]$ to $[\tau_{a_1}, a_2]$, where the length $a_1 - \tau_{a_1}$ of extension is proportional to the degree of the repair.

lowing a perfect repair, which is the most effective repair, the reduction and extension are both maximal, resulting in a useful life period that begins at the point of perfect repair; see Figure 4.4 (right). The conditional intensity does not change following a minimal repair.

CFR/IFR phase: Failures outside the DFR phase are governed by wear or damage accumulation, whose effect may be slight for a period of time (the useful life period), but accelerates system deterioration in time (the wear-out period). A general repair following such failures removes some of the accumulated wear, which improves the working condition of the system. Therefore, when the conditional intensity function is non-decreasing, we model the effect of a general repair such that the system behaves like a younger system with an *extended* useful life period, where a perfect repair returns the conditional intensity to the start of the extended useful life period. The length of the extended CFR phase is determined following the repairs performed in the DFR phase; see Figure 4.5.

The failures in the useful life period of the system are assumed to be chance and less due to the aging of the system. A general repair in this phase can be viewed as a form of maintenance that removes some of the accumulated damage and thus delays the wear-out period. We defined general repairs as repairs that can improve the working condition of the system, thus improving its reliability, which may decrease the number of future failures. Although following a general repair in the CFR phase does not

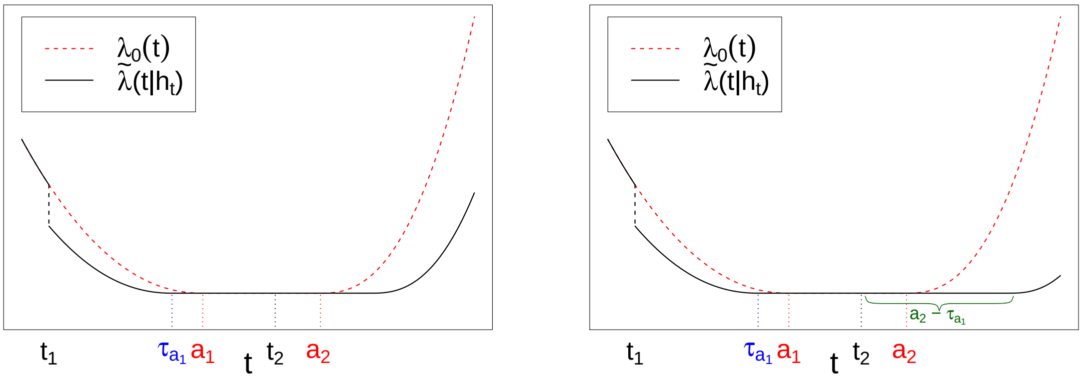


Figure 4.5: Example conditional intensity functions of a system whose failures are followed by general repairs. The repair at t_1 in both plots is imperfect with degree $\delta_1 = 0.2$. The repair at t_2 is: imperfect with degree $\delta_2 = 0.3$ (left), and perfect (right). Notice that, following the perfect repair at t_2, the conditional intensity is returned to the start of the extended useful life period which has length $a_2 - \tau_{a_1}$, where τ_{a_1} is a function of the first repair at t_1 and its degree δ_1.

change the conditional intensity immediately after the repair, it does push back the IFR phase, which results in an improvement in system reliability.

To simplify the modeling approach, we unify the components of the repair model corresponding to the decreasing and non-decreasing phases, and implement the general repair model as follows.

4.3.2 Model Implementation

To implement the general repair model, we define two functions: an *age modification function*, denoted by $A(.)$, and a *modified baseline intensity function*, denoted by $\lambda_1(.)$. Before the first failure, $A(t) = t$ and $\lambda_1(t) = \lambda_0(t)$, for all $t \geq 0$, where $\lambda_0(t) = r(t)$ is a BFR function with change-points $0 < a_1 < a_2$; see (4.1). Then, the effect of a general repair is implemented as changes in these functions.

The conditional intensity function of the failure process is comprised of the two functions, such that, at any time $t \geq 0$,

$$\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_1(A(t)) \ . \tag{4.18}$$

We now proceed to formally define the age modification and modified baseline intensity functions and illustrate the effect of changes in the two functions on the behavior of the conditional intensity.

4.3.2.1 The Age Modification Function

The age modification function, at any time t, for $t \geq 0$, is defined as follows:

$$A(t) = t + \sum_{i=1}^{N(t^-)} \delta_i \left[a_1 - A(T_i)\right] \mathbb{I}_{A(T_i)<a_1} - \sum_{j=1}^{N(t^-)} \delta_j \left[A(T_j) - a_1\right] \mathbb{I}_{A(T_j)\geq a_1} \ , \tag{4.19}$$

where $N(t^-)$ denotes the number of failures before time $t \geq 0$, $\mathbb{I}_{.}$ denotes an indicator random variable[I], and $A(T_i)$ denotes the modified age at the time of the i-th failure. The modified age at any point is stochastic, since it depends on earlier failure times and also the degrees, if random, of repairs performed following the failures. Here, we assume that the degrees of repair are given.

Let $\mathcal{T}_{a_1}$ and τ_{a_1} denote, respectively, the point at which the modified age reaches a_1 and its realization. That is,

$$\begin{aligned} \mathcal{T}_{a_1} &= \min\{t : A(t) \geq a_1\} \\ &= a_1 - \sum_{i=1}^{N(a_1^-)} \delta_i \left[a_1 - A(T_i)\right] \mathbb{I}_{A(T_i)<a_1} \ . \end{aligned} \tag{4.20}$$

$\mathcal{T}_{a_1}$ is the point at which the conditional intensity function first reaches its minimum; we will discuss this further in Section 4.3.2.3.

For $n \in \mathbb{N}_+$, the effect of the n-th general repair performed at time T_n is implemented in the age modification function as follows:

- When the modified age $A(T_n)$ is less than a_1, then the effect of the corresponding general repair is an increase in the modified age. A minimal repair does not change the modified age; a perfect repair increases the modified age to a_1; and an imperfect repair increases the modified age by a fraction δ_n of the distance $[a_1 - A(T_n)]$.

- When the modified age $A(T_n)$ is at least a_1 (i.e. when the time T_n of the failure is greater than or equal to $\mathcal{T}_{a_1}$), then the effect of the corresponding general repair is a decrease in the modified age. A minimal repair does not change the modified age; a

[I] An indicator random variable of any event $\mathcal{B}$ is defined as follows:

$$\mathbb{I}_{\mathcal{B}} = \begin{cases} 1 \ , & \text{if } \mathcal{B} \text{ occurs} \\ 0 \ , & \text{if } \mathcal{B} \text{ does not occur} \ . \end{cases}$$

perfect repair decreases the modified age to a_1; and an imperfect repair decreases the modified age by a fraction δ_n of the distance $[A(T_n) - a_1]$; see Figure 4.6.

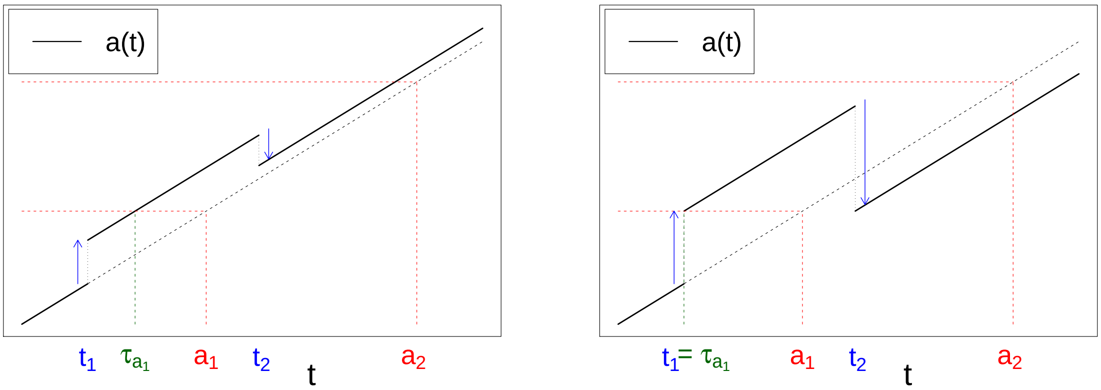


Figure 4.6: Examples of the age modification function, where the two failures at times t_1 and t_2 are rectified by: imperfect repair ($\delta_1 = 0.5$, $\delta_2 = 0.5$) (left); and perfect repair (right).

Although, the age modification function can be expressed with a single summation term as follows:

$$A(t) = t - \sum_{i=1}^{N(t^-)} \delta_i \left[A(T_j) - a_1 \right] \quad , \tag{4.21}$$

we have expressed it in a form closest to its description. It is easy to see that, when $a_1 = 0$ (i.e. when the system does not have a burn-in phase), (4.21) reduces to an age reduction function; see Section 3.2.3.

In a typical age reduction model, the conditional intensity at any time t is given by $\lambda_0(A(t))$, where $\lambda_0(.)$ is the original baseline intensity function. Here, however, this results in reaching the useful life period faster (i.e. aging the system), which is not the effect proposed by the general repair model. Therefore, we define a new baseline intensity function.

4.3.2.2 The Modified Baseline Intensity Function

The modified baseline intensity function $\lambda_1(.)$ is the original baseline intensity function $\lambda_0(.)$ modified to have a delayed IFR phase (i.e. an extended CFR phase). Therefore, the first change-point of $\lambda_1(.)$ is a_1, and we denote its second change-point by A_2', where $A_2' \geq a_2$ (note that, A_2' is a random variable). Then, the modified baseline intensity function $\lambda_1(.)$ is formally defined as follows:

$$\lambda_1(s) = \begin{cases} r_1(s) \ , & 0 \leq s \leq a_1 \\ r_2(s) \ , & a_1 \leq s \leq A_2' \\ r_3(s - (A_2' - a_2)) \ , & s \geq A_2' \ , \end{cases} \tag{4.22}$$

where the functions $r_i(.)$, $i \in \{1, 2, 3\}$, are the same functions used to define the original baseline intensity (or failure rate) function given in (4.1). For all $s \in [a_1, A_2']$, we have

$$r_2(s) = r_1(a_1) = r_3(A_2') \ . \tag{4.23}$$

The CFR phase $[a_1, A_2']$ of the function $\lambda_1(.)$ is of a stochastic nature, and its length depends on the random variable A_2', which is defined as follows:

$$\begin{aligned} A_2' &= a_2 + \sum_{i=1}^{N(a_1^-)} \delta_i \, [a_1 - A(T_i)] \, \mathbb{I}_{A(T_i) < a_1} \\ &= a_2 + (a_1 - \mathcal{T}_{a_1}) \ > \ a_2 \ . \end{aligned} \tag{4.24}$$

This change-point is the original change-point a_2 shifted by the total increase in the modified age resulting from repairs performed during the DFR phase, i.e. repairs performed while the modified age is less than a_1; see Figure 4.7.

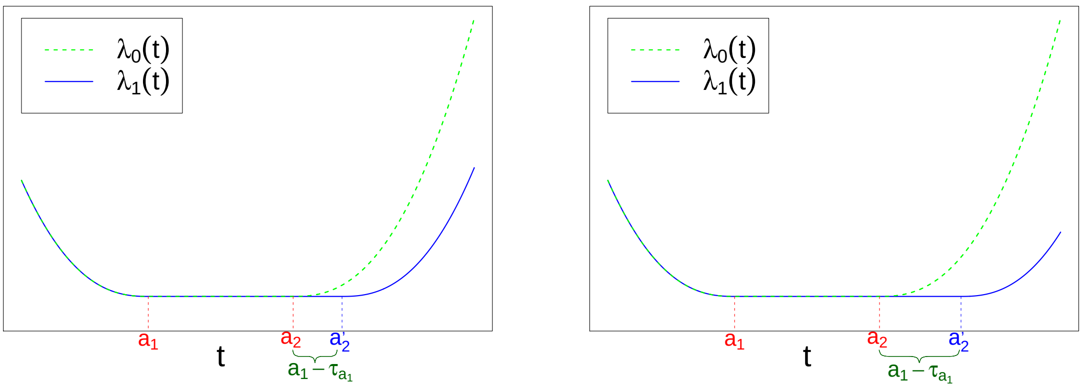


Figure 4.7: Illustration of the modified baseline intensity functions corresponding to the plots in Figure 4.6. The value of the change-point A_2' depends only on the time of the failure in the DFR phase and the degree of the corresponding repair. That is, $a_2' = a_2 + a_1 - \tau_{a_1}$, where τ_{a_1} is a function of t_1 and δ_1.

For $n \in \mathbb{N}_+$, the effect of the n-th general repair at time T_n is implemented in the modified baseline intensity function as follows:

- When the modified age $A(T_n)$ at the time of failure is less than a_1, then the effect of the corresponding general repair is an additional delay of length $\delta_n\,[a_1 - A(T_n)]$ in the IFR phase of the baseline intensity function.

- When the modified age $A(T_n)$ is at least a_1 (i.e. when the time T_n of the failure is at least $\mathcal{T}_{a_1}$), then the corresponding general repair does not affect the baseline intensity function. In other words, the value of the new change-point A_2' is unaffected by general repairs performed after $\mathcal{T}_{a_1}$; see Figure 4.7.

When all repairs in $(0, a_1]$ are minimal (or no failure occurs in this period), then $A_2' = a_2$ and $\lambda_1 = \lambda_0$, otherwise $A_2' > a_2$ and $\lambda_1(t) = \lambda_0(t)$, for all $t \leq a_2$, and $\lambda_1(t) < \lambda_0(t)$, for all $t > a_2$; see Figure 4.7.

The changes made in the age modification and modified baseline intensity functions following repairs result in the proposed effects described in Section 4.3.1, and observed in the conditional intensity functions plotted in Figures 4.4 and 4.5.

4.3.2.3 The Conditional Intensity Function

As mentioned earlier, for the proposed repair model, the conditional intensity function of the process at time $t \geq 0$ is given by $\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_1(A(t))$.

When all repairs are minimal, $A(t) = t$ and $\lambda_1 = \lambda_0$, and therefore, the conditional intensity function reduces to

$$\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_0(t) \ , \tag{4.25}$$

which is the conditional intensity function of a Poisson (or minimal repair) process with intensity $\lambda_0(.)$.

When all repairs are perfect, the failure process is not equivalent to a replacement (renewal) process; see Section 4.2.1. The conditional intensity function of the replacement process is given by $\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_0(t - T_{N(t^-)})$; see Section 3.2.1. When $a_1 = 0$, then the proposed model reduces to an age reduction model, where the perfect repair process is equivalent to the replacement process.

The proposed approach should not be confused with typical virtual age models. Here, the original baseline intensity function is altered following repairs performed in the DFR phase. In virtual age models, however, the baseline intensity function remains unchanged and the effects of repairs are described as changes in the virtual age of the system alone. Therefore, for a given virtual age function $V(.)$ and baseline intensity function $\lambda_0(.)$, the conditional intensity function of a virtual age model, at time $t \geq 0$, is given by $\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_0(V(t))$.

Example 3. Suppose that the first three failures of the system, at times t_1, t_2 and t_3, are followed by general repair. To illustrate the proposed model, we select the failure times such that one general repair is performed in each of the three phases. In Figure 4.8, we have plotted the conditional intensity function, and the corresponding age modification and modified baseline intensity functions for the following three cases: (top row) the repair at t_1 is minimal ($\delta_1 = 0$); (middle row) the repair at t_1 is imperfect ($\delta_1 = 0.3$); and (bottom row) the repair at t_1 is perfect ($\delta_1 = 1$). The failure times t_1, t_2 and t_3, and the other degrees δ_2 and δ_3 are fixed, and only δ_1 varies (row-wise). Note that, $a(.)$ and a_2' denote the realizations of the age modification function $A(.)$ and the change-point A_2', respectively.

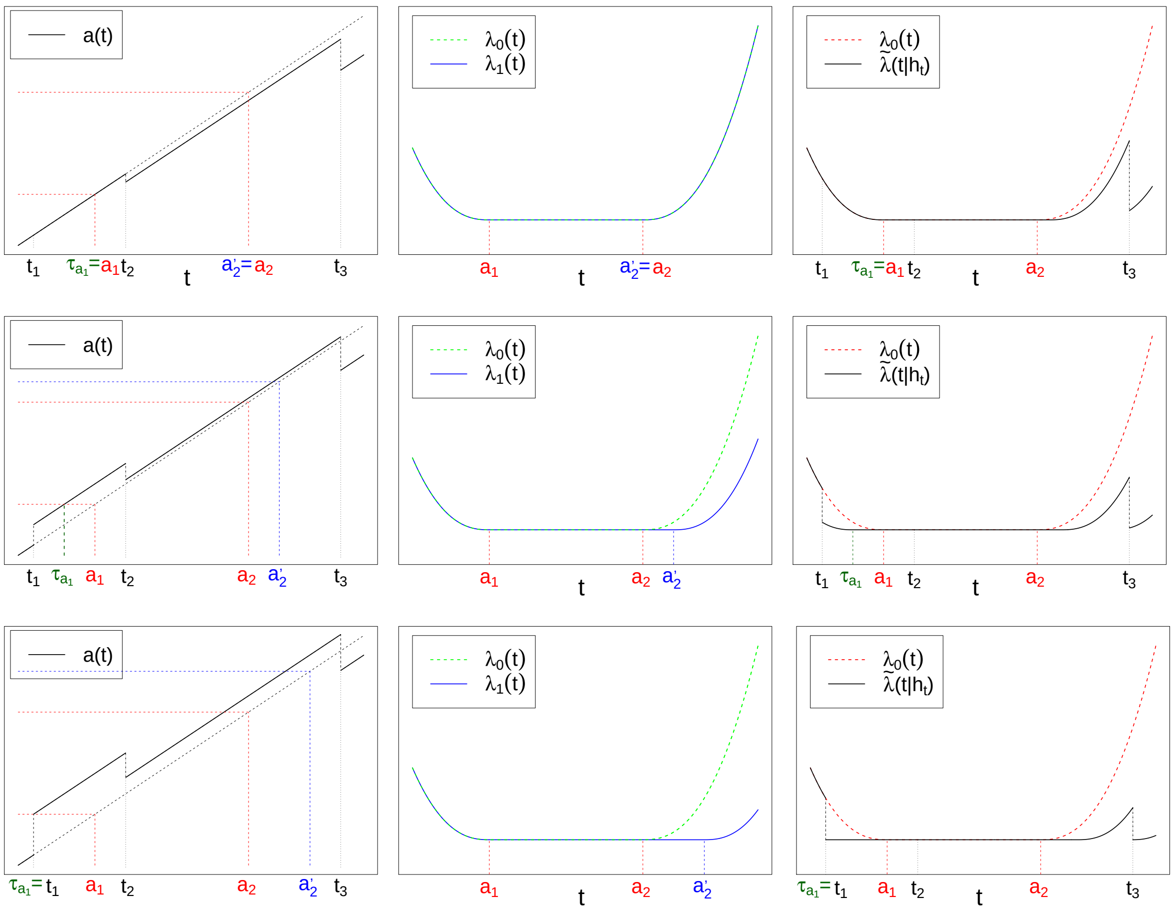


Figure 4.8: Example conditional intensity functions (right column), with corresponding age modification functions (left column) and modified baseline intensity functions (middle column). General repairs are performed at t_1, t_2 and t_3 with fixed degrees $\delta_2 = 0.4$ and $\delta_3 = 0.2$, and varying degree δ_1: $\delta_1 = 0$ (top row); $\delta_1 = 0.5$ (middle row); and $\delta_1 = 1$ (bottom row).

In Figure 4.8 (right column), at no point the sample conditional intensity is greater than the original baseline intensity (or failure rate of the original lifetime); i.e. in all three cases,

$$\tilde{\lambda}(t|h_t) = \lambda_1(a(t)) \leq \lambda_0(t) \ . \tag{4.26}$$

This implies that, the system following a general repair at any point is at least as reliable as a minimally-repaired system (or one that has not failed). Let $h_{t,0}$, $h_{t,\delta}$ and $h_{t,1}$ denote the realizations of the histories for the three cases (rows), respectively. Looking at the conditional intensity functions in the three rows, it is evident that the ordering in expression (4.11) holds, since

$$\tilde{\lambda}(t|h_{t,0}) \ \geq \ \tilde{\lambda}(t|h_{t,\delta}) \ \geq \ \tilde{\lambda}(t|h_{t,1}) \ , \qquad \forall \ t \ . \tag{4.27}$$

That is, for each $t \geq 0$, the conditional intensity is a decreasing function of the degree δ_1 of the first repair, given that all other function parameters are fixed.

As mentioned earlier, $\mathcal{T}_{a_1}$ is the point at which the modified age reaches a_1. This point is also the time at which the DFR phase is exited, i.e.

$$\mathcal{T}_{a_1} = \ \min\{t : \tilde{\lambda}(t|\mathcal{H}_t) = \lambda_0(a_1)\} \ . \tag{4.28}$$

This is visible in the age modification functions in the left column of Figure 4.8 and the corresponding conditional intensity functions in the right column.

In Figure 4.8 (middle column), notice that, when the repair at t_1 is minimal (top row), then the two baseline intensity functions $\lambda_1(.)$ and $\lambda_0(.)$ are equal. When the repair at t_1 is non-minimal (middle and bottom rows), the only change in $\lambda_1(.)$ when compared to $\lambda_0(.)$ is the delayed start of the IFR phase. Before a_2, the two functions coincide. Also, the length of the extension in the CFR phase of $\lambda_1(.)$ is $|a_2' - a_2| \ = \ a_1 - \tau_{a_1} \ = \ \delta_1 \ (a_1 - t_1)$, which is proportional to the degree δ_1, since t_1 is fixed for all plots. □

4.3.3 Model Properties

The reliability function conditional on the history of the failure process, is given in (4.16). The corresponding MRL function at time t is given by

$$\mu(t|\mathcal{H}_t) = \int_0^\infty \bar{F}_t(t+x|\mathcal{H}_t)\ dx = \int_0^\infty e^{-\int_t^{t+x} \tilde{\lambda}(s|\mathcal{H}_s)\ ds}\ dx \ , \tag{4.29}$$

where $\mathcal{H}_s = \mathcal{H}_t \cup \{N(t+s) - N(t) = 0\}$, for $s > t$; see (4.16) for the reliability function.

When n failures have occurred prior to t, then the above functions reduce to the reliability and MRL functions corresponding to the $(n+1)$-th residual lifetime defined in (4.7), i.e.

$$\bar{F}_t(t+x|h_{t;n}) = \mathrm{P}\{T_{n+1} > t+x \mid T_{n+1} > t, N(t) = n, T_1 = t_1, \ldots, T_n = t_n; \delta_1, \ldots, \delta_n\} \ ; \tag{4.30}$$

$$\mu(t|h_{t;n}) = E[T_{n+1} - t \mid T_{n+1} > t, N(t) = n, T_1 = t_1, \ldots, T_n = t_n; \delta_1, \ldots, \delta_n] \ , \tag{4.31}$$

for $n \in \mathbb{N}_+$ and $t > t_n$. Note that, the relevant information in the history of the process is the number and the times of the failures before time t, and also, the degrees of the corresponding general repairs.

The conditional reliability and MRL functions are both increasing in each degree of repair when all other parameters are fixed, since, for each $t > t_n$, the conditional intensity function is decreasing in each δ_i, $i \in \{1, \ldots, n\}$, when all other parameters are fixed (as illustrated in Example 3 on page 71). This implies that, the order of residual lifetimes in (4.10) holds, when the partial ordering is in terms of conditional reliability ('ST') or mean residual ('MR') lifetime; see Section 4.2.1.

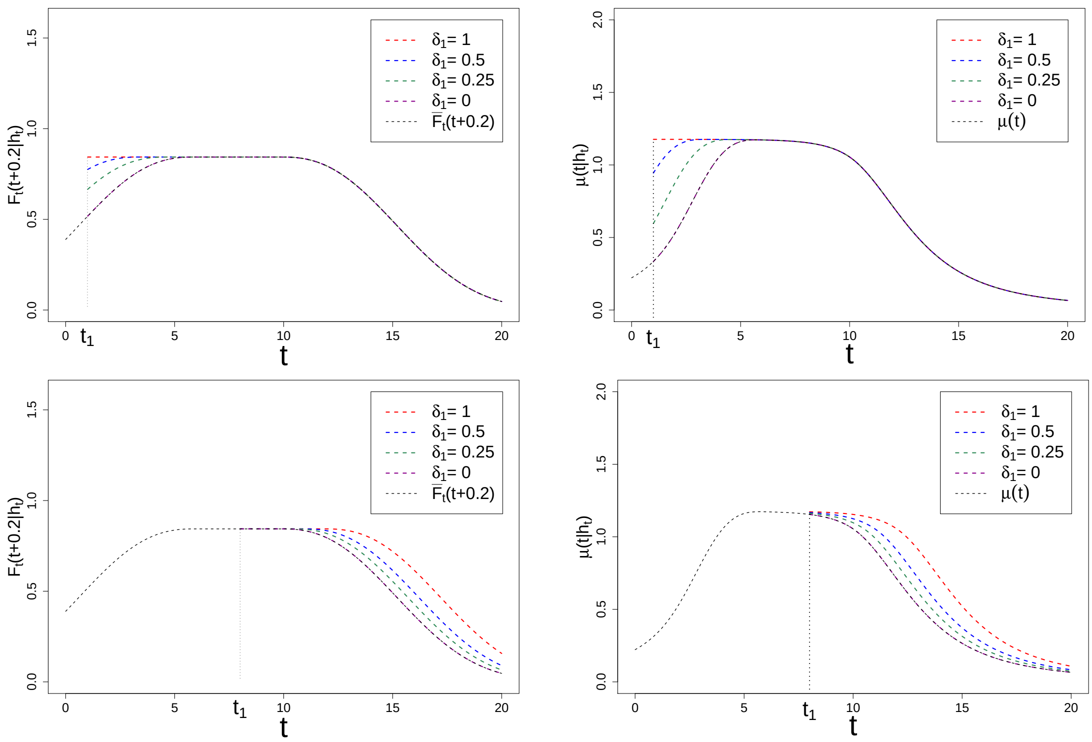


Figure 4.9: Example conditional reliability and MRL functions, given in (4.30) and (4.31), where the failure of the system is at time t_1 and followed by a general repair of degree δ_1. The conditional functions are plotted for $t > t_1$.

Example 4. To illustrate the effect of repairs on the conditional reliability and MRL functions, suppose that one failure has occurred in the interval $[0, 20]$ and it is followed by a general repair. In Figure 4.9, the conditional reliability function $\bar{F}_t(t + x|h_t)$ given in (4.30) and the MRL function $\mu(t|h_t)$ given in (4.31) are plotted for various values of the degree δ_1, for $t > t_1$ and $x = 0.2$. The conditional reliability and MRL functions of the original lifetime (which is equivalent to the case where the repair is minimal) are plotted over $[0, 20]$ (black dashed line). In the top row, $t_1 = 1$ is in the DFR phase, and in the bottom row, $t_1 = 8$ is outside the DFR phase. The chosen change-points are $a_1 = 6$ and $a_2 = 10$.

In Figure 4.9, notice that both the conditional reliability function and the MRL function are increasing in the degree δ_1 of the general repair at t_1. When $t_1 < a_1$ (top row), the improvement is in the DFR phase, and when $t_1 > a_1$ (bottom row), the improvement is in the CFR and IFR phases. □

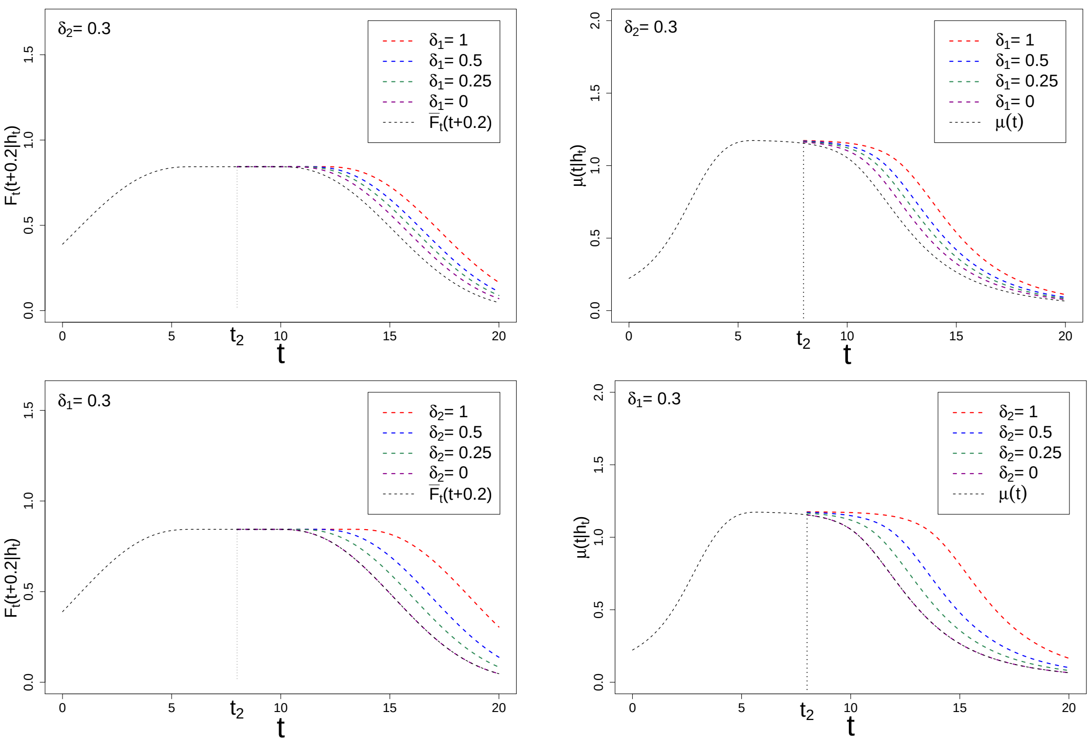


Figure 4.10: Example conditional reliability and MRL functions, given in (4.30) and (4.31), where the failures of the system, at times t_1 and t_2, are followed by general repair. The conditional functions are plotted for $t > t_2$.

Example 5. Now suppose that two failures have occurred in the interval $[0, 20]$, one in the DFR phase, at time $t_1 = 1$, and the other outside the DFR phase, at time $t_2 = 8$. In Figure 4.10, the conditional reliability and the MRL functions are plotted for various values of the degrees δ_1 and δ_2, for $t > t_2$ and $x = 0.2$. These functions, for the original lifetime (which is equivalent to the case where both repairs are minimal), are plotted over $[0, 20]$ (black dashed line). In the top row, δ_2 is fixed, and δ_1 varies, and in the bottom row, δ_1 is fixed, and δ_2 varies.

In Figure 4.10, notice that the conditional reliability functions and the MRL functions are increasing in each of the degrees of repair, when the other is held fixed. In each plot, the lowest curve corresponds to the case where both repairs are minimal (i.e. $\delta_1 = \delta_2 = 0$). □

4.3.3.1 A Note on the Age Modification Function

With the age modification function proposed in Section 4.3.2, there is no restriction on the amount of wear that can be removed following a general repair outside the DFR phase. Other age modification functions can be defined restricting the amount of damage that can be undone.

Suppose that, in the non-decreasing phases, a repair can at most remove damage accumulated since the last repair. Then, an appropriate age modification function can be defined as follows:

$$A(t) = t + \sum_{i=1}^{N(t^-)} \delta_i \left[a_1 - A(T_i)\right] \mathbb{I}_{A(T_i) < a_1} - \sum_{i=1}^{N(t^-)} \delta_i \left[A(T_i) - \max(a_1, A(T_{i-1}^+))\right] \mathbb{I}_{A(T_i) \geq a_1} \,, \tag{4.32}$$

where $A(T_i^+)$ is the modified age immediately after the i-th repair. Note that, the difference between the two models is only in the effect of repairs performed outside the DFR phase of the conditional intensity function [cf. (4.19)]. The following example illustrates the difference between the two models.

Example 6. Suppose that two failures have occurred while the conditional intensity function is still decreasing and two have occurred while it is non-decreasing. The two age modification functions and the corresponding modified baseline intensity and conditional intensity functions are plotted in Figure 4.11. The values of the failure times and the degrees of the corresponding general repairs are the same in both rows.

In Figure 4.11, notice that, for all $t > t_4$, the conditional intensity corresponding to the second age modification function (bottom row, right) is greater than that of the first age modification function (top row, right). This is because the amount of wear that can be removed is restricted in the second repair model. Also, the modified baseline intensity functions for both models are the same, since the difference in the models is only in the effect of repairs performed outside the DFR phase of the conditional intensity function (which do not influence the second change-point of the modified baseline intensity function). □

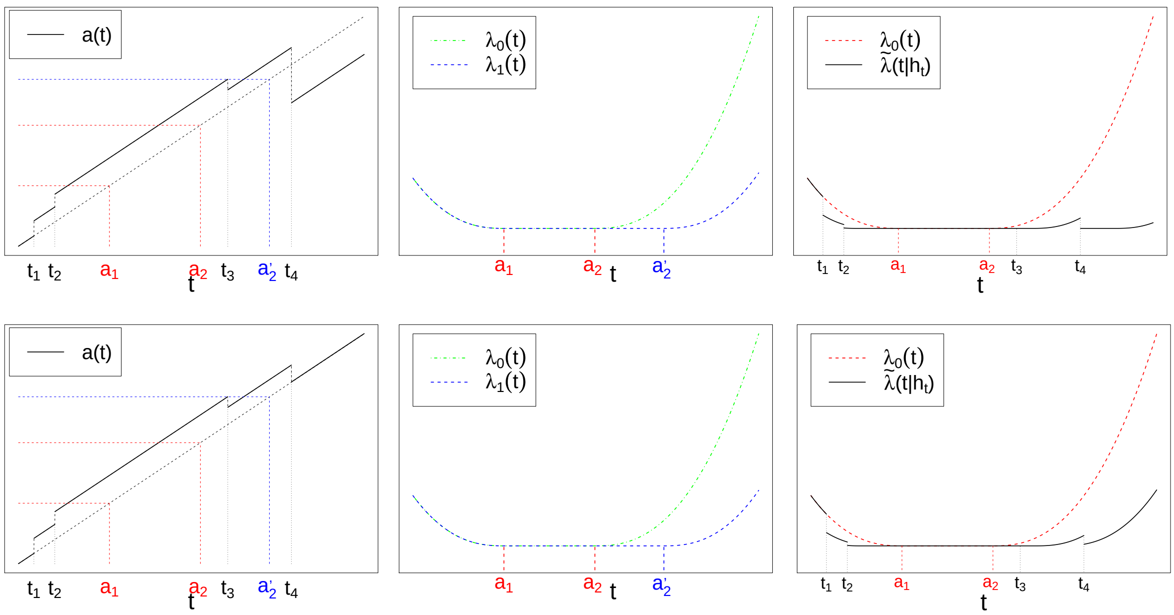


Figure 4.11: An illustration of the difference between the age modification function defined in (4.19) (top row, left) and the age modification function defined in (4.32) (bottom row, left). Corresponding modified baseline intensity and conditional intensity functions are plotted in the middle and right columns, respectively.

4.4 The Failure (or General Repair) Process

The proposed failure process is the sequence of consecutive system failures, where each failure is followed by a general repair and the effect of the general repair is modeled as proposed in Section 4.3. Since we assume that each failure is followed immediately by an instantaneous general repair, the number of failures is equal to the number of general repairs. Therefore, the associated counting process $\{N(t); t \in \mathbb{R}_+\}$ is the sequence of the numbers of general repairs in each interval $(0, t]$, for $t \geq 0$.

The distribution of the count $N(t)$, $t \geq 0$, can be derived using the distributions of the failure times through the following equation:

$$\mathrm{P}\{N(t) = n\} = \mathrm{P}\{T_n \leq t\} - \mathrm{P}\{T_{n+1} \leq t\} \ . \tag{4.33}$$

The expected number of failures in the interval $(0, t]$, can also be derived from the distributions of the failure times, using the relationship

$$E[N(t)] = \sum_{n=1}^{\infty} \mathrm{P}\{T_n \leq t\} \ ; \tag{4.34}$$

see Section 3.1.1.

In the following sections, we derive the distribution and reliability functions of the failure times and the inter-failure lifetimes of the proposed general repair process.

4.4.1 Distribution of Failure Times

Let $F_n(.)$, $\bar{F}_n(.)$ and $f_n(.)$ denote, respectively, the cumulative distribution, reliability and density functions of the n-th failure time T_n, for $n \in \mathbb{N}_+$. For the original lifetime T_1 $(= X_1)$, these functions are defined as follows:

$$F_1(t) = \mathrm{P}\{T_1 \le t\} = 1 - \mathrm{P}\{N(t) = 0\} = 1 - e^{-\int_0^t \lambda_0(s)\, ds} \quad ; \tag{4.35}$$

$$\bar{F}_1(t) = \mathrm{P}\{T_1 > t\} = \mathrm{P}\{N(t) = 0\} = e^{-\int_0^t \lambda_0(s)\, ds} \quad ; \tag{4.36}$$

$$f_1(t) = \frac{d}{dt} F_1(t) = -\frac{d}{dt} \bar{F}_1(t) = \lambda_0(t)\, e^{-\int_0^t \lambda_0(s)\, ds} \quad . \tag{4.37}$$

Let $a_n(t)$ denote the realization of the age modification function at time t, given that n failures have occurred prior to time t, for $n \in \mathbb{N}_+$. The subscript n is used to emphasize the dependence of this function on the n previous failure times $\{t_1, \ldots, t_n\}$ and the corresponding degrees of repair $\{\delta_1, \ldots, \delta_n\}$. Before the first failure of the system, $a(t) = a_0(t) = t$.

Let, for $n \in \mathbb{N}_+$, $\bar{F}_{n+1}(.\,|t_1, \ldots, t_n)$ denote the conditional reliability function of the failure time T_{n+1}, given the n previous failure times. Then, for $t > t_n$,

$$\begin{aligned} \bar{F}_{n+1}(t|t_1, \ldots, t_n) &= \mathrm{P}\{T_{n+1} > t \mid T_1 = t_1, \ldots, T_n = t_n\} \\ &= \mathrm{P}\{N(t) - N(t_n) = 0 \mid T_1 = t_1, \ldots, T_n = t_n\} \\ &= e^{-\int_{t_n}^t \tilde{\lambda}(s|h_s)\, ds} = e^{-\int_{t_n}^t \lambda_1(a_n(s))\, ds} \quad ; \end{aligned} \tag{4.38}$$

we discussed a generalization of this conditional reliability function (associated with the residual lifetimes) in Section 4.3.3. Next, the corresponding conditional distribution function is, for $t > t_n$, given by

$$\begin{aligned} F_{n+1}(t|t_1, \ldots, t_n) &:= \mathrm{P}\{T_{n+1} \le t \mid T_1 = t_1, \ldots, T_n = t_n\} \\ &= 1 - \mathrm{P}\{T_{n+1} > t \mid T_1 = t_1, \ldots, T_n = t_n\} \\ &= 1 - \bar{F}_{n+1}(t|t_1, \ldots, t_n) \quad . \end{aligned} \tag{4.39}$$

Note that, at $t = t_n^+$, we have $F_{n+1}(t_n^+|t_1,\dots,t_n) = 0$ and $\bar{F}_{n+1}(t_n^+|t_1,\dots,t_n) = 1$.

The corresponding conditional density function is simply the derivative of the conditional distribution function:

$$\begin{aligned} f_{n+1}(t|t_1,\dots,t_n) &:= \frac{d}{dt}F_{n+1}(t|t_1,\dots,t_n) = -\frac{d}{dt}\bar{F}_{n+1}(t|t_1,\dots,t_n) \\ &= \tilde{\lambda}(t|h_t)\ e^{-\int\limits_{t_n}^{t}\tilde{\lambda}(s|h_s)\ ds} = \lambda_1(a_n(t))\ e^{-\int\limits_{t_n}^{t}\lambda_1(a_n(s))\ ds} \quad , \end{aligned} \tag{4.40}$$

which is defined for $0 < t_1 < \dots < t_n < t$, $n \in \mathbb{N}_+$. The exponent term represents the conditional probability of having no failure between t_n and t, and the conditional intensity is such that the product $\tilde{\lambda}(t|h_t)\ dt = \lambda_1(a_n(t))\ dt$ represents the probability of having a failure in $(t, t+dt]$, given the history $\mathcal{H}_t = h_t$ of the process.

Given the functions in (4.38), (4.39) and (4.40), we can now derive the (unconditional) distributions. The distribution function of the $(n+1)$-th failure time is derived by removing the conditioning on (4.39):

$$F_{n+1}(t) = \int\limits_0^t \dots \int\limits_0^{t_2} F_{n+1}(t|t_1,\dots,t_n)\ f_n(t_1,\dots,t_n)\ dt_1 \dots dt_n \quad , \tag{4.41}$$

where $f_n(t_1,\dots,t_n)$ denotes the joint density function of the first n failure times at $(t_1,\dots,t_n)$, and $F_{n+1}(0) = 0$. In general, for $n \in \mathbb{N}_+$ and $t > t_n$, this joint density function is a product of the conditional density functions in (4.40):

$$\begin{aligned} f_n(t_1,\dots,t_n) &= f_n(t_n|t_1,\dots,t_{n-1}) \dots f_2(t_2|t_1)\ f_1(t_1) \\ &= \lambda_1(a_{n-1}(t_n))\ e^{-\int\limits_{t_{n-1}}^{t_n}\lambda_1(a_{n-1}(s))\ ds} \dots \lambda_1(a_1(t_2))\ e^{-\int\limits_{t_1}^{t_2}\lambda_1(a_1(s))\ ds}\ \lambda_0(t_1)\ e^{-\int\limits_0^{t_1}\lambda_0(s)\ ds} \quad , \end{aligned} \tag{4.42}$$

since, before the first failure, $\lambda_1 = \lambda_0$– after the first general repair, the second change-point a_2' of $\lambda_1(.)$ may be shifted depending on the degrees of the repairs before τ_{a_1}; see Section 4.3.2.

The (unconditional) reliability function of the $(n+1)$-th failure time is derived as follows:

$$\begin{aligned} \bar{F}_{n+1}(t) &= \int\limits_0^\infty \dots \int\limits_0^{t_2} \bar{F}_{n+1}(t|t_1,\dots,t_n)\ f_n(t_1,\dots,t_n)\ dt_1 \dots dt_n \\ &= \bar{F}_n(t) + \int\limits_0^t \dots \int\limits_0^{t_2} \bar{F}_{n+1}(t|t_1,\dots,t_n)\ f_n(t_1,\dots,t_n)\ dt_1 \dots dt_n \quad , \end{aligned} \tag{4.43}$$

where the last expression is derived by splitting the outer integral (the integral with respect to t_n), so that

$$\bar{F}_{n+1}(t) = \mathrm{P}\{T_{n+1} > t,\ T_n > t\} + \mathrm{P}\{T_{n+1} > t,\ T_n \leq t\} \ . \tag{4.44}$$

The relationship between the reliability and distribution functions is $\bar{F}_{n+1} = 1 - F_{n+1}$, for all $n \in \mathbb{N}_+$, and when $t = 0$, $F_{n+1}(0) = 0$ and $\bar{F}_{n+1}(0) = 1 - F_{n+1}(0) = 1$.

Finally, for $t \geq 0$, the density function of the $(n+1)$-th failure time T_{n+1}, is given by

$$f_{n+1}(t) = \int\limits_0^t \ldots \int\limits_0^{t_2} f_{n+1}(t|t_1,\ldots,t_n)\ f_n(t_1,\ldots,t_n)\ dt_1 \ldots dt_n \ . \tag{4.45}$$

Note that, this density function can be derived by differentiating the corresponding distribution or reliability function. Using the distribution function, this is straight-forward, so we show this using the reliability function[II]:

$$\begin{aligned}
f_{n+1}(t) &= \frac{d}{dt}\ F_{n+1}(t) = -\frac{d}{dt}\ \bar{F}_{n+1}(t) \\
&= -\frac{d}{dt}\ \bar{F}_n(t) - \frac{d}{dt}\ \left(\int\limits_0^t \ldots \int\limits_0^{t_2} \bar{F}_{n+1}(t|t_1,\ldots,t_n)\ f_n(t_1,\ldots,t_n)\ dt_1 \ldots dt_n \right) \\
&= f_n(t) + \int\limits_0^t \int\limits_0^{t_n} \ldots \int\limits_0^{t_2} \left(-\frac{d}{dt}\ \bar{F}_{n+1}(t|t_1,\ldots,t_{n-1},t_n) \right)\ f_n(t_1,\ldots,t_{n-1},t_n)\ dt_1 \ldots dt_{n-1}\ dt_n \\
&\quad - \left[\int\limits_0^{t_n} \ldots \int\limits_0^{t_2} \bar{F}_{n+1}(t|t_1,\ldots,t_{n-1},t_n)\ f_n(t_1,\ldots,t_{n-1},t_n)\ dt_1 \ldots dt_{n-1} \right]_{t_n = t} \\
&= f_n(t) + \int\limits_0^t \int\limits_0^{t_n} \ldots \int\limits_0^{t_2} \left(-\frac{d}{dt}\ \bar{F}_{n+1}(t|t_1,\ldots,t_{n-1},t_n) \right)\ f_n(t_1,\ldots,t_{n-1},t_n)\ dt_1 \ldots dt_{n-1}\ dt_n \\
&\quad - \int\limits_0^t \ldots \int\limits_0^{t_2} f_n(t_1,\ldots,t_{n-1},t_n = t)\ dt_1 \ldots dt_{n-1} \\
&= \int\limits_0^t \int\limits_0^{t_n} \ldots \int\limits_0^{t_2} f_{n+1}(t|t_1,\ldots,t_{n-1},t_n)\ f_n(t_1,\ldots,t_{n-1},t_n)\ dt_1 \ldots dt_{n-1}\ dt_n \ ,
\end{aligned} \tag{4.46}$$

since, when $t_n = t$, the conditional reliability function $\bar{F}_{n+1}(t|t_1,\ldots,t_{n-1},t_n)$ is unity, and the joint density of $(T_1,\ldots,T_n)$, integrated over all possible values of the first $n-1$ failure

[II]To differentiate the expression with the variable in the limits of the integral the Leibniz integral rule is used:

$$\frac{d}{dx} \int\limits_{u(x)}^{v(x)} h(x,y)\ dy = \int\limits_{u(x)}^{v(x)} \frac{d}{dx}\ h(x,y)\ dy + h(x,v(x))\ \frac{d}{dx}\ v(x) - h(x,u(x))\ \frac{d}{dx}\ u(x) \ .$$

times, reduces to the marginal density of T_n, i.e.

$$\int_0^t \dots \int_0^{t_2} f_n(t_1, \dots, t_{n-1}, t)\, dt_1 \dots dt_{n-1} = f_n(t) \ . \tag{4.47}$$

4.4.2 Distribution of Inter-Failure Lifetimes

The distributions of the inter-failure lifetimes of the process can be derived using the conditional distributions of the failure times.

For $n \in \mathbb{N}_+$, the $(n+1)$-th inter-failure lifetime is $X_{n+1} = T_{n+1} - T_n$. Let $G_n(.)$, $\bar{G}_n(.)$ and $g_n(.)$ denote respectively the distribution, reliability and density functions of the n-th inter-failure lifetime X_n, for $n \in \mathbb{N}_+$. These functions, for the first lifetime $X_1 = T_1$, are defined in (4.35), (4.36) and (4.37). For $n \in \mathbb{N}_+$, the distribution function of the $(n+1)$-th inter-failure lifetime is derived as follows:

$$\begin{aligned} G_{n+1}(x) &= \mathrm{P}\{X_{n+1} \le x\} = \mathrm{P}\{T_{n+1} \le T_n + x\} \\ &= \int_0^\infty \dots \int_0^{t_2} F_{n+1}(t_n + x | t_1, \dots, t_n)\, f_n(t_1, \dots, t_n)\, dt_1 \dots dt_n \ , \end{aligned} \tag{4.48}$$

defined for $x > 0$, where the conditional distribution function is given in (4.39) and the joint density is defined in (4.42). Therefore, the probability that the $(n+1)$-th inter-failure lifetime is less than x units of time is equal to the probability of the $(n+1)$-th failure occurring within x units of the n-th failure. The corresponding reliability function is given by

$$\begin{aligned} \bar{G}_{n+1}(x) &= 1 - G_{n+1}(x) \\ &= 1 - \int_0^\infty \dots \int_0^{t_2} F_{n+1}(t_n + x | t_1, \dots, t_n)\, f_n(t_1, \dots, t_n)\, dt_1 \dots dt_n \\ &= \int_0^\infty \dots \int_0^{t_2} \big(1 - F_{n+1}(t_n + x | t_1, \dots, t_n)\big)\, f_n(t_1, \dots, t_n)\, dt_1 \dots dt_n \\ &= \int_0^\infty \dots \int_0^{t_2} \bar{F}_{n+1}(t_n + x | t_1, \dots, t_n)\, f_n(t_1, \dots, t_n)\, dt_1 \dots dt_n \ , \end{aligned} \tag{4.49}$$

since integrating the joint density function of the first n failure points over its entire support is unity; see (4.38) for the condition reliability function. Note that, the distributions of the inter-failure lifetimes depend on the degrees of all previous general repairs.

The density function of the $(n+1)$-th inter-failure lifetime is derived by differentiating the distribution function:

$$
\begin{aligned}
g_{n+1}(x) &= \frac{d}{dx}\, G_{n+1}(x) = -\frac{d}{dx}\, \bar{G}_{n+1}(x) \\
&= \int_0^\infty \dots \int_0^{t_2} \frac{d}{dx}\, F_{n+1}(t_n + x | t_1, \dots, t_n)\; f_n(t_1, \dots, t_n)\; dt_1 \dots dt_n \\
&= \int_0^\infty \dots \int_0^{t_2} f_{n+1}(t_n + x | t_1, \dots, t_n)\; f_n(t_1, \dots, t_n)\; dt_1 \dots dt_n \;\; ,
\end{aligned}
\tag{4.50}
$$

where the conditional density function is given in (4.40).

Sequence of minimal repairs. When all failures of the system are followed by minimal repair, the system behaves as though it has not failed, i.e. $a(t) = t$ and $\lambda_1(t) = \lambda_0(t)$, for all $t \geq 0$. Therefore, the conditional reliability function of the $(n+1)$-th failure time T_{n+1} becomes

$$
\begin{aligned}
\bar{F}_{n+1}(t | t_1, \dots, t_n) &= e^{-\int_{t_n}^{t} \lambda_1(a_n(s))\, ds} = e^{-\int_{t_n}^{t} \lambda_0(s)\, ds} \\
&= \frac{\bar{F}_1(t)}{\bar{F}_1(t_n)} = \mathrm{P}\{T_1 > t \mid T_1 > t_n\} \;\; ,
\end{aligned}
\tag{4.51}
$$

defined for $t > t_n$, $n \in \mathbb{N}_+$. This conditional reliability is equivalent to the probability that the first failure of the system is after time t, given that the system is in an operational state at time t_n.

Then, the conditional reliability function of the $(n+1)$-th inter-failure lifetime, when all repairs are minimal, is simply

$$
\begin{aligned}
\mathrm{P}\{X_{n+1} > x \mid T_1 = t_1, \dots, T_n = t_n\} &= \mathrm{P}\{T_{n+1} > t_n + x \mid T_1 = t_1, \dots, T_n = t_n\} \\
&= e^{-\int_{t_n}^{t_n + x} \lambda_0(s)\, ds} \;\; ,
\end{aligned}
\tag{4.52}
$$

defined for $x > 0$ and $n \in \mathbb{N}_+$. The (unconditional) reliability function of the inter-failure lifetime X_{n+1}, given in (4.49), then reduces to

$$
\begin{aligned}
\bar{G}_{n+1}(x) &= \int_0^\infty e^{-\int_{t_n}^{t_n + x} \lambda_0(s)\, ds} \left(\int_0^{t_n} \dots \int_0^{t_2} f_n(t_1, \dots, t_{n-1}, t_n)\; dt_1 \dots dt_{n-1} \right) dt_n \\
&= \int_0^\infty e^{-\int_{t_n}^{t_n + x} \lambda_0(s)\, ds} f_n(t_n)\; dt_n \;\; ,
\end{aligned}
\tag{4.53}
$$

since the conditional reliability function depends only on the time t_n of the last failure before

$t_n + x$, for $x > 0$ and $n \in \mathbb{N}_+$. This is the reliability function of the $(n+1)$-th inter-failure lifetime of a Poisson process with intensity function $\lambda_0(.)$.

Sequence of replacements. As previously discussed, here, the process of consecutive perfect repairs is not equivalent to the replacement (or renewal) process. However, when $a_1 = 0$, then the age modification function $A(.)$ reduces to an age reduction function, and the modified baseline intensity function $\lambda_1(.)$ reduces to the original baseline intensity function $\lambda_0(.)$. Therefore, when $a_1 = 0$ and all repairs are perfect, the conditional intensity function becomes

$$\tilde{\lambda}(t|\mathcal{H}_t) = \lambda_0(A(t)) = \lambda_0(t - T_{N(t^-)}) \ , \tag{4.54}$$

for all $t \geq 0$, which is the conditional intensity function of a renewal process in one dimension; see Section 4.3.2.

When $a_1 > 0$ (i.e. when the lifetime distribution has a DFR phase), for the consecutive replacements of the failed system, the conditional reliability function of the $(n+1)$-th failure time needs to be replaced by

$$\bar{F}_{n+1}(t|t_1, \ldots, t_n) = \bar{F}_1(t - t_n) \ , \tag{4.55}$$

for all $t > t_n$ and $n \in \mathbb{N}_+$. Then, the conditional reliability function for the $(n+1)$-th inter-failure lifetime, for $x > 0$, becomes

$$\begin{aligned} \mathrm{P}\{X_{n+1} > x \mid T_1 = t_1, \ldots, T_n = t_n\} &= \bar{F}_{n+1}(t_n + x|t_1, \ldots, t_n) \\ &= \bar{F}_1(x) \ . \end{aligned} \tag{4.56}$$

Then, the reliability function in (4.49), for consecutive replacements of the system, becomes

$$\begin{aligned} \bar{G}_{n+1}(x) &= \int_0^\infty \ldots \int_0^{t_2} \bar{F}_1(x)\ f_n(t_1, \ldots, t_n)\ dt_1 \ldots dt_n \\ &= \bar{F}_1(x) \left(\int_0^\infty \ldots \int_0^{t_2} f_n(t_1, \ldots, t_n)\ dt_1 \ldots dt_n \right) \\ &= \bar{F}_1(x) \ , \end{aligned} \tag{4.57}$$

which implies that the inter-failure lifetimes are independent and identically distributed, as is the case with the renewal process; see Section 3.2.1.

4.5 Chapter Conclusion

In this chapter, we proposed a new approach to model the effect of general repairs performed on a system whose original lifetime distribution is assumed to have a bathtub-shaped failure rate function. We discussed and illustrated the properties of the model through multiple examples, and derived the distributions of the failure times and inter-failure lifetimes of the associated failure process.

Here, the effect of a general repair is incorporated in the conditional intensity function of the failure process. It is assumed that the system following a general, non-minimal repair is in a better working condition than a minimally-repaired system, and therefore, the effect of a non-minimal repair is modeled as a possible improvement in the reliability of the system.

The degree of a general repair reflects the effectiveness of the repair and a repair that is more effective is expected to result in greater reliability improvement. Illustrations of the model showed that the conditional reliability and mean residual lifetime functions are both increasing in each degree of repair, when all other parameters of the functions are fixed.

In reliability literature, the Poisson process has been suggested for modeling consecutive minimal repairs and the renewal process is used to model consecutive replacements. The proposed failure process, when all repairs are minimal, reduces to a Poisson process whose intensity function is given by the failure rate function of the original lifetime. Here, perfect repair is not modeled as a replacement (or renewal). However, when the first change-point is set to zero (i.e. the original failure rate is non-decreasing), the model reduces to an age reduction (or virtual age) model, where perfect repair is equivalent to replacement.

Chapter 5

Warranty Servicing Strategies for a System with a Bathtub-Shaped Failure Rate Function

In this chapter, we suggest warranty servicing strategies for a system whose lifetime is modeled with a distribution having a bathtub-shaped failure rate (BFR) function. We apply the general repair model suggested in Chapter 4 when deriving the expected total warranty servicing cost for each servicing strategy. We suggest a class of cost functions to model the cost of an individual general repair. We then provide numerical illustrations of the proposed strategies.

This chapter is arranged as follows. In Section 5.1, we suggest possible cost functions for an individual general repair. In Section 5.2, we present the warranty servicing strategies. In Section 5.3, we provide numerical illustrations of the strategies and compute the expected costs for various cost functions. In Section 5.4, we conclude with a summary of the chapter.

5.1 Cost of an Individual General Repair

The cost of a repair is often assumed to be proportional to the degree of the repair. This is a reasonable assumption, since the degree of the repair reflects its effectiveness, and repairs that are more effective lead to greater improvement in the reliability of the system following the repair. The cost of a repair can also depend on the time of the repair, since the cost of labor and material are often time-dependent. Here, we assume that the cost of a repair is both a function of the degree of the repair and the time of the repair.

Let $c(T;\delta_T)$ denote the cost of an individual general repair at time T with degree $\delta_T \in [0,1]$. This cost is a random variable, since the failure time T is random. We use $c(t;\delta_t)$ to denote its realization for $T = t$.

Since perfect repair is the most effective general repair and minimal repair is the least effective, this cost is assumed to be increasing in the degree δ_t of repair, such that, for each $t \geq 0$,

$$c(t;0) \quad \leq \quad c(t;\delta_t) \quad \leq \quad c(t;1) \quad , \tag{5.1}$$

where $\delta_t \in (0,1)$ is the degree of an imperfect repair. In order to formulate the cost functions, we first model the costs of the two extremes, i.e. minimal and perfect repairs.

A minimal repair does not change the working condition of the system; we therefore assume that its cost is a function of time alone, such that

$$c(t;0) = c_0(t) \quad , \tag{5.2}$$

where $c_0(t)$ is some increasing function of t. This cost is also the minimum cost of a repair at time $t \geq 0$. The maximum cost of a repair at time t is the cost of a perfect repair, which we model as follows:

$$c(t;1) = c_0(t) + c_p(t) =: c_1(t) \quad , \tag{5.3}$$

where $c_p(.)$ is the difference between the cost of a minimal repair and a perfect repair. The function $c_p(.)$ can be defined in various ways depending on the repair model: for instance, it can reflect the amount of additional effort involved in performing a perfect repair in comparison to a minimal repair. We assume that all cost functions are positive.

Given the cost functions in (5.2) and (5.3), we define the cost of any general repair of degree $\delta_t \in [0,1]$ performed at time t as follows:

$$\begin{aligned} c(t;\delta_t) &= c_0(t) + (\delta_t)^q \left[c_1(t) - c_0(t)\right] \\ &= c_0(t) + (\delta_t)^q \left[c_0(t) + c_p(t) - c_0(t)\right] \\ &= c_0(t) + (\delta_t)^q \, c_p(t) \quad , \end{aligned} \tag{5.4}$$

where $q \geq 0$. The first component in the above cost function is the minimum cost of a repair at time t, and the second component is the cost for the improvement made or damage undone resulting from a general repair of degree δ_t. When $\delta_t = 0$, we get the cost of a minimal repair defined in (5.2), and when $\delta_t = 1$, we get the cost of a perfect repair defined in (5.3).

In (5.4), the parameter q determines how the cost of an imperfect repair changes in relation to the costs of perfect and minimal repairs, and is included to provide flexibility. When all parameters are fixed, the cost $c(t;\delta_t)$ is decreasing in q, since as q increases $(\delta_t)^q\ c_p(t)$ approaches zero. Therefore, for values of q close to 0, the cost of an imperfect repair at time t is closer to that of a perfect repair at that time (since $(\delta_t)^q$ approaches 1 as q approaches 0); and as q increases, the cost of the imperfect repair moves away from the cost of a perfect repair (and closer to the cost of a minimal repair).

Note that, the cost function in (5.4) is an increasing function of the degree of repair, and therefore, the cost of an imperfect repair is always bounded between the cost of a minimal repair and the cost of a perfect repair.

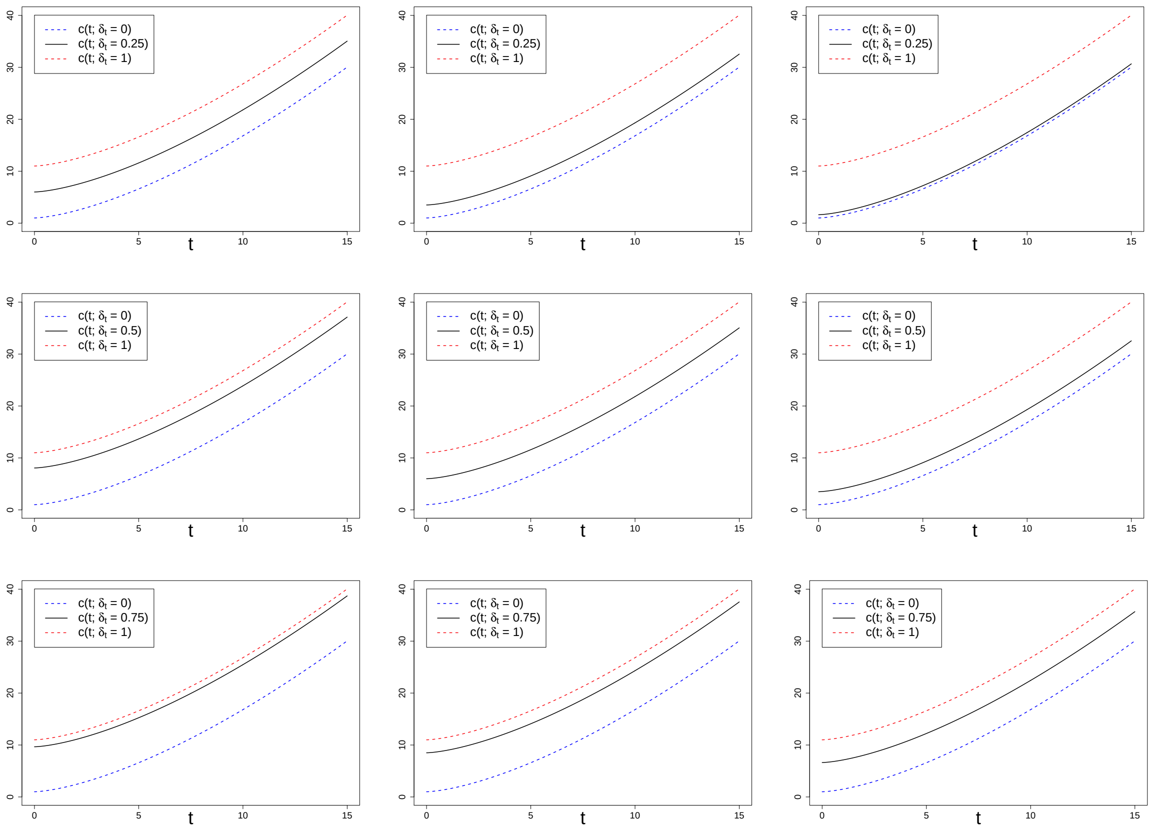


Figure 5.1: Illustrations of the example cost function $c(t;\delta_t)$ in (5.6), plotted over $t \in [0, 15]$, for $b_0 = 1$; $b_1 = b_2 = 0.5$; $m_1 = m_2 = 1.5$; and various values of δ_t (see plot legends); and $q = 0.5$ (left column), $q = 1$ (middle column), $q = 2$ (right column).

The minimum cost of a repair (i.e. the cost of a minimal repair) can be any non-decreasing function of time. Here, for instance, we will use the following cost function in our numerical illustrations:

$$c(t;0) = c_0(t) = b_0 + b_1 \, t^{m_1} \ , \tag{5.5}$$

where $b_0 > 0$ and $b_1, m_1 \geq 0$ are parameters of the cost function; b_0 is the cost of a minimal repair performed following a failure at the start of the lifetime of the system. We will now present some examples of the additional cost $c_p(.)$.

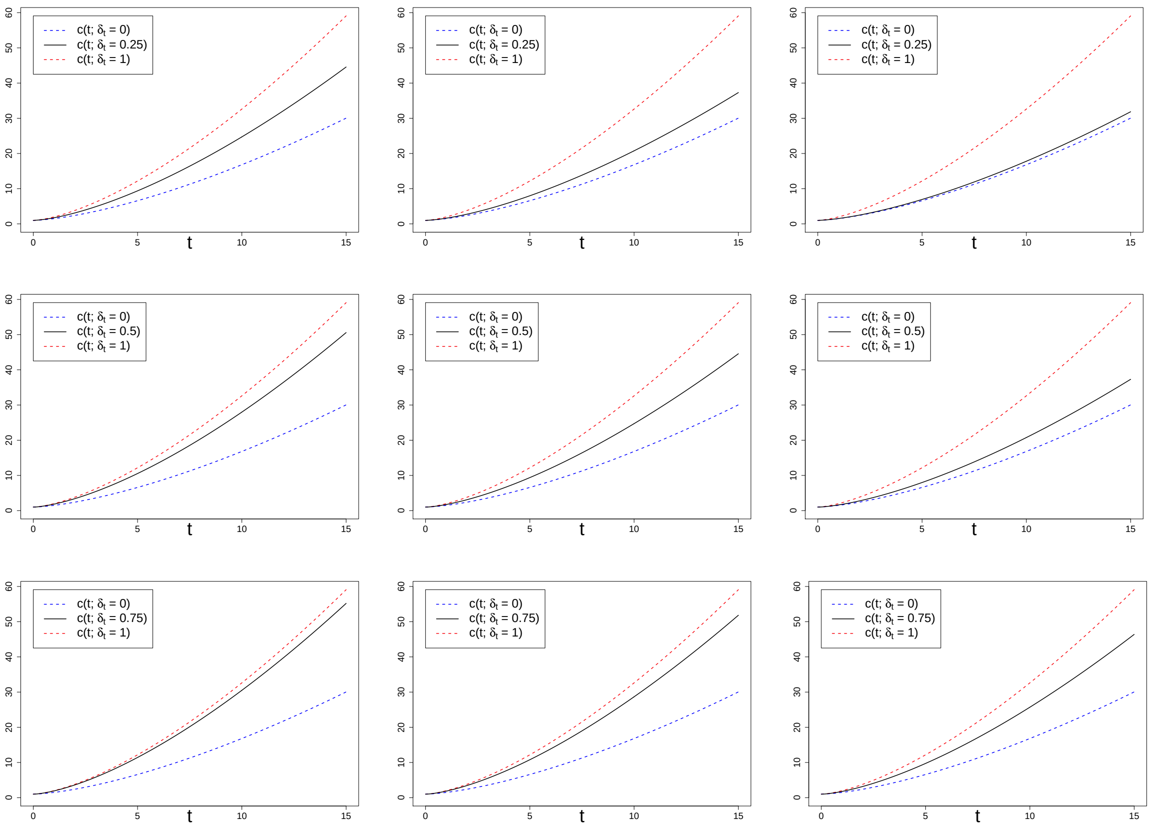


Figure 5.2: Illustrations of the example cost function $c(t;\delta_t)$ in (5.7), plotted over $t \in [0, 15]$, for $b_0 = 1$; $b_1 = b_2 = 0.5$; $m_1 = m_2 = 1.5$; and various values of δ_t (see plot legends); and $q = 0.5$ (left column), $q = 1$ (middle column), $q = 2$ (right column).

Example 1. The additional cost $c_p(.)$ can be constant, i.e. $c_p(t) \equiv c_p$, for all $t \geq 0$, so that the cost of a perfect repair at any time is always c_p units higher than the cost of a minimal repair at that time. Then, using the cost of a minimal repair in (5.5), the cost of a general repair of degree $\delta_t \in [0, 1]$, becomes

$$c(t;\delta_t) = b_0 + b_1 \, t^{m_1} + \left(\delta_t\right)^q \, c_p \ , \tag{5.6}$$

for $c_p \geq 0$; see Figure 5.1. In Figure 5.1, in each row, the degree of the imperfect repair is constant across the three columns. Notice that, in each row, as q increasing from 0.5 (left) to 2 (right), the cost of the imperfect repair (black solid line) moves closer to the cost of a minimal repair (blue dashed line) or away from the cost of the perfect repair (red dashed line). $\square$

Example 2. The additional cost $c_p(.)$ can be an increasing function of time, for instance, $c_p(t) = b_2\ t^{m_2}$, so that the difference between the cost of a perfect repair and the cost of a minimal repair increases over time. Then, using the cost of a minimal repair in (5.5), the cost of a general repair of degree $\delta_t \in [0,1]$, becomes

$$c(t;\delta_t) = b_0 + b_1\ t^{m_1} + \left(\delta_t\right)^q\ b_2\ t^{m_2}\ \ , \tag{5.7}$$

for $b_0 > 0$, and $b_1, b_2, m_1, m_2 \geq 0$; see Figure 5.2. $\square$

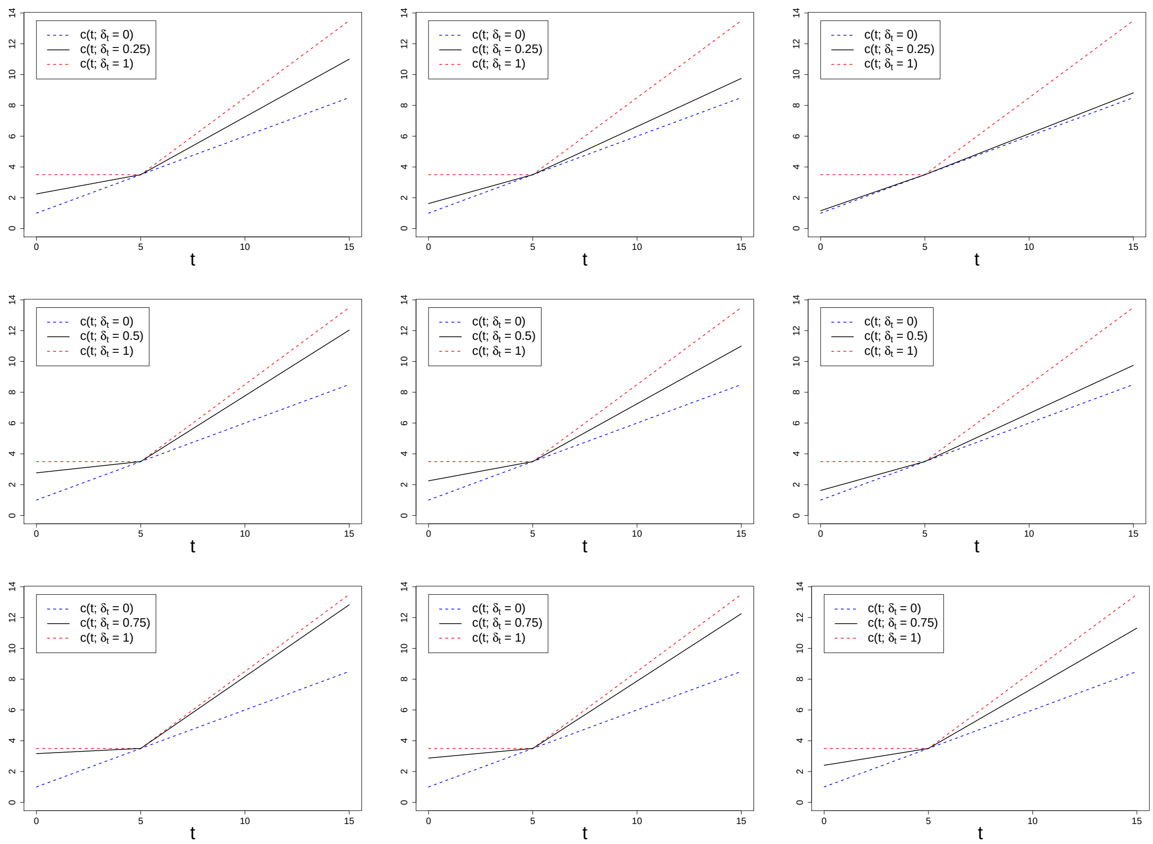


Figure 5.3: Illustrations of the example cost function $c(t;\delta_t)$ in (5.8), plotted over $t \in [0, 15]$, for $b_0 = 1$; $b_1 = b_2 = 0.5$; $m_1 = m_2 = 1$; and various values of δ_t (see plot legends); and $q = 0.5$ (left column), $q = 1$ (middle column), $q = 2$ (right column).

Example 3. The additional cost $c_p(.)$ can reflect directly the amount of improvement (or damage removal) using the definition of the general repair model. Recall that, according to the repair model, a perfect repair at time t returned the modified age $A(t)$ to a_1, which resulted in the maximum reliability improvement possible at that time. Therefore, one possible measure of system improvement following a perfect repair is the difference $|A(t) - a_1|$. Then, using the cost of a minimal repair in (5.5), the cost of a general repair of degree $\delta_t \in [0, 1]$, becomes

$$c(t; \delta_t) = b_0 + b_1\, t^{m_1} + \left(\delta_t\right)^q\, b_2\, \left(|A(t) - a_1|\right)^{m_2} \ , \tag{5.8}$$

for $b_0 > 0$, and $b_1, m_1, b_2, m_2 \geq 0$. Note that, since this cost is a function of $A(t)$, it depends on all failure times and degrees of repair before time t. Since the second cost component is a function of $|A(t) - a_1|$, when $A(t) = a_1$, the cost of a general repair at t, regardless of its degree, is equal to the cost of a minimal repair at this point. According to the proposed general repair model, the effectiveness of any general repair performed around the end of the DFR phase is minimal; see Section 4.3.1.

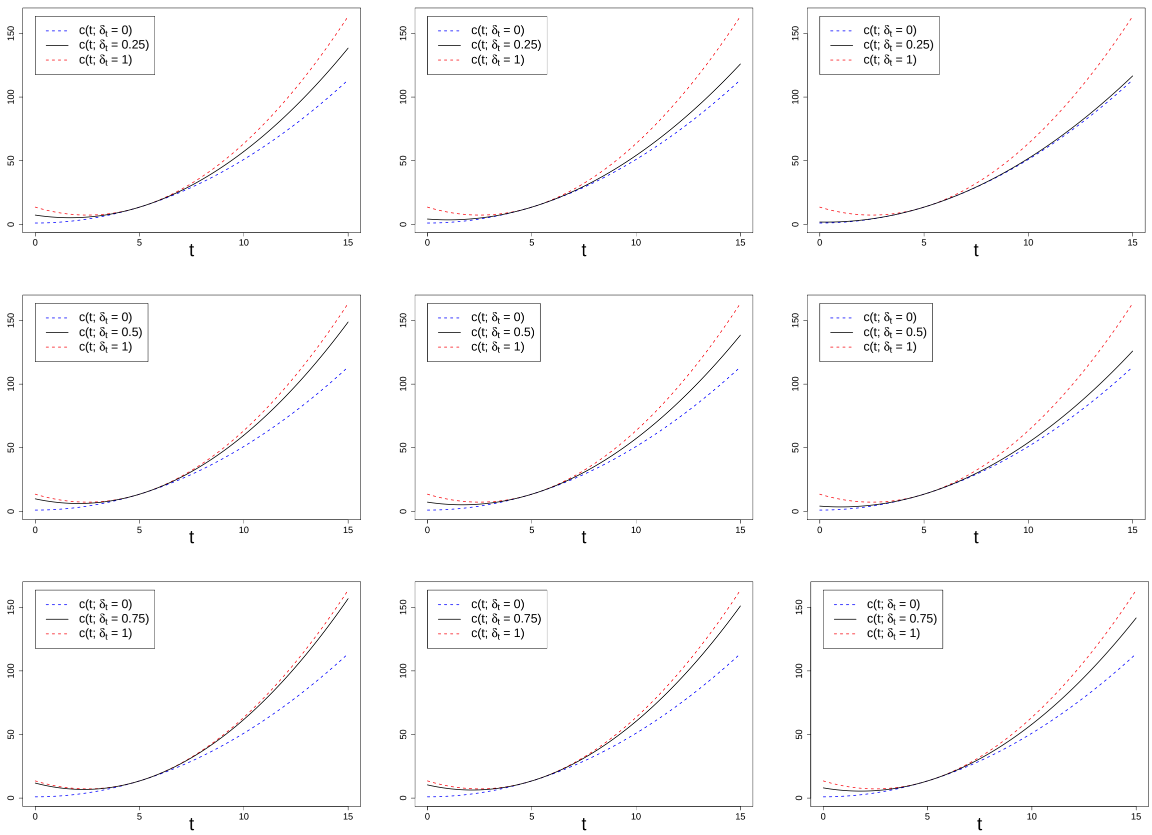


Figure 5.4: Illustrations of the example cost function $c(t; \delta_t)$ in (5.8), plotted over $t \in [0, 15]$, for $b_0 = 1$; $b_1 = b_2 = 0.5$; $m_1 = m_2 = 2$; and various values of δ_t (see plot legends); and $q = 0.5$ (left column), $q = 1$ (middle column), $q = 2$ (right column).

In Figures 5.3 and 5.4, we have plotted this cost function for the cost of the first repair at time $T_1 = t$, over $t \in (0, 15]$, for $m_1 = m_2 = 1$ and $m_1 = m_2 = 2$, respectively. Notice that, the cost of all general repairs at time $t = a_1$ is equal to the cost of the minimal repair at this point, since before the first failure $|a(t) - a_1| = |t - a_1|$, which is 0 when $t = a_1$, and therefore, $c_p(a_1) = 0$.

In Figure 5.3 and 5.4, notice that the cost of a minimal repair (blue dashed line) is simply an increasing function of time. The cost of an imperfect repair (solid black line) at any given point is an increasing function of the degree of repair. Also, the cost of an imperfect repair is always bounded between the cost of a minimal repair and the cost of a perfect repair (dashed red line). $\square$

5.2 Warranty Servicing Strategies

In this section, we propose warranty servicing strategies for a system whose lifetime is modeled with a distribution having a BFR function.

Let w denote the end of the warranty period. We propose two warranty servicing strategies based on the following cases:

(i) The useful life period of the original system is long and the warranty ends during the useful life period, i.e. $a_1 \leq w < a_2$.

(ii) The useful life period of the original system is short compared to its wear-out period and the warranty ends in the wear-out period, i.e. $w > a_2$.

The type of warranty considered is a non-renewing, free-repair warranty (FRW) policy, where the warranty period $(0, w]$ is fixed and the manufacturer repairs the failed system with no cost to the consumer; see Section 2.3.

We make the following assumptions in modeling the process of warranty claims: (a) each failure of the system is followed immediately by a warranty claim; (b) all claims are legitimate; (c) the time to process a claim is negligible with respect to the operating time of the system and set equal to zero; and (d) all repairs are immediate and instantaneous. Therefore, modeling the process of failures of the system is equivalent to modeling the process of warranty claims; see Sections 2.2 and 2.4.

5.2.1 Warranty Servicing Strategy I

For this strategy, we assume that the warranty period ends when the system is still in its useful life (CFR) period, i.e. before system wear-out. Since the warranty ends before system wear-out begins, a non-minimal repair in the CFR phase is an unnecessary cost to the manufacturer– delaying the wear-out period of the system does not affect the rate of failures before the end of the warranty period; see Section 4.3 for the repair model.

Therefore, for a warranted system whose warranty expires before it begins to wear out, we suggest the following warranty servicing strategy.

Strategy I. *The first failure of the system while the conditional intensity function is* ***decreasing*** *is rectified with a general, non-minimal repair and all remaining failures are rectified with minimal repair.*

Therefore, according to Strategy I, the first repair in $(0, a_1]$ can be a non-minimal repair with degree $\delta_1 \in (0, 1]$, and all remaining repairs are minimal.

5.2.1.1 Expected Number of Failures (Claims)

Let T_1 denote the time of the first failure of the system and let δ_1 denote the corresponding degree of repair. For strategy I, the expected number of system failures during the warranty period $(0, w]$ is derived as follows. We consider two cases: (i) at least one failure occurs in the DFR phase $(0, a_1]$ (i.e. $T_1 \leq a_1$); and (ii) no failure occurs in the DFR phase (i.e. $T_1 > a_1$).

Then, for case (i), the conditional expected number of failures is given by

$$E[N(w)|T_1 = t_1 \leq a_1] = 1 + \int_{t_1}^{w} \tilde{\lambda}(s|h_{s,1})\, ds = 1 + \int_{t_1}^{w} \lambda_1(a_1(s))\, ds \ , \tag{5.9}$$

where $a_1(s) = s + \delta_1\ (a_1 - t_1)$ denotes the realization of the modified age when one non-minimal repair has been performed prior to time s; $\lambda_1(.)$ denotes the modified baseline intensity function; and $\tilde{\lambda}(.|h_s)$ denotes the conditional intensity function of the process given the history $\mathcal{H}_s = h_s$; see Section 4.3. Note that, in general, the notation $h_{s,n}$ is used to denote the history of the process when n non-minimal repairs have been performed prior to time s, for $n \in \mathbb{N}$.

The conditional expected number of failures for case (ii), where no failure has occurred in $(0, a_1]$, is given by

$$E[N(w)|T_1 = t_1 > a_1] = \int\limits_{a_1}^{w} \tilde{\lambda}(s|h_{s,0})\ ds = \int\limits_{a_1}^{w} \lambda_1(a_0(s))\ ds = \int\limits_{a_1}^{w} \lambda_0(s)\ ds\ , \tag{5.10}$$

since, when no non-minimal repair is performed before time s (also, before the first failure of the system), the modified age is given by $a_0(s) = s$, and the modified baseline intensity function $\lambda_1(.)$ is equal to the original baseline intensity function $\lambda_0(.)$; see Section 4.3.2.2.

On removing the conditioning on the expected numbers of failures in (5.9) and (5.10), we get the (unconditional) expected number of failures during the warranty period $(0, w]$:

$$\begin{aligned} E[N(w)] &= \int\limits_{0}^{a_1} E[N(w)|T_1 = t_1 \leq a_1]\ dF_1(t_1) + \int\limits_{a_1}^{\infty} E[N(w)|T_1 = t_1 > a_1]\ dF_1(t_1) \\ &= \int\limits_{0}^{a_1} \left[1 + \int\limits_{t_1}^{w} \lambda_1(a_1(s))\ ds\right] f_1(t_1)\ dt_1 + \left[\int\limits_{a_1}^{w} \lambda_0(s)\ ds\right] \int\limits_{a_1}^{\infty} f_1(t_1)\ dt_1 \\ &= \int\limits_{0}^{a_1} \left[1 + \int\limits_{t_1}^{w} \lambda_1(a_1(s))\ ds\right] f_1(t_1)\ dt_1 + \left[\int\limits_{a_1}^{w} \lambda_0(s)\ ds\right] \bar{F}_1(a_1)\ , \end{aligned} \tag{5.11}$$

where $\bar{F}_1(.)$ is the reliability function of the time T_1 to first failure, which is given by

$$\bar{F}_1(t) = \mathrm{P}\{T_1 > t\} = \mathrm{P}\{N(t) = 0\} = e^{-\int\limits_0^t \lambda_0(s)\ ds}\ ; \tag{5.12}$$

and the corresponding density function is given by

$$f_1(t) = -\frac{d}{dt}\ \bar{F}_1(t) = \lambda_0(t)\ e^{-\int\limits_0^t \lambda_0(x)\ dx}\ ; \tag{5.13}$$

see Section 4.4.1. Note that, the expected number of failures is different for each $\delta_1 \in [0, 1]$, and therefore, we will also use the notation $E[N(w \mid \delta_1)]$ to refer to (5.11) for a particular δ_1.

5.2.1.2 Expected Total Warranty Servicing Cost

To derive the expected total warranty servicing cost for Strategy I, we use the cost functions introduced in Section 5.1, where the cost of a repair performed at time t with degree $\delta_t \in [0, 1]$ is denoted by $c(t; \delta_t)$.

As with the expected number of failures, the expected cost in $(0, w]$ is derived by considering the two cases: (i) $T_1 \leq a_1$; and (ii) $T_1 > a_1$. Then, for the first case, the conditional expected cost is given by

$$E[C(w)|T_1 = t_1 \leq a_1] = c(t_1;\delta_1) + \int_{t_1}^{w} c(s;0)\ \tilde{\lambda}(s|h_{s,1})\ ds = c(t_1;\delta_1) + \int_{t_1}^{w} c(s;0)\ \lambda_1(a_1(s))\ ds\ , \tag{5.14}$$

where $c(t_1;\delta_1)$ is the cost of the general, non-minimal repair at time t_1 with degree δ_1. Note that, since all repairs after t_1 are minimal, the process after t_1 is a Poisson process, and therefore, integrals of the form

$$\int_{l}^{u} c(s;0)\ \tilde{\lambda}(s|h_s)\ ds\ \equiv\ E\left[\sum_{i=N(l)+1}^{N(u)} c(T_i;0)\right] \tag{5.15}$$

represent the expected cost of minimal repairs in the interval $(l, u]$; refer to Boland [44] for more on (5.15). When all repairs are minimal, the conditional intensity does not change following each repair, and hence, the intensity function is equivalent to the conditional intensity function of the corresponding Poisson (or minimal repair) process over the defined interval.

The conditional expected cost for case (ii), where no failure has occurred in the interval $(0, a_1]$, is given by

$$E[C(w)|T_1 = t_1 > a_1] = \int_{a_1}^{w} c(s;0)\ \tilde{\lambda}(s|h_{s,0})\ ds = \int_{a_1}^{w} c(s;0)\ \lambda_1(a_0(s))\ ds = \int_{a_1}^{w} c(s;0)\ \lambda_0(s)\ ds\ . \tag{5.16}$$

On removing the conditioning on the expected costs in (5.14) and (5.16), we get the (unconditional) expected total warranty servicing cost for Strategy I:

$$\begin{aligned} E[C(w)] &= \int_{0}^{a_1} E[C(w)|T_1 = t_1 \leq a_1]\ dF_1(t_1) + \int_{a_1}^{\infty} E[C(w)|T_1 = t_1 > a_1]\ dF_1(t_1) \\ &= \int_{0}^{a_1} \left[c(t_1;\delta_1) + \int_{t_1}^{w} c(s;0)\ \lambda_1(a_1(s))\ ds\right] f_1(t_1)\ dt_1 + \left[\int_{a_1}^{w} c(s;0)\ \lambda_0(s)\ ds\right] \int_{a_1}^{\infty} f_1(t_1)\ dt_1 \\ &= \int_{0}^{a_1} \left[c(t_1;\delta_1) + \int_{t_1}^{w} c(s;0)\ \lambda_1(a_1(s))\ ds\right] f_1(t_1)\ dt_1 + \left[\int_{a_1}^{w} c(s;0)\ \lambda_0(s)\ ds\right] \bar{F}_1(a_1)\ , \end{aligned} \tag{5.17}$$

where the distribution functions of T_1 are given in (5.12) and (5.13); see Section 4.4.1 for the baseline intensity functions, the age modification function and the conditional intensity function.

Notice that, when all the cost functions in (5.17) are replaced by 1, the expected cost reduces to the expected number of failures in (5.11).

The expected total warranty servicing cost $E[C(w)]$ is different for each $\delta_1 \in [0,1]$, and therefore, we will also use the notation $E[C(w \mid \delta_1)]$ to refer to (5.17) for a particular δ_1.

5.2.2 Warranty Servicing Strategy II

For this strategy, we assume that the useful life period of the system is short, and therefore, the warranty period ends during the wear-out period of the original system. Since the warranty period may include part of the wear-out period (the length depends on previous repairs), with this strategy we aim to extend the useful life period of the system and delay its wear-out period, by having two possible general repairs of degree greater than 0 (i.e. non-minimal repairs). Since the system reliability is improved during the warranty period, the cost of servicing the warranty for the manufacturer may be reduced (with the appropriate choice of the degrees of repair).

Therefore, for a warranted system whose warranty expires during its wear-out period, we suggest the following warranty servicing strategy.

Strategy II. *When the conditional intensity function of the system is* ***decreasing****, the first failure of the system is rectified with a non-minimal repair and all remaining failures are rectified with minimal repair. When the conditional intensity function of the system is* ***non-decreasing****, the first failure of the system is rectified with a non-minimal repair and all remaining failures are rectified with minimal repair.*

In other words, only the first repair in the DFR phase and the first repair outside the DFR phase of the conditional intensity function are general, non-minimal repairs, and all remaining repairs under warranty are minimal.

According to the general repair model proposed in Section 4.3, a non-minimal repair in the DFR phase results in extending the useful life period such that it begins earlier. Therefore, the start of the useful life period (i.e. the end of the DFR phase) depends on the time and degree of the non-minimal repair performed in the interval $(0, a_1]$. Let $\mathcal{T}_{a_1}$ denote the end of the DFR phase, and let τ_{a_1} denote its realization; see Section 4.3.2.1. If no failure has occurred in the DFR phase $(0, a_1]$, then $\tau_{a_1} = a_1$. If at least one failure has occurred in $(0, a_1]$, then $\tau_{a_1} \leq a_1$ depending on the degree of the performed repairs. The end of the DFR phase, given the time t_1 of the first failure and the corresponding degree of repair δ_1, is given by

$$\tau_{a_1} = \begin{cases} a_1 \ , & t_1 \geq a_1 \\ a_1 - \delta_1 \ (a_1 - t_1), & t_1 < a_1 \end{cases} . \tag{5.18}$$

Therefore, the subintervals of the warranty strategy are characterized by the random variable $\mathcal{T}_{a_1}$. Then, given the warranty period $(0, w]$, this servicing strategy can be expressed as follows:

- *The first repair in the DFR phase $(0, \mathcal{T}_{a_1}]$ of the conditional intensity function is non-minimal with degree δ_1 and all remaining repairs in this interval are minimal.*
- *The first repair in the non-decreasing phase $(\mathcal{T}_{a_1}, w]$ of the conditional intensity function is non-minimal with degree δ_1' and all remaining repairs in this interval are minimal.*

5.2.2.1 Expected Number of Failures (Claims)

As before, let T_1 denote the time to first failure of the system, with reliability and density functions given in (5.12) and (5.13), respectively. Let T_1' denote the time of the first failure of the system outside the DFR phase, i.e. in $(\mathcal{T}_{a_1}, w]$, and let t_1' denote its realization. Then, to derive the expected number of failures for Strategy II, we must consider the following four possible cases:

(i) at least one failure has occurred in each of the subintervals $(0, \mathcal{T}_{a_1}]$ and $(\mathcal{T}_{a_1}, w]$, i.e. $T_1 \leq a_1$ and $T_1' \leq w$;

(ii) no failure has occurred in the first subinterval $(0, \mathcal{T}_{a_1}]$ and at least one failure has occurred in the second subinterval $(\mathcal{T}_{a_1}, w]$, i.e. $T_1 > a_1$ and $T_1' \leq w$ (here, $\mathcal{T}_{a_1} = a_1$, and $T_1' \stackrel{\mathrm{d}}{=} T_1$);

(iii) at least one failure has occurred in the first subinterval $(0, \mathcal{T}_{a_1}]$ and no failure has occurred in the second subinterval $(\mathcal{T}_{a_1}, w]$, i.e. $T_1 \leq a_1$ and $T_1' > w$;

(iv) no failure has occurred in either of the two subintervals, i.e. $T_1' \equiv T_1 > w$.

The conditional expected number of failures for the last case is zero, since no failure has occurred during the warranty period $(0, w]$.

As before, let the history $h_{s,n}$, for $n \in \{1, 2\}$, denote the case where n non-minimal repairs have been performed before time s. Here, the histories for both cases (ii) and (iii) include one prior non-minimal repair; however, the two histories are not the same. Therefore, we use $h'_{s,1}$ and $h_{s,1}$ to distinguish between the two. Similarly, for the corresponding realizations of the age modification functions, we use $a_1(s;\delta'_1)$ and $a_1(s;\delta_1)$. Then, the conditional expected number of failures for case (i) is given by

$$\begin{aligned} E[N(w)|T_1 = t_1 \le a_1, T'_1 = t'_1 \le w] &= 1 + \int_{t_1}^{\tau_{a_1}} \tilde{\lambda}(s|h_{s,1})\, ds + 1 + \int_{t'_1}^{w} \tilde{\lambda}(s|h_{s,2})\, ds \\ &= 2 + \int_{t_1}^{\tau_{a_1}} \lambda_1(a_1(s;\delta_1))\, ds + \int_{t'_1}^{w} \lambda_1(a_2(s;\delta_1,\delta'_1))\, ds \ , \end{aligned} \tag{5.19}$$

where $a_2(s;\delta_1,\delta'_1)$ denotes the realization of the modified age at time s after the two non-minimal repairs. Note that, since we have conditioned on the events that at least one failure has occurred in each subinterval, the conditional expected number of failures counts at least two failures.

The conditional expected number of failures for cases (ii) and (iii) are similarly derived as follows:

$$E[N(w)|T_1 = t_1 > a_1, T'_1 = t'_1 \le w] = 1 + \int_{t'_1}^{w} \tilde{\lambda}(s|h'_{s,1})\, ds = 1 + \int_{t'_1}^{w} \lambda_1(a_1(s;\delta'_1))\, ds \ ; \tag{5.20}$$

$$E[N(w)|T_1 = t_1 \le a_1, T'_1 = t'_1 > w] = 1 + \int_{t_1}^{\tau_{a_1}} \tilde{\lambda}(s|h_{s,1})\, ds = 1 + \int_{t_1}^{\tau_{a_1}} \lambda_1(a_1(s;\delta_1))\, ds \ . \tag{5.21}$$

We used the notations $a_1(s;\delta_1)$, $a_1(s;\delta'_1)$ and $a_2(s;\delta_1,\delta'_1)$ to specify which degrees of repair apply. The degree of repair in the first subinterval $(0,\tau_{a_1}]$ is δ_1 and in the second subinterval $(\tau_{a_1}, w]$ is δ'_1. Therefore, the realizations of the age modification functions are given by

$$a_1(s;\delta_1) = s + \delta_1\ (a_1 - t_1) \ , \tag{5.22}$$

defined for all $s \in (t_1, t'_1]$;

$$a_1(s;\delta'_1) = s - \delta'_1\ (t'_1 - a_1) \ , \tag{5.23}$$

defined for all $s \in (t'_1, w]$, given that $t_1 > a_1$; and

$$a_2(s;\delta_1,\delta'_1) = s + \delta_1\ (a_1 - t_1) - \delta'_1(a_1(t'_1;\delta_1) - a_1) \ , \tag{5.24}$$

defined for $s \in (t_1', w]$, given that $t_1 \le a_1$ and $t_1' \le w$.

Using (5.19) – (5.21), and on removing the conditioning, we derive the expected number of failures for Strategy II:

$$\begin{aligned}
E[N(w)] = &\int_0^{a_1}\int_{\tau_{a_1}}^{w} \left(2 + \int_{t_1}^{\tau_{a_1}} \tilde{\lambda}(s|h_{s,1})\, ds + \int_{t_1'}^{w} \tilde{\lambda}(s|h_{s,2})\, ds\right) f_{T_1'|T_1}(t_1'|t_1)\, f_1(t_1)\, dt_1'\, dt_1 \\
&+ \int_{a_1}^{w} \left(1 + \int_{t_1'}^{w} \tilde{\lambda}(s|h_{s,1}')\, ds\right) f_1(t_1')\, dt_1' \\
&+ \int_0^{a_1}\int_{w}^{\infty} \left(1 + \int_{t_1}^{\tau_{a_1}} \tilde{\lambda}(s|h_{s,1})\, ds\right) f_{T_1'|T_1}(t_1'|t_1)\, f_1(t_1)\, dt_1'\, dt_1 \\
= &\int_0^{a_1}\int_{\tau_{a_1}}^{w} \left(2 + \int_{t_1}^{\tau_{a_1}} \lambda_1(a_1(s;\delta_1))\, ds + \int_{t_1'}^{w} \lambda_1(a_2(s;\delta_1,\delta_1'))\, ds\right) f_{T_1'|T_1}(t_1'|t_1)\, f_1(t_1)\, dt_1'\, dt_1 \\
&+ \int_{a_1}^{w} \left(1 + \int_{t_1'}^{w} \lambda_1(a_1(s;\delta_1'))\, ds\right) f_1(t_1')\, dt_1' \\
&+ \int_0^{a_1}\int_{w}^{\infty} \left(1 + \int_{t_1}^{\tau_{a_1}} \lambda_1(a_1(s;\delta_1))\, ds\right) f_{T_1'|T_1}(t_1'|t_1)\, f_1(t_1)\, dt_1'\, dt_1 \quad .
\end{aligned} \tag{5.25}$$

The function $f_{T_1'|T_1}(.|t_1)$ is the density function of T_1' given $T_1 = t_1$, and is derived by differentiating the corresponding conditional probability function:

$$\begin{aligned}
F_{T_1'|T_1}(s|t_1) &= \mathrm{P}\{T_1' \le s|T_1 = t_1\} = 1 - \mathrm{P}\{T_1' > s|T_1 = t_1\} = 1 - \mathrm{P}\{T_2 > s|T_2 > \tau_{a_1}, T_1 = t_1\} \\
&= 1 - \frac{\mathrm{P}\{T_2 > s|T_1 = t_1\}}{\mathrm{P}\{T_2 > \tau_{a_1}|T_1 = t_1\}} = 1 - e^{-\int_{\tau_{a_1}}^{s} \tilde{\lambda}(x|h_{x,1})\, dx} = 1 - e^{-\int_{\tau_{a_1}}^{s} \lambda_1(a_1(x;\delta_1))\, dx} \quad ,
\end{aligned} \tag{5.26}$$

which is defined for $s > \tau_{a_1}$; see Section 4.4.1. Since all repairs between $T_1 = t_1$ and τ_{a_1} are minimal, the system behaves as though it has not failed in the interval $(t_1, \tau_{a_1}]$. Then, the distribution of T_1' (which is the time of the first failure after τ_{a_1}) is the same as the distribution of the time T_2 of the second failure, conditional on T_2 happening after τ_{a_1} [cf. (4.38)]. The corresponding conditional density function is then given by

$$f_{T_1'|T_1}(s|t_1) = \frac{d}{ds} F_{T_1'|T_1}(s|t_1) = \tilde{\lambda}(s|h_{s,1})\, e^{-\int_{\tau_{a_1}}^{s} \tilde{\lambda}(x|h_{x,1})\, dx} = \lambda_1(a_1(s;\delta_1))\, e^{-\int_{\tau_{a_1}}^{s} \lambda_1(a_1(x;\delta_1))\, dx} \quad . \tag{5.27}$$

Note that, products of the form $\tilde{\lambda}(t|h_t)\ e^{-\int_x^t \tilde{\lambda}(s|h_s)\,ds}\ dt$ are the approximate probability of having a failure at time t, given that no failure has occurred in the interval $(x, t]$, given the history of the failure process.

The third summand in (5.25) can be further simplified– since the conditional expected number of failures is not a function of t_1', we can replace the summand by

$$\int_0^{a_1} \left(1 + \int_{t_1}^{\tau_{a_1}} \tilde{\lambda}(s|h_{s,1})\ ds\right) \bar{F}_{T_1'|T_1}(w|t_1)\ f_1(t_1)\ dt_1\ \ , \tag{5.28}$$

where $\tilde{\lambda}(s|h_{s,1}) = \lambda_1(a_1(s;\delta_1))$, and the conditional reliability function is given by

$$\bar{F}_{T_1'|T_1}(w|t_1) := 1 - F_{T_1'|T_1}(w|t_1) = e^{-\int_{\tau_{a_1}}^{w} \tilde{\lambda}(x|h_{x,1})\,dx} = e^{-\int_{\tau_{a_1}}^{w} \lambda_1(a_1(x;\delta_1))\,dx} = \int_w^{\infty} f_{T_1'|T_1}(t_1'|t_1)\ dt_1'\ . \tag{5.29}$$

Since the expected number of failures is a function of δ_1 and δ_1', we will also use the notation $E[N(w \mid \delta_1, \delta_1')]$ to refer to (5.25).

5.2.2.2 Expected Total Warranty Servicing Cost

The expected total warranty servicing cost for Strategy II, is derived by considering the four possible cases mentioned in the previous section; see page 96. The conditional expected cost corresponding to case (iv) is zero (since no failure has occurred during the warranty period $(0, w]$. The conditional expected costs corresponding to case (i) is given by

$$\begin{aligned}
&E[C(w)|T_1 = t_1 \le a_1, T_1' = t_1' \le w] \\
&\quad = c(t_1;\delta_1) + \int_{t_1}^{\tau_{a_1}} c(s;0)\ \tilde{\lambda}(s|h_{s,1})\ ds + c(t_1';\delta_1') + \int_{t_1'}^{w} c(s;0)\ \tilde{\lambda}(s|h_{s,2})\ ds \\
&\quad = c(t_1;\delta_1) + \int_{t_1}^{\tau_{a_1}} c(s;0)\ \lambda_1(a_1(s;\delta_1))\ ds + c(t_1';\delta_1') + \int_{t_1'}^{w} c(s;0)\lambda_1(a_2(s;\delta_1,\delta_1'))\ ds\ \ ,
\end{aligned} \tag{5.30}$$

where $c(s;0)$ is the cost of a minimal repair at time s, $c(t_1;\delta_1)$ is the cost of the first repair in the DFR phase at time $t_1(\le a_1)$ with degree δ_1, and $c(t_1';\delta_1')$ is the cost of the first repair after the DFR phase at time $t_1'(> \tau_{a_1})$ with degree δ_1'; see page 97 for the realizations of the age modification functions. Similarly, the conditional expected cost for case (ii) is given by

$$\begin{aligned} E[C(w)|T_1 = t_1 > a_1, T_1' = t_1' \le w] &= c(t_1';\delta_1') + \int_{t_1'}^{w} c(s;0)\ \tilde{\lambda}(s|h_{s,1}')\ ds \\ &= c(t_1';\delta_1') + \int_{t_1'}^{w} c(s;0)\ \lambda_1(a_1(s;\delta_1'))\ ds\ \ ; \end{aligned} \tag{5.31}$$

and the conditional expected cost for case (iii) is given by

$$\begin{aligned} E[C(w)|T_1 = t_1 \le a_1, T_1' = t_1' > w] &= c(t_1;\delta_1) + \int_{t_1}^{\tau_{a_1}} c(s;0)\ \tilde{\lambda}(s|h_{s,1})\ ds \\ &= c(t_1;\delta_1) + \int_{t_1}^{\tau_{a_1}} c(s;0)\ \lambda_1(a_1(s;\delta_1))\ ds\ \ . \end{aligned} \tag{5.32}$$

Using the above conditional costs, upon removing the conditioning, we derive the expected total warranty servicing cost for Strategy II as follows:

$$\begin{aligned} E[C(w)] = &\int_0^{a_1}\int_{\tau_{a_1}}^{w} \left(c(t_1;\delta_1) + \int_{t_1}^{\tau_{a_1}} c(s;0)\ \tilde{\lambda}(s|h_{s,1})\ ds + c(t_1';\delta_1') + \int_{t_1'}^{w} c(s;0)\ \tilde{\lambda}(s|h_{s,2})\ ds \right) \\ &\qquad \times f_{T_1'|T_1}(t_1'|t_1)\ f_1(t_1)\ dt_1'\ dt_1 \\ &+ \int_{a_1}^{w} \left(c(t_1';\delta_1') + \int_{t_1'}^{w} c(s;0)\ \tilde{\lambda}(s|h_{s,1}')\ ds \right) f_1(t_1')\ dt_1' \\ &+ \int_0^{a_1}\int_{w}^{\infty} \left(c(t_1;\delta_1) + \int_{t_1}^{\tau_{a_1}} c(s;0)\ \tilde{\lambda}(s|h_{s,1})\ ds \right) f_{T_1'|T_1}(t_1'|t_1)\ f_1(t_1)\ dt_1'\ dt_1\ \ , \end{aligned} \tag{5.33}$$

where $f_{T_1'|T_1}(.|t_1)$ is the conditional density function of T_1', given $T_1 = t_1$, which is defined in (5.27). Note that, in the last summand of (5.33), the conditional cost does not depend on t_1', and therefore, the last summand can be replaced by

$$\int_0^{a_1} \left(c(t_1;\delta_1) + \int_{t_1}^{\tau_{a_1}} c(s;0)\ \tilde{\lambda}(s|h_{s,1})\ ds \right) \bar{F}_{T_1'|T_1}(w|t_1)\ f_1(t_1)\ dt_1\ \ , \tag{5.34}$$

where $\bar{F}_{T_1'|T_1}(.|t_1)$ is the conditional reliability function of T_1' given $T_1 = t_1$, and is defined in (5.29).

On substituting for the conditional intensity functions in (5.33) and further simplifying the expressions, we get the following expected total warranty servicing cost for Strategy II:

$$E[C(w)] = \int_0^{a_1}\int_{\tau_{a_1}}^{w} \left(c(t_1;\delta_1) + \int_{t_1}^{\tau_{a_1}} c(s;0)\ \lambda_1(a_1(s;\delta_1))\ ds + c(t_1';\delta_1') + \int_{t_1'}^{w} c(s;0)\ \lambda_1(a_2(s;\delta_1,\delta_1'))\ ds \right)$$

$$\times f_{T_1'|T_1}(t_1'|t_1)\ f_1(t_1)\ dt_1'\ dt_1$$

$$+ \int_{a_1}^{w} \left(c(t_1';\delta_1') + \int_{t_1'}^{w} c(s;0)\ \lambda_1(a_1(s;\delta_1'))\ ds \right) f_1(t_1')\ dt_1'$$

$$+ \int_0^{a_1} \left(c(t_1;\delta_1) + \int_{t_1}^{\tau_{a_1}} c(s;0)\ \lambda_1(a_1(s;\delta_1))\ ds \right) \bar{F}_{T_1'|T_1}(w|t_1)\ f_1(t_1)\ dt_1 \ . \tag{5.35}$$

Note that, when all cost functions in (5.35) are replaced by 1, then the expected total warranty servicing cost reduces to the expected number of failures derived in (5.25).

Since the expected total warranty serving cost for Strategy II is a function of the degrees of repair δ_1 and δ_1', we will also use the notation $E[C(w \mid \delta_1, \delta_1')]$ to refer to (5.35).

The derived expected total warranty servicing costs for both strategies are conditional on the degrees of the non-minimal repairs being δ_1 and δ_1'. If the degrees are random, then these expected costs are conditional on $D_1 = \delta_1$ and $D_1' = \delta_1'$; then to derive the expected costs, the conditioning needs to be removed using the distributions of D_1 and D_1'. For simplicity, we assume that the degrees δ_1 and δ_1' of the non-minimal repairs are preassigned.

5.3 Numerical Illustrations

In this section, we provide numerical illustrations of the two warranty servicing strategies. To compute the expected total warranty servicing costs for each strategy, we apply the general repair model proposed in Section 4.3 along with the cost function from Example 3 (on page 90).

The original baseline intensity (or failure rate) function used in the numerical illustrations of the proposed servicing strategies is the following BFR function [39]:

$$\lambda_0(t) = \begin{cases} \lambda + \alpha_1\ \beta_1\ (a_1 - t)^{\beta_1 - 1}\ , & t \le a_1 \\ \lambda\ , & a_1 \le t \le a_2 \\ \lambda + \alpha_2\ \beta_2\ (t - a_2)^{\beta_2 - 1}\ , & t \ge a_2\ , \end{cases}$$

where $\lambda > 0$, $\beta_1, \beta_2 > 0$, and $\alpha_1, \alpha_2 > 0$.

Parameter values for Strategy I. To illustrate Strategy I, we have chosen change-points $a_1 = 4$ and $a_2 = 15$. The other parameters are: $\lambda = 0.4$, $\alpha_1 \beta_1 = 0.45$, $\alpha_2 \beta_2 = 0.35$, $\beta_1 = 3.3$, and $\beta_2 = 3.6$; see Figure 5.5 (left).

Parameter values for Strategy II. To illustrate Strategy II, we have chosen change-points $a_1 = 3$ and $a_2 = 6$. The other parameter values are: $\lambda = 0.15$, $\alpha_1\ \beta_1 = 0.6$, $\alpha_2\ \beta_2 = 0.5$, $\beta_1 = 3.4$, and $\beta_2 = 3.1$; see Figure 5.5 (right).

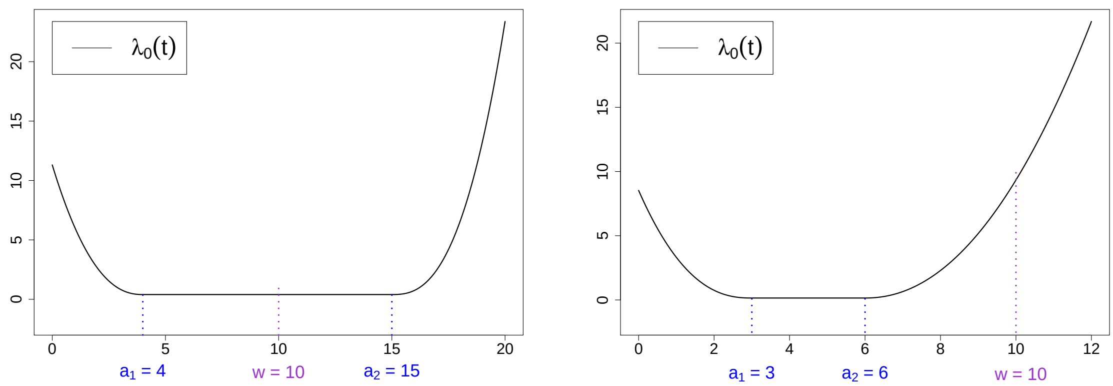


Figure 5.5: The original baseline intensity function $\lambda_0(t)$ used in the illustration of: Strategy I (left) and Strategy II (right).

Note that, the warranty period for both illustrations is set to $(0, w] = (0, 10]$; however, the change-points a_1 and a_2 for the two baseline intensity functions are chosen such that, for Strategy I, $a_1 < w < a_2$, and for Strategy II, $w > a_2$. Recall that, Strategy I was suggested for a system with a long useful life period and Strategy II was suggested for a system with a short useful life period; see Figure 5.5.

5.3.1 Expected Number of Failures for Strategies I and II

The expected number of failures can be used to examine the behavior of the general repair model. A good repair model will stay true to the definitions of the various types of general repair, resulting in the following property: as the degree of any of the non-minimal repairs increases, given that the degrees of all other repairs remain fixed, the corresponding expected number of failures decreases. This is because as the degree of the repair increases, so does the conditional reliability of the repaired system.

To illustrate the behavior of the general repair model, in Figure 5.6, we have plotted the expected number of failures for Strategies I and II as a function of the degree(s) of the non-minimal repair(s). The expected number of failures for the two strategies were derived in (5.11) and (5.25), respectively.

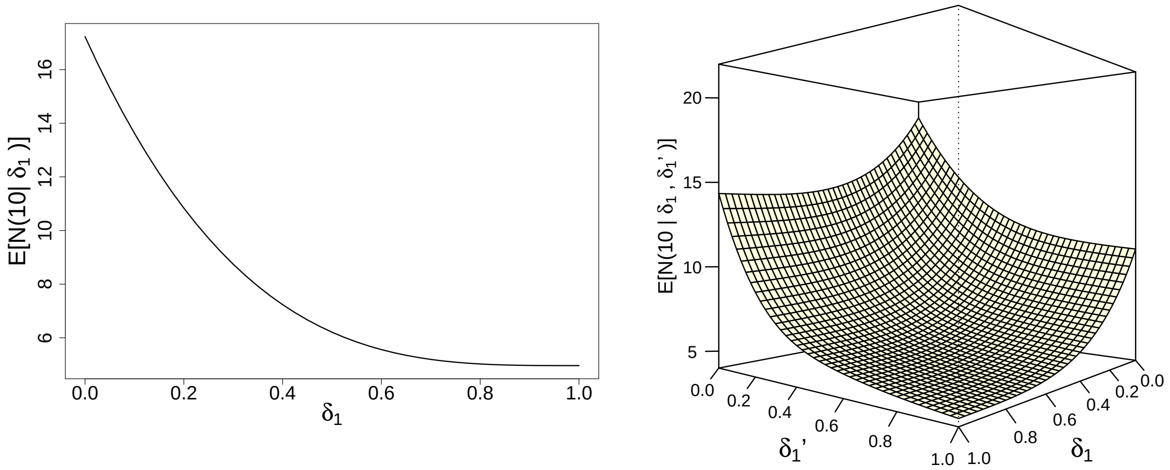


Figure 5.6: The expected number of failures $E[N(10 \mid \delta_1)]$ (left) and $E[N(10 \mid \delta_1, \delta_1')]$ (right), for degrees $\delta_1, \delta_1' \in [0, 1]$.

Notice that, for both strategies, the expected number of failures is a decreasing function of the degree(s) of the non-minimal repair(s). In Figure 5.6, for instance, for any fixed δ_1, the expected number of failures decreases as δ_1' increases, and for any fixed δ_1', the expected number of failures decreases as δ_1 increases (given that all other repairs are minimal). This indicates that, according to the general repair model, the reliability of the system increases as the degree of any non-minimal repair increases, given that the degrees of all other repairs remain fixed.

5.3.2 Expected Total Warranty Servicing Costs for Strategy I

To illustrate Strategy I, we use the example cost function given in (5.8), and compute the expected total warranty servicing cost $E[C(10 \mid \delta_1)]$ for $\delta_1 \in [0, 1]$. The expected total warranty servicing cost $E[C(w \mid \delta_1)]$ for Strategy I was derived in (5.17).

To compute the expected costs, we choose the following parameter values for the example cost function in (5.8): we fix $m_1 = m_2 = 2$ and $b_0 = 0.01$, and compute the expected costs for various values of the parameters b_1, b_2 and q: $b_1 \in \{0.5, 1.0, 1.5, 2.0, 5.0\}$; $b_2 \in \{0.5, 1.0, 2.5, 5.0, 7.5\}$; and $q \in \{1, 2\}$.

Let $\delta_1^\star$ denote the degree of the non-minimal repair that minimizes the expected total warranty servicing cost, and therefore, $E[C(10 \mid \delta_1^\star)]$ represents the minimum, over all $\delta_1 \in [0,1]$, expected total warranty servicing cost over the warranty period $(0,10]$. In Tables 5.1 and 5.2, the minimum expected total costs are tabulated for $q = 1$ and $q = 2$ respectively, for various values of the cost parameters b_1 and b_2. Also included in each table are the corresponding expected costs when all repairs are minimal, i.e. $\delta_1 = 0$, and the expected costs when the non-minimal repair is perfect, i.e. $\delta_1 = 1$.

Table 5.1: Minimum expected total warranty servicing costs $E[C(10 \mid \delta_1^\star)]$, for $q = 1$.

b_1	b_2	$\delta_1^\star$	$E[C(10 \mid \delta_1 = \delta_1^\star)]$	$E[C(10 \mid \delta_1 = 0)]$	$E[C(10 \mid \delta_1 = 1)]$
0.5	0.5	0.36	70.3579	75.9536	74.3114
	1.0	0.24	72.5922		81.9472
	2.5	0.05	75.6992		104.8546
	5.0	0.01	75.9535		143.0336
	7.5	0.01	75.9535		181.2127
1.0	0.5	0.46	137.6500	151.9072	140.9869
	1.0	0.36	140.7158		148.6228
	2.5	0.2	146.8492		171.5302
	5.0	0.05	151.3985		209.7092
	7.5	0.01	151.9069		247.8883
1.5	0.5	0.51	204.6426	227.8608	207.6625
	1.0	0.42	208.1364		215.2983
	2.5	0.27	215.8275		238.2057
	5.0	0.14	223.4811		276.3848
	7.5	0.05	227.0977		314.5638
2.0	0.5	0.54	271.5255	303.8144	274.3381
	1.0	0.46	275.3000		281.9739
	2.5	0.32	284.0101		304.8813
	5.0	0.20	293.6983		343.0604
	7.5	0.12	299.6093		381.2394
5.0	0.5	0.63	672.1569	759.5360	674.3915
	1.0	0.57	676.7079		682.0273
	2.5	0.46	688.2501		704.9347
	5.0	0.36	703.5790		743.1138
	7.5	0.29	715.8480		781.2928

Note that, column 5 (where $\delta_1 = 0$) and column 6 (where $\delta_1 = 1$) in Table 5.1 (when $q = 1$) are identical to the corresponding columns in Table 5.2 (when $q = 2$). This is because the parameter q of the cost function describes only how the cost of an imperfect repair changes with respect to the cost of a perfect repair, and has no effect on the cost of a minimal or perfect repair. In the cost function, when all other parameters are fixed, as q increases, the cost of an imperfect repair moves farther away from that of a perfect repair and closer to the cost of a minimal repair. The effect of this on the expected total warranty servicing cost can be seen with a row-wise comparison of the two tables. The minimum expected total warranty servicing costs in column 2 of Table 5.1 are lower than the corresponding expected costs in column 2 of Table 5.2.

When b_1 is fixed, $\delta_1^\star$ decreases as b_2 increases. That is, as the cost of imperfect repair increases, an imperfect repair of a lower degree results in the minimum expected total war-

Table 5.2: Minimum expected total warranty servicing costs $E[C(10 \mid \delta_1^\star)]$, for $q = 2$.

b_1	b_2	$\delta_1^\star$	$E[C(10 \mid \delta_1 = \delta_1^\star)]$	$E[C(10 \mid \delta_1 = 0)]$	$E[C(10 \mid \delta_1 = 1)]$
0.5	0.5	0.39	68.5676	75.9536	74.3114
	1.0	0.32	69.5120		81.94718
	2.5	0.22	71.0965		104.8546
	5.0	0.16	72.3989		143.0336
	7.5	0.12	73.1156		181.2127
1	0.5	0.46	135.7533	151.9072	140.9869
	1.0	0.39	137.1352		148.6228
	2.5	0.29	139.7373		171.5302
	5.0	0.22	142.1931		209.7092
	7.5	0.18	143.7192		247.8883
1.5	0.5	0.51	202.7344	227.8608	207.6625
	1.0	0.44	204.4022		215.2983
	2.5	0.34	207.7174		238.2057
	5.0	0.26	211.0781		276.3848
	7.5	0.22	213.2896		314.5638
2	0.5	0.53	269.6271	303.8144	274.3381
	1.0	0.46	271.5066		281.9739
	2.5	0.37	275.3743		304.8813
	5.0	0.29	279.4746		343.0604
	7.5	0.25	282.2741		381.2394
5	0.5	0.61	670.3639	759.5360	674.3915
	1.0	0.55	672.9425		682.0273
	2.5	0.46	678.7664		704.9347
	5.0	0.39	685.6762		743.1138
	7.5	0.35	690.8903		781.2928

ranty servicing cost. When b_2 is fixed, $\delta_1^\star$ increases as b_1 increases. That is, as the cost of minimal repair increases, an imperfect repair of a higher degree results in the minimum expected total warranty servicing cost. This trend is clearly observed in Figures 5.7 and 5.8, where we have plotted the expected cost function $E[C(10 \mid \delta_1)]$ over $\delta_1 \in [0, 1]$, for $q = 1$ and $q = 2$ respectively, and for various values of the cost parameters b_1 and b_2. In both figure, first b_1 is kept fixed and b_2 is increased (top rows), and then b_2 is kept fixed and b_1 is increased (bottom rows).

In Figures 5.7 and 5.8, notice that in all plots, after a certain degree of repair, the initially decreasing expected total warranty servicing cost begins to increase. This is because there is a trade-off between the effect of the cost of a general repair on the expected total warranty servicing cost and the effect of the resulting improvement in system reliability on the expected total warranty servicing cost.

5.3.3 Expected Total Warranty Servicing Costs for Strategy II

To illustrate Strategy II, we again use the example cost function given in (5.8), and compute the expected total warranty servicing cost $E[C(10 \mid \delta_1, \delta_1')]$ for $(\delta_1, \delta_1') \in [0, 1]^2$. The expected total warranty servicing cost $E[C(w \mid \delta_1, \delta_1')]$ for Strategy II was derived in (5.35).

To compute the expected costs, we choose the following parameter values for the ex-

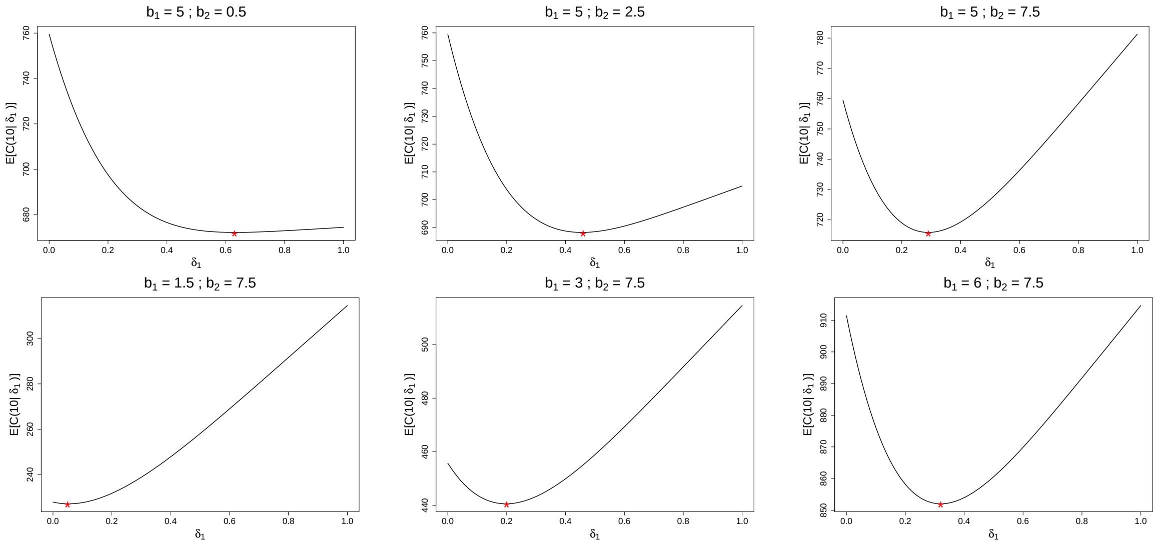


Figure 5.7: Expected cost function $E[C(10 \mid \delta_1)]$ plotted over $\delta_1 \in [0,1]$, for $q = 1$, and for various values of b_1 and b_2.

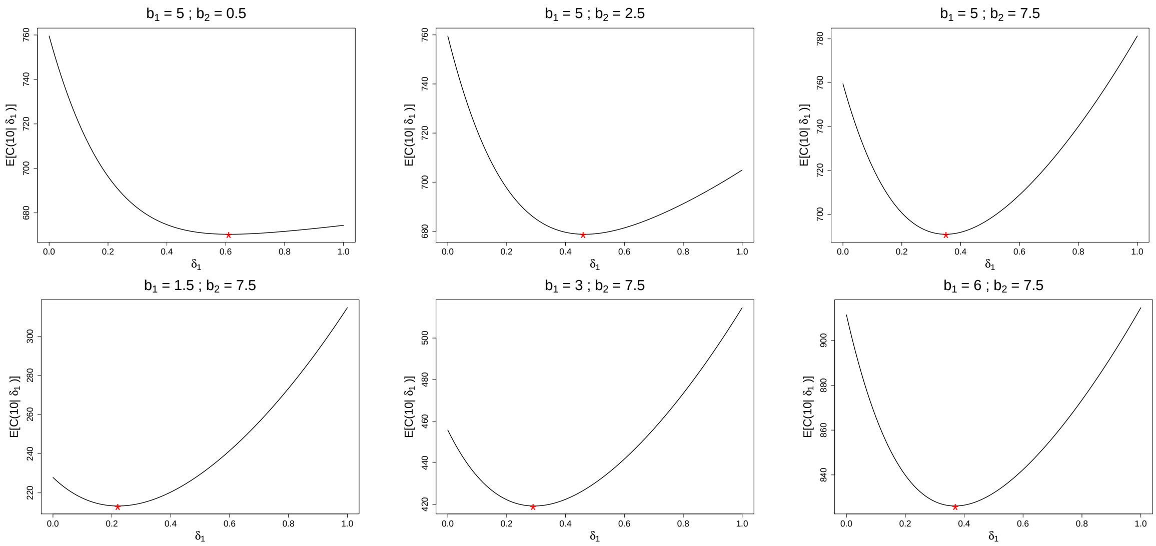


Figure 5.8: Expected cost function $E[C(10 \mid \delta_1)]$ plotted over $\delta_1 \in [0,1]$, for $q = 2$, and for various values of b_1 and b_2.

ample cost function in (5.8): we fix $m_1 = m_2 = 1$ and $b_0 = 0.01$, and compute the expected costs for various values of the parameters b_1, b_2 and q: $b_1 \in \{0.5, 1.0, 1.5, 2.0\}$; $b_2 \in \{3, 4, 5, 7, 10, 20\}$; and $q \in \{0.5, 1\}$.

Let $\delta_1^\star$ and $\delta_1'^\star$ denote the degrees of the two non-minimal repairs that minimize the expected total warranty servicing cost, and therefore, $E[C(10 \mid \delta_1^\star, \delta_1'^\star)]$ represents the minimum, over all $(\delta_1, \delta_1') \in [0,1]^2$, expected total warranty servicing cost over the warranty period $(0, 10]$. In Tables 5.3 and 5.4, the minimum expected total costs are tabulated for

$q = 0.5$ and $q = 1$ respectively, for various values of the cost parameters b_1 and b_2. Also included in each table are the corresponding expected costs when all repairs are minimal, i.e. $\delta_1 = \delta_1' = 0$, and the expected costs when the two non-minimal repairs are perfect, i.e. $\delta_1 = \delta_1' = 1$.

Table 5.3: Minimum expected total warranty servicing costs $\boldsymbol{E[C(10 \mid \delta_1^\star, \delta_1'^\star)]}$, for $q = 0.5$.

b_1	b_2	$\delta_1^\star$	$\delta_1'^\star$	$E[C(10 \mid \delta_1 = \delta_1^\star, \delta_1' = \delta_1'^\star)]$	$E[C(10 \mid \delta_1 = 0, \delta_1' = 0)]$	$E[C(10 \mid \delta_1 = 1, \delta_1' = 1)]$
0.5	3	0.200	1.000	30.5618	59.7742	34.2423
	4	0.075	1.000	32.4433		41.5965
	5	0.025	1.000	33.9725		48.9507
	7	0.000	0.975	36.8458		63.6591
	10	0.000	0.925	41.1559		85.7217
	20	0.000	0.675	53.8835		159.2638
1.0	3	0.525	1.000	53.3625	119.5484	46.4220
	4	0.375	1.000	56.3467		53.7762
	5	0.275	1.000	58.8907		61.1304
	7	0.125	1.000	63.1079		75.8388
	10	0.000	0.950	67.9449		97.9014
	20	0.000	0.875	82.3116		171.4435
1.5	3	0.950	1.000	74.2629	179.3225	58.6017
	4	0.600	1.000	78.3329		65.9559
	5	0.475	1.000	81.6310		73.3101
	7	0.300	1.000	87.1222		88.0185
	10	0.150	1.000	93.7057		110.0811
	20	0.000	0.975	109.1007		183.6232
2.0	3	1.000	1.000	93.9877	239.0967	70.7814
	4	0.950	1.000	99.0171		78.1356
	5	0.650	1.000	103.2224		85.4898
	7	0.450	1.000	109.8481		100.1982
	10	0.275	1.000	117.7813		122.2608
	20	0.000	1.000	135.8897		195.8028

As observed in the illustration of Strategy I, as the value of q increases from 0.5 (Table 5.3) to 1 (Table 5.4), the expected total warranty servicing costs decrease, since a larger q results in the cost of an imperfect repair of fixed degree moving away from the cost of a perfect repair and closer to the cost of a minimal repair. The minimum expected total warranty servicing costs in Table 5.3 are higher than the corresponding minimum expected costs in Table 5.4. Also, notice that, columns 6 and 7 in both tables are identical, since the value of the parameter q does not affect the cost of a minimal repair or a perfect repair; see Example 3 on page 90.

The other trends in the two tables are similar to those observed in the illustration of Strategy I. When b_1 is fixed, both $\delta_1^\star$ and $\delta_1'^\star$ decrease (or do not increase) as b_2 increases. When the cost of imperfect repair increases, imperfect repairs of lower degrees yield the lowest expected costs. When b_2 is fixed, both $\delta_1^\star$ and $\delta_1'^\star$ increase (or do not decrease) as b_1 increases. That is, as the cost of minimal repair increases, the degrees of the non-minimal repairs that result in the minimum expected total warranty servicing cost increases.

In Figures 5.9 and 5.10, we have plotted the expected cost $E[C(10 \mid \delta_1, \delta_1')]$ as a function

Table 5.4: Minimum expected total warranty servicing costs $E[C(10 \mid \delta_1^\star, \delta_1'^\star)]$, for $q = 1$.

b_1	b_2	$\delta_1^\star$	$\delta_1'^\star$	$E[C(10 \mid \delta_1 = \delta_1^\star, \delta_1' = \delta_1'^\star)]$	$E[C(10 \mid \delta_1 = 0, \delta_1' = 0)]$	$E[C(10 \mid \delta_1 = 1, \delta_1' = 1)]$
	3	0.000	0.825	24.8901		34.2423
	4	0.000	0.750	27.4725		41.5965
0.5	5	0.000	0.675	29.8276	59.7742	48.9507
	7	0.000	0.600	34.0648		63.6591
	10	0.000	0.500	39.5521		85.7217
	20	0.000	0.275	52.5053		159.2638
	3	0.175	1.000	39.7657		46.4220
	4	0.100	0.950	43.6183		53.7762
1.0	5	0.050	0.875	46.9128	119.5484	61.1304
	7	0.000	0.775	52.4278		75.8388
	10	0.000	0.675	59.6551		97.9014
	20	0.000	0.500	79.1042		171.4435
	3	0.300	1.000	53.2145		58.6017
	4	0.200	1.000	57.5836		65.9559
1.5	5	0.150	1.000	61.6488	179.3225	73.3101
	7	0.050	0.900	68.8012		88.0185
	10	0.000	0.800	77.3479		110.0811
	20	0.000	0.600	100.2048		183.6232
	3	0.375	1.000	66.2623		70.7814
	4	0.300	1.000	70.9526		78.1356
2.0	5	0.225	1.000	75.3590	239.0967	85.4898
	7	0.125	1.000	83.4991		100.1982
	10	0.050	0.875	93.8257		122.2608
	20	0.000	0.675	119.3102		195.8028

of δ_1 and δ_1', for $q = 0.5$ and $q = 1$ respectively. In the top row of each figure, b_1 is kept fixed and b_2 is increased along the columns, and in the bottom row, b_2 is kept fixed and b_1 is increased along the columns. The trends in Tables 5.3 and 5.4 are better seen in the figures.

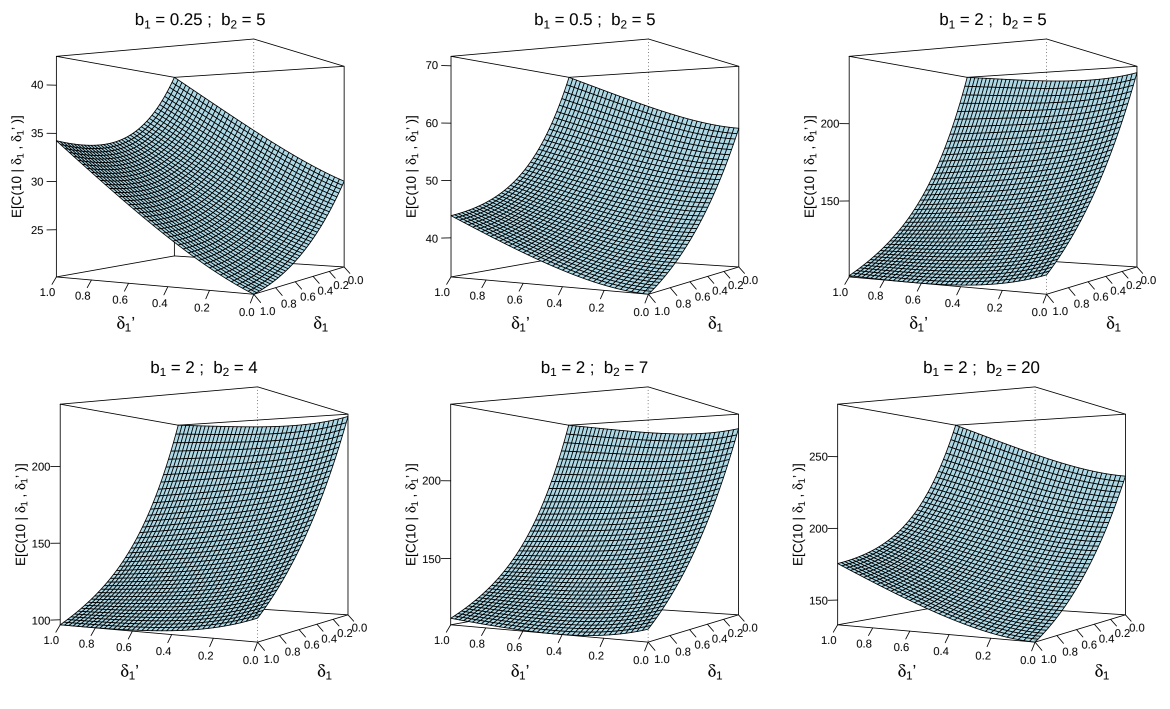


Figure 5.9: Expected cost function $E[C(10 \mid \delta_1, \delta_1')]$ plotted over $(\delta_1, \delta_1') \in [0,1]^2$, for $q = 0.5$, and for various values of b_1 and b_2.

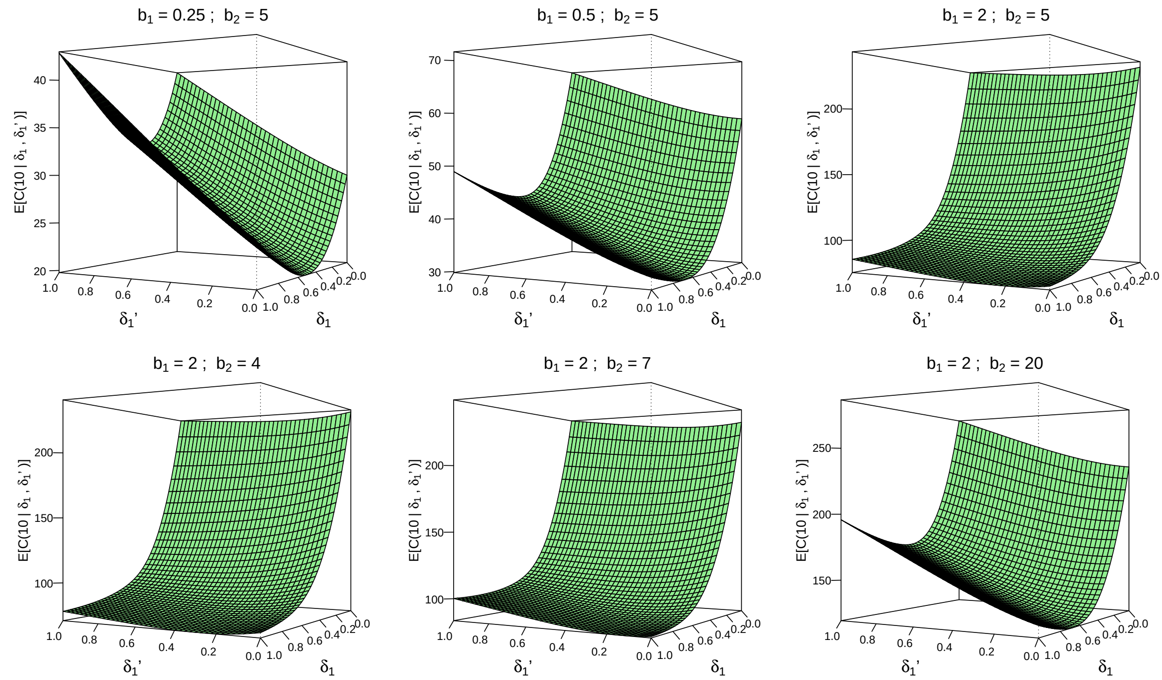


Figure 5.10: Expected cost function $E[C(10 \mid \delta_1, \delta_1')]$ plotted over $(\delta_1, \delta_1') \in [0,1]^2$, for $q = 1$, and for various values of b_1 and b_2.

Since computing the expected costs for Strategy II is computationally intensive, a coarser grid (for the degrees δ_1 and δ_1') was chosen to illustrate this strategy. The values of the expected cost function were interpolated between the grid points.

5.4 Chapter Conclusion

In this chapter, we proposed warranty servicing strategies for a system whose lifetime is modeled with a distribution having a bathtub-shaped failure rate function. We derived the expected number of failures and the expected total warranty servicing costs for each of the strategies, using the general repair model proposed in the previous chapter.

We suggested possible cost functions for modeling the cost of an individual general repair, where the cost of repair is a function of both the time of the repair and the degree of the repair.

We provided numerical illustrations of the suggested strategies. We computed the expected number of failures and the expected total warranty servicing costs for each strategy using an example cost function. The illustrations showed that as the degree of any given general repair increases (while others remain fixed), the expected number of failures decreases, since the conditional reliability of the system is further improved. As discussed in the previous chapter, the degree of repair indicates the effectiveness of the repair, and a repair of higher degree results in greater reliability improvement. We also observed that there is a tradeoff between the cost of a repair and the improvement in system reliability following the repair, in terms of their effect on the expected total warranty servicing cost.

Part III

Modeling Repairs in Two Dimensions with Applications in Warranty Analysis

Chapter 6

Failure Modeling in Two Dimensions

In this chapter, we provide a brief review of fundamental concepts used in modeling consecutive failures of a repairable system whose lifetime is modeled as a bivariate random variable, and we review existing models of general repairs in two dimensions.

This chapter is organized as follows. In Section 6.1, we provide some background for failure modeling in two dimensions. In Section 6.2, we provide a review of the literature on modeling general repairs in two dimensions. In Section 6.3, we conclude with a brief chapter summary.

6.1 Fundamental Concepts

6.1.1 Stochastic Counting Processes

A stochastic process is a family of random variables, denoted by $\{Z(\mathbf{t}); \mathbf{t} \in \mathcal{T}\}$, where $\mathcal{T}$ is the parameter index set of the process, and $\mathbf{t}$ is the vector of parameter indexes (arguments). Stochastic processes are distinguished based on three factors: (i) their state space, denoted by $\mathcal{S}$; (ii) their index parameter set $\mathcal{T}$; and (iii) the relationship between the random variables $Z(\mathbf{t})$.

Let $\mathcal{R}^d$ denote the real d-dimensional Euclidean space, where $d \geq 1$. Then, the counting process $\{N(\mathbf{t}); \mathbf{t} \in \mathcal{T}\}$ in d dimensions is a stochastic process whose state space is the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ and whose index parameter set is $\mathcal{T} \subseteq \mathcal{R}^d$. For every stochastic counting process, there exists a stochastic point process, which is equivalent to the counting process.

Consecutive failures of a system (that is either repaired or replaced upon each failure) can be modeled as a stochastic process. We refer to this process as the *failure process*, which could mean either the counting process or the corresponding point process.

In the context of warranty analysis, the failures of the system are often characterized by the age variable (time) and a usage variable (such as mileage). For clarity, in explaining the failure process, we assume that the index parameters of the counting process are time and usage. Therefore, the index parameter set is $\mathcal{T} = \mathcal{R}_+^2 = [0, \infty) \times [0, \infty)$.

When counting system failures, the relationship between the counts $N(\mathbf{t})$ depend on the type (and effect) of rectification actions performed following each failure of the system.

When at most one failure can occur in an infinitesimally small region, the failure process in two dimensions can be modeled by the point process $\{(T_n, U_n); n \in \mathbb{N}_+\}$, where T_n denotes the time of the n-th failure, U_n denotes the usage at the n-th failure, and $\mathbb{N}_+ = \{1, 2, \dots\}$. For this point process, the failure points are scattered in the two-dimensional space $\mathbb{R}_+^2$, such that

$$\begin{aligned} &0 < T_1 < T_2 < \cdots < T_n < \dots \;\; ; \\ &0 < U_1 < U_2 < \cdots < U_n < \dots \;\; . \end{aligned} \tag{6.1}$$

The corresponding counting process is denoted by the sequence $\{N(t, u); t, u \in \mathbb{R}_+\}$, and satisfies the following properties:

- $N(0, 0) = 0$;
- $N(t, u) \in \mathbb{N}$ for all $t > 0$;
- $N(t, u) = \inf\limits_{(s,v) \in \mathbb{R}_+^2 \setminus [0,t] \times [0,u]} N(s, v)$ for all $t, u \in \mathbb{R}_+$, or $N(t, u) \leq N(s, v)$ for all $t < s$ and $u < v$;

For simplicity, we have used the notation $N(t, u)$ to denote the number of failures in sets of the form $(0, t] \times (0, u]$, i.e. $N(t, u) \equiv N((0, t] \times (0, u])$, for $t, u > 0$. Note that, for $t < s$ and $u < v$, the process increment $N(s, v) - N(t, u)$ does not count the number of failures that occur in $(t, s] \times (u, v]$; see Figure 6.1. The number of failures in $(t, s] \times (u, v]$ is given by

$$N(s, v) - N(s, u) - N(t, v) + N(t, u) =: N((t, s] \times (u, v]) \;\; . \tag{6.2}$$

The point process $\{(T_n, U_n); n \in \mathbb{N}_+\}$ and the counting process $\{N(t, u); t, u \in \mathbb{R}_+\}$ contain the same information, and can be used interchangeably.

Given the ordering in (6.1), the number of failures in the set $(0, t] \times (0, u]$ can be expressed

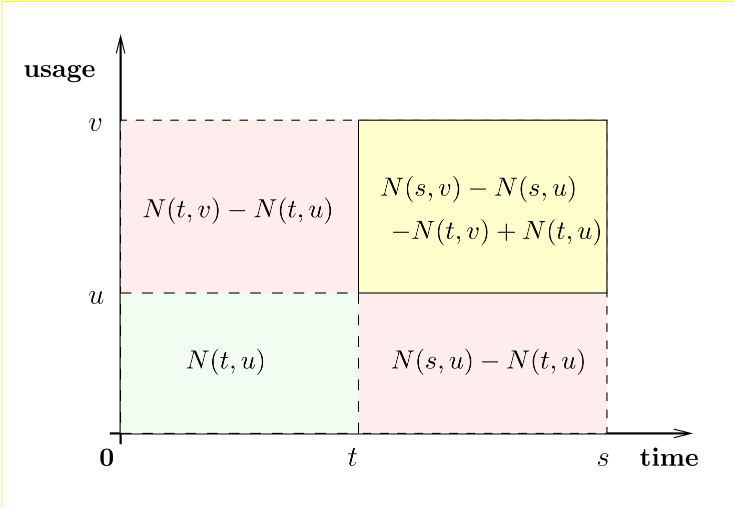


Figure 6.1: An illustration of the increments of the failure process $\{N(t,u); t,u \in \mathbb{R}_+\}$.

as

$$N(t,u) = \max\{n : T_n \le t \text{ and } U_n \le u, \text{for } n \in \mathbb{N}_+\} = \sum_{n=1}^{\infty} \mathbb{I}_{\{T_n \le t, U_n \le u\}} \ , \tag{6.3}$$

where the indicator random variable is defined as follows:

$$\mathbb{I}_{\{T_n \le t, U_n \le u\}} = \begin{cases} 1 \ , & \text{if } T_n \le t \text{ and } U_n \le u \ \ \big(\text{or } (T_n, U_n) \in (0,t] \times (0,u]\big) \\ 0 \ , & \text{otherwise} \ . \end{cases} \tag{6.4}$$

Let $\{N_X(t); t \in \mathbb{R}_+\}$ and $\{N_Y(u); u \in \mathbb{R}_+\}$ denote the two marginal counting processes with respect to time and usage respectively, where

$$N_X(t) = \sum_{n=1}^{\infty} \mathbb{I}_{\{T_n \le t\}} \qquad \text{and} \qquad N_Y(u) = \sum_{n=1}^{\infty} \mathbb{I}_{\{U_n \le u\}} \ . \tag{6.5}$$

Then, the failure process in two dimensions can also be represented in terms of the marginal processes, since

$$N(t,u) = \min\{N_X(t), N_Y(u)\} \ . \tag{6.6}$$

The failure process can also be represented by the sequence $\{(X_n, Y_n); n \in \mathbb{N}_+\}$ of the non-negative bivariate inter-failure lifetimes, where, for $n \in \mathbb{N}_+$, $X_{n+1} = T_{n+1} - T_n$ is the $(n+1)$-th inter-failure time and $Y_{n+1} = U_{n+1} - U_n$ is the $(n+1)$-th inter-failure usage. Therefore, for $n \in \mathbb{N}_+$,

$$(T_n, U_n) = \left(\sum_{j=1}^{n} X_j, \sum_{j=1}^{n} Y_j \right) \ , \tag{6.7}$$

where $(T_1, U_1) = (X_1, Y_1)$ represents the bivariate lifetime of the original system. As discussed earlier, repair times are assumed to be negligible and set equal to zero, and therefore, the system is in an operational state between failures; see Section 2.2.

6.1.1.1 Cumulative Intensity Functions

The expected number of failures in $(0,t] \times (0,u]$, $t,u \in \mathbb{R}_+$, is denoted by the function $\Lambda(.,.)$, and is referred to as the *mean function* or the *cumulative intensity function* corresponding to the stochastic counting process $\{N(t,u); t,u \in \mathbb{R}_+\}$ [16, 14]. This function is defined as

$$\Lambda(t,u) = E[N(t,u)] = \sum_{n=0}^{\infty} n \, \mathrm{P}\{N(t,u) = n\} \ . \tag{6.8}$$

The distribution of the random variables $N(t,u)$, $t,u > 0$, can be determined using the distributions of the failure points. Let $F_n(.,.)$ denote the distribution function of the n-th failure point (T_n, U_n), for $n \in \mathbb{N}_+$. Then since

$$\begin{aligned} N(t,u) \geq n \quad & \Rightarrow \quad \min\{N_X(t), N_Y(u)\} \geq n \\ & \Rightarrow \quad \max\{N_X(t), N_Y(u)\} \geq n \\ & \Rightarrow \quad N_X(t) \geq n \text{ and } N_Y(u) \geq n \ , \end{aligned} \tag{6.9}$$

we can express the probabilities $\mathrm{P}\{N(t,u) = n\}$, for $n \in \mathbb{N}_+$, in terms of the distributions of the failure points as follows:

$$\begin{aligned} \mathrm{P}\{N(t,u) = n\} =& \mathrm{P}\{N(t,u) \geq n\} - \mathrm{P}\{N(t,u) \geq n+1\} \\ =& \mathrm{P}\{N_X(t) \geq n, N_Y(u) \geq n\} - \mathrm{P}\{N_X(t) \geq n+1, N_Y(u) \geq n+1\} \\ =& \mathrm{P}\{T_n \leq t, U_n \leq u\} - \mathrm{P}\{T_{n+1} \leq t, U_{n+1} \leq u\} \\ =& F_n(t,u) - F_{n+1}(t,u) \ . \end{aligned} \tag{6.10}$$

Note that, the probability of no failure occurring in the region $(0,t] \times (0,u]$ is given by

$$\mathrm{P}\{N(t,u) = 0\} = 1 - \mathrm{P}\{N(t,u) \geq 1\} = 1 - \mathrm{P}\{T_1 \leq t, U_1 \leq u\} = 1 - F_1(t,u) \ . \tag{6.11}$$

The expected number of failures can also be expressed in terms of the distribution functions $F_n(.,.)$, $n \in \mathbb{N}_+$. Using the definition in (6.3), we have

$$\Lambda(t,u) = \sum_{n=1}^{\infty} E[\mathbb{I}_{\{T_n \leq t, U_n \leq u\}}] = \sum_{n=1}^{\infty} P\{T_n \leq t, U_n \leq u\} = \sum_{n=1}^{\infty} F_n(t,u) \ . \tag{6.12}$$

Alternatively, this can be derived using the definition in (6.8) as follows:

$$\begin{aligned}
\Lambda(t,u) &= \sum_{k=0}^{\infty} k \, \mathrm{P}\{N(t,u)=k\} = \sum_{k=1}^{\infty}\sum_{n=1}^{k} \mathrm{P}\{N(t,u)=k\} \\
&= \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \mathrm{P}\{N(t,u)=k\} = \sum_{n=1}^{\infty} \mathrm{P}\{N(t,u)\geq n\} \\
&= \sum_{n=1}^{\infty} \mathrm{P}\{T_n \leq t, U_n \leq u\} = \sum_{n=1}^{\infty} F_n(t,u) \ .
\end{aligned} \tag{6.13}$$

6.1.1.2 Intensity Functions

When the stochastic process is orderly, at most one failure can occur in an infinitesimally small region. Consider the region $(t, t+dt] \times (u, u+du]$, for $dt \to 0$ and $du \to 0$; see Figure 6.2. The counting process increment $N((t, t+dt] \times (u, u+du])$ is a binary random variable, taking values in $\{0, 1\}$, such that

$$N((t,t+dt]\times(u,u+du]) = \begin{cases} 1 \ , & \text{with approx. probability } \lambda(t,u)\, dt\, du \\ 0 \ , & \text{with approx. probability } 1-\lambda(t,u)\, dt\, du \ , \end{cases} \tag{6.14}$$

where $\lambda(t,u)$ is the *rate of occurrence of failures* (ROCOF) function of the process at time t and usage u [14]. Formally, the ROCOF function is defined as follows:

$$\lambda(t,u) = \lim_{dt,du\to 0} \frac{\mathrm{P}\{N((t,t+dt]\times(u,u+du])=1\}}{dt\ du} \ . \tag{6.15}$$

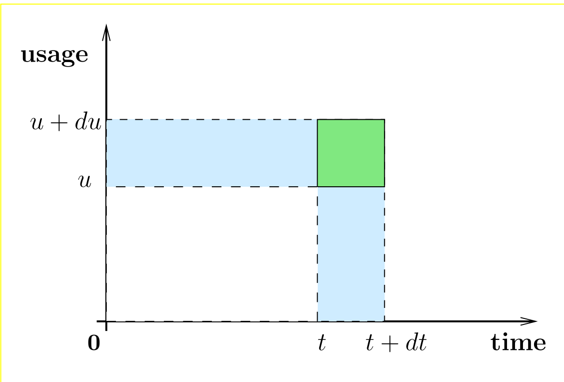


Figure 6.2: Illustration of the set $(t, t+dt] \times (u, u+du]$.

Since failures do not occur simultaneously, and the probability of having more than one failure in the small region $(t, t+dt] \times (u, u+du]$ is of order $dt\ du$ or less, the ROCOF function is equal to the *intensity function* of the process (we will use the terms interchangeably). That is,

$$\lim_{dt,du\to 0} \frac{\mathrm{P}\{N((t,t+dt]\times(u,u+du])=1\}}{dt\ du} = \lim_{dt,du\to 0} \frac{\mathrm{P}\{N((t,t+dt]\times(u,u+du])\geq 1\}}{dt\ du} \ . \tag{6.16}$$

The intensity function of an orderly stochastic process is the derivative of the cumulative intensity function of the process. That is,

$$\begin{aligned}
\lambda(t,u) &= \lim_{dt,du\to 0} \frac{\mathrm{P}\{N((t,t+dt]\times(u,u+du])\geq 1\}}{dt\ du} \\
&= \lim_{dt,du\to 0} \frac{\mathrm{P}\{N((t,t+dt]\times(u,u+du])=1\}}{dt\ du} = \lim_{dt,du\to 0} \frac{E[N((t,t+dt]\times(u,u+du])]}{dt\ du} \\
&= \lim_{dt,du\to 0} \frac{E[N(t+dt,u+du)]-E[N(t+dt,u)]-E[N(t,u+du)]+E[N(t,u)]}{dt\ du} \\
&= \lim_{dt\to 0} \frac{1}{dt}\left[\lim_{du\to 0} \frac{E[N(t+dt,u+du)]-E[N(t+dt,u)]}{du} - \lim_{du\to 0} \frac{E[N(t,u+du)]-E[N(t,u)]}{du}\right] \\
&= \lim_{dt\to 0} \frac{1}{dt}\left[\frac{\partial}{\partial u}E[N(t+dt,u)] - \frac{\partial}{\partial u}E[N(t,u)]\right] \\
&= \frac{\partial^2}{\partial t\ \partial u}E[N(t,u)] = \frac{\partial^2}{\partial t\ \partial u}\Lambda(t,u) \ .
\end{aligned} \tag{6.17}$$

Therefore, the expected number of failures in $(t,t+dt]\times(u,u+du]$, expressed in terms of the intensity function, is

$$\Lambda(t,u) = \int_0^t \int_0^u \lambda(s,v)\ dv\ ds \ . \tag{6.18}$$

Let $\mathcal{H}_{t,u}$ denote the history of the process at the point (t,u). The history $\mathcal{H}_{t,u}$ is stochastic and includes the trajectory of the failure process up to the point (t,u). Conditional on this history, the *conditional intensity function* of an orderly process in two dimensions is defined as follow:

$$\tilde{\lambda}(t,u|\mathcal{H}_{t,u}) = \lim_{dt,du\to 0} \frac{\mathrm{P}\{N((t,t+dt]\times(u,u+du])=1 \mid \mathcal{H}_{t,u}\}}{dt\ du} \ , \tag{6.19}$$

where the product $\tilde{\lambda}(t,u|\mathcal{H}_{t,u})\ dt\ du$ is the approximate probability of a failure occurring in the region $(t,t+dt]\times(u,u+du]$, conditional on the history of the process at time t and usage u, for $t,u>0$ [16]. The conditional intensity function at any point is stochastic, since the history of the process is stochastic.

Note that, the intensity function can be viewed as the expected value of the conditional intensity function with respect to the history of the process.

6.1.2 Failure/Hazard Rate Functions

The *(instantaneous) failure rate* or *hazard rate* function, often denoted by r or h, characterizes the first failure of the system, i.e. the original lifetime of the system $(T_1, U_1) = (X_1, Y_1)$. The concept of failure rate functions for bivariate (or multivariate) distributions is more complex, and unlike the univariate case, the approach to failure rate functions is not unique [cf. Section 3.1.1.3].

In this section, we review two of the most commonly-used failure (or hazard) rate functions suggested for bivariate lifetimes: (i) the bivariate failure rate function by Basu [45]; and (ii) the hazard gradient vector by Johnson & Kotz [46]. Other multivariate generalizations of the univariate failure rate function have been proposed; see for example Shaked & Shanthikumar [47] and Cox [48].

6.1.2.1 Bivariate Failure Rate Function

The bivariate conditional intensity function of the failure process before the first failure of the system is given by the *instantaneous bivariate failure rate* function $r(.,.)$. Let $(T, U) \equiv (T_1, U_1)$, denote the original lifetime (or the point of first failure) of the system. Then, the bivariate failure rate function is defined as follows:

$$\begin{aligned} r(t,u) &= \lim_{dt,du \to 0} \frac{\mathrm{P}\{T \leq t+dt, U \leq u+du \mid T > t, U > u\}}{dt\ du} \\ &= \lim_{dt,du \to 0} \frac{\mathrm{P}\{t < T \leq t+dt, u < U \leq u+du\}}{dt\ du\ \mathrm{P}\{T > t, U > u\}} \quad , \end{aligned} \tag{6.20}$$

for $t, u > 0$. Therefore, before the first failure,

$$\tilde{\lambda}(t, u|\mathcal{H}_{t,u}) = r(t,u) \ . \tag{6.21}$$

The quantity $r(t,u)\ dt\ \ du + o(dt\ \ du)$ can be viewed as the conditional probability of the system failing for the first time in the rectangle $(t, t+dt] \times (u, u+du]$, given that the system is in an operational state at time t and usage u [49, 50].

This bivariate generalization of the univariate failure rate function was introduced by Basu [45].

Let $f(.,.)$, $F(.,.)$, and $\bar{F}(.,.)$ denote respectively the density, distribution and reliability functions of the point of first failure (T, U). When the density function exists, then for $t, u >$

0, it is defined as follow:

$$f(t,u) = \frac{\partial^2}{\partial t\, \partial u} F(t,u) = \frac{\partial^2}{\partial t\, \partial u} \bar{F}(t,u) \ . \tag{6.22}$$

Then, the above bivariate failure rate function becomes

$$\begin{aligned} r(t,u) &= \lim_{dt,du \to 0} \frac{\mathrm{P}\{t < T \le t+dt, u < U \le u+du\}}{dt\ du\ \mathrm{P}\{T > t, U > u\}} \ , \\ &= \lim_{dt,du \to 0} \frac{F(t+dt,u+du) - F(t,u+du) - F(t+dt,u) + F(t,u)}{dt\ du\ \bar{F}(t,u)} \ , \\ &= \frac{1}{\bar{F}(t,u)} \frac{\partial^2}{\partial t\, \partial u}\ F(t,u) \\ &= \frac{f(t,u)}{\bar{F}(t,u)} \ , \end{aligned} \tag{6.23}$$

which resembles the univariate failure rate function[I]. The bivariate distribution and reliability functions, in terms of the density function, are:

$$F(t,u) = \int_0^t \int_0^u f(s,u)\ du\ ds \ ; \tag{6.24}$$

$$\bar{F}(t,u) = \int_t^\infty \int_u^\infty f(s,u)\ du\ ds \ . \tag{6.25}$$

Note that, $r(t,u)\ dt\ du$ is not the probability of having a system failure in $(t, t+dt] \times (u, u+du]$, given that no failure has occurred in $(0,t] \times (0,u]$ (i.e. $N(t,u) = 0$). The two events $\{N(t,u) = 0\}$ and $\{T > t, U > u\}$ are not equivalent, since

$$\begin{aligned} \mathrm{P}\{N(t,u) = 0\} &= 1 - \mathrm{P}\{N(t,u) \ge 1\} \\ &= 1 - \mathrm{P}\{T \le t, U \le u\} = 1 - F(t,u) \\ &\ne \mathrm{P}\{T > t, U > u\} = \bar{F}(t,u) \ . \end{aligned} \tag{6.26}$$

This makes bivariate failure modeling less straight-forward than the univariate case, for which $F = 1 - \bar{F}$. For bivariate probability distributions, $\bar{F}(t,u) \ne 1 - F(t,u)$ (see Figure 6.3), instead

$$\bar{F}(t,u) = 1 - F(t,\infty) - F(\infty,u) + F(t,u) \ ; \tag{6.27}$$

or equivalently

$$F(t,u) = 1 - \bar{F}(t,0) - \bar{F}(0,u) + \bar{F}(t,u). \tag{6.28}$$

[I] The univariate failure rate function is defined as $r(t) = f(t)/\bar{F}(t)$, where f and $\bar{F}$ are the density and reliability functions of the time to first failure; see Section 3.1.1.3.

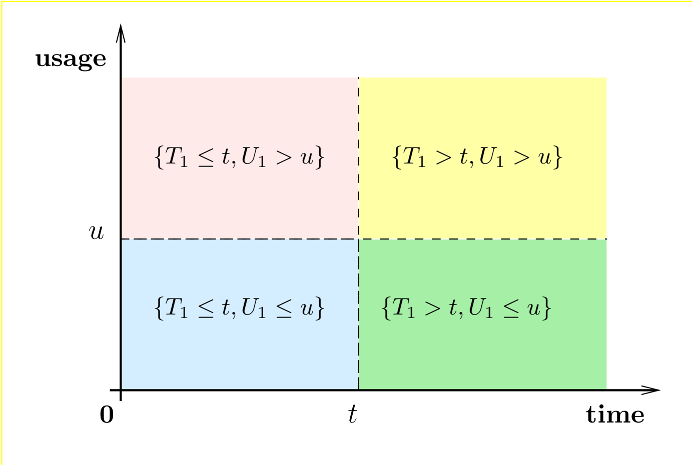


Figure 6.3: Illustration of the regions of the time-usage space with respect to the point (t, u).

Note that, the marginal distribution and reliability functions of T and U, appearing in (6.27) and (6.28), are

$$F_T(t) := F(t, \infty) = \mathrm{P}\{T \le t\} \;; \qquad \bar{F}_T(t) := \bar{F}(t, 0) = \mathrm{P}\{T > t\} \;; \tag{6.29}$$

$$F_U(u) := F(\infty, u) = \mathrm{P}\{U \le u\} \;; \qquad \bar{F}_U(u) := \bar{F}(0, u) = \mathrm{P}\{U > u\} \;. \tag{6.30}$$

In the univariate case, the distribution of the time to first failure can be defined in terms of the failure rate function. In the bivariate case however, the solution $F(.,.)$ to equation (6.20), given the bivariate failure rate function $r(.,.)$, is not known. In the univariate case the failure rate function uniquely determines the distribution. In the bivariate case this is in general not the case; see Navarro [51] who discusses the conditions under which the bivariate failure rate function $r(.,.)$ can uniquely determine the distribution $F(.,.)$.

6.1.2.2 The Hazard Gradient

One multivariate generalization of the univariate failure rate, which appears often in the literature, is the vector-valued hazard rate proposed by Johnson and Kotz [46].

In the univariate case, the failure rate (or hazard rate) function is defined as $r(t) = -\frac{d}{dt} \ln \bar{F}(t)$, over the set $\{t : \bar{F}(t) > 0\}$, where $\bar{F} = 1 - F$ is the univariate reliability function of the time to first failure [52]. The univariate cumulative failure rate function, denoted by $R(.)$, is then given by

$$R(t) = \int_0^t r(s)\, ds = -\ln \bar{F}(t) \;; \tag{6.31}$$

see Section 3.1.1.3. Note that, in the univariate case, the distribution function $F(.)$ is uniquely determined by the failure rate function $r(.)$.

Given the bivariate reliability function $\bar{F}(.,.)$, one bivariate analog of the univariate cumulative failure rate function is defined as follows:

$$H(t,u) = -\ln \bar{F}(t,u) \ , \tag{6.32}$$

over the set $\{(t,u) : \bar{F}(t,u) > 0\}$, and is referred to as the *bivariate cumulative hazard function*. It is assumed that $\bar{F}(0,0) = 1$ (i.e. the random variables T and U are positive with probability one), and hence, $H(0,0) = 0$. Since the reliability function $\bar{F}(t,u)$ is non-increasing in its arguments, the cumulative hazard function $H(t,u)$ is non-decreasing in $t,u \geq 0$ (i.e. its partial derivatives with respect to t and u are both non-negative, for all $t,u \geq 0$) [46, 52].

The *hazard gradient*, introduced by Johnson and Kotz [46], is the gradient vector of the bivariate cumulative hazard function $H(.,.)$ and is denoted by $\mathbf{h}(.,.)$, i.e. $\mathbf{h} = \nabla H$. Defined over $\{(t,u) : \bar{F}(t,u) > 0\}$, the hazard gradient is given by

$$\mathbf{h}(t,u) = \left(\frac{\partial}{\partial t} H(t,u), \frac{\partial}{\partial u} H(t,u) \right) \ . \tag{6.33}$$

The hazard gradient is a vector of the conditional failure rate functions $h_T(.,.)$ and $h_U(.,.)$, where

$$\begin{aligned} h_T(t,u) :=& \lim_{dt \to 0} \frac{\mathrm{P}\{T \leq t + dt \mid T > t, U > u\}}{dt} = \lim_{dt \to 0} \frac{\bar{F}(t,u) - \bar{F}(t+dt,u)}{dt \ \bar{F}(t,u)} \\ =& \frac{1}{\bar{F}(t,u)} \lim_{dt \to 0} \frac{\bar{F}(t,u) - \bar{F}(t+dt,u)}{dt} = -\frac{1}{\bar{F}(t,u)} \frac{\partial}{\partial t} \bar{F}(t,u) \\ =& -\frac{\partial}{\partial t} \ln \bar{F}(t,u) = \frac{\partial}{\partial t} H(t,u) \ ; \end{aligned} \tag{6.34}$$

and similarly,

$$h_U(t,u) := \lim_{du \to 0} \frac{\mathrm{P}\{U \leq u + du \mid T > t, U > u\}}{du} = \frac{\partial}{\partial u} H(t,u) \ . \tag{6.35}$$

Therefore, $h_T(t,u) \ dt$ can be interpreted as the approximate probability of the system failing for the first time at time t, given that the system usage is greater than u units; see Figure 6.4 (left). Similarly, $h_U(t,u) \ du$ can be interpreted as the approximate probability that the system fails for the first time at usage u, given that it has not failure prior to time t; see Figure 6.4 (right).

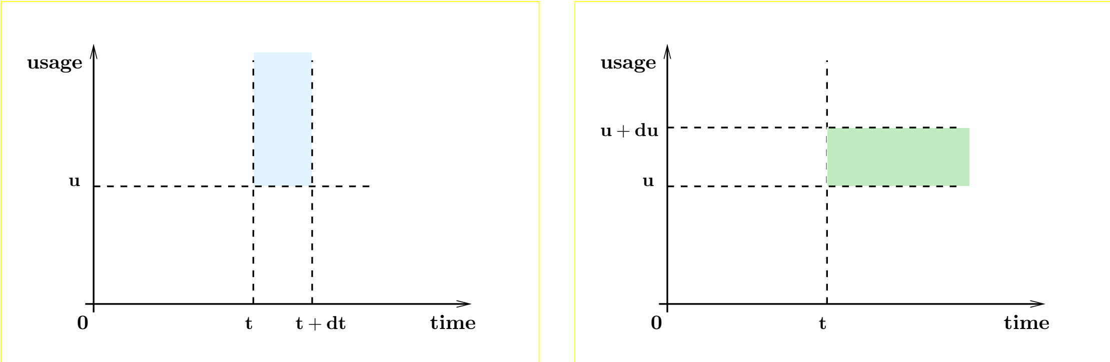


Figure 6.4: Illustration of the sets associated with the components of the hazard gradient vector, given in (6.34) and (6.35).

6.1.3 Marginal, Conditional and Bivariate Distributions

The hazard gradient uniquely determines the bivariate distribution– the bivariate lifetime distribution can be constructed using the components of the hazard gradient. In the following sections, we will see that the marginal and conditional distributions (used to define the bivariate distribution) can be derived from the hazard gradient vector.

6.1.3.1 Marginal Distributions

The marginal failure rate functions of the variables T and U can be derived from the components of the hazard gradient vector [52, 53].

Let $\bar{F}_T(.)$ and $\bar{F}_U(.)$ denote the marginal reliability functions of T and U respectively; see (6.29) and (6.30). Then, the corresponding marginal failure rate functions, denoted by $r_T(.)$ and $r_U(.)$ respectively, are given by

$$\begin{aligned} r_T(t) &= \frac{-\frac{\partial}{\partial t}\bar{F}_T(t)}{\bar{F}_T(t)} = -\frac{\partial}{\partial t}\ln \bar{F}_T(t) \\ &= -\frac{\partial}{\partial t}\ln \bar{F}(t,0) = \frac{\partial}{\partial t}H(t,0) = h_T(t,0) \ ; \end{aligned} \tag{6.36}$$

and similarly,

$$r_U(u) = \frac{-\frac{\partial}{\partial u}\bar{F}_U(u)}{\bar{F}_U(u)} = h_U(0,u) \ . \tag{6.37}$$

When the corresponding densities, denoted by $f_T(.)$ and $f_U(.)$ exist, then the marginal failure rate functions can be expressed in terms of the marginal densities as follows:

$$r_T(t) = f_T(t)/\bar{F}_T(t) \quad \text{and} \quad r_U(u) = f_U(u)/\bar{F}_U(u) \ . \tag{6.38}$$

6.1.3.2 Conditional Distributions

The two components of the hazard gradient vector represent the failure rates of the conditional random variables $[T|U > u]$ and $[U|T > t]$.

Let $f_T(.|U > u)$, $\bar{F}_T(.|U > u)$ and $r_T(.|U > u)$ denote the density, reliability and failure rate functions of T conditional on $U > u$. Similarly, let $f_U(.|T > t)$, $\bar{F}_U(.|T > t)$ and $r_U(.|T > t)$ denote the corresponding functions for U conditional on $T > t$.

When the corresponding densities exist, then the conditional failure rate functions can be expressed in terms of these functions as follows:

$$\begin{aligned} r_T(t|U > u) &= \frac{f_T(t|U > u)}{\bar{F}_T(t|U > u)} = \frac{\frac{\partial}{\partial t} F_T(t|U > u)}{\bar{F}_T(t|U > u)} \\ &= \frac{1}{\bar{F}_T(t|U > u)} \lim_{dt \to 0} \frac{\mathrm{P}\{t \le T \le t + dt|U > u\}}{dt} \\ &= \lim_{dt \to 0} \frac{\frac{\bar{F}(t,u)}{\bar{F}_U(u)} - \frac{\bar{F}(t+dt,u)}{\bar{F}_U(u)}}{dt\, \frac{\bar{F}(t,u)}{\bar{F}_U(u)}} = \lim_{dt \to 0} \frac{\bar{F}(t,u) - \bar{F}(t+dt,u)}{dt\, \bar{F}(t,u)} \\ &= -\frac{\partial}{\partial t} \ln \bar{F}(t,u) = h_T(t,u) \ ; \end{aligned} \tag{6.39}$$

and similarly,

$$r_U(u|T > t) = \frac{f_U(u|T > t)}{\bar{F}_U(u|T > t)} = h_U(t,u) \ . \tag{6.40}$$

Then, the corresponding conditional reliability functions can be defined as follows:

$$\mathrm{P}\{T > t|U > u\} = \bar{F}_T(t|U > u) = e^{-\int\limits_0^t r_T(s|U>u)\, ds} = e^{-\int\limits_0^t h_T(s,u)\, ds} \ , \tag{6.41}$$

and

$$\mathrm{P}\{U > u|T > t\} = \bar{F}_U(u|T > t) = e^{-\int\limits_0^u r_U(v|T>t)\, dv} = e^{-\int\limits_0^u h_U(t,v)\, dv} \ . \tag{6.42}$$

6.1.3.3 Bivariate Distributions

The reliability function $\bar{F}(.,.)$ of the bivariate random variable (T, U) can be constructed using the relationship between the marginal failure rate functions and the components of the hazard gradient vector [52]. From (6.39) and (6.40), we have

$$r_T(t|U > u)\, dt = h_T(t,u)\, dt \approx \mathrm{P}\{t \le T \le t + dt|T > t, U > u\} \ , \tag{6.43}$$

and

$$r_U(u|T > t)\ du = h_U(t, u)\ du \approx \mathrm{P}\{u \le U \le u + du|T > t, U > u\}\ . \tag{6.44}$$

From the relationships between the components of the hazard gradient and the conditional and marginal distributions, it follows that

$$\begin{aligned}\bar{F}(t, u) = \bar{F}_T(t|U > u)\ \bar{F}_U(u) &= e^{-\int\limits_0^t r_T(s|U>u)\ ds}\ e^{-\int\limits_0^u r_U(v)\ dv} \\ &= e^{-\left(\int\limits_0^t h_T(s,u)\ ds + \int\limits_0^u h_U(0,v)\ dv\right)}\ ;\end{aligned} \tag{6.45}$$

or equivalently,

$$\begin{aligned}\bar{F}(t, u) = \bar{F}_U(u|T > t)\ \bar{F}_T(t) &= e^{-\int\limits_0^u r_U(v|T>t)\ dv}\ e^{-\int\limits_0^t r_T(s)\ ds} \\ &= e^{-\left(\int\limits_0^u h_U(t,v)\ dv + \int\limits_0^t h_T(s,0)\ ds\right)}\ .\end{aligned} \tag{6.46}$$

Therefore, the hazard gradient vector completely determines the bivariate distribution [46]. Note that, from the definitions of the hazard gradient components, it follows that:

$$\frac{\partial}{\partial u}\ h_T(t, u) = \frac{\partial}{\partial t}\ h_U(t, u) = -\frac{\partial^2}{\partial t\ \partial u}\ \ln \bar{F}(t, u)\ . \tag{6.47}$$

Since $H(t, u) = -\ln \bar{F}(t, u)$, and therefore, $\bar{F}(t, u) = \exp\big\{-H(t, u)\big\}$, the bivariate cumulative hazard function $H(.,.)$ can be expressed in terms of the components of the hazard gradient as follows:

$$H(t, u) = \int\limits_0^t h_T(s, u)\ ds + \int\limits_0^u h_U(0, v)\ dv = \int\limits_0^u h_U(t, v)\ dv + \int\limits_0^t h_T(s, 0)\ ds\ . \tag{6.48}$$

Note that, (6.47) is the second order mixed partial derivative of the bivariate cumulative hazard function $H(.,.)$; refer to Marshall [52] for more on the hazard gradient.

6.1.4 Stochastic Aging Classification

Many bivariate aging concepts analogous to the univariate aging classes have appeared in the reliability literature, and in this section, we will discuss concepts relevant to our study.

As in the univariate case, bivariate aging classes can be based on failure rate, conditional reliability or mean residual life. Although most classes have been defined for multi-component systems, we will adapt the definitions to suit reliability classes for a single component system whose lifetime is defined in terms of its age and usage.

6.1.4.1 Classes based on Conditional Reliability

A bivariate analogue of the univariate conditional reliability function $\bar{F}(t+x)/\bar{F}(t)$ is given by

$$\frac{\bar{F}(t+s,u+v)}{\bar{F}(t,u)} = \mathrm{P}\{T > t+s, U > u+v | T > t, U > u\} \ , \tag{6.49}$$

for $t, u \geq 0$ and $s, v > 0$. This function is interpreted as the probability of a system surviving s units of time and v units of usage, given that the system has survived up to time t and usage u. Note that, when $t = u = 0$, then the above function reduces to the reliability function of the system at time s and usage v, i.e. (6.49) becomes $\bar{F}(s,v)$, for $s, v \geq 0$. We will refer to this function as the BCR (bivariate conditional reliability) function.

In the univariate case, for an IFR (DFR) distribution, the corresponding conditional reliability function $\bar{F}(t+x)/\bar{F}(t)$ is decreasing (increasing) in t, for all $t \geq 0$ and each $x \geq 0$. An analogous definition of an IFR (DFR) distribution of a bivariate random variable, introduced by Harris [54], is as follows.

Definition 6.1. *A bivariate lifetime distribution is bivariate increasing (decreasing) failure rate (BIFR (BDFR)), iff the BCR function in (6.49) is decreasing (increasing) in t and u, for all $t, u \geq 0$ and each $s, v \geq 0$.*

Let $\varphi_{\bar{F}}$ denote the vector of the two conditional reliability functions derived from the BCR function in (6.49) by setting $v = 0$ and $s = 0$ respectively. That is, for $t, u \geq 0$ and $s, v \geq 0$, let

$$\varphi_{\bar{F}}(s,v;t,u) := \left(\frac{\bar{F}(t+s,u)}{\bar{F}(t,u)}, \frac{\bar{F}(t,u+v)}{\bar{F}(t,u)} \right) \ , \tag{6.50}$$

where

$$\frac{\bar{F}(t+s,u)}{\bar{F}(t,u)} = \mathrm{P}\{T > t+s | T > t, U > u\} \ ; \tag{6.51}$$

and

$$\frac{\bar{F}(t,u+v)}{\bar{F}(t,u)} = \mathrm{P}\{U > u+v | T > t, U > u\} \ . \tag{6.52}$$

If a distribution F is BIFR (BDFR), then it follows that these two conditional reliability functions are also decreasing (increasing) functions of t and u, for all $t, u \geq 0$ and each $s, v \geq 0$.

This, in turn, implies that the corresponding conditional probabilities of failure,

$$\mathrm{P}\{T \leq t+dt | T > t, U > u\} = 1 - \frac{\bar{F}(t+dt,u)}{\bar{F}(t,u)} \approx h_T(t,u)\; dt \quad , \tag{6.53}$$

and

$$\mathrm{P}\{U \leq u+du | T > t, U > u\} = 1 - \frac{\bar{F}(t,u+du)}{\bar{F}(t,u)} \approx h_U(t,u)\; du \quad , \tag{6.54}$$

where s and v have been replaced by dt and du, are increasing (decreasing) in t and u, for all $t, u \geq 0$ and each $dt, du \geq 0$. Therefore, for a BIFR (BDFR) distribution, the components $h_T(t,u)$ and $h_U(t,u)$ of the hazard gradient vector are both increasing (decreasing) in t and u, for all $t, u \geq 0$.

The above definition of BIFR (BDFR) class of distributions implies that the marginal distributions of T and U are both IFR (DFR), since when $u = v = 0$,

$$\frac{\bar{F}(t+s,u+v)}{\bar{F}(t,u)} = \frac{\bar{F}_T(t+s)}{\bar{F}_T(t)} \quad ; \tag{6.55}$$

and when $t = s = 0$,

$$\frac{\bar{F}(t+s,u+v)}{\bar{F}(t,u)} = \frac{\bar{F}_U(u+v)}{\bar{F}_U(u)} \quad . \tag{6.56}$$

The BIFR (BDFR) class in Definition 6.1, in the context of system reliability, can be used to model the reliability of systems that deteriorate (improve) with time and use. If a system deteriorates (improves) with time and use, then the system when it is new is in a better (worse) condition than a used system. This can be modeled using the following class of bivariate distributions (the condition is weaker than that of Definition 6.1).

Definition 6.2. *The class of bivariate lifetime distribution is bivariate new-better-than-used (BNBU) (bivariate new-worse-than-used (BNWU)), iff*

$$\frac{\bar{F}(t+s,u+v)}{\bar{F}(t,u)} \leq (\geq) \frac{\bar{F}(0+s,0+v)}{\bar{F}(0,0)} = \bar{F}(s,v) \quad , \tag{6.57}$$

for all $t, u \geq 0$ and each $s, v \geq 0$.

Note that, Definitions 6.1 and 6.2 are both based on the BCR function in (6.49). Definitions of bivariate IFR (DFR) distributions based on variants of (6.49) and also the conditional reliability vector in (6.50) have been suggested; see for instance Lai & Xie [18]. We will however use Definition 6.1, since it is suitable for this study. Henceforth, we will use BIFR/BDFR to refer exclusively to distributions in the class in Definition 6.1.

Basu's failure rate function and BIFR/BDFR classes. As mentioned above, the distribution being BIFR (BDFR) implies that the corresponding hazard gradient components are increasing (decreasing)– it does not however imply that the bivariate failure rate function $r(.,.)$ (as defined by Basu [45]) is increasing (decreasing) in $t, u \geq 0$. In terms of the BCR function in (6.49),

$$\begin{aligned} r(t,u)\ dt\ du &\approx \mathrm{P}\{T \leq t+dt, U \leq u+du | T > t, U > u\} \\ &= \frac{F(t+dt,u+du) - F(t+dt,u) - F(t,u+du) + F(t,u)}{\bar{F}(t,u)} \\ &= \frac{\bar{F}(t,u) - \bar{F}(t+dt,u) - \bar{F}(t,u+du) + \bar{F}(t+dt,u+du)}{\bar{F}(t,u)} \\ &= 1 - \frac{\bar{F}(t+dt,u)}{\bar{F}(t,u)} - \frac{\bar{F}(t,u+du)}{\bar{F}(t,u)} + \frac{\bar{F}(t+dt,u+du)}{\bar{F}(t,u)} \quad . \end{aligned} \tag{6.58}$$

When the BCR function is decreasing (i.e. F is BIFR), then

$$\frac{\bar{F}(t+dt,u+du)}{\bar{F}(t,u)} \leq \frac{\bar{F}(t+dt,u)}{\bar{F}(t,u)} \tag{6.59}$$

and

$$\frac{\bar{F}(t+dt,u+du)}{\bar{F}(t,u)} \leq \frac{\bar{F}(t,u+du)}{\bar{F}(t,u)} \quad , \tag{6.60}$$

for all $t, u \geq 0$ and each $dt, du \geq 0$. In other words, for any $t, u \geq 0$ and each $dt, du > 0$, the BCR function at any point is bounded from above by the minimum of the components of the conditional reliability vector $\boldsymbol{\varphi}_{\bar{F}}(dt, du; t, u)$:

$$\frac{\bar{F}(t+dt,u+du)}{\bar{F}(t,u)} \leq \min\left(\frac{\bar{F}(t+dt,u)}{\bar{F}(t,u)}, \frac{\bar{F}(t,u+du)}{\bar{F}(t,u)}\right) \quad . \tag{6.61}$$

Rearranging the terms in (6.58), and using (6.61), we get

$$\begin{aligned} r(t,u)\ dt\ du &\approx 1 - \frac{\bar{F}(t+dt,u)}{\bar{F}(t,u)} - \left[\frac{\bar{F}(t,u+du)}{\bar{F}(t,u)} - \frac{\bar{F}(t+dt,u+du)}{\bar{F}(t,u)}\right] \\ &\leq 1 - \frac{\bar{F}(t+dt,u)}{\bar{F}(t,u)} \approx h_T(t,u)\ dt \ ; \end{aligned} \tag{6.62}$$

and similarly,

$$r(t,u)\ dt\ du \leq 1 - \frac{\bar{F}(t,u+du)}{\bar{F}(t,u)} \approx h_U(t,u)\ du \ . \tag{6.63}$$

Therefore, we can conclude that, when F is BIFR, then for all $t, u \geq 0$,

$$r(t,u)\ dt\ du \leq \min\left(h_T(t,u)\ dt,\ h_U(t,u)\ du\right) \ . \tag{6.64}$$

When the BCR function is increasing (i.e. F is BDFR), then

$$\frac{\bar{F}(t+dt,u+du)}{\bar{F}(t,u)} \geq \frac{\bar{F}(t+dt,u)}{\bar{F}(t,u)} \tag{6.65}$$

and

$$\frac{\bar{F}(t+dt,u+du)}{\bar{F}(t,u)} \geq \frac{\bar{F}(t,u+du)}{\bar{F}(t,u)} \ , \tag{6.66}$$

for all $t,u \geq 0$ and each $dt,du \geq 0$. In other words, for any $t,u \geq 0$ and each $dt,du > 0$, the BCR function at any point is bounded from below by the maximum of the components of the conditional reliability vector $\varphi_{\bar{F}}(dt,du;t,u)$:

$$\frac{\bar{F}(t+dt,u+du)}{\bar{F}(t,u)} \geq \max\left(\frac{\bar{F}(t+dt,u)}{\bar{F}(t,u)}, \frac{\bar{F}(t,u+du)}{\bar{F}(t,u)}\right) \ . \tag{6.67}$$

Rearranging the terms in (6.58), and using (6.67), we get

$$\begin{aligned} r(t,u)\ dt\ du &\approx 1 - \frac{\bar{F}(t+dt,u)}{\bar{F}(t,u)} + \left[\frac{\bar{F}(t+dt,u+du)}{\bar{F}(t,u)} - \frac{\bar{F}(t,u+du)}{\bar{F}(t,u)}\right] \\ &\geq 1 - \frac{\bar{F}(t+dt,u)}{\bar{F}(t,u)} \approx h_T(t,u)\ dt \ ; \end{aligned} \tag{6.68}$$

and similarly,

$$r(t,u)\ dt\ du \geq 1 - \frac{\bar{F}(t,u+du)}{\bar{F}(t,u)} \approx h_U(t,u)\ du \ . \tag{6.69}$$

Therefore, we can conclude that, when F is BDFR, then for all $t,u \geq 0$,

$$r(t,u)\ dt\ du \geq \max\left(h_T(t,u)\ dt,\ h_U(t,u)\ du\right) \ . \tag{6.70}$$

6.1.4.2 Classes based on Failure Rate

The bivariate reliability function $\bar{F}(.,.)$ cannot in general be determined by the bivariate failure rate function $r(.,.)$ by Basu [45]. The hazard gradient vector, however, uniquely determines the distribution, and is often used when defining classes of distributions in terms of failure/hazard rate. In terms of the hazard gradient vector, we have the following class of distributions.

Definition 6.3. *The distribution F of the bivariate random variable (T,U) is bivariate increasing (decreasing) failure rate (BIFR2 (BDFR2)), iff the hazard gradient component $h_T(t,u)$ is, for each u, increasing in t, and the hazard gradient component $h_U(t,u)$ is, for each t, increasing in u, for all $t,u \geq 0$.*

That is, F is BIFR2 (BDFR2), iff for each $u \geq 0$,

$$\frac{\partial}{\partial t}\, h_T(t,u) = \frac{\partial^2}{\partial t^2}\, H(t,u) \quad \geq (\leq) \;\; 0 \;, \qquad\qquad \forall t \geq 0 \;; \tag{6.71}$$

and, for each $t \geq 0$,

$$\frac{\partial}{\partial u}\, h_U(t,u) = \frac{\partial^2}{\partial u^2}\, H(t,u) \quad \geq (\leq) \;\; 0 \;, \qquad\qquad \forall u \geq 0 \;. \tag{6.72}$$

As in the univariate case, this definition is equivalent to the components of the conditional reliability vector in (6.50) having the opposite monotonicity of the corresponding components of the hazard gradient vector, since

$$\left(1 - \frac{\bar{F}(t+dt,u)}{\bar{F}(t,u)},\, 1 - \frac{\bar{F}(t,u+du)}{\bar{F}(t,u)}\right) \approx \big(h_T(t,u)\; dt,\; h_U(t,u)\; du\big) \;\; ; \tag{6.73}$$

see Section 3.1.2 for the analogous univariate definitions.

As discussed earlier, F being BIFR (BDFR) implies that the components of the conditional reliability vector are decreasing (increasing) in both arguments. Therefore, F being BIFR (BDFR) implies that F is also BIFR2 (BDFR2).

The BCR function can be expressed in terms of the components of the hazard gradient vector as follows:

$$\begin{aligned}
\frac{\bar{F}(t+dt,u+du)}{\bar{F}(t,u)} &= \frac{e^{-H(t+dt,u+du)}}{e^{-H(t,u)}} = \frac{e^{-\left(\int\limits_0^{t+dt} h_T(s,u+du)\, ds + \int\limits_0^{u+du} h_U(0,v)\, dv\right)}}{e^{-\left(\int\limits_0^{t} h_T(s,u)\, ds + \int\limits_0^{u} h_U(0,v)\, dv\right)}} \\
&= e^{-\left(\int\limits_0^{t+dt} h_T(s,u+du)\, ds - \int\limits_0^{t} h_T(s,u)\, ds\right)}\; e^{-\left(\int\limits_0^{u+du} h_U(0,v)\, dv - \int\limits_0^{u} h_U(0,v)\, dv\right)} \\
&= e^{-\left(\int\limits_0^{t+dt} h_T(s,u+du)\, ds - \int\limits_0^{t} h_T(s,u)\, ds\right)}\; e^{-\int\limits_u^{u+du} h_U(0,v)\, dv} \;;
\end{aligned} \tag{6.74}$$

or equivalently,

$$\frac{\bar{F}(t+dt,u+du)}{\bar{F}(t,u)} = e^{-\left(\int\limits_0^{u+du} h_U(t+dt,v)\, dv - \int\limits_0^{u} h_U(t,v)\, dv\right)}\; e^{-\int\limits_t^{t+dt} h_T(s,0)\, ds} \;. \tag{6.75}$$

Note that, the distribution F being BIFR2 (BDFR2) (Definition 6.3) does not imply that F is BIFR (BDFR) (Definition 6.1).

6.1.4.3 Classes based on Mean Residual Life

Let $\boldsymbol{\mu}(.,.)$ denote the vector of conditional mean residual time and usage functions[II]. These functions correspond to the components of the conditional reliability vector in (6.50), and we denote them by $\mu_T(.,.)$ and $\mu_U(.,.)$ respectively. Then, for $t, u \geq 0$,

$$\boldsymbol{\mu}(t,u) := \big(\mu_T(t,u), \mu_U(t,u)\big) \ . \tag{6.76}$$

The function $\mu_T(t,u)$ is the conditional mean residual time (MRT) at t, given that the system is in an operational state at (t,u), i.e.

$$\begin{aligned}\mu_T(t,u) &= E[T-t|T>t, U>u] = \int_0^\infty \frac{\bar{F}(t+s,u)}{\bar{F}(t,u)}\, ds \\ &= \int_0^\infty \frac{e^{-\left(\int_0^{t+dt} h_T(x,u)\, dx + \int_0^u h_U(0,v)\, dv\right)}}{e^{-\left(\int_0^{t} h_T(x,u)\, dx + \int_0^u h_U(0,v)\, dv\right)}}\, ds = \int_0^\infty e^{-\int_t^{t+s} h_T(x,u)\, dx} ds = \int_0^\infty e^{-\left(H(t+s,u)-H(t,u)\right)} ds \ .\end{aligned} \tag{6.77}$$

Similarly, $\mu_U(t,u)$ is the conditional mean residual usage (MRU) at u, given that the system is in an operational state at (t,u), i.e.

$$\begin{aligned}\mu_U(t,u) &= E[U-u|T>t, U>u] = \int_0^\infty \frac{\bar{F}(t,u+v)}{\bar{F}(t,u)}\, dv \\ &= \int_0^\infty e^{-\int_u^{u+v} h_U(t,y)\, dy} dv = \int_0^\infty e^{-\left(H(t,u+v)-H(t,u)\right)} dv \ .\end{aligned} \tag{6.78}$$

The conditional MRL function for each variable reduces to its marginal MRL function when the other variable is simply greater than 0. That is, $\mu_T(t,0) = \mu_T(t)$ and $\mu_U(0,u) = \mu_U(u)$, where $\mu_T(.)$ and $\mu_U(.)$ denote the marginal MRT and MRU functions respectively.

As with the hazard gradient vector, the conditional MRL vector $\boldsymbol{\mu}(,,.)$ also uniquely determines the bivariate distribution [55, 53]. The relationship between the components of the hazard gradient vector and the corresponding components of the conditional MRL vector is the same as that of the univariate case, where one can be described solely in terms of the other; see Section 3.1.2.3 for the univariate case.

The derivatives of the conditional MRT and MRU functions in (6.77) and (6.78) are re-

[II]We may use the term "mean residual life" to refer to the expected residual of any non-negative random variable– for instance, usage.

spectively given by

$$
\begin{aligned}
\frac{\partial}{\partial t}\mu_T(t,u) &= \frac{\partial}{\partial t}\int_0^\infty e^{-\left(H(t+s,u)-H(t,u)\right)} ds \\
&= \int_0^\infty e^{-\left(H(t+s,u)-H(t,u)\right)}\ \frac{\partial}{\partial t}\left[-H(t+s,u)+H(t,u)\right]\ ds \\
&= \int_0^\infty e^{-\left(H(t+s,u)-H(t,u)\right)}\ \left[h_T(t,u)-h_T(t+s,u)\right]\ ds \\
&= h_T(t,u)\int_0^\infty e^{-\left(H(t+s,u)-H(t,u)\right)}\ ds - \int_0^\infty e^{-\left(H(t+s,u)-H(t,u)\right)}\ h_T(t+s,u)\ ds \\
&= h_T(t,u)\ \mu_T(t,u) - \frac{1}{e^{-H(t,u)}}\int_0^\infty e^{-H(t+s,u)}\ h_T(t+s,u)\ ds \\
&= h_T(t,u)\ \mu_T(t,u) + \frac{1}{e^{-H(t,u)}}\int_0^\infty \frac{\partial}{\partial t}\ e^{-H(t+s,u)}\ ds \\
&= h_T(t,u)\ \mu_T(t,u) + \frac{1}{e^{-H(t,u)}}\int_t^\infty \frac{\partial}{\partial x}\ e^{-H(x,u)}\ dx \\
&= h_T(t,u)\ \mu_T(t,u) + \frac{1}{e^{-H(t,u)}}\ \left(e^{-H(\infty,u)}-e^{-H(t,u)}\right) \\
&= h_T(t,u)\ \mu_T(t,u) - 1\ ,
\end{aligned}
\tag{6.79}
$$

and

$$
\frac{\partial}{\partial u}\mu_U(t,u) = h_U(t,u)\ \mu_U(t,u) - 1\ . \tag{6.80}
$$

Therefore, one can derive the hazard gradient vector $\mathbf{h}(.,.)$ from the components of the conditional MRL vector $\boldsymbol{\mu}(.,.)$ using the following relationships:

$$
\left(h_T(t,u),\ h_U(t,u)\right) = \left(\frac{1+\frac{\partial}{\partial t}\mu_T(t,u)}{\mu_T(t,u)},\ \frac{1+\frac{\partial}{\partial u}\mu_U(t,u)}{\mu_U(t,u)}\right)\ . \tag{6.81}
$$

Two classes of bivariate lifetime distributions based on the conditional MRL vector, analogous to the univariate classes, are as follows.

Definition 6.4. *The class of lifetime distribution is bivariate increasing (decreasing) mean residual life (BIMRL (BDMRL)), iff the two components $\mu_T(t,u)$ and $\mu_U(t,u)$ of the conditional MRL vector are increasing (decreasing) in t and u respectively, for all $t,u \geq 0$.*

Definition 6.5. *The class of distribution is bivariate new-better-than-used in expectation (BNBUE)* *(new-worse-than-used in expectation (BNWUE)), iff, for all $t, u \geq 0$ and each $s, v \geq 0$,*

$$\mu_T(t,u) = E[T-t|T>t, U>u] \ \leq (\geq)\ \mu_T(0,u) = E[T|U>u] \tag{6.82}$$

and

$$\mu_U(t,u) = E[U-u|T>t, U>u] \ \leq (\geq)\ \mu_U(t,0) = E[U|T>t] \ . \tag{6.83}$$

The chains of implications are similar to those defined in the univariate case [cf. (3.25); also see Figure 3.5]. That is,

$$\begin{aligned} F \in \text{ BIFR2 (BDFR2)} \ &\Rightarrow\ F \in \text{ BNBU (BNWU)} &&\Rightarrow\ F \in \text{ BNBUE (BNWUE)} \ ; \\ F \in \text{ BIFR2 (BDFR2)} \ &\Rightarrow\ F \in \text{ BDMRL (BIMRL)} &&\Rightarrow\ F \in \text{ BNBUE (BNWUE)} \ . \end{aligned} \tag{6.84}$$

Note that, $F \in$ BIFR (BDFR) implies $F \in$ BIFR2 (BDFR2), and therefore the other implications in (6.84) follow. We discussed some of these implications in the preceding sections (see Sections 6.1.4.1 and 6.1.4.2)– the proofs for the others follow from the relationships between the components of the hazard gradient vector, the conditional reliability vector and the MRL vector; see also the proofs for the univariate case in Section 3.1.2.3. Refer to Lai & Xie [18] for these and more on classes of bivariate distributions.

6.1.5 Partial Orderings of Distributions

In this section, we provide a brief review of partial orderings of bivariate lifetime distributions in terms of the BCR function, the conditional reliability vector, the hazard gradient and the conditional MRL vector.

6.1.5.1 Partial Ordering based on Conditional Reliability

A partial ordering of bivariate random variables can be defined in terms of the BCR function in (6.49) as follows.

Definition 6.6. *The bivariate random variable (T, U) with reliability function $\bar{G}(.,.)$ is stochastically smaller (larger) than the bivariate random variable (X, Y) with reliability function $\bar{F}(.,.)$, iff*

$$\frac{\bar{G}(t+s, u+v)}{\bar{G}(t,u)} \leq (\geq) \frac{\bar{F}(t+s, u+v)}{\bar{F}(t,u)} \ , \tag{6.85}$$

for all $t, u \geq 0$ and each $s, v \geq 0$.

A weaker partial ordering can be defined in terms of the conditional reliability vector in (6.50) as follows.

Definition 6.7. *The bivariate random variable* (T, U) *with reliability function* $\bar{G}(.,.)$ *is conditionally stochastically smaller (larger) than the bivariate random variable* (X, Y) *with reliability function* $\bar{F}(.,.)$*, iff*

$$\frac{\bar{G}(t+s,u)}{\bar{G}(t,u)} \leq (\geq) \frac{\bar{F}(t+s,u)}{\bar{F}(t,u)} \qquad \text{and} \qquad \frac{\bar{G}(t,u+v)}{\bar{G}(t,u)} \leq (\geq) \frac{\bar{F}(t,u+v)}{\bar{F}(t,u)} \ , \tag{6.86}$$

for all $t, u \geq 0$ *and each* $s, v \geq 0$.

6.1.5.2 Partial Ordering based on Hazard Rate

A partial ordering of bivariate random variables can be defined in terms of the components of the hazard gradient vector in (6.33) as follows.

Definition 6.8. *The bivariate random variable* (T, U)*, with hazard gradient components* $h_T(.,.)$ *and* $h_U(.,.)$*, is conditionally stochastically smaller (larger) in hazard rate ordering than the bivariate random variable* (X, Y)*, with hazard gradient components* $h_X(.,.)$ *and* $h_Y(.,.)$*, iff*

$$h_T(t,u) \geq (\leq)\ h_X(t,u) \quad \text{and} \quad h_U(t,u) \geq (\leq)\ h_Y(t,u) \ , \tag{6.87}$$

for all $t, u \geq 0$ *and each* $s, v \geq 0$.

See (6.34) and (6.35) for the definitions of the components of the hazard gradient vector.

6.1.5.3 Partial Ordering based on Mean Residual Life

A partial ordering of bivariate random variables can be defined in terms of the components of the conditional MRL vector in (6.76) as follows.

Definition 6.9. *The bivariate random variable* (T, U)*, with conditional MRL vector components* $\mu_T(.,.)$ *and* $\mu_U(.,.)$*, is conditionally stochastically smaller (larger) in mean residual ordering than the bivariate random variable* (X, Y)*, with conditional MRL vector components* $\mu_X(.,.)$ *and* $\mu_Y(.,.)$*, iff*

$$\mu_T(t,u) \leq (\geq)\ \mu_X(t,u) \quad \text{and} \quad \mu_U(t,u) \leq (\geq)\ \mu_Y(t,u) \ , \tag{6.88}$$

for all $t, u \geq 0$ *and each* $s, v \geq 0$.

See (6.77) and (6.78) for the definitions of the components of the conditional MRL vector.

6.2 Review of Repair Models

When modeling the effect of general repairs in two dimensions (here, time and usage), where one dimension is time and the other usage, one must take into account the relationship (dependence) between usage and time. Based on the type of relationship, two approaches have been suggested for modeling consecutive failures in two dimensions, where failures of the system are rectified by general repair: (i) the one-dimensional approach; and (ii) the two-dimensional approach.

In this section, we will briefly review each of the two approaches and provide an overview of models suggested for various types of general repair.

Recall that, general repairs are categorized as perfect, imperfect and minimal, with perfect repair being most effective, minimal repair being least effective and imperfect repair having effectiveness between those of the minimal and perfect repairs; see Section 2.2.

6.2.1 The One-dimensional Approach

The one dimensional approach involves reducing the problem of modeling consecutive failures of a system in two dimensions to modeling its failures in one dimension by assuming a relationship between the age and the usage of the system.

One suggested modeling method is to combine the two dimensions (time and usage) to form a new composite dimension, and define the failure process in this single dimension. This method is rarely used, since there is some loss of information when combining the two dimensions, as there may be more than one (time, usage) point yielding the same value in the new dimension and an equivalence is implied that requires assumptions [56, 57].

Another method, which makes up the bulk of the reliability literature on modeling consecutive failures rectified by general repair, is to assume that usage is a function of age with a usage rate parameter that is a random variable. By conditioning on the value of this usage rate (whose distribution is assumed to be known), the process of consecutive failures in two dimensions is effectively reduced to a process in one dimension, thus simplifying the modeling process. Univariate general repair models are then applied to model the effect of general repairs; see Section 3.2 for univariate repair models. The conditioning on the usage rate is later removed to derive the distributions associated with the unconditional failure process in two dimensions.

Let R denote this usage rate, whose distribution function we denote by $F_R(.)$. The relationship between time and usage is assumed to be linear, so that $M(t) = Rt$, where $M(t)$ denotes the cumulative usage of the system at calendar time t. Therefore, the associated point process (or sequence of failure points) is given by $\{(T_n, U_n) = (T_n, RT_n); n \in \mathbb{N}_+\}$, where $U_n \equiv M(T_n)$, for $n \in \mathbb{N}_+$.

Let $\{N(t,u); t, u \in \mathbb{R}_+\}$ denote the associated counting process in two dimensions, and let $\{N_X(t|r); t \in \mathbb{R}_+\}$ denote the counting process along the x-axis, given the usage rate $R = r$. Therefore, $N(t,u)$ counts the number of failures in the region $(0,t] \times (0,u]$, and $N_X(t|r)$ counts the number of failures in the interval $(0,t]$, for a system used at rate $R = r$. Then, the distribution of the count $N(t,u)$ is derived as follows:

$$\mathrm{P}\{N(t,u) = n\} = \int_0^{u/t} \mathrm{P}\{N_X(t|r) = n\}\, dF_R(r) + \int_{u/t}^{\infty} \mathrm{P}\{N_X(\tau|r) = n\}\, dF_R(r) \ , \tag{6.89}$$

where $\tau := u/r$ is the earliest time at which usage exceeds u units, when $r > u/t$; see Figure 6.5.

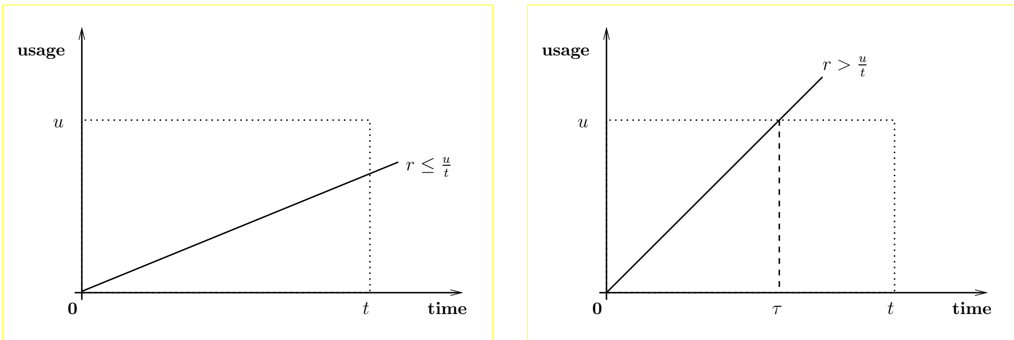


Figure 6.5: Illustration of realizations of the usage rate $R = r$, for $r \leq u/t$ (left) and $r > u/t$ (right).

The corresponding expected number of failures in the region $(0,t] \times (0,u]$ is derived as follows:

$$E[N(t,u)] = \int_0^{u/t} E[N_X(t|r)]\, dF_R(r) + \int_{u/t}^{\infty} E[N_X(\tau|r)]\, dF_R(r) \tag{6.90}$$

where $E[N_X(t|r)]$ is the expected number of failures in the interval $(0,t]$ for a system used at rate $R = r$; we will discuss this approach further in Chapter 8.

Let $\tilde{\lambda}_r(t|\mathcal{H}_t)$ denote the conditional intensity function of the process $\{N_X(t \mid r); t \in \mathbb{R}_+\}$ at time t, conditional on the process history $\mathcal{H}_t$. Then, the effect of a general repair can be incorporated into this conditional intensity function, since it uniquely determines the probability structure of the conditional counting process in one dimension; see Chapter 3 for univariate general repair models.

When all repairs are minimal, $\{N_X(t|r); t \in \mathbb{R}_+\}$ is a Poisson process with conditional intensity function equal to the conditional failure rate function of the original lifetime (time of first failure) of a system used at rate $R = r$. We denote this failure rate function by $\rho(.|r)$. Then, for the Poisson process,

$$\tilde{\lambda}_r(t|\mathcal{H}_t) = \rho(t|r) \ , \tag{6.91}$$

for all $t \geq 0$.

When all repairs are perfect (replacements), then $\{N_X(t|r); t \in \mathbb{R}_+\}$ is a renewal process in one dimension, with inter-failure lifetimes all having the following conditional distribution function:

$$F(t|r) = 1 - e^{-\int_0^t \rho(s|r)\,ds} \ , \tag{6.92}$$

where $\rho(.|r)$ is the failure rate function of the original system lifetime conditional on $R = r$. The conditional intensity function for this process is given by

$$\tilde{\lambda}_r(t|\mathcal{H}_t) = \rho(t - T_{N_X(t^-|r)}|r) \ , \tag{6.93}$$

where $T_{N_X(t^-|r)}$ is the last failure of the system before time t, given that the system is used at rate $R = r$.

This one-dimensional approach to modeling consecutive failures of a repairable system in two dimensions has been studied extensively; see Blishke & Murthy [2] for more on this approach.

One-dimensional approaches with other functional forms of the time-usage relationship have been suggested; see for instance Yang & Nachlas [58], Yang et al. [59] and Eliashberg et al. [60].

6.2.2 The Two-dimensional Approach

Where the one-dimensional approach assumes that usage is a function of age, the two-dimensional approach assumes only a correlation between age and usage. Then, the lifetime of the original system is modeled by some bivariate lifetime distribution which describes the correlation structure between age and usage. Following the first failure of the system, general repair models are applied to model the distribution of the successive bivariate inter-failure lifetimes of the repairable system.

The two-dimensional approach, due to its complexity, has not been investigated or applied as much as the one-dimensional approach, especially in modeling consecutive failures followed by imperfect repairs.

In this section, we review bivariate general repair models and stochastic point processes in two dimensions suggested for the three types of general repair.

6.2.2.1 Perfect Repair Process in Two Dimensions

The effect of a perfect repair is defined to be equivalent to that of a replacement of the failed system with a new and identical system, when the lifetime distribution is assumed to be BIFR (see Definition 6.1 for the BIFR (BDFR) class of distributions). The BIFR class of lifetime distributions is used to model the lifetime of a system whose working condition deteriorates with age and usage– therefore, the system is at its best working condition at the start of its lifetime (i.e. when its age and usage are both zero). Then, it is reasonable to assume that a perfect repair restores the working condition of the system to this point. Therefore, the consecutive failures of a repairable system having a BIFR lifetime distribution, where all failures are rectified by perfect repair, is modeled as a *renewal process* in two dimensions; see Hunter [61, 62].

The theory of renewal processes in both one dimension and two dimensions is well established. In this section, we will briefly review some basic results of renewal theory in two dimensions; see Section 3.2.1 for the process in one dimension.

Suppose that failures of a system are followed by immediate and instantaneous perfect repairs (or replacements). Note that, since the repairs are immediate and instantaneous, the failure points are the points at which the perfect repairs are performed. Then, the consecutive perfect repairs of the original system constitute a renewal process.

A stochastic counting process $\{N(t,u); t,u \in \mathbb{R}_+\}$ is a renewal process in two dimensions, if the sequence of bivariate inter-failure lifetimes $\{(X_n, Y_n); n \in \mathbb{N}_+\}$ are independent and identically distributed random variables with some distribution F. That is, $\mathrm{P}\{X_n \leq x, Y_n \leq y\} = F(x,y)$, for all $n \in \mathbb{N}_+$, where $F(.,.)$ denotes the distribution function of the bivariate lifetime of the original system (i.e. the point of first failure) [61]. For the sequence of failure points $\{(T_n, U_n); n \in \mathbb{N}_+\}$, the random variable

$$N(t,u) = \max\{n : T_n \leq t \text{ and } U_n \leq u; n \in \mathbb{N}_+\} \tag{6.94}$$

counts the number of renewals (perfect repairs) of the system in the region $(0,t] \times (0,u]$, and at $t = u = 0$, $N(0,0) = 0$. The distribution of the count $N(t,u)$ is derived using the distributions of the failure points $\{(T_n, U_n); n \in \mathbb{N}_+\}$ at the point (t,u), for $t,u \geq 0$; see Section 6.1.1.1.

As before, let $F_n(.,.)$ denote the distribution function of the failure point (T_n, U_n), for $n \in \mathbb{N}_+$. Then, the distribution function of the first failure point (T_1, U_1) is given by

$$F_1(t,u) = \mathrm{P}\{T_1 \leq t, U_1 \leq u\} = \mathrm{P}\{X_1 \leq t, Y_1 \leq u\} = F(t,u) \ . \tag{6.95}$$

Following the first perfect repair (or replacement) of the system at the point (T_1, U_1), the distribution of the second failure point (T_2, U_2) is given by

$$\begin{aligned} F_2(t,u) &= \mathrm{P}\{T_2 \leq t, U_2 \leq u\} = \mathrm{P}\{T_1 + X_2 \leq t, U_1 + Y_2 \leq u\} \\ &= \mathrm{P}\{X_2 \leq t - T_1, Y_2 \leq u - U_1\} \\ &= \int_0^t \int_0^u \mathrm{P}\{X_2 \leq t - t_1, Y_2 \leq u - u_1 | T_1 = t_1, U_1 = u_1\} \, dF_1(t_1, u_1) \\ &= \int_0^t \int_0^u F(t - t_1, u - u_1) \, dF_1(t_1, u_1) =: F^{**}F_1(t,u) \ , \end{aligned} \tag{6.96}$$

where $F^{**}F_1(.,.)$ denotes the convolution of F with $F_1(= F)$. In general, the distribution of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}) is derived as follows:

$$\begin{aligned} F_{n+1}(t,u) &= \mathrm{P}\{T_{n+1} \leq t, U_{n+1} \leq u\} = \mathrm{P}\{T_n + X_{n+1} \leq t, U_n + Y_{n+1} \leq u\} \\ &= \mathrm{P}\{X_{n+1} \leq t - T_n, Y_{n+1} \leq u - U_n\} \\ &= \int_0^t \int_0^u \mathrm{P}\{X_{n+1} \leq t - t_n, Y_{n+1} \leq u - u_n | T_n = t_n, U_n = u_n\} \, dF_n(t_n, u_n) \\ &= \int_0^t \int_0^u F(t - t_n, u - u_n) \, dF_n(t_n, u_n) := F^{**}F_n(t,u) \ , \end{aligned} \tag{6.97}$$

for $t,u \geq 0$, where $F^{**}F_n(.,.)$ denotes the convolution of F with F_n, for $n \in \mathbb{N}_+$. With the renewal process, the conditional distribution of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}), given all previous renewal (failure) points, depends only on the point $(T_n, U_n) = (t_n, u_n)$ of the last renewal before it, for $n \in \mathbb{N}_+$; see Hunter [61].

Note that, with the renewal process, the marginal point processes $\{T_n; n \in \mathbb{N}_+\}$ and $\{U_n; n \in \mathbb{N}_+\}$ are both renewal processes. The corresponding counting processes are denoted by $\{N_X(t); t \in \mathbb{R}_+\}$ and $\{N_Y(u); u \in \mathbb{R}_+\}$, respectively– $N_X(t)$ is the number of renewals in the time interval $(0,t]$, and $N_Y(u)$ is the number of renewals in the usage interval $(0,u]$; see Section 6.1.1.

The distribution functions of the failure points are used to determine the distribution of the number of failures and the expected number of failures; see (6.10) and (6.12). The expected number of failures or the cumulative intensity function $\Lambda(.,.) := E[N(.,.)]$ of the renewal process is referred to as the *bivariate renewal function* [61]. As in the univariate case (in Section 3.2.1), by conditioning on the original lifetime $(T_1, U_1) = (X_1, Y_1)$, the renewal function can be expressed as the following *renewal equation*:

$$\begin{aligned}\Lambda(t,u) &= \int_0^t \int_0^u E[N(t,u)|T_1 = t_1, U_1 = u_1] \, dF(t_1,u_1) \\ &= \int_0^t \int_0^u (1 + E[N(t-t_1, u-u_1)]) \, dF(t_1,u_1) \\ &= \int_0^t \int_0^u [1 + \Lambda(t-t_1, u-u_1)] \, dF(t_1,u_1) \\ &= F(t,u) + \int_0^t \int_0^u \Lambda(t-t_1, u-u_1) \, dF(t_1,u_1) \ .\end{aligned} \tag{6.98}$$

This follows from the definition of the renewal process, where the system is replaced by a new and identical system following each failure. The number of failures before time t and usage u, given the point $(T_1, U_1) = (t_1, u_1)$ of first failure, is equal in distribution to

$$N\big((t_1,t] \times (u_1,u]\big) \equiv N(t,u) - N(t_1,u) - N(t,u_1) + N(t_1,u_1) \stackrel{\text{d}}{=} N(t-t_1, u-u_1) \ , \tag{6.99}$$

since the system is renewed at (t_1, u_1); see Figure 6.6. When the distribution function $F(.,.)$ is given and continuous, then the renewal equation can sometimes be solved to get the renewal function $\Lambda(.,.)$; see Hunter [61, 62]

The conditional intensity function for the renewal process in two dimensions is given by

$$\tilde{\lambda}(t,u|\mathcal{H}_{t,u}) = \begin{cases} \lambda_0(t - T_n, u - U_n) \ , & \text{for } (t,u) \in (T_n, T_{n+1}] \times (U_n, U_{n+1}];\ 0 \le n \le N(t,u) \\ 0 \ , & \text{otherwise} \ , \end{cases} \tag{6.100}$$

where the initial (baseline) intensity function $\lambda_0(.,.)$ is the bivariate failure rate function $r(.,.)$ of the original system, i.e. $\lambda_0(t,u) = r(t,u)$, for $t, u \ge 0$ [63, 14]; see Sections 6.1.1.2 and 6.1.2.

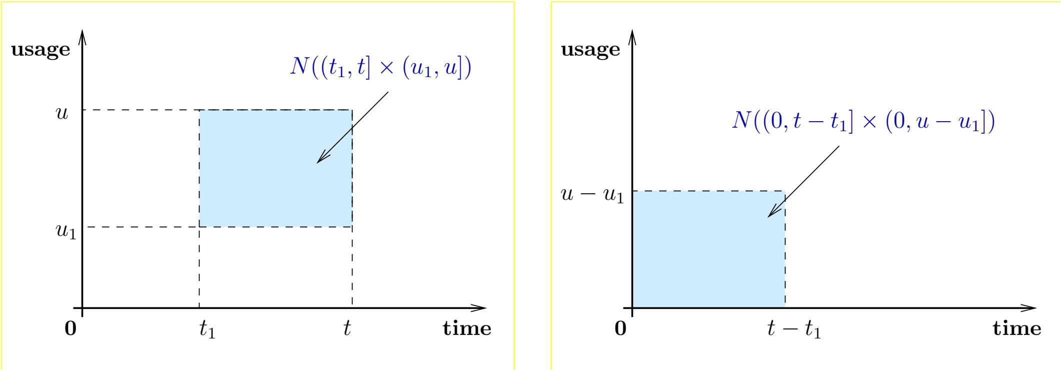


Figure 6.6: Illustration of the sets $(t_1, t] \times (u_1, u]$ and $(0, t - t_1] \times (0, u - u_1]$ corresponding to the counts in (6.99).

The renewal process in two dimensions takes into account the correlation between the age and usage of the system. If a functional relationship exists between the usage and the age, the two-dimensional process can effectively be reduced to a conditional one-dimensional process, where the renewal process in one dimension can be used to model consecutive perfect repairs. We discussed this one-dimensional approach in the previous section.

6.2.2.2 Minimal Repair Process in Two Dimensions

The working condition of a system following a minimal repair does not change. In other words, the working condition of the system immediately following a minimal repair is the same as its working condition just before system failure.

Consecutive failures of a repairable system with a univariate lifetime, where failures are rectified by minimal repair, can be modeled as a Poisson process in one dimension, because the conditional intensity function of the Poisson process is equal to the failure rate corresponding to the lifetime of the original system (i.e. does not change following each repair); see Chapter 3.

The d-dimensional generalization of the Poisson process– which counts the number of points randomly scattered in a subset of the d-dimensional space $\mathbb{R}^d$– is the spatial Poisson process. According to this process, the counts over disjoint subsets $\mathcal{A}_i \subset \mathbb{B}$, for $i \in \mathbb{N}_+$, are independent and have a Poisson distribution with mean measure (or cumulative intensity)

$$\Lambda(\mathcal{A}_i) := \int_{\mathcal{A}_i} \lambda(\mathbf{x})\, d\mathbf{x} \ , \tag{6.101}$$

where $\lambda : \mathbb{R}^d \to \mathbb{R}$ is the intensity function defined over $\mathbb{B} \subset \mathbb{R}^d$. This process does not take into account the ordering of points, and therefore, consecutive minimal repairs of a system with a bivariate lifetime cannot be modeled as a Poisson process in two dimensions; see Figure 6.7.

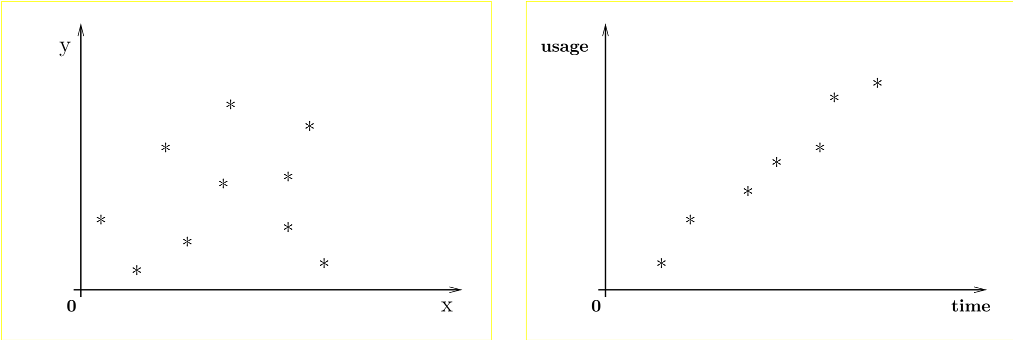


Figure 6.7: Illustration of an unordered scattering of points in the two-dimensional space $\mathbb{R}_+^2$ (left) versus a trajectory of a failure process where points are ordered (right) (see (6.1) for order).

Let $\{N(t,u); t,u \in \mathbb{R}_+\}$ denote the counting process, where $N(t,u)$ counts the number of minimal repairs performed in $(0,t] \times (0,u]$. The corresponding point process is denoted by $\{(T_n, U_n); n \in \mathbb{N}_+\}$. When the rectification action is a minimal repair, the bivariate inter-failure lifetimes, (X_n, Y_n), $n \in \mathbb{N}_+$, are neither independent nor identically distributed.

Baik et al [64] propose the following approach to model the process of consecutive minimal repairs in two dimensions. The probability of having n failures in the region $(0,t] \times (0,u]$ is derived by conditioning on the following event:

$$\{N(t',u')=n-1,\ N((t',t'+dt']\times(u',u'+du'])=1,\ N((t'+dt',t]\times(u'+du',u])=0\}\ , \tag{6.102}$$

i.e. $n-1$ failures occur in the region $\mathcal{A} = (0,t'] \times (0,u']$; the n-th failure occurs in the region $\mathcal{B} = (t',t'+dt'] \times (u',u'+du']$; and no failure occurs in the region $\mathcal{C} = (t'+dt',t] \times (u'+du',u]$; see Figure 6.8. To shorten the expression in (6.102), we use the sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$:

$$\{N(\mathcal{A})=n-1,\ N(\mathcal{B})=1,\ N(\mathcal{C})=0\}\ . \tag{6.103}$$

Then, the probability of the event in (6.103) can be derived as follows:

$$\begin{aligned}\mathrm{P}\{N(\mathcal{A})=n-1,\ N(\mathcal{B})=1,\ N(\mathcal{C})=0\} &= \mathrm{P}\{N(\mathcal{C})=0 \mid N(\mathcal{B})=1,\ N(\mathcal{A})=n-1\}\\ &\quad \mathrm{P}\{N(\mathcal{B})=1 \mid N(\mathcal{A})=n-1\}\ \mathrm{P}\{N(\mathcal{A})=n-1\}\ .\end{aligned} \tag{6.104}$$

When all repairs are minimal, the conditional distribution of the $(n+1)$-th failure point

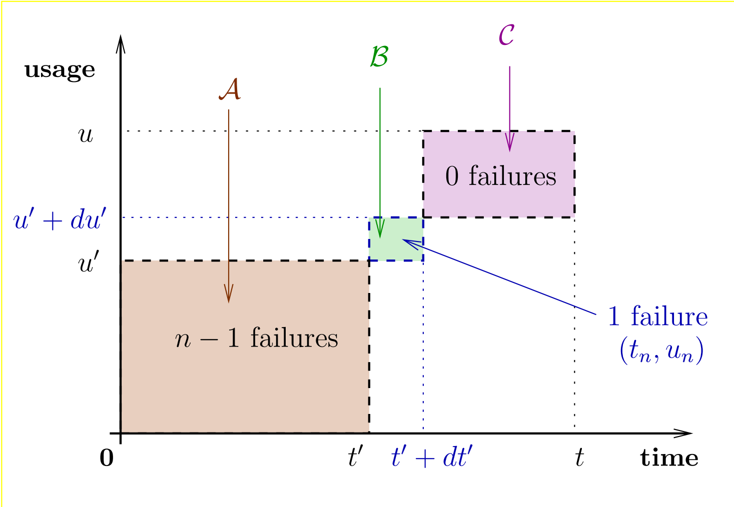


Figure 6.8: An illustration of the description of number of failures from Baik et al. [64].

(T_{n+1}, U_{n+1}), given all n previous failure points, depends only on the last failure point $(T_n, U_n) = (t_n, u_n)$, for $n \in \mathbb{N}_+$. This follows from the definition of a minimal repair, which is that the system following a minimal repair behaves like a system that has not failed, and therefore,

$$\begin{aligned}
&\mathrm{P}\{T_{n+1} \le t, U_{n+1} \le u \mid (T_1, U_1) = (t_1, u_1), \ldots, (T_n, U_n) = (t_n, u_n)\} \\
&\quad \overset{\text{def}}{=} \mathrm{P}\{T_1 \le u, U_1 \le u \mid T_1 > t_n, U_1 > u_n\} \\
&\quad = \frac{F(t,u) - F(t_n,u) - F(t,u_n) + F(t_n,u_n)}{\bar{F}(t_n,u_n)} \ ,
\end{aligned} \tag{6.105}$$

where (T_1, U_1) is the original lifetime with distribution and reliability functions $F_1 = F$ and $\bar{F}_1 = \bar{F}$ respectively.

Upon passing the first conditional probability in (6.104) through the limit (shrinking set $\mathcal{B}$), we get

$$\begin{aligned}
&\lim_{dt',du'\to 0} \mathrm{P}\{N(\mathcal{C})=0 \mid N(\mathcal{B})=1,\ N(\mathcal{A})=n-1\} \\
&\quad = 1 - \lim_{dt',du'\to 0} \mathrm{P}\{N(\mathcal{C}) \ge 1 \mid N(\mathcal{B})=1,\ N(\mathcal{A})=n-1\} \\
&\quad = 1 - \mathrm{P}\{T_{n+1} \le u, U_{n+1} \le u \mid (T_n, U_n) = (t'^+, u'^+),\ N(\mathcal{A})=n-1\} \\
&\quad = 1 - \frac{F(t,u) - F(t',u) - F(t,u') + F(t',u')}{\bar{F}(t',u')} \ ;
\end{aligned} \tag{6.106}$$

since F is continuous, $F(x^+, y^+) = F(x, y)$. Note that, the event that at least one failure occurs in $\mathcal{C}$ is equivalent to the event that the $(n+1)$-th failure point (T_{n+1}, U_{n+1}) is in $\mathcal{C}$, given that n failures have occurred in $\mathcal{A} \cup \mathcal{B}$. The second to last expression in (6.106) follows from passing through the limit, i.e.

$$\begin{aligned}\lim_{dt',du' \to 0} \{N(\mathcal{B})=1,\ N(\mathcal{A})=n-1\} &= \lim_{dt',du' \to 0} \{(T_n, U_n) \in \mathcal{B},\ N(\mathcal{A})=n-1\} \\ &\equiv \{(T_n, U_n) = (t'^{+}, u'^{+}),\ N(\mathcal{A})=n-1\} \ ,\end{aligned} \tag{6.107}$$

where $\mathcal{B} = (t', t'+dt'] \times (u', u'+du']$.

Since the process is orderly (i.e. simultaneous failures do not occur), the second conditional probability in (6.104) becomes

$$\begin{aligned}\mathrm{P}\{N(\mathcal{B})=1 \mid N(\mathcal{A})=n-1\} &= \mathrm{P}\{N((t', t'+dt'] \times (u', u'+du'])=1 \mid N(t', u')=n-1\} \\ &\approx r(t', u')\ dt'\ du' \ ,\end{aligned} \tag{6.108}$$

where $r(.,.)$ is the bivariate failure rate function defined in (6.20). This follows from the conditional intensity function of the minimal repair process in two dimensions being

$$\tilde{\lambda}(t, u|\mathcal{H}_{t,u}) = \begin{cases} \lambda_0(t, u) \ , & \text{for } (t, u) \in (T_n, T_{n+1}] \times (U_n, U_{n+1}];\ 0 \le n \le N(t, u) \\ 0 \ , & \text{otherwise} \ , \end{cases} \tag{6.109}$$

where the baseline intensity function $\lambda_0(.,.)$ is equal to the failure rate function $r(.,.)$.

Then, given (6.106) and (6.108), the probability of n failures occurring in $(0, t] \times (0, u]$ can be derived from the following recursive expression:

$$\begin{aligned}\mathrm{P}\{N(t, u)=n\} = &\int_0^t \int_0^u \left(1 - \frac{F(t, u) - F(t', u) - F(t, u') + F(t', u')}{\bar{F}(t', u')}\right) \mathrm{P}\{N(t', u')=n-1\} \\ &\times r(t', u')\ dt'\ du' \ ,\end{aligned} \tag{6.110}$$

for $n \in \mathbb{N}_+$, where $\mathrm{P}\{N(t, u)=0\} = 1 - F(t, u)$ [50, 64].

6.2.2.3 Imperfect Repair Processes in Two Dimensions

When consecutive failures of a repairable system are rectified by imperfect repair, the bivariate inter-failure lifetimes $\{(X_n, Y_n); n \in \mathbb{N}_+\}$ are, as in the minimal repair case, neither independent nor identically distributed (analogous to the univariate imperfect repair models). Often the distribution of each of the failure points (T_n, U_n), for $n \in \mathbb{N}_+$, depends on the effect of all imperfect repairs performed prior to that point, which makes the modeling process less straight-forward.

Therefore, imperfect maintenance models in two dimensions often assume that the bivariate inter-failure lifetimes are independent. Consider, for instance, the *quasi-renewal process* in two dimensions. Let $\{(X'_n, Y'_n); n \in \mathbb{N}_+\}$ denote the sequence of bivariate inter-failure lifetimes of a renewal process in two dimensions. Then, the sequence $\{(X_n, Y_n); n \in \mathbb{N}_+\}$ forms a quasi-renewal process in two dimensions, iff

$$(X_n, Y_n) \stackrel{\mathrm{d}}{=} \left(\alpha^{n-1} X'_n,\ \beta^{n-1}\ Y'_n\right) \ , \tag{6.111}$$

for all $n \in \mathbb{N}_+$, where α and β, $\alpha, \beta > 0$, are parameters of the quasi-renewal process in two dimensions; see Section 3.2.3.3 for the quasi-renewal process in one dimension.

Let, for $n \in \mathbb{N}_+$, the function $G_n(.,.)$ denote the distribution function of the n-th bivariate inter-failure lifetime (X_n, Y_n). Also, let $F(.,.)$ denote the common distribution of the bivariate inter-failures lifetimes of the associated renewal process. Then, $G_1(x, y) = F(x, y)$, for all $x, y \geq 0$, and the distribution of the $(n+1)$-th bivariate inter-failure lifetime of the quasi-renewal process in two dimensions is, for $n \in \mathbb{N}_+$, given by

$$\begin{aligned} G_{n+1}(x, y) &= \mathrm{P}\{X_{n+1} \leq x, Y_{n+1} \leq y\} \\ &= \mathrm{P}\{\alpha^n X'_{n+1} \leq x, \beta^n\ Y'_{n+1} \leq y\} \\ &= \mathrm{P}\{X'_n \leq \alpha^{-n} x, Y'_n \leq \beta^{-n} y\} = F\left(\frac{x}{\alpha^n}, \frac{y}{\beta^n}\right) \ . \end{aligned} \tag{6.112}$$

Note that, when $\alpha = \beta = 1$, the quasi renewal process in two dimensions reduces to the (ordinary) renewal process in two dimensions. The marginal processes $\{X_n; n \in \mathbb{N}_+\}$ and $\{Y_n; n \in \mathbb{N}_+\}$ are increasing sequences when $\alpha, \beta > 1$ and decreasing sequences when $\alpha, \beta < 1$. This process, with $\alpha, \beta \in (0, 1]$, is used to model consecutive failures of a system where each failure is followed by imperfect repair.

The quasi-renewal process is a generalization of the renewal process in two dimensions, but not of the minimal repair process in two dimensions; refer to Gülay [65] for more on the quasi-renewal process.

In the next chapter, we propose a generalization of the renewal process in two dimensions to model the general repair process, where the inter-failure lifetimes are not independent. We will show that the proposed model, as its special cases, includes both the renewal process in Section 6.2.2.1 and the minimal repair process in Section 6.2.2.2.

6.3 Chapter Summary

In this chapter, we provided a brief review of concepts (such as, stochastic counting processes, intensity processes and failure rate functions) necessary in modeling consecutive failures of systems in two dimensions.

We also discussed various bivariate aging classes and partial orderings of bivariate distributions in terms of the bivariate conditional reliability, the hazard gradient vector and the mean residual life vector.

We reviewed various bivariate general repair models and the associated failure processes in two dimensions.

Chapter 7

Modeling Repairs in Two Dimensions

In this chapter, we propose a new approach to model the effect of a general repair performed on a system whose lifetime is modeled as a bivariate random variable. We then develop a failure (or general repair) process to model consecutive failures of the system, where failures are rectified by general repair.

This chapter is arranged as follows. In Section 7.1, we describe the type of system considered in this study. In Section 7.2, we introduce the general repair model. In Section 7.3, we derive the distribution and reliability functions of the consecutive failure points and the bivariate inter-failure lifetimes of the proposed failure process. In section 7.4, we discuss constructing bivariate increasing failure rate distributions. In Section 7.5, we conclude with a chapter summary.

7.1 The System

For a system whose lifetime is modeled as a bivariate random variable, the two variables are often assumed to be time (or age) and some measure of the usage of the system (such as, mileage, number of flights, etc.). We will use the generic terms 'age' (or time) and 'usage' for the two lifetime variables. Here, the lifetime variables are assumed to be correlated but not functionally related.

We consider a system whose ability to perform its intended function decreases as its age and usage increase, given that it is in an operational state. That is, if the system has not failed, its working condition deteriorates with time and use. Therefore, to model the distribution of the original bivariate lifetime, or the time and usage at first failure, we must choose a lifetime distribution that displays this pattern of degradation.

In the univariate case, lifetime distributions having an increasing failure rate (IFR) function have been used to model the lifetime of a deteriorating system. These lifetime distributions are characterized by a decreasing conditional reliability function, which implies a decreasing mean residual lifetime (MRL) function. Both the conditional reliability function and the MRL function can be viewed as probabilistic measures of the working condition of the system, and therefore, when decreasing, can model positive aging (or degradation).

A bivariate increasing failure rate concept analogous to the univariate case is the decreasing *bivariate conditional reliability* (BCR) function, which was discussed in Section 6.1.4. For a bivariate lifetime (T, U), this function is defined as

$$\frac{\bar{F}(t+s, u+v)}{\bar{F}(t,u)} = \frac{\mathrm{P}\{T > t+s, U > u+v\}}{\mathrm{P}\{T > t, U > u\}} \,, \tag{7.1}$$

for $t, u \geq 0$ and $s, v \geq 0$. Then, when this function is decreasing in t and u, for all $t, u \geq 0$ and each $s, v \geq 0$, the associated lifetime distribution F is said to have the *bivariate increasing failure rate* (BIFR) property. For brevity, we will say that the distribution is BIFR when we mean the bivariate lifetime distribution has a decreasing BCR function.

The BCR function is interpreted as the probability that the system survives an additional s units of time and v units of usage, given that it is in an operational state at the point (t, u), i.e. its first failure is after the point (t, u). An appealing consequence of defining the IFR property through this conditional reliability function is that this definition implies that the hazard gradient components (which together uniquely determine the lifetime distribution) are increasing in their arguments; see Section 6.1.4.

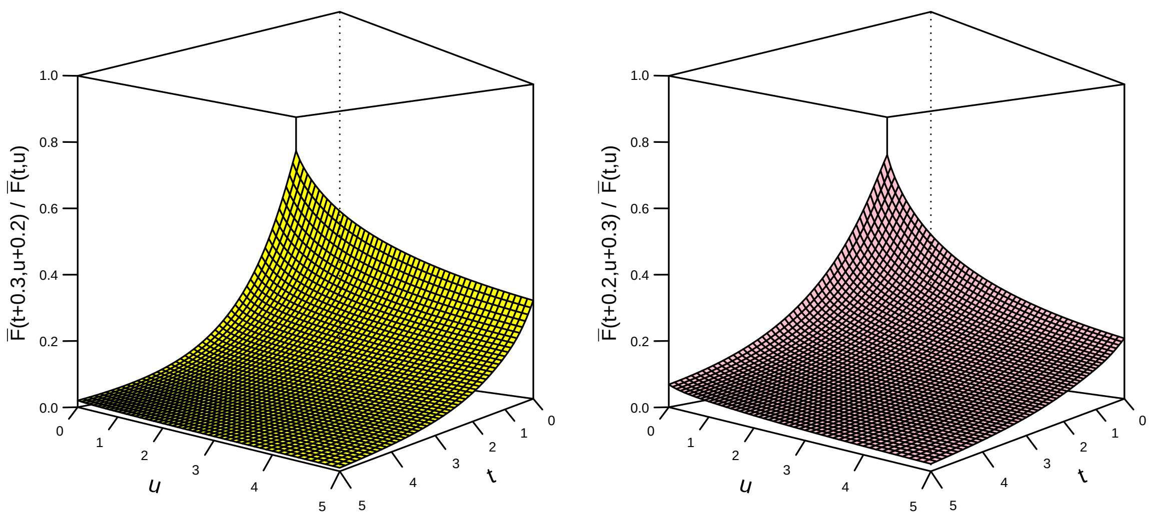


Figure 7.1: Plots illustrating a decreasing bivariate conditional reliability function with Weibull marginals, plotted for: (i) $s = 0.3$ and $v = 0.2$ (left) and (ii) $s = 0.2$ and $v = 0.3$ (right).

In Figure 7.1, for various values of $s, v \geq 0$, we have plotted an example of a BCR function which is decreasing in $t, u \geq 0$.

7.2 The General Repair Model

Following each failure, the system is rectified by general repair. The effectiveness of a general repair is described by its degree, which we denote by the vector (δ_n, γ_n) for the n-th repair, where $(\delta_n, \gamma_n) \in [0,1]^2$ and $n \in \mathbb{N}_+$. A general repair can be categorized as one of the following three types: (i) minimal repair, which is assumed to be least effective; (ii) perfect repair, which is assumed to be most effective; and (iii) imperfect repair, which is assumed to be more effective than a minimal repair but less effective than a perfect repair. We let $(0,0)$ represent the degree of a minimal repair; $(1,1)$ the degree of a perfect repair; and $(\delta, \gamma) \in [0,1]^2 \setminus \{(0,0),(1,1)\}$ the degree of an imperfect repair.

The two components of the degree (δ_n, γ_n) represent the repair effectiveness in terms of time and usage respectively. Although, the components δ_n and γ_n can each vary in the range $[0,1]$, given either, some values of the other may be more likely. It makes sense to consider modeling the degrees of repair (δ_n, γ_n), $n \in \mathbb{N}_+$, as bivariate random variables. For the purpose of this study, we have assumed that the degrees of repair are given (fixed/preassigned).

Since the system is deteriorating with age and use, we can model a perfect repair as a replacement of the system with a new and identical system, i.e. perfect repairs can be modeled as system renewals. Here, we generalize this approach to model general, imperfect repairs. We propose that, *a general repair performed following a system failure be modeled as a replacement of the system with an identical system at some younger age and lower usage, where the age and usage of the replacement system depend on the effectiveness of the general repair.*

The associated failure process includes the renewal process in two dimensions (used to model consecutive perfect repairs, when the system lifetime distribution is BIFR) and the minimal repair process in two dimensions; see Section 6.2.2.

7.2.1 The Virtual Age and Usage Processes

In order to implement the above general repair model, we define two marginal processes $\{A(t,u); t,u \in \mathbb{R}_+\}$ and $\{B(t,u); t,u \in \mathbb{R}_+\}$, where $A(t,u)$ and $B(t,u)$ denote the *virtual age* and the *virtual usage* of the system at the point $(t,u) \in \mathbb{R}_+^2$. The virtual age and virtual usage at any point are functions of all previous failure points and also the degrees of the corresponding repairs. Based on the assumptions of a repair model, various virtual age and usage functions can be defined.

Here, we make the following assumptions in modeling the effect of general repairs:

(a) Failures of the system are followed immediately by a general repair;

(b) The time to repair the failed system is assumed to be negligible compared to its operating time, and therefore, assumed to be equal to zero;

(c) A perfect repair performed on the failed system can remove system degradation or undo damage accumulated, in terms of both age and usage of the system, since the start of its lifetime (i.e. perfect repair is equivalent to replacement).

Let (T_n, U_n) denote the time and usage at the n-th failure of the system, $n \in \mathbb{N}_+$. Then, based on the above assumptions, we define the virtual age and usage functions, at the point (t, u), as follows:

$$\begin{aligned} A(t,u) &= t - \sum_{i=1}^{N(t^-,u^-)} \delta_i \; A(T_i, U_i) \;\; ; \\ B(t,u) &= u - \sum_{i=1}^{N(t^-,u^-)} \gamma_i \; B(T_i, U_i) \;\; , \end{aligned} \tag{7.2}$$

respectively, where $N(t^-, u^-)$ denotes the number of failures in $[0, t) \times [0, u)$, $t, u \geq 0$, with $N(0, 0) = 0$. Before the first failure, the virtual age and usage of the system are equal to its actual age and usage; therefore, at $(t, u) = (0, 0)$, we have $A(0, 0) = B(0, 0) = 0$. Note that, the failure points (T_n, U_n), $n \in \mathbb{N}_+$, are ordered, such that

$$\begin{aligned} &0 < T_1 < T_2 < \cdots < T_n < \dots \;\; ; \\ &0 < U_1 < U_2 < \cdots < U_n < \dots \;\; . \end{aligned} \tag{7.3}$$

Therefore, $N(t^-, u^-) = \max\{n : T_n < t \text{ and } U_n < u, n \in \mathbb{N}_+\}$.

If a general repair can only undo damage accumulated since the previous repair [cf. assumption (c)], then the virtual age and usage functions can be defined as follows:

$$\begin{aligned} A(t,u) &= t - \sum_{i=1}^{N(t^-,u^-)} \delta_i \; \left[A(T_i, U_i) - A(T_{i-1}^+, U_{i-1}^+)\right] \;\; ; \\ B(t,u) &= u - \sum_{i=1}^{N(t^-,u^-)} \gamma_i \; \left[B(T_i, U_i) - B(T_{i-1}^+, U_{i-1}^+)\right] \;\; , \end{aligned} \tag{7.4}$$

where, in general, $A(T_n^+, U_n^+)$ and $B(T_n^+, U_n^+)$ denote the virtual age and virtual usage immediately after the n-th repair, for $n \in \mathbb{N}_+$.

For this study, we will use the functions in (7.2); however, the proposed repair model is general enough to work with other virtual age and usage functions that are increasing in $t, u \geq 0$ and decreasing in each δ_n and γ_n, for $n \in \mathbb{N}_+$.

Sequence of perfect repairs. When all repairs are perfect, i.e. when $(\delta_n, \gamma_n) = (1, 1)$, for all $n \in \mathbb{N}_+$, then, the virtual age function at any point (t, u) reduces to

$$\begin{aligned} A(t,u) &= t - \sum_{i=1}^{N(t^-,u^-)} A(T_i, U_i) \\ &= t - \sum_{i=1}^{N(t^-,u^-)-1} A(T_i, U_i) - A(t_{N(t^-,u^-)}, u_{N(t^-,u^-)}) \\ &= t - \sum_{i=1}^{N(t^-,u^-)-1} A(T_i, U_i) - T_{N(t^-,u^-)} + \sum_{i=1}^{N(t^-,u^-)-1} A(T_i, U_i) \\ &= t - T_{N(t^-,u^-)} \ . \end{aligned} \tag{7.5}$$

Therefore, immediately after the n-th repair, the virtual age is $A(T_n^+, U_n^+) = T_n^+ - T_n = 0$. Similarly, the corresponding virtual usage at the point (t, u) is given by

$$B(t,u) = u - U_{N(t^-,u^-)} \ , \tag{7.6}$$

and the virtual usage immediately after the n-th perfect repair is $B(T_n^+, U_n^+) = 0$.

Sequence of minimal repairs. When all failures of the system are followed by minimal repair, i.e. when $(\delta_n, \gamma_n) = (0, 0)$, for all $n \in \mathbb{N}_+$, then, the virtual age function at the point (t, u) is given by

$$A(t,u) = t - \sum_{i=1}^{N(t^-,u^-)} 0 \ A(T_i, U_i) = t \ , \tag{7.7}$$

Therefore, immediately after the n-th minimal repair, the virtual age is $A(T_n^+, U_n^+) = T_n^+$. Similarly, the corresponding virtual usage at the point (t, u) is given by

$$B(t,u) = u \ , \tag{7.8}$$

and the virtual usage immediately after the n-th minimal repair is $B(T_n^+, U_n^+) = U_n^+$.

The virtual age and usage vector $(A(t, u), B(t, u))$ is stochastic because of its dependence on the number and the points of failures that occur at random in the rectangle $[0, t) \times [0, u)$, $t, u \geq 0$. We will use $a(t, u)$ and $b(t, u)$ to denote the realizations of the two random variables

$A(t, u)$ and $B(t, u)$, respectively.

These virtual age and usage vectors are novel extensions, from one dimension to two dimensions, of the virtual age models introduced by Kijima [27]; see Section 3.2.3.

7.2.2 The Effect of General Repairs

Given the virtual age and usage vector, we now proceed to define the effect of general repairs on the distribution of succeeding failures of the system.

We can describe the effect of repairs in terms of the conditional reliability function, or equivalently, in terms of the conditional distribution function of the failure points. We begin with the conditional reliability function, since it may be more intuitive to describe general repairs in terms of their effect on system survival (or reliability).

7.2.2.1 Conditional Reliability Function

Let $\bar{F}_{n+1}(.,.|\boldsymbol{t_n}, \boldsymbol{u_n})$, where $\boldsymbol{t_n} = (t_1, \ldots, t_n)$ and $\boldsymbol{u_n} = (u_1, \ldots, u_n)$, denote the reliability function of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}), for $n \in \mathbb{N}_+$, given all previous failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$. Formally,

$$\bar{F}_{n+1}(t, u|\boldsymbol{t_n}, \boldsymbol{u_n}) = \mathrm{P}\{T_{n+1} > t, U_{n+1} > u \mid (T_1, U_1) = (t_1, u_1), \ldots, (T_n, U_n) = (t_n, u_n)\} \ , \tag{7.9}$$

for $t > t_n$ and $u > u_n$; see Figure 7.2.

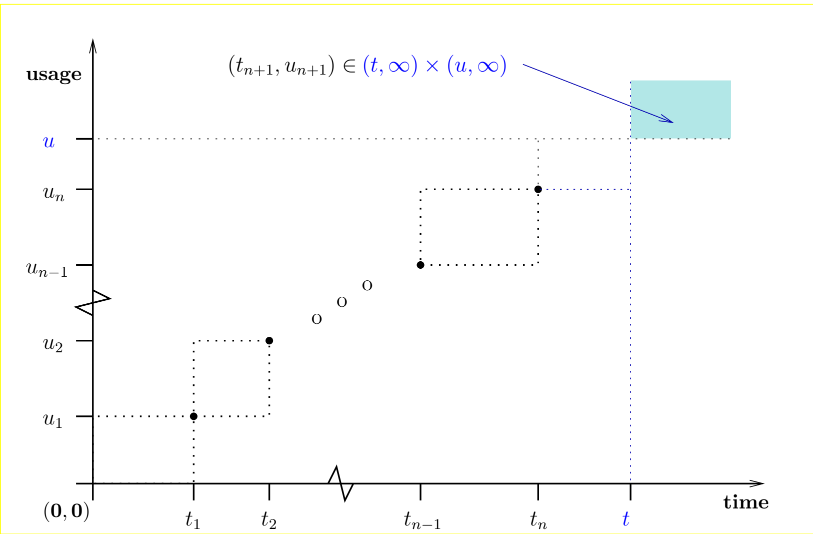


Figure 7.2: Illustration of the n failure points prior to (t, u) for the conditional reliability function $\bar{F}_{n+1}(t, u|\boldsymbol{t_n}, \boldsymbol{u_n})$, defined for $t > t_n$ and $u > u_n$.

This reliability function represents the probability that the $(n+1)$-th failure of the system, given previous failure points, is after the point (t,u). In other words, $\bar{F}_{n+1}(t,u|\boldsymbol{t_n},\boldsymbol{u_n})$ is the conditional probability that the system survives $t-t_n$ units of time and $u-u_n$ units of usage, following the n-th general repair at the point (t_n,u_n).

The repair model in terms of the reliability function. We define the effect of a general repair on the conditional reliability function of the $(n+1)$-th failure point, for $n \in \mathbb{N}_+$, as follows:

$$\begin{aligned}
\bar{F}_{n+1}(t,u|\boldsymbol{t_n},\boldsymbol{u_n}) &= \mathrm{P}\{T_{n+1} > t, U_{n+1} > u \mid (T_1,U_1) = (t_1,u_1),\ldots,(T_n,U_n) = (t_n,u_n)\} \\
&= \mathrm{P}\{T_1 > a(t,u), U_1 > b(t,u) \mid T_1 \geq a(t_n^+,u_n^+), U_1 \geq b(t_n^+,u_n^+)\} \\
&= \frac{\mathrm{P}\{T_1 > a(t,u), U_1 > b(t,u), T_1 \geq a(t_n^+,u_n^+), U_1 \geq b(t_n^+,u_n^+)\}}{\mathrm{P}\{T_1 \geq a(t_n^+,u_n^+), U_1 \geq b(t_n^+,u_n^+)\}} \\
&= \frac{\mathrm{P}\{T_1 > a(t,u), U_1 > b(t,u)\}}{\mathrm{P}\{T_1 \geq a(t_n^+,u_n^+), U_1 \geq b(t_n^+,u_n^+)\}} \\
&= \frac{\bar{F}(a(t,u),b(t,u))}{\bar{F}(a(t_n^+,u_n^+),b(t_n^+,u_n^+))}\ ,
\end{aligned} \tag{7.10}$$

for $t > t_n$ and $u > u_n$, where $\bar{F}(.,.)$ denotes the reliability function of the lifetime of the original system (i.e. point of first failure); see Figure 7.3. Note that, since the virtual age and usage functions $A(.,.)$ and $B(.,.)$ are increasing in their arguments (when all other parameters are fixed), their realizations are ordered, i.e. $a(t,u) \geq a(t_n^+,u_n^+)$ and $b(t,u) \geq b(t_n^+,u_n^+)$, for all $t \geq t_n^+$ and $u \geq u_n^+$.

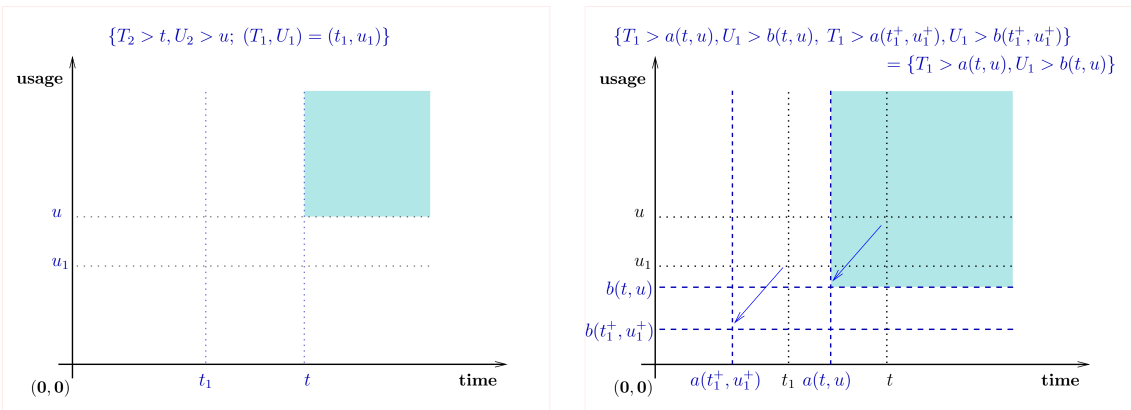


Figure 7.3: Illustrations of the sets (actual and virtual) used in defining $\bar{F}_2(t,u|t_1,u_1)$, where $\delta_1 = 0.45$ and $\gamma_1 = 0.6$.

Therefore, according to the proposed general repair model, the probability of the system failing for the $(n+1)$-th time after the point (t,u), given all previous failure points

$\{(t_1, u_1), \ldots, (t_n, u_n)\}$, is equal to the probability of an identical system failing for the first time after the point $\big(a(t,u), b(t,u)\big)$, given that it is operating at the point $\big(a(t_n^+, u_n^+), b(t_n^+, u_n^+)\big)$. Note that, simultaneous reductions in the two functions $a(.,.)$ and $b(.,.)$ take place only at the points (t_n^+, u_n^+), $n \in \mathbb{N}_+$, and therefore, the vector $\big(a(t_n^+, u_n^+), b(t_n^+, u_n^+)\big)$ represents the virtual age and usage of the system immediately following the n-th general repair.

Sequence of perfect repairs. When all repairs are perfect, then $a(t_n^+, u_n^+) = b(t_n^+, u_n^+) = 0$, for all $n \in \mathbb{N}_+$; see page 151. Therefore, the conditional reliability function in (7.10) reduces to

$$\begin{aligned} \bar{F}_{n+1}(t, u | \boldsymbol{t_n}, \boldsymbol{u_n}) &= \frac{\bar{F}(a(t,u), b(t,u))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \\ &= \frac{\bar{F}(t - t_n, u - u_n)}{\bar{F}(0,0)} = \bar{F}(t - t_n, u - u_n) \ , \end{aligned} \tag{7.11}$$

since $\bar{F}(0,0) = 1$; see Figure 7.4 (left). This is the conditional reliability function of the renewal process in two dimensions, introduced by Hunter [61].

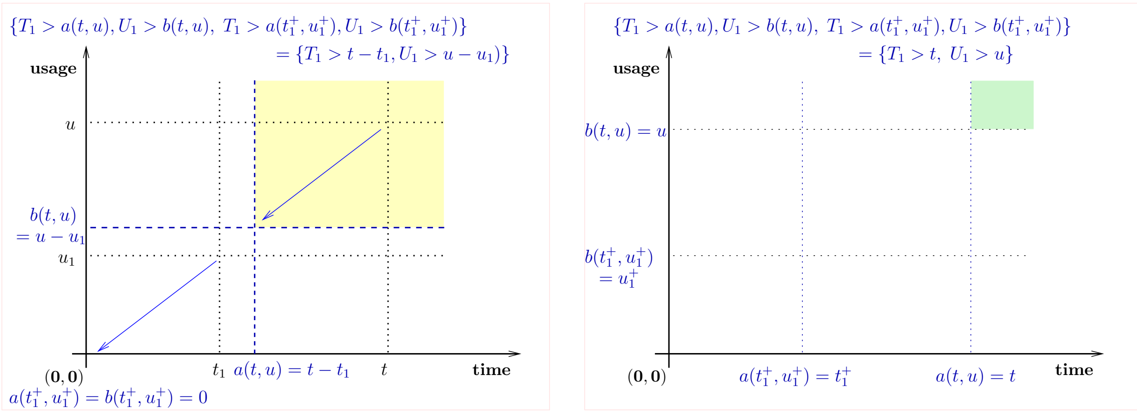


Figure 7.4: Illustrations of the (virtual) sets used in defining $\bar{F}_2(t, u | t_1, u_1)$: for $(\delta_1, \gamma_1) = (1,1)$ (left) and $(\delta_1, \gamma_1) = (0,0)$ (right).

Sequence of minimal repairs. When all repairs are minimal, then $a(t_n^+, u_n^+) = t_n^+$ and $b(t_n^+, u_n^+) = u_n^+$, for all $n \in \mathbb{N}_+$. Therefore, the conditional reliability function in (7.10) becomes

$$\begin{aligned} \bar{F}_{n+1}(t, u | \boldsymbol{t_n}, \boldsymbol{u_n}) &= \frac{\bar{F}(a(t,u), b(t,u))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \\ &= \frac{\bar{F}(t,u)}{\bar{F}(t_n^+, u_n^+)} = \frac{\bar{F}(t,u)}{\bar{F}(t_n, u_n)} \ , \end{aligned} \tag{7.12}$$

since $\bar{F}(.,.)$ is absolutely continuous; see Figure 7.4 (right). This function is the conditional reliability function corresponding to the minimal repair process in two dimensions; refer to Baik et al [64].

The proposed general repair model is a two-dimensional analog of the virtual age model in one dimension (proposed by Kijima [27]), with conditional reliability function given by

$$\mathrm{P}\{T_{n+1} > t | T_1 = t_1, \ldots, T_n = t_n\} = \mathrm{P}\{X > v_n + x | X > v_n\} = \frac{\bar{F}(v_n + x)}{\bar{F}(v_n)} \ , \tag{7.13}$$

for $n \in \mathbb{N}_+$ and $x = t - t_n$, where v_n is the virtual age immediately after the n-th repair, X is the original univariate lifetime and $\bar{F}(.)$ is the reliability function of the original lifetime; see Section 3.2.3.

7.2.2.2 Conditional Distribution Function

We can also define the effect of a general repair in terms of the conditional distribution functions of the failure points. Let the function $F_{n+1}(.,.|\boldsymbol{t_n}, \boldsymbol{u_n})$ denote the distribution function of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}), for $n \in \mathbb{N}_+$, given all previous failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$. Formally,

$$F_{n+1}(t, u | \boldsymbol{t_n}, \boldsymbol{u_n}) = \mathrm{P}\{T_{n+1} \le t, U_{n+1} \le u \mid (T_1, U_1) = (t_1, u_1), \ldots, (T_n, U_n) = (t_n, u_n)\} \ , \tag{7.14}$$

for $t > t_n$ and $u > u_n$; see Figure 7.5.

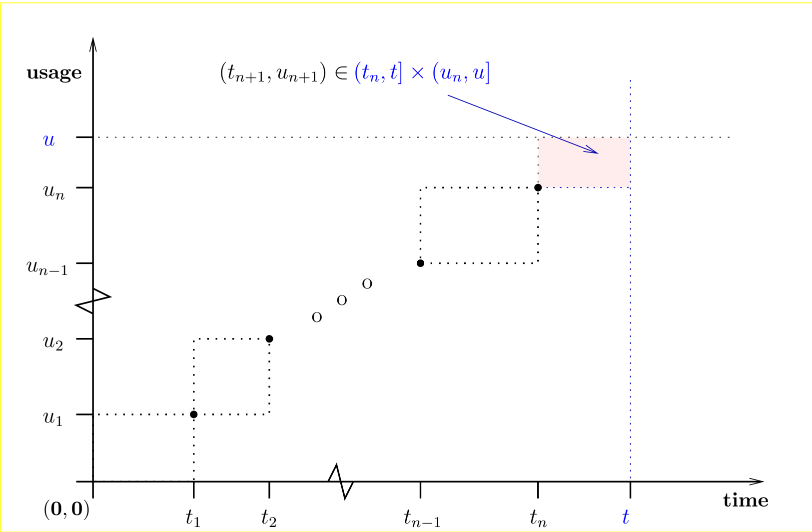


Figure 7.5: Illustration of the n failure points prior to (t, u) for the conditional distribution function $F_{n+1}(t, u|\boldsymbol{t_n}, \boldsymbol{u_n})$, defined for $t > t_n$ and $u > u_n$.

Notation. To simplify the expressions appearing in the remainder of this chapter, we adopt the following notation from Nelson [66]. Let G denote a non-decreasing real function of two variables, such that $G : S_1 \times S_2 \to \mathbb{R}$, where S_1 and S_2 are non-empty subsets of the real line $\mathbb{R} = (-\infty, \infty)$. Then, the *G-volume of set* $\mathcal{B}$, where $\mathcal{B} = [x_1, x_2] \times [y_1, y_2] \subseteq S_1 \times S_2$, is defined as

$$\begin{aligned} V_G(\mathcal{B}) &\equiv V_G([x_1, x_2] \times [y_1, y_2]) \\ &:= G(x_2, y_2) - G(x_2, y_1) - G(x_1, y_2) + G(x_1, y_1) \ . \end{aligned} \tag{7.15}$$

When G is a lifetime probability distribution function, then the G-volume of a set is interpreted as the probability of a failure occurring in that set (or region); see Figure 7.6.

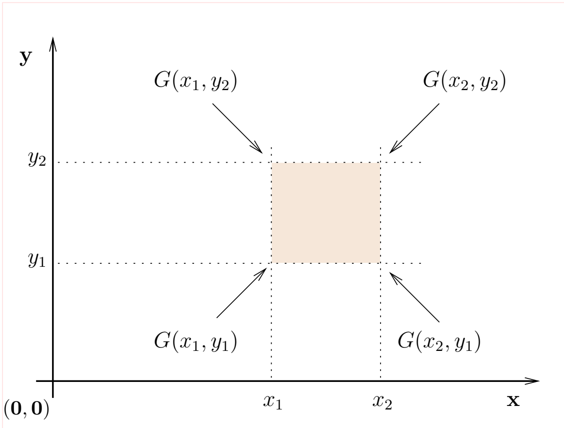


Figure 7.6: Illustration of the points of evaluation for the G-volume of the set $[x_1, x_2] \times [y_1, y_2]$.

The repair model in terms of the distribution function. Using the above notation, the effect of a general repair in terms of the conditional distribution function of the $(n+1)$-th failure point, for $n \in \mathbb{N}_+$, given all previous failure points, is given by

$$\begin{aligned} F_{n+1}(t, u | \boldsymbol{t_n}, \boldsymbol{u_n}) &= \mathrm{P}\{T_{n+1} \le t, U_{n+1} \le u \mid (T_1, U_1) = (t_1, u_1), \ldots, (T_n, U_n) = (t_n, u_n)\} \\ &= \mathrm{P}\{T_1 \le a(t, u), U_1 \le b(t, u) \mid T_1 \ge a(t_n^+, u_n^+), U_1 \ge b(t_n^+, u_n^+)\} \\ &= \frac{V_F([a(t_n^+, u_n^+), a(t, u)] \times [b(t_n^+, u_n^+), b(t, u)])}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \ , \end{aligned} \tag{7.16}$$

for $t \ge t_n^+$ and $u \ge u_n^+$, where $F(.,.)$ and $\bar{F}(.,.)$ are the distribution and reliability functions of the original bivariate lifetime.

Therefore, the probability of the system failing for the $(n+1)$-th time before time t and usage u, given all previous failure points, is equal to the probability of an identical system (or the original system) failing for the first time on or before the point $(a(t, u), b(t, u))$ given that it is still operating at the point $(a(t_n^+, u_n^+), b(t_n^+, u_n^+))$; see Figure 7.7. This condition (i.e. "still operating") is **not** equivalent to the event that the system does not fail prior to the point $(a(t_n^+, u_n^+), b(t_n^+, u_n^+))$, the probability of which is given by $1 - F(a(t_n^+, u_n^+), b(t_n^+, u_n^+))$. For bivariate distributions, $\bar{F} \neq 1 - F$; see Section 6.1.2.

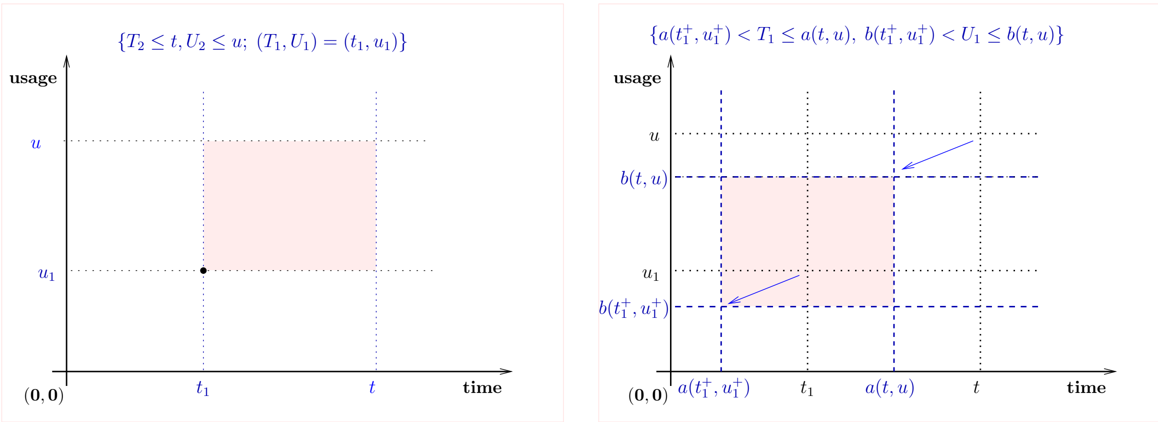


Figure 7.7: Illustrations of the sets (actual and virtual) used in defining $F_2(t, u|t_1, u_1)$, where $\delta_1 = 0.6$ and $\gamma_1 = 0.3$.

In (7.16), the support of the function $F(.,.)$ is the set $S_1 \times S_2 = \mathbb{R}_+^2$, and the F-volume of the virtual set is

$$\begin{aligned} &V_F([a(t_n^+, u_n^+), a(t,u)] \times [b(t_n^+, u_n^+), b(t,u)]) \\ &= F(a(t,u), b(t,u)) - F(a(t_n^+, u_n^+), b(t,u)) - F(a(t,u), b(t_n^+, u_n^+)) + F(a(t_n^+, u_n^+), b(t_n^+, u_n^+)) \\ &= \bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+)) - \bar{F}(a(t_n^+, u_n^+), b(t,u)) - \bar{F}(a(t,u), b(t_n^+, u_n^+)) + \bar{F}(a(t,u), b(t,u)) \ . \end{aligned} \tag{7.17}$$

Substituting (7.17) in (7.16), we can express the conditional distribution function in terms of the conditional reliability function defined in (7.10) as follows:

$$\begin{aligned} F_{n+1}(t, u|\boldsymbol{t_n}, \boldsymbol{u_n}) &= 1 - \frac{\bar{F}(a(t_n^+, u_n^+), b(t,u)) + \bar{F}(a(t,u), b(t_n^+, u_n^+)) - \bar{F}(a(t,u), b(t,u))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \\ &= 1 - \frac{\bar{F}(a(t_n^+, u), b(t_n^+, u)) + \bar{F}(a(t, u_n^+), b(t, u_n^+)) - \bar{F}(a(t,u), b(t,u))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \\ &= 1 - \bar{F}_{n+1}(t_n^+, u|\boldsymbol{t_n}, \boldsymbol{u_n}) - \bar{F}_{n+1}(t, u_n^+|\boldsymbol{t_n}, \boldsymbol{u_n}) + \bar{F}_{n+1}(t, u|\boldsymbol{t_n}, \boldsymbol{u_n}) \ . \end{aligned} \tag{7.18}$$

The second expression follows from the definitions of the virtual age and usage functions $A(.,.)$ and $B(.,.)$, given in (7.2). The proof if straight-forward: when n failures have occurred prior to the point (t, u) and given the n failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$, the virtual age at the point (t, u) depends only on the time variables, and the virtual usage at the point (t, u) depends only on the usage variables. Therefore, for all $t_n^+ \leq t < T_{n+1}$ and $u_n^+ \leq u < U_{n+1}$, the virtual age at the point (t, u) is equal to the virtual age at the point (t, u_n^+), and the virtual usage at the point (t, u) is equal to the virtual usage at the point (t_n^+, u). That is, the following equalities hold:

$$\begin{aligned} a(t_n^+, u_n^+) &= a(t_n^+, u) \ ; \qquad & a(t,u) &= a(t, u_n^+); \\ b(t_n^+, u_n^+) &= b(t, u_n^+) \ ; \qquad & b(t,u) &= b(t_n^+, u) \ . \end{aligned} \tag{7.19}$$

Sequence of perfect repairs. When all repairs are perfect, $a(t, u) = t - t_n$ and $b(t, u) = u - u_n$, for all $t > t_n$ and $u > u_n$. Therefore, the virtual age and usage immediately after a repair are both equal to zero, i.e. $a(t_n^+, u_n^+) = b(t_n^+, u_n^+) = 0$, for all $n \in \mathbb{N}_+$. Then, the conditional distribution function of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}), given in (7.16), reduces to

$$\begin{aligned} F_{n+1}(t, u | \boldsymbol{t_n}, \boldsymbol{u_n}) &= \frac{V_F([a(t_n^+, u_n^+), a(t,u)] \times [b(t_n^+, u_n^+), b(t,u)])}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \\ &= \frac{V_F([0, t - t_n] \times [0, u - u_n])}{\bar{F}(0,0)} \\ &= \frac{F(t - t_n, u - u_n) - F(t - t_n, 0) - F(0, u - u_n) + F(0,0)}{\bar{F}(0,0)} \\ &= F(t - t_n, u - u_n) \ , \end{aligned} \tag{7.20}$$

since $F(t - t_n, 0) = F(0, u - u_n) = F(0,0) = 0$ and $\bar{F}(0,0) = 1$; see Figure 7.8 (left). This is the conditional distribution function corresponding to the $(n+1)$-th renewal point of the renewal process in two dimensions [61].

Sequence of minimal repairs. When all repairs are minimal, then $a(t, u) = t$ and $b(t, u) = u$, for all $t > t_n$ and $u > u_n$, $n \in \mathbb{N}_+$. Then, the conditional distribution function of the $(n+1)$-th failure point, given in (7.16), becomes

$$\begin{aligned} F_{n+1}(t, u | \boldsymbol{t_n}, \boldsymbol{u_n}) &= \frac{V_F([a(t_n^+, u_n^+), a(t,u)] \times [b(t_n^+, u_n^+), b(t,u)])}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \\ &= \frac{V_F([t_n^+, t] \times [u_n^+, u])}{\bar{F}(t_n^+, u_n^+)} \\ &= \frac{F(t,u) - F(t_n^+, u) - F(t, u_n^+) + F(t_n^+, u_n^+)}{\bar{F}(t_n^+, u_n^+)} \\ &= \frac{F(t,u) - F(t_n, u) - F(t, u_n) + F(t_n, u_n)}{\bar{F}(t_n, u_n)} \ , \end{aligned} \tag{7.21}$$

since $F(.,.)$ and $\bar{F}(.,.)$ are absolutely continuous; see Figure 7.8 (right). This function is the conditional distribution function corresponding to the $(n+1)$-th failure point of the minimal repair process in two dimensions [64].

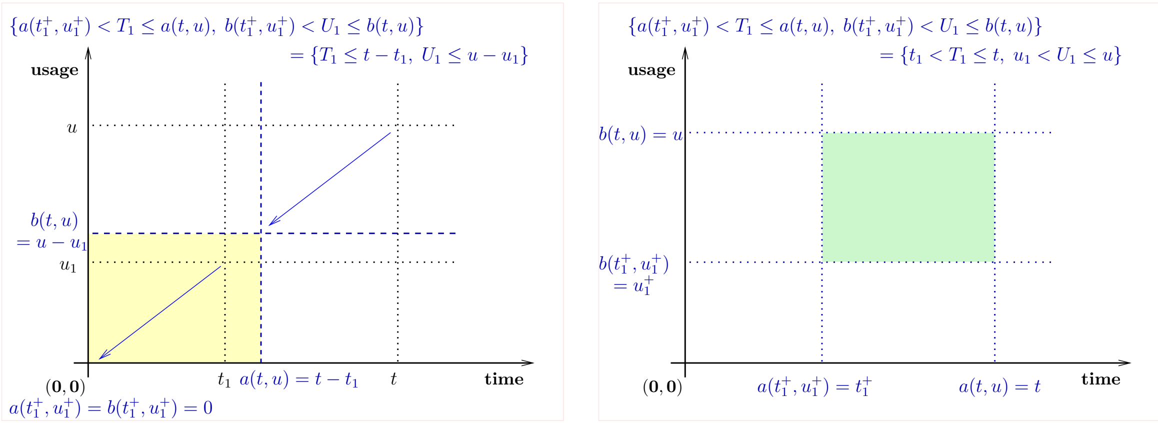


Figure 7.8: Illustrations of the (virtual) sets used in defining $F_2(t, u|t_1, u_1)$: for $(\delta_1, \gamma_1) = (1, 1)$ (left) and $(\delta_1, \gamma_1) = (0, 0)$ (right).

The conditional density function. The conditional density function of the $(n+1)$-th failure point, corresponding to the distribution and reliability functions in (7.16) and (7.10), is derived as follows:

$$\begin{aligned} f_{n+1}(t, u|\boldsymbol{t_n}, \boldsymbol{u_n}) &:= \frac{\partial^2}{\partial t\, \partial u} F_{n+1}(t, u|\boldsymbol{t_n}, \boldsymbol{u_n}) \\ &= \frac{\partial^2}{\partial t\, \partial u} \bar{F}_{n+1}(t, u|\boldsymbol{t_n}, \boldsymbol{u_n}) \\ &= \frac{1}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \frac{\partial^2}{\partial t\, \partial u} \bar{F}(a(t,u), b(t,u)) \\ &= \frac{f(a(t,u), b(t,u))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))}, \end{aligned} \tag{7.22}$$

for $t > t_n$ and $u > u_n$, where $f(.,.)$ is the bivariate density function of the original lifetime. Note that, by definition, $a(t, u)$ is of the form $t - c$ and $b(t, u)$ is of the form $u - c$, where c is constant with respect to t and u. Therefore, the second order mixed partial derivative of $\bar{F}(t - c, u - c)$, with respect to t and u, is simply $f(t - c, u - c)$.

7.3 The Failure (or General Repair) Process

The failure process $\{N(t, u); t, u \in \mathbb{R}_+\}$ is distinguished from other stochastic counting processes based on the relationships between the inter-failure lifetimes (X_n, Y_n), $n \in \mathbb{N}_+$. With a renewal process, the inter-failure lifetimes are independent and identically distributed bivariate random variables. Here, however, these lifetimes are neither independent nor identically distributed, and therefore, the distribution of any lifetime depends on all prior failure points. The increments of the counting process are not independent.

Since the processes $\{N(t,u); t,u \in \mathbb{R}_+\}$ and $\{(T_n, U_n); n \in \mathbb{N}_+\}$ are equivalent, the distributions of the numbers of failures can be derived from the distributions of the failure points. In this section, we derive the distributions of the failure points and the bivariate inter-failure lifetimes for the proposed failure (or general repair) process.

7.3.1 Distribution of Failure Points

In Section 7.2.2, we defined the effect of general repairs on the conditional distribution and reliability functions of succeeding failure points of the system, given all previous failure points. We now proceed to deriving the (unconditional) distribution and reliability functions of the failure points.

Let $F_n(.,.)$, $\bar{F}_n(.,.)$ and $f_n(.,.)$ denote respectively the (unconditional) distribution, reliability and density functions of the n-th failure point (T_n, U_n), where $n \in \mathbb{N}_+$. Then, the distribution function of the first failure point is given by $F_1(t,u) = F(t,u)$, and the corresponding reliability function is $\bar{F}_1(t,u) = \bar{F}(t,u)$, where F and $\bar{F}$ are the distribution and reliability functions of the bivariate lifetime of the original system. When the density function exists, it is given by

$$f_1(t,u) =: f(t,u) = \frac{\partial^2}{\partial t\, \partial u} F(t,u) = \frac{\partial^2}{\partial t\, \partial u} \bar{F}(t,u) \quad . \tag{7.23}$$

The distribution function of the second failure point (T_2, U_2) depends on that of the first, and is derived as follows:

$$\begin{aligned} F_2(t,u) &= \mathrm{P}\{T_2 \le t, U_2 \le u\} \\ &= \int_0^t \int_0^u F_2(t,u|t_1,u_1)\, dF_1(t_1,u_1) \\ &= \int_0^t \int_0^u \frac{V_F([a(t_1^+,u_1^+), a(t,u)] \times [b(t_1^+,u_1^+), b(t,u)])}{\bar{F}(a(t_1^+,u_1^+), b(t_1^+,u_1^+))}\, dF_1(t_1,u_1) \quad , \end{aligned} \tag{7.24}$$

where the conditional distribution function $F_2(.,.|t_1,u_1)$ is defined in (7.16), the F-volume $V_F(.)$ is defined in (7.15), and

$$dF_1(t,u) := f_1(t,u)\, dt\, du \quad . \tag{7.25}$$

The distribution function of the third failure point (T_3, U_3) depends on the previous two failure points, and is derived as follows:

$$\begin{aligned} F_3(t,u) &= \mathrm{P}\{T_3 \le t, U_3 \le u\} \\ &= \int_0^t \int_0^u \int_0^{t_2} \int_0^{u_2} F_3(t,u|\boldsymbol{t_2},\boldsymbol{u_2})\ f_2(t_2,u_2|t_1,u_1)\ f_1(t_1,u_1)\ du_1\ dt_1\ du_2\ dt_2 \\ &= \int_0^t \int_0^u \int_0^{t_2} \int_0^{u_2} \frac{V_F([a(t_2^+,u_2^+),a(t,u)] \times [b(t_2^+,u_2^+),b(t,u)])}{\bar{F}(a(t_2^+,u_2^+),b(t_2^+,u_2^+))}\ f_2(\boldsymbol{t_2},\boldsymbol{u_2})\ du_1\ dt_1\ du_2\ dt_2\ , \end{aligned} \tag{7.26}$$

where $f_2(\boldsymbol{t_2},\boldsymbol{u_2})$ denotes the joint density of the first two failure points at (t_1, u_1) and (t_2, u_2); and $f_2(.,.|t_1,u_1)$ denotes the conditional density function of the second failure point (T_2, U_2) given the first point $(T_1, U_1) = (t_1, u_1)$; see (7.22).

In general, for $n \in \mathbb{N}_+$ and $t, u \ge 0$, the (unconditional) distribution function $F_{n+1}(.,.)$ of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}) is given by

$$\begin{aligned} F_{n+1}(t,u) &= \mathrm{P}\{T_{n+1} \le t, U_{n+1} \le u\} \\ &= \int_0^t \int_0^u \ldots \int_0^{t_2} \int_0^{u_2} F_{n+1}(t,u|\boldsymbol{t_n},\boldsymbol{u_n})\ f_n(t_n,u_n|\boldsymbol{t_{n-1}},\boldsymbol{u_{n-1}}) \ldots f_1(t_1,u_1)\ du_1\ dt_1\ \ldots\ du_n\ dt_n \\ &= \int_0^t \int_0^u \ldots \int_0^{t_2} \int_0^{u_2} \frac{V_F([a(t_n^+,u_n^+),a(t,u)] \times [b(t_n^+,u_n^+),b(t,u)])}{\bar{F}(a(t_n^+,u_n^+),b(t_n^+,u_n^+))}\ f_n(\boldsymbol{t_n},\boldsymbol{u_n})\ du_1\ dt_1\ \ldots \\ &\qquad du_n\ dt_n\ , \end{aligned} \tag{7.27}$$

where $F_{n+1}(0,0) = F_{n+1}(t,0) = F_{n+1}(0,u) = 0$. Note that, the joint density of the first n failure points, denoted by $f_n(\boldsymbol{t_n},\boldsymbol{u_n})$ at the points $\{(t_1,u_1),\ldots,(t_n,u_n)\}$, is given by

$$f_n(\boldsymbol{t_n},\boldsymbol{u_n}) = f_n(t_n,u_n|\boldsymbol{t_{n-1}},\boldsymbol{u_{n-1}})\ \ldots\ f_2(t_2,u_2|t_1,u_1)\ f_1(t_1,u_1)\ , \tag{7.28}$$

defined for $0 < t_1 < t_2 < \ldots < t_n < \ldots$ and $0 < u_1 < u_2 < \ldots < u_n < \ldots$, where $\boldsymbol{t_n} = (t_1, \ldots, t_n)$ and $\boldsymbol{u_n} = (u_1, \ldots, u_n)$, for $n \in \mathbb{N}_+$; see (7.22) for the conditional density functions.

We can use the above distribution functions to derive the (unconditional) reliability functions using the following relationship: for $n \in \mathbb{N}_+$, the reliability function of the $(n+1)$-th failure point in terms of the corresponding distribution function is

$$\bar{F}_{n+1}(t,u) := 1 - F_{n+1}(t,\infty) - F_{n+1}(\infty,u) + F_{n+1}(t,u)\ , \tag{7.29}$$

for $t, u \geq 0$, where $F_{n+1}(t, \infty) =: F_{T_{n+1}}(t)$ and $F_{n+1}(\infty, u) =: F_{U_{n+1}}(u)$ are the marginal distribution functions of the $(n+1)$-th failure time and the $(n+1)$-th failure usage, respectively.

Alternatively, to derive the reliability function of (T_{n+1}, U_{n+1}) using the conditional reliability function defined in (7.10), we need to consider the possible positions of the previous failure point (T_n, U_n), for $n \in \mathbb{N}_+$; see Figure 7.9.

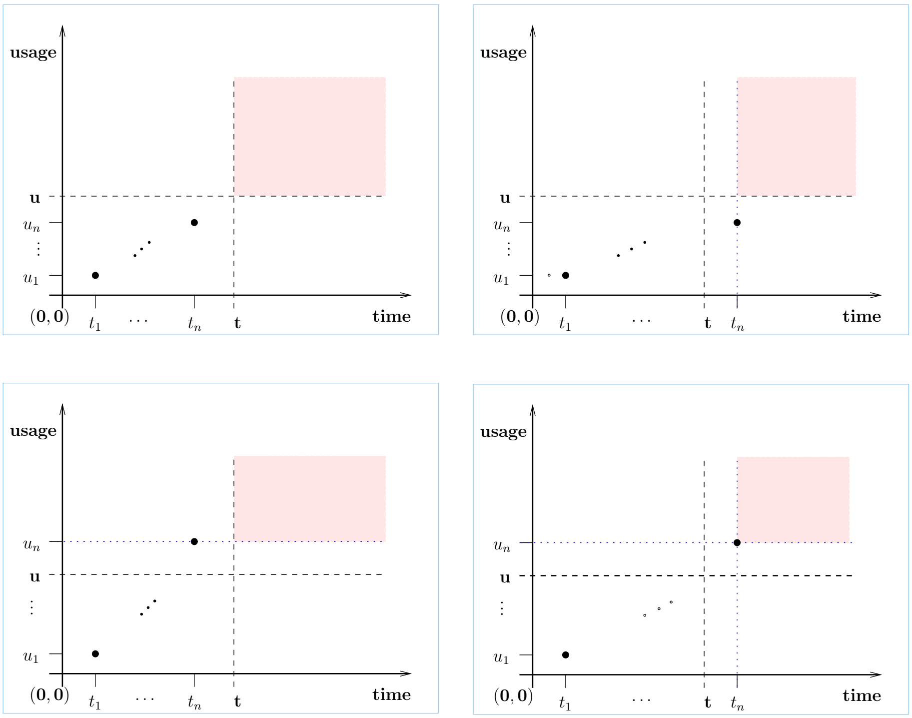


Figure 7.9: Illustration of the possible trajectories of the failure process, in terms of the n-th failure point (t_n, u_n). In each plot, the shaded area is the set corresponding to the conditional reliability function of (T_{n+1}, U_{n+1}) at the point (t, u).

The conditional reliability function of (T_{n+1}, U_{n+1}) in (7.10) was defined for $t > t_n$ and $u > u_n$, where (t_n, u_n) is the realization of the n-th failure point. To derive the reliability function, we need to determine the conditional reliability function for the four cases in Figure 7.9. Since the failure points are ordered, we have:

(i) when the n-th failure is before time t and usage u, the conditional reliability function is $\bar{F}_{n+1}(t, u|\boldsymbol{t_n}, \boldsymbol{u_n})$;

(ii) when the n-th failure is before time t but after usage u, the conditional reliability function is $\bar{F}_{n+1}(t, u_n|\boldsymbol{t_n}, \boldsymbol{u_n})$;

(iii) when the n-th failure is before usage u but after time t, the conditional reliability function is $\bar{F}_{n+1}(t_n, u|\boldsymbol{t_n}, \boldsymbol{u_n})$;

(iv) when the n-th failure is after time t and usage u, the conditional reliability function is $\bar{F}_{n+1}(t_n, u_n | \boldsymbol{t_n}, \boldsymbol{u_n}) = 1$.

Therefore, for $n \in \mathbb{N}_+$, the reliability function for the $(n+1)$-th failure point (T_{n+1}, U_{n+1}) can be derived as follows:

$$
\begin{aligned}
\bar{F}_{n+1}(t,u) &= \mathrm{P}\{T_{n+1} > t, U_{n+1} > u\} \\
&= \int_0^\infty \int_0^\infty \cdots \int_0^{t_2} \int_0^{u_2} \bar{F}_{n+1}(\max(t, t_n^+), \max(u, u_n^+) | \boldsymbol{t_n}, \boldsymbol{u_n})\, f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\, du_1\, dt_1 \ldots du_n\, dt_n \\
&= \int_0^t \int_0^u \cdots \int_0^{t_2} \int_0^{u_2} \frac{\bar{F}(a(t,u), b(t,u))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))}\, f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\, du_1\, dt_1 \ldots du_n\, dt_n \\
&\quad + \int_0^t \int_u^\infty \cdots \int_0^{t_2} \int_0^{u_2} \frac{\bar{F}(a(t, u_n^+), b(t, u_n^+))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))}\, f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\, du_1\, dt_1 \ldots du_n\, dt_n \\
&\quad + \int_t^\infty \int_0^u \cdots \int_0^{t_2} \int_0^{u_2} \frac{\bar{F}(a(t_n^+, u), b(t_n^+, u))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))}\, f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\, du_1\, dt_1 \ldots du_n\, dt_n \\
&\quad + \int_t^\infty \int_u^\infty \cdots \int_0^{t_2} \int_0^{u_2} \frac{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))}\, f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\, du_1\, dt_1 \ldots du_n\, dt_n \\
&= \mathrm{P}\{T_{n+1} > t,\ U_{n+1} > u,\ T_n \le t,\ U_n \le u\} + \mathrm{P}\{T_{n+1} > t,\ U_{n+1} > u,\ T_n \le t,\ U_n > u\} \\
&\quad + \mathrm{P}\{T_{n+1} > t,\ U_{n+1} > u,\ T_n > t,\ U_n \le u\} + \mathrm{P}\{T_n > t,\ U_n > u\}\ .
\end{aligned}
\tag{7.30}
$$

This function, when simplified, reduces to the reliability function in (7.29), which is defined in terms of the distribution function. We can derive the associated marginal reliability functions from the above reliability function as follows: $\bar{F}_{T_{n+1}}(t) := \bar{F}_{n+1}(t, 0)$ and $\bar{F}_{U_{n+1}}(u) := \bar{F}_{n+1}(0, u)$.

The density function of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}), for $n \in \mathbb{N}_+$, is given by

$$
\begin{aligned}
f_{n+1}(t,u) &= \int_0^t \int_0^u \cdots \int_0^{t_2} \int_0^{u_2} f_{n+1}(t, u | \boldsymbol{t_n}, \boldsymbol{u_n})\, f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\, du_1\, dt_1\ \ldots du_n\, dt_n \\
&= \int_0^t \int_0^u \cdots \int_0^{t_2} \int_0^{u_2} \frac{f(a(t,u), b(t,u))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))}\, f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\, du_1\, dt_1\ \ldots du_n\, dt_n\ ,
\end{aligned}
\tag{7.31}
$$

for $t, u \ge 0$, where the conditional density function is defined in (7.22).

The distribution functions derived here are functions of the bivariate degrees of repair (through the virtual age and usage functions), which we assume are preassigned.

The associated failure (or general repair) process is a generalization, in terms of the effectiveness of repairs, of the perfect repair (or renewal) process and the minimal repair process in two dimensions. Therefore, by selecting the appropriate degrees of repair, we can derive the distributions of the failure times of the perfect and minimal repair processes.

7.3.1.1 Failures Followed by Perfect Repair

When all failures of the system are followed by perfect repair, the virtual age and usage at any point depend only on the last renewal before that point. That is, when n failures have occurred before the point (t, u), $a(t, u) = t - t_n$ and $b(t, u) = u - u_n$, for $t > t_n$ and $u > u_n$, and $n \in \mathbb{N}_+$; see Section 7.2.1. Therefore, for $n \in \mathbb{N}_+$, using the corresponding conditional distribution function in (7.20), the distribution function in (7.27) reduces to

$$\begin{aligned} F_{n+1}(t,u) &= \int_0^t \int_0^u \dots \int_0^{t_2} \int_0^{u_2} F_{n+1}(t,u|\boldsymbol{t_n},\boldsymbol{u_n})\, f_n(\boldsymbol{t_n},\boldsymbol{u_n})\, du_1\, dt_1\, \dots\, du_n\, dt_n \\ &= \int_0^t \int_0^u F(t-t_n,u-u_n) \left(\int_0^{t_n} \int_0^{u_n} \dots \int_0^{t_2} \int_0^{u_2} f_n(\boldsymbol{t_n},\boldsymbol{u_n})\, du_1\, dt_1\, \dots du_{n-1}\, dt_{n-1} \right) du_n\, dt_n \\ &= \int_0^t \int_0^u F(t-t_n,u-u_n)\, f_n(t_n,u_n)\, du_n\, dt_n \\ &= \int_0^t \int_0^u F(t-t_n,u-u_n)\, dF_n(t_n,u_n) =: F^{**}F_n(t,u)\ , \end{aligned} \tag{7.32}$$

when all repairs are perfect, where $F(.,.)$ is the distribution function of the original bivariate lifetime; $F^{**}F_n$ denotes the convolution of F with F_n; and $dF_n(t_n, u_n) := f_n(t_n, u_n)\, dt_n\, du_n$. The corresponding density function is derived by substituting for the virtual age and usage functions in (7.31). Then, for $n \in \mathbb{N}_+$ and $t, u \geq 0$, we get

$$f_{n+1}(t,u) = \int_0^t \int_0^u \dots \int_0^{t_2} \int_0^{u_2} f(t-t_n,u-u_n)\, f_n(\boldsymbol{t_n},\boldsymbol{u_n})\, du_1\, dt_1\, \dots\, du_n\, dt_n\ , \tag{7.33}$$

since $a(t_n^+, u_n^+) = b(t_n^+, u_n^+) = 0$, for all $n \in \mathbb{N}$, and therefore, $\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+)) = \bar{F}(0, 0) = 1$. Note that, $f(.,.)$ is the density function of the original bivariate lifetime.

These functions are the distribution and density functions of the $(n+1)$-th replacement point of a renewal process in two dimensions; refer to Hunter [61]. Therefore, the proposed general repair process includes the renewal process as a special case, i.e. when all degrees of repair are $(1, 1)$.

7.3.1.2 Failures Followed by Minimal Repair

When all failures of the system are followed by minimal repair, the virtual age and usage at any point are equal to the actual age and usage; see Section 7.2.1. Then, the corresponding conditional distribution function of (T_{n+1}, U_{n+1}), defined in (7.21), is only a function of the last minimal repair at point (t_n, u_n). Therefore, when all repairs are minimal, for $n \in \mathbb{N}_+$ and $t, u \geq 0$, the distribution function in (7.27) reduces to

$$\begin{aligned}
F_{n+1}(t,u) &= \int_0^t \int_0^u \cdots \int_0^{t_2} \int_0^{u_2} F_{n+1}(t,u|\boldsymbol{t_n},\boldsymbol{u_n})\, f_n(\boldsymbol{t_n},\boldsymbol{u_n})\, du_1\, dt_1\, \ldots\, du_n\, dt_n \\
&= \int_0^t \int_0^u \frac{V_F([t_n^+,t]\times[u_n^+,u])}{\bar{F}(t_n^+,u_n^+)} \left(\int_0^{t_n} \int_0^{u_n} \cdots \int_0^{t_2} \int_0^{u_2} f_n(\boldsymbol{t_n},\boldsymbol{u_n})\, du_1\, dt_1\, \ldots du_{n-1}\, dt_{n-1} \right) du_n\, dt_n \\
&= \int_0^t \int_0^u \frac{F(t,u) - F(t_n,u) - F(t,u_n) + F(t_n,u_n)}{\bar{F}(t_n,u_n)}\, f_n(t_n,u_n)\, du_n\, dt_n \ ,
\end{aligned} \tag{7.34}$$

where $F(.,.)$ and $\bar{F}(.,.)$ denote the distribution and reliability functions of the original lifetime. The corresponding density function is given by substituting for the virtual age and usage functions in (7.31). For $n \in \mathbb{N}_+$ and $t, u \geq 0$, the density function of (T_{n+1}, U_{n+1}) is given by

$$f_{n+1}(t,u) = \int_0^t \int_0^u \cdots \int_0^{t_2} \int_0^{u_2} \frac{f(t,u)}{\bar{F}(t_n,u_n)}\, f_n(\boldsymbol{t_n},\boldsymbol{u_n})\, du_1\, dt_1\, \ldots\, du_n\, dt_n \ , \tag{7.35}$$

when all repairs are minimal, since $a(t,u) = t$ and $b(t,u) = u$, for all $t, u \geq 0$.

These functions are the distribution and density functions of the $(n+1)$-th failure point of the minimal repair process in two dimensions; refer to Baik et al. [64]. Therefore, the proposed general repair process reduces to the minimal repair process when all degrees of repair are set to $(0,0)$.

7.3.2 Distribution of Bivariate Inter-failure Lifetimes

In this section, we derive the distribution of the bivariate inter-failure lifetimes (X_n, Y_n), for $n \in \mathbb{N}_+$. The corresponding conditional distribution functions will later be used to simulate the proposed failure (or general repair) process.

The first bivariate inter-failure lifetime is equal to the point of first failure, i.e. $(X_1, Y_1) = (T_1, U_1)$. For $n \in \mathbb{N}_+$, the inter-failure time X_{n+1} is defined as the time between the n-th and

$(n+1)$-th failures, and the inter-failure usage Y_{n+1} is defined as the usage between the n-th and $(n+1)$-th failures, i.e.

$$(X_{n+1}, Y_{n+1}) = \big(T_{n+1} - T_n,\ U_{n+1} - U_n\big) \ . \tag{7.36}$$

Let $G_n(.,.)$ and $\bar{G}_n(.,.)$ denote the probability distribution and reliability functions of the n-th bivariate inter-failure lifetime (X_n, Y_n), for $n \in \mathbb{N}_+$. Then, the distribution function of the first bivariate inter-failure lifetime is given by

$$G_1(x,y) = F_1(x,y) = F(x,y) \ , \tag{7.37}$$

for $x, y \geq 0$, where $F_1 = F$ is the distribution corresponding to the lifetime of the original system, i.e. the point of first failure.

The distribution of the $(n+1)$-th bivariate inter-failure lifetime (X_{n+1}, Y_{n+1}) depends on the distributions of the n previous failure points $\{(T_1, U_1), \dots, (T_n, U_n)\}$, for $n \in \mathbb{N}_+$. Given all previous failure points, the probability that $X_{n+1} \leq x$ and $Y_{n+1} \leq y$ is equal to the probability of having the $(n+1)$-th failure in the region $(t_n, t_n + x] \times (u_n, u_n + y]$, for $n \in \mathbb{N}_+$ and $x, y > 0$, where (t_n, u_n) denotes the realization of the n-th failure point (T_n, U_n). Therefore, for $n \in \mathbb{N}_+$ and $x, y > 0$, the conditional distribution function of the $(n+1)$-th inter-failure lifetime is derived as follows:

$$\begin{aligned}
G_{n+1}(x, y | \boldsymbol{t_n}, \boldsymbol{u_n}) &:= \mathrm{P}\{X_{n+1} \leq x, Y_{n+1} \leq y \mid (T_1, U_1) = (t_1, u_1), \dots, (T_n, U_n) = (t_n, u_n)\} \\
&= \mathrm{P}\{T_{n+1} - T_n \leq x, U_{n+1} - U_n \leq y \mid (T_1, U_1) = (t_1, u_1), \dots, (T_n, U_n) = (t_n, u_n)\} \\
&= \mathrm{P}\{T_{n+1} \leq t_n + x, U_{n+1} \leq u_n + y \mid (T_1, U_1) = (t_1, u_1), \dots, (T_n, U_n) = (t_n, u_n)\} \\
&= F_{n+1}(t_n + x, u_n + y | \boldsymbol{t_n}, \boldsymbol{u_n}) \ ,
\end{aligned} \tag{7.38}$$

where $F_{n+1}(.,.|\boldsymbol{t_n}, \boldsymbol{u_n})$ is the conditional distribution function of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}), defined in (7.16). Given these conditional distribution functions, we can now derive the unconditional distribution functions $G_{n+1}(.,.)$, for $n \in \mathbb{N}_+$.

The distribution function of the second bivariate inter-failure lifetime (X_2, Y_2) is given by

$$\begin{aligned}
G_2(x,y) &= \mathrm{P}\{X_2 \leq x, Y_2 \leq y\} = \mathrm{P}\{T_2 \leq T_1 + x, U_2 \leq U_1 + y\} \\
&= \int_0^\infty \int_0^\infty F_2(t_1 + x, u_1 + y \mid t_1, u_1)\, f_1(t_1, u_1)\, du_1\, dt_1 \ .
\end{aligned} \tag{7.39}$$

Similarly, the distribution function of the third bivariate inter-failure lifetime (X_3, Y_3) is given by

$$\begin{aligned} G_3(x,y) &= \mathrm{P}\{X_3 \leq x, Y_3 \leq y\} = \mathrm{P}\{T_3 \leq T_2 + x, U_3 \leq U_2 + y\} \\ &= \int_0^{\infty}\int_0^{\infty}\int_0^{t_2}\int_0^{u_2} F_3(t_2 + x, u_2 + y|\boldsymbol{t_2}, \boldsymbol{u_2})\ f_2(\boldsymbol{t_2}, \boldsymbol{u_2})\ du_1\ dt_1\ du_2\ dt_2\ \ , \end{aligned} \tag{7.40}$$

where $f_2(\boldsymbol{t_2}, \boldsymbol{u_2})$ is the joint density of the first two failure points at the points (t_1, u_1) and (t_2, u_2)– this joint density function is defined in (7.28). Note that, the only difference between these distribution functions and those of the corresponding failure points, is that the value of the last observed failure point is not bounded from above (see end limits of the two outer integrals).

In general, the distribution function of the $(n+1)$-th bivariate inter-failure lifetime, for $n \in \mathbb{N}_+$ and $x, y > 0$, is given by

$$\begin{aligned} G_{n+1}(x,y) &= \mathrm{P}\{X_{n+1} \leq x, Y_{n+1} \leq y\} = \mathrm{P}\{T_{n+1} \leq T_n + x, U_{n+1} \leq U_n + y\} \\ &= \int_0^{\infty}\int_0^{\infty}\dots\int_0^{t_2}\int_0^{u_2} F_{n+1}(t_n + x, u_n + y|\boldsymbol{t_n}, \boldsymbol{u_n})\ f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\ du_1\ dt_1\ \dots\ du_n\ dt_n\ \ , \end{aligned} \tag{7.41}$$

where the conditional distribution and the joint density functions of the failure points are given in (7.16) and (7.28) respectively.

The corresponding reliability function can be derived in a similar manner, and is, for $n \in \mathbb{N}_+$ and $x, y > 0$, given by

$$\begin{aligned} \bar{G}_{n+1}(x,y) &= \mathrm{P}\{X_{n+1} > x, Y_{n+1} > y\} = \mathrm{P}\{T_{n+1} > T_n + x, U_{n+1} > U_n + y\} \\ &= \int_0^{\infty}\int_0^{\infty}\dots\int_0^{t_2}\int_0^{u_2} \bar{F}_{n+1}(t_n + x, u_n + y|\boldsymbol{t_n}, \boldsymbol{u_n})\ f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\ du_1\ dt_1\ \dots\ du_n\ dt_n\ \ , \end{aligned} \tag{7.42}$$

since the conditional reliability function of (X_{n+1}, Y_{n+1}) is given by

$$\begin{aligned} \bar{G}_{n+1}(x,y|\boldsymbol{t_n}, \boldsymbol{u_n}) &:= \mathrm{P}\{X_{n+1} > x, Y_{n+1} > y \mid (T_1, U_1) = (t_1, u_1), \dots, (T_n, U_n) = (t_n, u_n)\} \\ &= \mathrm{P}\{T_{n+1} > t_n + x, U_{n+1} > u_n + y \mid (T_1, U_1) = (t_1, u_1), \dots, (T_n, U_n) = (t_n, u_n)\} \\ &= \bar{F}_{n+1}(t_n + x, u_n + y|\boldsymbol{t_n}, \boldsymbol{u_n})\ \ . \end{aligned} \tag{7.43}$$

The conditional reliability function $\bar{F}_{n+1}(.,.|\boldsymbol{t_n}, \boldsymbol{u_n})$ of the failure point (T_{n+1}, U_{n+1}) is defined in (7.10). Note that, to derive the reliability functions of the bivariate inter-failure lifetimes, unlike for the failure points, we do not need to consider the four possible regions in which the last failure can occur [cf. (7.30); also see Figure 7.9].

Before we derive the corresponding density functions, we need to determine the derivatives of the virtual age and usage functions, which appear in the conditional distribution and reliability functions, with respect to x and y. The virtual age and usage functions of interest, for $x, y > 0$, are:

$$\begin{aligned} a(t_n + x, u_n + y) &= t_n + x - \sum_{i=1}^{n} \delta_i \, a(t_i, u_i) \ ; \\ b(t_n + x, u_n + y) &= u_n + y - \sum_{i=1}^{n} \gamma_i \, b(t_i, u_i) \ . \end{aligned} \tag{7.44}$$

Note that, since x and y are strictly greater than 0, the n-th failure is counted in evaluating the virtual age and usage. Also, the derivatives of the virtual age and usage functions above, with respect to x and y respectively, are both 1. Then, the density function of the $(n+1)$-th inter-failure lifetime, which we denote by $g_{n+1}(.,.)$, can be derived as follows:

$$\begin{aligned} g_{n+1}(x,y) &= \frac{\partial^2}{\partial x \, \partial y} \, G_{n+1}(x,y) = \frac{\partial^2}{\partial x \, \partial y} \, \bar{G}_{n+1}(x,y) \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^{t_2} \int_0^{u_2} \frac{\partial^2}{\partial x \, \partial y} \, \bar{F}_{n+1}(t_n + x, u_n + y | \boldsymbol{t_n}, \boldsymbol{u_n}) \, f_n(\boldsymbol{t_n}, \boldsymbol{u_n}) \, du_1 \, dt_1 \, \ldots \, du_n \, dt_n \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^{t_2} \int_0^{u_2} \frac{\partial^2}{\partial x \, \partial y} \frac{\bar{F}(a(t_n + x, u_n + y), b(t_n + x, u_n + y))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \, f_n(\boldsymbol{t_n}, \boldsymbol{u_n}) \, du_1 \, dt_1 \, \ldots \, du_n \, dt_n \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^{t_2} \int_0^{u_2} \frac{f(a(t_n + x, u_n + y), b(t_n + x, u_n + y))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \, f_n(\boldsymbol{t_n}, \boldsymbol{u_n}) \, du_1 \, dt_1 \, \ldots \, du_n \, dt_n \end{aligned} \tag{7.45}$$

for $x, y > 0$, where we have used the conditional reliability function in (7.10). Note that, the conditional density function in (7.45) can be replaced by

$$g_{n+1}(x, y | \boldsymbol{t_n}, \boldsymbol{u_n}) := \frac{f(a(t_n^+ + x, u_n^+ + y), b(t_n^+ + x, u_n^+ + y))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \ , \tag{7.46}$$

for $x, y \geq 0$ (i.e. equality included) [cf. (7.22)].

The distribution of a bivariate inter-failure lifetime depends, through the virtual age and usage functions, on the degrees of the general repairs performed prior to the start of that lifetime. In the following sections, we will show that by choosing the appropriate degrees of repair, the functions derived here reduce to those corresponding to the perfect and minimal repair processes in two dimensions.

7.3.2.1 Inter-failure Lifetimes following Perfect Repairs

When all repairs are perfect, then $a(t_n^+ + x, u_n^+ + y) = t_n^+ + x - t_n = x$, and similarly, $b(t_n^+ + x, u_n^+ + y) = y$. Then, immediately following the n-th repair, we have virtual age $a(t_n^+, u_n^+) = 0$ and virtual usage $b(t_n^+, u_n^+) = 0$. Therefore, the conditional distribution function in (7.38) reduces to

$$\begin{aligned} G_{n+1}(x, y|\boldsymbol{t_n}, \boldsymbol{u_n}) &= F_{n+1}(t_n + x, u_n + y|\boldsymbol{t_n}, \boldsymbol{u_n}) \\ &= \frac{V_F([a(t_n^+, u_n^+), a(t_n^+ + x, u_n^+ + y)] \times [b(t_n^+, u_n^+), b(t_n^+ + x, u_n^+ + y)])}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \\ &= \frac{V_F([0, x] \times [0, y])}{\bar{F}(0, 0)} = F(x, y) \ , \end{aligned} \tag{7.47}$$

for $x, y \geq 0$ and $n \in \mathbb{N}_+$ [cf. (7.20)]; see (7.17) for the definition of the F-volume $V_F(.)$. The corresponding (unconditional) distribution function in (7.41) becomes

$$\begin{aligned} G_{n+1}(x, y) &= \int_0^\infty \int_0^\infty \ldots \int_0^{t_2} \int_0^{u_2} F(x, y)\ f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\ du_1\ dt_1\ \ldots\ du_n\ dt_n \\ &= F(x, y) \int_0^\infty \int_0^\infty f_n(t_n, u_n)\ du_n\ dt_n = F(x, y) \ , \end{aligned} \tag{7.48}$$

when all repairs are perfect, since, for $n \in \mathbb{N}_+$, the integral of the density function $f_n(t_n, u_n)$ over its entire support is equal to $F_n(\infty, \infty) = 1$. This implies that the bivariate inter-failure lifetimes are independent and identically distributed random variables with a common distribution F, as is the case with the renewal process in two dimensions; refer to Hunter [61]. We can also prove this in terms of either the reliability function or the density function.

The conditional reliability function in (7.43), when all repairs are perfect, reduces to

$$\begin{aligned} \bar{G}_{n+1}(x, y|\boldsymbol{t_n}, \boldsymbol{u_n}) &= \bar{F}_{n+1}(t_n^+ + x, u_n^+ + y \mid \boldsymbol{t_n}, \boldsymbol{u_n}) \\ &= \frac{\bar{F}(a(t_n^+ + x, u_n^+ + y), b(t_n^+ + x, u_n^+ + y))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \\ &= \frac{\bar{F}(x, y)}{\bar{F}(0, 0)} = \bar{F}(x, y) \ , \end{aligned} \tag{7.49}$$

for $x, y \geq 0$ and $n \in \mathbb{N}_+$, since $\bar{F}(0, 0) = 1$. Then, the corresponding (unconditional) reliability function, given in (7.42), becomes

$$\begin{aligned}\bar{G}_{n+1}(x,y) &= \int_0^\infty \int_0^\infty \cdots \int_0^{t_2} \int_0^{u_2} \bar{F}(x,y)\, f_n(\boldsymbol{t_n},\boldsymbol{u_n})\, du_1\, dt_1\, \ldots\, du_n\, dt_n \\ &= \bar{F}(x,y) \int_0^\infty \int_0^\infty f_n(t_n,u_n)\, du_n\, dt_n = \bar{F}(x,y) \ .\end{aligned} \tag{7.50}$$

Similarly, the conditional density function defined in (7.46), when all repairs are perfect, reduces to

$$\begin{aligned}g_{n+1}(x,y|\boldsymbol{t_n},\boldsymbol{u_n}) &= \frac{f(a(t_n^+ + x, u_n^+ + y), b(t_n^+ + x, u_n^+ + y))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))} \\ &= \frac{f(x,y)}{\bar{F}(0,0)} = f(x,y) \ ,\end{aligned} \tag{7.51}$$

for $x, y \geq 0$. Then, the corresponding density function, given in (7.45), becomes

$$\begin{aligned}g_{n+1}(x,y) &= \int_0^\infty \int_0^\infty \cdots \int_0^{t_2} \int_0^{u_2} g_{n+1}(x,y|\boldsymbol{t_n},\boldsymbol{u_n})\, f_n(\boldsymbol{t_n},\boldsymbol{u_n})\, du_1\, dt_1\, \ldots\, du_n\, dt_n \\ &= f(x,y) \int_0^\infty \int_0^\infty \cdots \int_0^{t_2} \int_0^{u_2} f_n(\boldsymbol{t_n},\boldsymbol{u_n})\, du_1\, dt_1\, \ldots\, du_n\, dt_n = f(x,y) \ ,\end{aligned} \tag{7.52}$$

since the integral of the joint density function over its entire support is unity.

Therefore, the proposed failure process becomes a renewal process in two dimensions, when all degrees of repair are set to $(1, 1)$.

7.3.2.2 Inter-failure Lifetimes following Minimal Repairs

When all system failures are followed by minimal repairs, then $a(t,u) = t$ and $b(t,u) = u$, for all $t, u \geq 0$. Therefore, when all repairs are minimal, the conditional distribution function in (7.38) reduces to

$$\begin{aligned}G_{n+1}(x,y|\boldsymbol{t_n},\boldsymbol{u_n}) &= F_{n+1}(t_n^+ + x, u_n^+ + y|\boldsymbol{t_n},\boldsymbol{u_n}) \\ &= \frac{V_F([t_n^+, t_n^+ + x] \times [u_n^+, u_n^+ + y])}{\bar{F}(t_n^+, u_n^+)} \\ &= \frac{F(t_n + x, u_n + y) - F(t_n, u_n + y) - F(t_n + x, u_n) + F(t_n, u_n)}{\bar{F}(t_n, u_n)} \ ,\end{aligned} \tag{7.53}$$

for $x, y \geq 0$ and $n \in \mathbb{N}_+$ [cf. (7.21)]. Then, since the conditional distribution function depends only on the last failure point (t_n, u_n), the corresponding (unconditional) distribution function becomes

$$G_{n+1}(x,y) = \int_0^\infty \int_0^\infty \dots \int_0^{t_2} \int_0^{u_2} \frac{V_F([t_n^+, t_n^+ + x] \times [u_n^+, u_n^+ + y])}{\bar{F}(t_n^+, u_n^+)} f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\, du_1\, dt_1 \dots du_n\, dt_n$$
$$= \int_0^\infty \int_0^\infty \frac{V_F([t_n^+, t_n^+ + x] \times [u_n^+, u_n^+ + y])}{\bar{F}(t_n^+, u_n^+)} f_n(t_n, u_n)\, du_n\, dt_n \ , \tag{7.54}$$

for $x, y \geq 0$.

Similarly, when all repairs are minimal, the reliability function in (7.42) reduces to

$$\bar{G}_{n+1}(x,y) = \int_0^\infty \int_0^\infty \dots \int_0^{t_2} \int_0^{u_2} \frac{\bar{F}(t_n + x, u_n + y)}{\bar{F}(t_n, u_n)} f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\, du_1\, dt_1 \dots du_n\, dt_n$$
$$= \int_0^\infty \int_0^\infty \frac{\bar{F}(t_n + x, u_n + y)}{\bar{F}(t_n, u_n)} f_n(t_n, u_n)\, du_n\, dt_n \ , \tag{7.55}$$

since the corresponding conditional reliability function is given by

$$\bar{G}_{n+1}(x, y | \boldsymbol{t_n}, \boldsymbol{u_n}) = \bar{F}_{n+1}(t_n^+ + x, u_n^+ + y | \boldsymbol{t_n}, \boldsymbol{u_n}) = \frac{\bar{F}(t_n + x, u_n + y)}{\bar{F}(t_n, u_n)} \ , \tag{7.56}$$

for $x, y \geq 0$ and $n \in \mathbb{N}_+$ [cf. (7.12)].

Finally, the density function of (X_{n+1}, Y_{n+1}), given in (7.45), when all repairs are minimal, becomes

$$g_{n+1}(x,y) = \int_0^\infty \int_0^\infty \dots \int_0^{t_2} \int_0^{u_2} g_{n+1}(x, y | \boldsymbol{t_n}, \boldsymbol{u_n})\, f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\, du_1\, dt_1 \dots du_n\, dt_n$$
$$= \int_0^\infty \int_0^\infty \frac{f(t_n + x, u_n + y)}{\bar{F}(t_n, u_n)} \left(\int_0^{t_n} \int_0^{u_n} \dots \int_0^{t_2} \int_0^{u_2} f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\, du_1\, dt_1 \dots du_{n-1}\, dt_{n-1} \right) du_n\, dt_n$$
$$= \int_0^\infty \int_0^\infty \frac{f(t_n + x, u_n + y)}{\bar{F}(t_n, u_n)} f_n(t_n, u_n)\, du_n\, dt_n \ , \tag{7.57}$$

for $n \in \mathbb{N}_+$ and $x, y \geq 0$, since the corresponding conditional density function is given by

$$g_{n+1}(x, y | \boldsymbol{t_n}, \boldsymbol{u_n}) = \frac{f(a(t_n^+ + x, u_n^+ + y), b(t_n^+ + x, u_n^+ + y))}{\bar{F}(a(t_n^+, u_n^+), b(t_n^+, u_n^+))}$$
$$= \frac{f(t_n^+ + x, u_n^+ + y)}{\bar{F}(t_n^+, u_n^+)} = \frac{f(t_n + x, u_n + y)}{\bar{F}(t_n, u_n)} \ . \tag{7.58}$$

The derived functions are the distribution, density and reliability functions of the bivariate inter-failure lifetimes corresponding to the minimal repair process in two dimensions; refer to Baik et al. [64] (also, see Section 6.2.2). Therefore, when all degrees of repair are set to $(0,0)$ (i.e. when all repairs are minimal), the proposed failure process becomes the minimal repair process in two dimensions.

7.3.3 Distribution of Number of Failures

The distribution of the numbers (or counts) of failures occurring in given subsets of the space $\mathbb{R}^2_+$ can be derived using the distribution functions of the failure points (T_n, U_n), $n \in \mathbb{N}_+$, which we derived in Section 7.3.1.

For $n \in \mathbb{N}_+$, the probability of n failures occurring in the region $(0,t] \times (0,u]$, for $t, u > 0$, is given by

$$\begin{aligned}\mathrm{P}\{N(t,u) = n\} =& \mathrm{P}\{N(t,u) \geq n\} - \mathrm{P}\{N(t,u) \geq n+1\} \\ =& \mathrm{P}\{T_n \leq t, U_n \leq u\} - \mathrm{P}\{T_{n+1} \leq t, U_{n+1} \leq u\} \\ =& F_n(t,u) - F_{n+1}(t,u) \ ,\end{aligned} \tag{7.59}$$

and the probability of no failure occurring in the region $(0,t] \times (0,u]$ is given by

$$\begin{aligned}\mathrm{P}\{N(t,u) = 0\} &= 1 - \mathrm{P}\{N(t,u) \geq 1\} \\ &= 1 - \mathrm{P}\{T_1 \leq t, U_1 \leq u\} \\ &= 1 - F(t,u) \neq \bar{F}(t,u) \ .\end{aligned} \tag{7.60}$$

The expected number of failures in a given region $(0,t] \times (0,u]$ can also be computed using the distribution functions of the failure points, i.e.

$$E[N(t,u)] = \sum_{n=1}^{\infty} F_n(t,u) \ ; \tag{7.61}$$

see Section 6.1.1 for details.

7.4 Constructing BIFR Distributions

In order to simulate and illustrate the proposed general repair process in two dimensions, we need a distribution function that satisfies the bivariate increasing failure rate (BIFR) property discussed in Section 7.1. A distribution F is said to be BIFR, if the bivariate conditional reliability (BCR) function $\bar{F}(x+s, y+v)/\bar{F}(x, y)$ is decreasing in both $x \geq 0$ and $y \geq 0$, for each $s, v \geq 0$. Here, "increasing" and "decreasing" are not used in the strict sense (i.e. they are synonymous with "non-decreasing" and "non-increasing" respectively).

Bivariate distributions can be constructed from univariate distributions. Various transformations have been suggested for constructing bivariate reliability functions from known marginals (see Balakrishnan and Lai [67] for a comprehensive review). Consider, for instance, the bivariate exponential distribution by Gumbel [68]:

$$\bar{F}(x, y) = e^{-x-y-\theta xy} \quad , \tag{7.62}$$

where $\theta > 0$ and $x, y \geq 0$, which has exponential marginals with reliability functions $\bar{F}_X(x) = e^{-x}$ and $\bar{F}_Y(y) = e^{-y}$ (scale parameters can be introduced to generalize this model). This distribution is BIFR, since

$$\frac{\bar{F}(x+s, y+v)}{\bar{F}(x, y)} = e^{-s-v-\theta(sy+vx+sv)} \quad , \tag{7.63}$$

is decreasing in $x, y \geq 0$, for each $s, v \geq 0$.

Note that, the bivariate reliability function in (7.62) is of the form

$$\bar{F}(x, y) = \bar{F}_X(x)\ \bar{F}_Y(y)\ e^{-\theta\, K(x,y)} \quad , \tag{7.64}$$

where $\bar{F}_X(.)$ and $\bar{F}_Y(.)$ are the marginal reliability functions of the lifetime variables X and Y respectively, and the function $K(.,.)$ is chosen such that $K(x, 0) = K(0, y) = K(0, 0) = 0$, for all $x, y \geq 0$, and $K(x, y)$ is increasing in $x, y \geq 0$. For the purpose of this study, we will work with the model in (7.64) due to its simplicity; see Lu and Bhattacharyya [69] for more on this and other types of transformation.

For (7.64) to be a reliability function, the following boundary conditions need to be satisfied:

(i) $\bar{F}(0, 0) = 1$;

(ii) $\bar{F}(x, 0) = \bar{F}_X(x)$, for $x \geq 0$, and $\bar{F}(0, y) = \bar{F}_Y(y)$, for $y \geq 0$;

(iii) $\bar{F}(x, \infty) = \bar{F}(\infty, y) = \bar{F}(\infty, \infty) = 0$, for $x, y \geq 0$.

In addition, model parameters and the function $K(.,.)$ must be chosen such that the density– assuming that the second order mixed partial derivative of $\bar{F}(.,.)$ exists– at any point is non-negative, i.e. for all $x, y > 0$, $f(x, y) = \partial^2\, \bar{F}(x, y)/\partial x \partial y \geq 0$.

The bivariate cumulative hazard function, denoted by $H(.,.)$, for the above model is given by

$$H(x,y) = -\ln \bar{F}(x,y) = -\ln \bar{F}_X(x) - \ln \bar{F}_Y(y) + \theta\, K(x,y) \ , \tag{7.65}$$

for $x, y \geq 0$, and therefore the hazard gradient vector is given by

$$\begin{aligned} \boldsymbol{h}(x,y) &= \left(-\frac{\partial}{\partial x} \ln \bar{F}_X(x) + \theta \frac{\partial}{\partial x} K(x,y) \ , \ -\frac{\partial}{\partial y} \ln \bar{F}_Y(y) + \theta \frac{\partial}{\partial y} K(x,y) \right) \\ &= \left(r_X(x) + \theta\, k_x(x,y) \ , \ r_Y(y) + \theta\, k_y(x,y) \right) \\ &= \left(h_X(x,y), h_Y(x,y) \right) \ , \end{aligned} \tag{7.66}$$

where $k_x(x,y) := \partial K(x,y)/\partial x$ and $k_y(x,y) := \partial K(x,y)/\partial y$. Therefore, when $y = 0$, we get $h_X(x,0) = r_X(x)$, and when $x = 0$, we get $h_Y(0,y) = r_Y(y)$, where $r_X(.)$ and $r_Y(.)$ are the marginal (univariate) failure rate functions of X and Y. For Gumbel's exponential distribution in (7.62), the hazard gradient vector is given by

$$\boldsymbol{h}(x,y) = (1 + \theta y \ , 1 + \theta x) \tag{7.67}$$

If a distribution is BIFR, then the components of the associated hazard gradient vector are both increasing in $x, y \geq 0$; see Section 6.1.4. This can be easily verified for the hazard gradient vector in (7.67).

It has been shown that the BCR function $\bar{F}(x+s, y+v)/\bar{F}(x,y)$, for $s = v = \Delta$, is decreasing in x and y, for each $\Delta > 0$, if the partial derivatives

$$\frac{\frac{\partial}{\partial x} \bar{F}(x+\Delta, y+\Delta)}{\bar{F}(x+\Delta, y+\Delta)} = \frac{\partial}{\partial x} \ln \bar{F}(x+\Delta, y+\Delta) \tag{7.68}$$

and

$$\frac{\frac{\partial}{\partial y} \bar{F}(x+\Delta, y+\Delta)}{\bar{F}(x+\Delta, y+\Delta)} = \frac{\partial}{\partial y} \ln \bar{F}(x+\Delta, y+\Delta) \tag{7.69}$$

(assuming that they exist) are decreasing in $\Delta > 0$, for each $x, y \geq 0$ (i.e. if the two components of the hazard gradient vector $\boldsymbol{h}(x+\Delta, y+\Delta)$, which are the negatives of the partial derivatives in (7.68) and (7.69), are increasing in Δ, for each $x, y \geq 0$); refer to Block [70]. This can be verified for the Gumbel exponential distribution which is BIFR, where the components of the hazard gradient vector

$$\boldsymbol{h}(x+\Delta, y+\Delta) = (1 + \theta\, (y+\Delta) \ , 1 + \theta\, (x+\Delta)) \tag{7.70}$$

are increasing in Δ, for each $x, y \geq 0$. We note that this also applies in the case where $s = \Delta_1$ and $v = \Delta_2$ are not necessarily equal.

Theorem 1. *The BCR function $\bar{F}(x+\Delta_1, y+\Delta_2)/\bar{F}(x,y)$ is decreasing in $x, y \geq 0$, for each $\Delta_1, \Delta_2 > 0$, if the two components $h_X(x+\Delta_1, y+\Delta_2)$ and $h_Y(x+\Delta_1, y+\Delta_2)$ of the hazard gradient vector $\boldsymbol{h}(x+\Delta_1, y+\Delta_2)$ are increasing in Δ_1 and Δ_2, for each $x, y \geq 0$.*

Proof. The proof is similar to that given in Block [70] (where $\Delta_1 = \Delta_2 = \Delta$). If the first component $h_X(x+\Delta_1, y+\Delta_2)$ is increasing in $\Delta_1, \Delta_2 > 0$, for each $x, y \geq 0$, then the corresponding partial derivative is decreasing in $\Delta_1, \Delta_2 > 0$, for each $x, y \geq 0$. That is, for fixed x, y,

$$
\begin{aligned}
& h_X(x+\Delta_1, y+\Delta_2) = -\frac{\partial}{\partial x} \ln \bar{F}(x+\Delta_1, y+\Delta_2) \text{ is increasing in } \Delta_1, \Delta_2 > 0 \\
\Leftrightarrow \quad & \frac{\frac{\partial}{\partial x}\bar{F}(x+\Delta_1, y+\Delta_2)}{\bar{F}(x+\Delta_1, y+\Delta_2)} = \frac{\partial}{\partial x} \ln \bar{F}(x+\Delta_1, y+\Delta_2) \text{ is decreasing in } \Delta_1, \Delta_2 > 0 \\
\Rightarrow \quad & \frac{\frac{\partial}{\partial x}\bar{F}(x,y)}{\bar{F}(x,y)} \geq \frac{\frac{\partial}{\partial x}\bar{F}(x+\Delta_1, y+\Delta_2)}{\bar{F}(x+\Delta_1, y+\Delta_2)} \\
\Rightarrow \quad & \frac{\frac{\partial}{\partial x}\bar{F}(x,y)}{\bar{F}(x,y)} - \frac{\frac{\partial}{\partial x}\bar{F}(x+\Delta_1, y+\Delta_2)}{\bar{F}(x+\Delta_1, y+\Delta_2)} \geq 0 \\
\Rightarrow \quad & \frac{\bar{F}(x+\Delta_1, y+\Delta_2)\,\frac{\partial}{\partial x}\bar{F}(x,y) - \bar{F}(x,y)\,\frac{\partial}{\partial x}\bar{F}(x+\Delta_1, y+\Delta_2)}{\bar{F}(x,y)\,\bar{F}(x+\Delta_1, y+\Delta_2)} \geq 0 \\
\Rightarrow \quad & \frac{\bar{F}(x,y)\,\frac{\partial}{\partial x}\bar{F}(x+\Delta_1, y+\Delta_2) - \bar{F}(x+\Delta_1, y+\Delta_2)\,\frac{\partial}{\partial x}\bar{F}(x,y)}{\bar{F}^2(x,y)} \leq 0 \\
\Rightarrow \quad & \frac{\partial}{\partial x}\frac{\bar{F}(x+\Delta_1, y+\Delta_2)}{\bar{F}(x,y)} \leq 0\ .
\end{aligned}
\tag{7.71}
$$

Therefore, for fixed $x, y \geq 0$, if the first component $h_X(x+\Delta_1, y+\Delta_2)$ of the hazard gradient vector is increasing in $\Delta_1, \Delta_2 > 0$, then the BCR function is decreasing in x, for each $\Delta_1, \Delta_2 > 0$. It can be similarly shown that, if the second component $h_Y(x+\Delta_1, y+\Delta_2)$ of the hazard gradient vector is increasing in $\Delta_1, \Delta_2 > 0$, for fixed $x, y \geq 0$, then the BCR function is decreasing in y, for each $\Delta_1, \Delta_2 > 0$. Therefore, if both components are increasing in $\Delta_1, \Delta_2 > 0$, for fixed $x, y \geq 0$, the BCR function is decreasing in $x, y \geq 0$, for each $\Delta_1, \Delta_2 > 0$. □

Therefore, to determine if a distribution is BIFR, it is enough to check whether the components of the associated hazard gradient vector satisfy the property in Theorem 1. This is useful, since: (i) the components of the hazard gradient vector often have simpler forms than the bivariate reliability function $\bar{F}(.,.)$ and are therefore easier to differentiate; and (ii) one can define a hazard gradient vector with the above property and then construct a reliability function (and BIFR distribution) using the following relationship [52]:

$$\bar{F}(x,y) = e^{-\int_0^x h_X(s,y)\,ds - \int_0^y h_Y(0,v)\,dv} = e^{-\int_0^y h_Y(x,v)\,dv - \int_0^x h_X(s,0)\,ds} . \tag{7.72}$$

This allows us to construct BIFR distributions of the form in (7.64) with other marginal distributions; for instance, Weibull marginals, which have been widely used in reliability studies to model the distribution of lifetime variables. To construct a bivariate distribution from known marginal distributions, the functional form of $K(.,.)$ in (7.64) must be defined. For the distribution to be BIFR, it is enough that:

(C1) the failure rates $r_X(x)$ and $r_Y(y)$ are increasing in $x \geq 0$ and $y \geq 0$;

(C2) the partial derivatives $k_x(x+\Delta_1, y+\Delta_2)$ and $k_y(x+\Delta_1, y+\Delta_2)$ of the function $K(x+\Delta_1, y+\Delta_2)$ are both increasing in $\Delta_1, \Delta_2 > 0$, for each $x, y \geq 0$.

Note that, the hazard gradient vector should be defined such that the following equality (derived from (7.72)) is satisfied for all $x, y \geq 0$:

$$\begin{aligned} & \int_0^x h_X(s,y)\,ds - \int_0^y h_Y(0,v)\,dv = \int_0^y h_Y(x,v)\,dv - \int_0^x h_X(s,0)\,ds \\ \Leftrightarrow \quad & \int_0^x \big(h_X(s,y) - r_X(s)\big)\,ds = \int_0^y \big(h_Y(x,v) - r_Y(v)\big)\,dv \ . \end{aligned} \tag{7.73}$$

A BIFR Distribution. Let $R_X(.)$ and $R_Y(.)$ denote the cumulative marginal failure rate functions of X and Y respectively. Then, when $K(x,y) = R_X(x)\ R_Y(y)$, the bivariate reliability function in (7.64) becomes

$$\begin{aligned} \bar{F}(x,y) &= \bar{F}_X(x)\ \bar{F}_Y(y)\ e^{-\ \theta\ R_X(x)\ R_Y(y)} \\ &= e^{-R_X(x)\left[1+\theta R_Y(y)\right]}\ e^{-R_Y(y)} = e^{-R_Y(y)\left[1+\theta R_X(x)\right]}\ e^{-R_X(x)} \ , \end{aligned} \tag{7.74}$$

for $x, y \geq 0$. Is $\bar{F}(.,.)$ a reliability function? All boundary conditions are satisfied. We know that, the cumulative failure rate functions are such that $R_X(0) = R_Y(0) = 0$ and as increasing functions, when $x, y \to \infty$, $R_X(x), R_Y(y) \to \infty$. The marginal reliability functions are such that $\bar{F}_X(0) = \bar{F}_Y(0) = 1$ and $\bar{F}_X(\infty) = \bar{F}_Y(\infty) = 0$. Then, we have

(i) $\bar{F}(0,0) = \bar{F}_X(0)\ \bar{F}_Y(0)\ e^0 = 1$;

(ii) when $y = 0$, $\bar{F}(x,0) = \bar{F}_X(x)\ \bar{F}_Y(0)\ e^0 = \bar{F}_X(x)$, for $x \geq 0$, and when $x = 0$, $\bar{F}(0,y) =$

$\bar{F}_X(0)\ \bar{F}_Y(y)\ e^0 = \bar{F}_Y(y)$, for $y \geq 0$;

(iii) $\bar{F}(x,\infty) = \bar{F}_X(x)\ \bar{F}_Y(\infty)\ e^{-\infty} = 0$, for $x \geq 0$, and similarly, $\bar{F}(\infty,y) = 0$, for $y \geq 0$, and $\bar{F}(\infty,\infty) = 0$.

The second order mixed partial derivative of the function $\bar{F}(.,.)$ in (7.74) is given by

$$\begin{aligned}
\frac{\partial^2}{\partial x \partial y}\bar{F}(x,y) &= e^{-\theta\, R_X(x)\, R_Y(y)}\Big\{f_X(x)f_Y(y) + \bar{F}_X(x)\ \bar{F}_Y(y)\theta^2\ R_X(x)\ R_Y(y)r_X(x)\ r_Y(y) \\
&\quad + \theta\bar{F}_X(x)f_Y(y)R_Y(y)r_X(x) + \theta\bar{F}_Y(y)r_Y(y)\big(R_X(x)f_X(x) - r_X(x)\bar{F}_X(x)\big)\Big\} \\
&= e^{-\theta\, R_X(x)\, R_Y(y)}\Big\{f_X(x)f_Y(y) + f_X(x)f_Y(y)\theta^2\ R_X(x)\ R_Y(y) \\
&\quad + \theta f_X(x)f_Y(y)R_Y(y) + \theta f_X(x)f_Y(y)R_X(x) - \theta f_X(x)f_Y(y)\Big\} \\
&= f_X(x)f_Y(y)e^{-\theta\, R_X(x)\, R_Y(y)}\Big\{1 + \theta\ \big(R_X(x) + R_Y(y) + \theta\ R_X(x)\ R_Y(y) - 1\big)\Big\}\ ,
\end{aligned} \tag{7.75}$$

which is non-negative for all $x, y \geq 0$, when $0 \leq \theta \leq 1$. Therefore, for $\theta \in [0,1]$, the joint density function $f(x,y) = \partial^2\bar{F}(x,y)/\partial x\partial y \geq 0$, for all $x, y \geq 0$, and $\bar{F}(.,.)$ is a reliability function. Note that, in (7.74), $\theta = 0$ when the two variables X and Y are independent.

This distribution was defined for uniform marginals in Barnett [71], and is sometimes referred to as the Gumbel-Barnett Copula. The uniform marginals can be replaced by any marginal distribution; refer to Balakrishnan and Lai [67].

Next, one needs to determine when this distribution is BIFR. To test the distribution for the BIFR property, we can use Theorem 1 (on page 175) or equivalently we can check for conditions C1 and C2 (on page 176).

The partial derivatives of the function $K(x,y)$ with respect to x and y are given by

$$k_x(x,y) = \frac{\partial}{\partial x}\ K(x,y) = R_Y(y)\ r_X(x) \tag{7.76}$$

and

$$k_y(x,y) = \frac{\partial}{\partial y}\ K(x,y) = R_X(x)\ r_Y(y)\ , \tag{7.77}$$

respectively. Therefore, the corresponding hazard gradient vector is given by

$$\boldsymbol{h}(x,y) = \Big(r_X(x) + \theta\ R_Y(y)\ r_X(x)\ \ ,\ r_Y(y) + \theta\ R_X(x)\ r_Y(y)\Big)\ . \tag{7.78}$$

The partial derivatives of the hazard gradient vector $\boldsymbol{h}(x+\Delta_1, y+\Delta_2)$ with respect to Δ_1

and Δ_2 are given by

$$\frac{\partial}{\partial\Delta_1}\boldsymbol{h}(x+\Delta_1,y+\Delta_2) = \left(\frac{\partial}{\partial\Delta_1}r_X(x+\Delta_1)+\frac{\partial}{\partial\Delta_1}k_x(x+\Delta_1,y+\Delta_2)\ ,\ \frac{\partial}{\partial\Delta_1}k_y(x+\Delta_1,y+\Delta_2)\right)$$
$$= \left((1+\theta R_Y(y+\Delta_2))\frac{\partial}{\partial\Delta_1}r_X(x+\Delta_1)\ ,\ r_Y(y+\Delta_2)\ \theta\ \frac{\partial}{\partial\Delta_1}R_X(x+\Delta_1)\right)\ , \tag{7.79}$$

and

$$\frac{\partial}{\partial\Delta_2}\boldsymbol{h}(x+\Delta_1,y+\Delta_2) = \left(\frac{\partial}{\partial\Delta_2}k_x(x+\Delta_1,y+\Delta_2)\ ,\ \frac{\partial}{\partial\Delta_2}r_Y(y+\Delta_2)+\frac{\partial}{\partial\Delta_2}k_y(x+\Delta_1,y+\Delta_2)\right)$$
$$= \left(r_X(x+\Delta_1)\ \theta\ \frac{\partial}{\partial\Delta_2}R_Y(y+\Delta_2)\ ,\ (1+\theta\ R_X(x+\Delta_1))\frac{\partial}{\partial\Delta_2}r_Y(y+\Delta_2)\right)\ . \tag{7.80}$$

If the components of the vectors in (7.79) and (7.80) are non-negative, then this distribution is BIFR (from Theorem 1). This is equivalent to conditions C1 and C2 (defined on page 176) being satisfied.

C1? If the marginal failure rates $r_X(x)$ and $r_Y(y)$ are increasing in x and y respectively, then $r_X(x+\Delta_1)$ is increasing in Δ_1 for fixed x and $r_Y(y+\Delta_2)$ is increasing in Δ_2 for fixed y. In other words, their derivatives, which appear in (7.79) and (7.80), are non-negative.

C2? We know that the cumulative failure rate is an increasing function of its argument, i.e. $R_X(x)$ and $R_Y(y)$ are increasing in x and y respectively. This implies that $R_X(x+\Delta_1)$ is increasing in Δ_1 for fixed x and $R_Y(y+\Delta_2)$ is increasing in Δ_2 for fixed y. That is, their derivatives, which appear in (7.79) and (7.80), are non-negative. We also know that, the marginal failure rates and the parameter θ are all non-negative. Therefore, the partial derivatives

$$\begin{aligned}
\frac{\partial}{\partial\Delta_1}\ k_x(x+\Delta_1,y+\Delta_2) &= R_Y(y+\Delta_2)\frac{\partial}{\partial\Delta_1}r_X(x+\Delta_1)\\
\frac{\partial}{\partial\Delta_2}\ k_x(x+\Delta_1,y+\Delta_2) &= r_X(x+\Delta_1)\frac{\partial}{\partial\Delta_2}R_Y(y+\Delta_2)\\
\frac{\partial}{\partial\Delta_1}\ k_y(x+\Delta_1,y+\Delta_2) &= r_Y(y+\Delta_2)\frac{\partial}{\partial\Delta_1}R_X(x+\Delta_1)\\
\frac{\partial}{\partial\Delta_2}\ k_y(x+\Delta_1,y+\Delta_2) &= R_X(x+\Delta_1)\frac{\partial}{\partial\Delta_2}r_Y(y+\Delta_2)
\end{aligned} \tag{7.81}$$

are also non-negative, for each $x, y \geq 0$.

Note that, for this particular distribution, conditions C1 and C2 are equivalent, since the signs of the derivatives in (7.81) depend on the monotonicity of the failure rate functions $r_X(.)$ and $r_Y(.)$ (all remaining terms are non-negative). Therefore, to construct a BIFR distribution of the form (7.74), it is enough to define marginal failure rate functions that are increasing (or non-decreasing) in their respective arguments.

The BCR function for the Barnett-Gumbel distribution (with known marginals) is given by

$$\frac{\bar{F}(x+s, y+v)}{\bar{F}(x,y)} = e^{-\left[R_X(x+s)-R_X(x)\right]} e^{-\left[R_Y(y+v)-R_Y(y)\right]} e^{-\theta\left[R_X(x+s)R_Y(y+v)-R_X(x)R_Y(y)\right]}, \quad (7.82)$$

which is decreasing in $x, y \geq 0$, for each $s, v \geq 0$.

Example 1. For exponential marginals with scale parameters $\alpha_1, \alpha_2 > 0$, we have the following bivariate reliability function:

$$\bar{F}(x,y) = e^{-\left(\frac{x}{\alpha_1}\right)} e^{-\left(\frac{y}{\alpha_2}\right)} e^{-\theta\left(\frac{x}{\alpha_1}\right)\left(\frac{y}{\alpha_2}\right)}. \quad (7.83)$$

When the marginals are standard exponential distributions, then the Barnett-Gumbel distribution reduces to Gumbel's bivariate exponential distribution discussed earlier. Here, the failure rate functions are $r_X(x) = 1/\alpha_1$ and $r_Y(y) = 1/\alpha_2$, which are both non-decreasing.

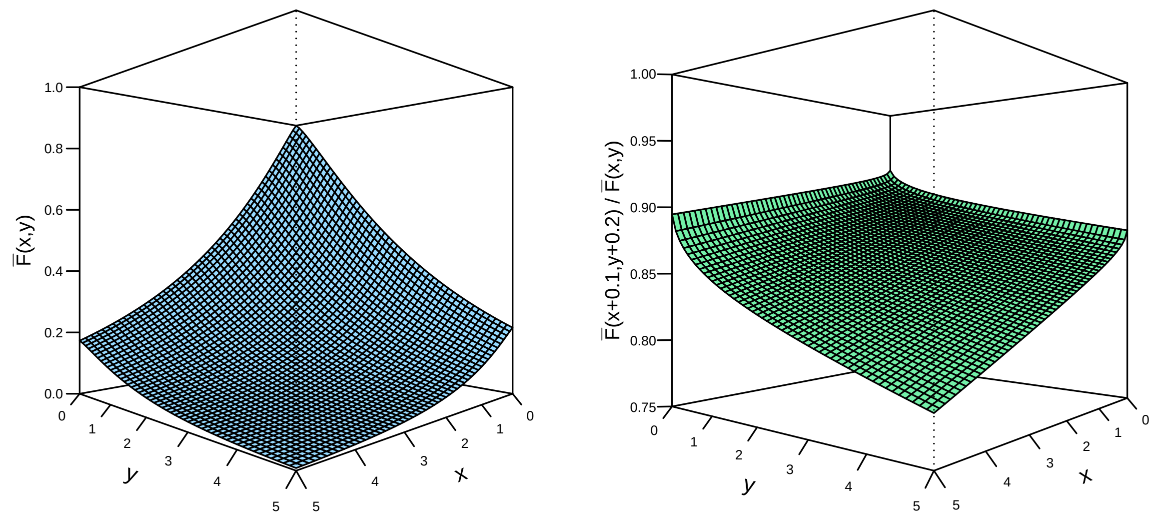


Figure 7.10: Illustrations of the bivariate reliability function (left) and the corresponding BCR function (right), with Weibull marginals, plotted over $x, y \in [0,5]$, for $s = 0.1$ and $v = 0.2$, and parameter values: $\theta = 0.7$, $\alpha_1 = 3$, $\beta_1 = 1.1$, $\alpha_2 = 3.5$ and $\beta_2 = 1.2$.

Example 2. For two-parameter Weibull marginals with scale parameters $\alpha_1, \alpha_2 > 0$ and shape parameters $\beta_1, \beta_2 \geq 0$, we have the following bivariate reliability function:

$$\bar{F}(x,y) = e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}} e^{-\left(\frac{y}{\alpha_2}\right)^{\beta_2}} e^{-\theta\left(\frac{x}{\alpha_1}\right)^{\beta_1}\left(\frac{y}{\alpha_2}\right)^{\beta_2}}. \quad (7.84)$$

Here, the marginal failure rate functions are

$$r_X(x) = \frac{\beta_1}{\alpha_1}\left(\frac{x}{\alpha_1}\right)^{\beta_1 - 1} \qquad \text{and} \qquad r_Y(y) = \frac{\beta_2}{\alpha_2}\left(\frac{y}{\alpha_2}\right)^{\beta_2 - 1}, \tag{7.85}$$

which are increasing in x and y respectively, for $\beta_1, \beta_2 > 1$. Therefore, for $\beta_1, \beta_2 > 1$, the distribution is BIFR.

We will use this distribution for our illustrations. In Figure 7.10, we have plotted the reliability and BCR functions of the Barnett-Gumbel distribution for Weibull marginals.

7.5 Chapter Conclusion

In this chapter, we proposed a new failure (or general repair) process to model consecutive failures (followed by general repair) of a system whose lifetime is modeled as a bivariate random variable. The lifetime distribution is assumed to be bivariate increasing failure rate (BIFR), and therefore, the effect of a general repair is modeled as a possible decrease in the virtual age and usage of the system following the repair. Therefore, the system after a general repair behaves like a younger (in terms of both age and usage), identical system.

We derived the distribution functions of the failure points and the bivariate inter-failure lifetimes, and showed that by choosing the appropriate degrees of repair, the proposed failure process reduces to the minimal and perfect repair processes in two dimensions.

We also investigated an approach to construct bivariate lifetime distributions with the BIFR property, using the marginal distributions of the two variables. We will use the Barnett-Gumbel distribution with Weibull marginals in our numerical illustrations later in this study.

In the following chapters, we will discuss some properties of the general repair models and suggest a simulation procedure to simulate trajectories of the associated general repair process and estimate the expected number of failures.

Chapter 8

Modeling Repairs in Two Dimensions: Model Properties

In this chapter, we discuss various properties of the general repair process in two dimensions proposed in the previous chapter– in particular, we examine the behavior of the proposed general repair model with respect to the degrees of repair.

This chapter is arranged as follows. In Section 8.1, we discuss the effect of the degrees of repair on the reliability of the system conditional on previous failure times. In Section 8.2, we discuss the effect of the degrees of repair on the ordering of the bivariate residual lifetime distributions. In Section 8.3, we show that when a functional relationship is imposed on the failure times and usages, then the proposed two-dimensional approach reduces to the one-dimensional approach to failure modeling in two dimensions, as defined by Blischke & Murthy [2]. In Section 8.4, we conclude with a summary of the chapter.

8.1 Conditional Reliability and Degrees of Repair

A general repair model must have the following property: the reliability of the system following a repair should improve as the degree of the repair increases. In this section, we discuss the relationship between the degrees of the general repairs performed on the system and the system reliability following these repairs.

We have assumed that the system lifetime distribution is BIFR, which translates to a bivariate conditional reliability (BCR) function that is decreasing in both time and usage; see Section 7.1. That is, given that a system has no prior failure, a younger (in terms of age and usage) system is more reliable than an older, identical system. Therefore, we model the effect of a general repair as reductions in the virtual age and virtual usage of the system following the repair.

Let the realizations of the virtual age and usage functions at the point (t,u), when n failures have occurred before this point, be denoted by $a_n(t,u)$ and $b_n(t,u)$ respectively, for $t,u \geq 0$ and $n \in \mathbb{N}_+$. Given the n failure points prior to the point (t,u), the virtual age $a_n(t,u)$ is a function of the failure times $\boldsymbol{t_n} = (t_1,\ldots,t_n)$ and the corresponding components $\boldsymbol{\delta_n} = (\delta_1,\ldots,\delta_n)$ of the bivariate degrees of repair. Similarly, the virtual usage $b_n(t,u)$ is a function of the failure usages $\boldsymbol{u_n} = (u_1,\ldots,u_n)$ and the corresponding components $\boldsymbol{\gamma_n} = (\gamma_1,\ldots,\gamma_n)$ of the degrees of repair. We have used the subscript n here to emphasize that these functions are functions of the n previous failure points and the corresponding degrees of repair. Note that, before the first failure (or when all repairs are minimal), the virtual age and usage functions are equal to the actual age and usage: $a_0(t,u) = t$ and $b_0(t,u) = u$.

Before the first failure, we used the BCR function $\bar{F}(t+x,u+y)/\bar{F}(t,u)$ as an indicator of the working condition of the system. Now suppose that n failures have occurred prior to the point (t,u). Then, for $n \in \mathbb{N}_+$, the following conditional reliability function can be viewed as an indicator of the working condition of the system at the point (t,u):

$$\begin{aligned}
&\mathrm{P}\{T_{n+1}>t+x, U_{n+1}>u+y \mid T_{n+1}>t, U_{n+1}>u, (T_1,U_1)=(t_1,u_1),\ldots,(T_n,U_n)=(t_n,u_n)\} \\
&\quad = \frac{\mathrm{P}\{T_{n+1}>t+x, U_{n+1}>u+y \mid (T_1,U_1)=(t_1,u_1),\ldots,(T_n,U_n)=(t_n,u_n)\}}{\mathrm{P}\{T_{n+1}>t, U_{n+1}>u \mid (T_1,U_1)=(t_1,u_1),\ldots,(T_n,U_n)=(t_n,u_n)\}} \\
&\quad = \frac{\bar{F}_{n+1}(t+x,u+y|\boldsymbol{t_n},\boldsymbol{u_n})}{\bar{F}_{n+1}(t,u|\boldsymbol{t_n},\boldsymbol{u_n})} = \frac{\bar{F}(a_n(t+x,u+y), b_n(t+x,u+y))}{\bar{F}(a_n(t,u),b_n(t,u))} \\
&\quad = \frac{\bar{F}(x+a_n(t,u), y+b_n(t,u))}{\bar{F}(a_n(t,u),b_n(t,u))} \;,
\end{aligned} \tag{8.1}$$

for $x,y > 0$, where the last line follows from the definitions of the virtual age and usage functions given in Section (7.2.1). When no failure has occurred in the region $(t,t+x] \times (u,u+y]$, then $a_n(t+x,u+y) = x + a_n(t,u)$ and $b_n(t+x,u+y) = y + b_n(t,u)$. When $t = t_n^+$ and $u = u_n^+$, (8.1) reduces to the conditional reliability function of the $(n+1)$-th failure point (T_{n+1},U_{n+1}), which is given in (7.10) on page 153.

Notice that, the conditional reliability function in (8.1) is now of the same form as the BCR function $\bar{F}(t+x,u+y)/\bar{F}(t,u)$, where t and u have been respectively replaced by the virtual age and usage at the point (t,u). We know that, the virtual age $a_n(t,u)$ is decreasing in each δ_j, for $j \in \{1,\ldots,n\}$, when all other parameters in this function are fixed, and similarly, the virtual usage $b_n(t,u)$ is decreasing in each γ_j, for $j \in \{1,\ldots,n\}$, when all other parameters in this function are fixed (see (7.2) on page 150 for the definitions of the virtual age and usage functions). Since the lifetime distribution F is assumed to be BIFR, the BCR function $\bar{F}(t+x, u+y)/\bar{F}(t,u)$ is decreasing in $t, u \geq 0$, for each $x, y \geq 0$. Therefore, the conditional reliability function in (8.1) is increasing in each δ_j and each γ_j, for $j \in \{1, \ldots, n\}$, when all remaining parameters (arguments and degrees) remain fixed.

The conditional reliability in (8.1) is the conditional probability of the system surviving x units of time and y units of usage, conditional on the system operating at time t and usage u, given that n failures have occurred prior to the point (t, u). Therefore, (8.1) is the reliability corresponding to the *conditional bivariate residual lifetime* of the system at time t and usage u:

$$\left[T_{n+1} - t, U_{n+1} - u \mid T_{n+1} > t, U_{n+1} > u, (T_1, U_1) = (t_1, u_1), \ldots, (T_n, U_n) = (t_n, u_n);\ \boldsymbol{\delta_n}, \boldsymbol{\gamma_n}\right] , \tag{8.2}$$

which is the bivariate analogue of the univariate residual lifetime conditional on all previous failure times and the corresponding repair degrees. When $t = t_n^+$ and $u = u_n^+$, then (8.2) is equal in distribution to the conditional bivariate inter-failure lifetime

$$\left[X_{n+1}, Y_{n+1} \mid (T_1, U_1) = (t_1, u_1), \ldots, (T_n, U_n) = (t_n, u_n);\ \boldsymbol{\delta_n}, \boldsymbol{\gamma_n}\right] , \tag{8.3}$$

which is also a function of the n previous failure times and degrees of repair, for $n \in \mathbb{N}_+$. Note that, the conditional reliability function of (8.3) is derived from (8.1) by setting $t = t_n^+$ and $u = u_n^+$. For any δ_i, $i \in \{1, \ldots, n\}$, when all other parameters are fixed, (8.3) with $\delta_i = \delta'$ is stochastically larger than (8.3) with $\delta_i = \delta$, when $\delta' \geq \delta$ (using Definition 6.6 on page 133). Similarly, for any γ_i, $i \in \{1, \ldots, n\}$, when all other parameters are fixed, (8.3) with $\gamma_i = \gamma'$ is stochastically larger than (8.3) with $\gamma_i = \gamma$, when $\gamma' \geq \gamma$. This follows from the conditional reliability in (8.1) being an increasing function of each δ_i and each γ_i, for $i \in \{1, \ldots, n\}$, when all other parameters are fixed.

The above results imply that, *at any point, the conditional reliability of the system following an imperfect repair is bounded between the conditional reliability following a minimal repair and the conditional reliability following a perfect repair.* Note that, in the bivariate case, this ordering of the degrees of repair based on the effectiveness of the repairs is not a complete ordering. For instance, it is not implied that an imperfect repair of degree $(0.2, 0.6)$ is more (or less) effective than an imperfect repair of degree $(0.6, 0.2)$. Both repairs are bounded (in effectiveness) between a minimal repair and a perfect repair, but the ordering of the reliabilities following the two imperfect repairs depends on the lifetime distribution.

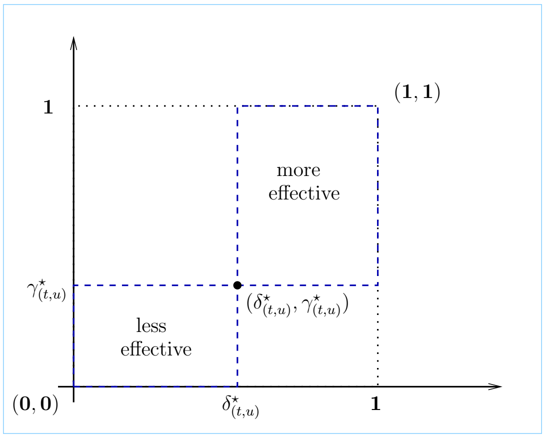


Figure 8.1: Illustration of the ordering of repairs in terms of their bivariate degree of repair (i.e. their effectiveness in improving the reliability of the repaired system).

With general repair models in one dimension, we have complete ordering independent of the lifetime distribution, i.e. the effectiveness (reliability improvement) of a general repair at any point increases as its univariate degree δ increases from 0 to 1. For the proposed general repair model in two dimensions, a partial ordering which applies to any BIFR lifetime distribution is as follows. A general repair of degree $(\delta_{(t,u)}, \gamma_{(t,u)})$ performed at the point (t, u), when compared to an imperfect repair of degree $(\delta^\star_{(t,u)}, \gamma^\star_{(t,u)})$ performed at the same point, is

- less effective, if $\delta_{(t,u)} < \delta^\star_{(t,u)}$ and $\gamma_{(t,u)} < \gamma^\star_{(t,u)}$; and
- more effective, if $\delta_{(t,u)} > \delta^\star_{(t,u)}$ and $\gamma_{(t,u)} > \gamma^\star_{(t,u)}$; see Figure 8.1 for an illustration.

8.2 Ordering of Conditional Residual Lifetimes

In this section, we discuss the partial ordering of bivariate distributions based on the conditional reliability vector, the hazard gradient vector and the mean residual vector; see Section 6.1.5 for more on partial orderings of bivariate lifetime distributions.

Let, as before, (T_n, U_n) and (X_{n+1}, Y_{n+1}) denote the n-th failure point and the $(n+1)$-th bivariate inter-failure lifetime, for $n \in \mathbb{N}_+$. Also, let $F(.,.)$ denote the distribution of the original lifetime $(T_1, U_1) = (X_1, Y_1)$. Recall that, for the original lifetime, the conditional reliability vector with components corresponding to the residual random variables $[X_1 - x | X_1 > x, Y_1 > y]$ and $[Y_1 - y | X_1 > x, Y_1 > y]$ is given by

$$\varphi_{\bar{F}_1}(s, v; x, y) = \left(\frac{\bar{F}(x+s, y)}{\bar{F}(x, y)}, \frac{\bar{F}(x, y+v)}{\bar{F}(x, y)} \right) \; ; \tag{8.4}$$

see Section 6.1.4.1. The corresponding conditional mean residual lifetime (MRL) vector is given by

$$\boldsymbol{\mu_1}(x,y) = \left(\mu_1^X(x,y),\ \mu_1^Y(x,y)\right) = \left(E[X_1 - x|X_1 > x, Y_1 > y],\ E[Y_1 - y|X_1 > x, Y_1 > y]\right)\ , \tag{8.5}$$

where the conditional mean residual time (MRT) and the conditional mean residual usage (MRU) functions can be expressed in terms of the components of the conditional reliability vector [53]. That is,

$$\mu_1^X(x,y) = \int_0^\infty \mathrm{P}\{X_1 > x+s|X_1 > x, Y_1 > y\}\ ds = \int_0^\infty \frac{\bar{F}(x+s,y)}{\bar{F}(x,y)}\ ds\ , \tag{8.6}$$

and

$$\mu_1^Y(x,y) = \int_0^\infty \mathrm{P}\{Y_1 > y+v|X_1 > x, Y_1 > y\}\ dv = \int_0^\infty \frac{\bar{F}(x,y+v)}{\bar{F}(x,y)}\ dv\ ; \tag{8.7}$$

see Section 6.1.4.3.

Also recall that the hazard gradient vector corresponding to the lifetime of the original system is

$$\mathbf{h_1}(x,y) = \left(h_{X_1}(x,y),\ h_{Y_1}(x,y)\right)\ , \tag{8.8}$$

where the components of the hazard gradient are such that

$$h_{X_1}(x,y)\ dx \approx \mathrm{P}\{X_1 \le x+dx|X_1 > x, Y_1 > y\} = 1 - \frac{\bar{F}(x+dx,y)}{\bar{F}(x,y)}\ , \tag{8.9}$$

and

$$h_{Y_1}(x,y)\ dy \approx \mathrm{P}\{Y_1 \le y+dy|X_1 > x, Y_1 > y\} = 1 - \frac{\bar{F}(x,y+dy)}{\bar{F}(x,y)}\ ; \tag{8.10}$$

see Section 6.1.4.2. Note that, as in the univariate case, the hazard gradient vector corresponding to the residual time and usage $([X_1 - x|X_1 > x, Y_1 > y], [Y_1 - y|X_1 > x, Y_1 > y])$ is the same as the hazard gradient vector corresponding to the original lifetime (X_1, Y_1).

When the lifetime distribution F is BIFR, then: (i) $\bar{F}(x+s,y)/\bar{F}(x,y)$ is decreasing in $x \ge 0$, for each $s, y \ge 0$ (or equivalently $h_{X_1}(x,y)$ is increasing in $x \ge 0$); and (ii) $\bar{F}(x,y+v)/\bar{F}(x,y)$ is decreasing in $y \ge 0$, for each $v, x \ge 0$ (or equivalently $h_{Y_1}(x,y)$ is increasing in $y \ge 0$). This implies that the components of the conditional MRL vector are also decreasing in the corresponding argument, i.e. $\mu_1^X(x,y)$ is decreasing in $x \ge 0$, for each $y \ge 0$, and $\mu_1^Y(x,y)$ is decreasing in $y \ge 0$, for each $x \ge 0$; see Section 6.1.4.

8.2.1 General Repairs and Conditional Reliability Vectors

The vectors in (8.4), (8.5) and (8.8) are all associated with the original bivariate lifetime (X_1, Y_1). Now, suppose that n failures have occurred before the point (t, u), where $n \in \mathbb{N}_+$ and $t, u > 0$. Let the conditional residual time of the system at the point (t, u), given that the system has been repaired n times prior to (t, u), $n \in \mathbb{N}_+$, be denoted by

$$\begin{aligned} &X_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_n}, \boldsymbol{\gamma_n}) \\ &:= [X_{n+1} - s \mid X_{n+1} > s, Y_{n+1} > v, (T_1, U_1) = (t_1, u_1), \ldots, (T_n, U_n) = (t_n, u_n);\ \boldsymbol{\delta_n}, \boldsymbol{\gamma_n}] \\ &\overset{\mathrm{d}}{=} [T_{n+1} - t \mid T_{n+1} > t, U_{n+1} > u, (T_1, U_1) = (t_1, u_1), \ldots, (T_n, U_n) = (t_n, u_n);\ \boldsymbol{\delta_n}, \boldsymbol{\gamma_n}]\ , \end{aligned} \tag{8.11}$$

and, similarly, the conditional residual usage is denoted by

$$\begin{aligned} &Y_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_n}, \boldsymbol{\gamma_n}) \\ &:= [Y_{n+1} - v \mid X_{n+1} > s, Y_{n+1} > v, (T_1, U_1) = (t_1, u_1), \ldots, (T_n, U_n) = (t_n, u_n);\ \boldsymbol{\delta_n}, \boldsymbol{\gamma_n}] \\ &\overset{\mathrm{d}}{=} [U_{n+1} - u \mid T_{n+1} > t, U_{n+1} > u, (T_1, U_1) = (t_1, u_1), \ldots, (T_n, U_n) = (t_n, u_n);\ \boldsymbol{\delta_n}, \boldsymbol{\gamma_n}]\ , \end{aligned} \tag{8.12}$$

where $s = t - t_n^+ \geq 0$ and $v = u - u_n^+ \geq 0$. The residual time $X_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_n}, \boldsymbol{\gamma_n})$ is the time until the $(n+1)$-th failure of the system, given that the system is in an operational state at the point (t, u) and has been repaired n times prior to this point. Similarly, the residual usage $Y_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_n}, \boldsymbol{\gamma_n})$ is the usage until the $(n+1)$-th failure of the system, given that the system is in an operational state at the point (t, u) and has been repaired n times prior to this point. The above notations are used to emphasize the dependence on previous failure points and the corresponding degrees of repair.

When $n = 0$, (8.11) and (8.12) reduce to the conditional residual time and usage, respectively, of the original bivariate lifetime (X_1, Y_1). When $t = t_n^+$ and $u = u_n^+$, then (8.11) and (8.12) become, respectively, the $(n+1)$-th conditional inter-failure time and the $(n+1)$-th conditional inter-failure usage, given previous failure points.

Sequence of minimal repairs. When all n repairs are minimal, the conditional residual time and usage at (t, u) are equal in distribution to those corresponding to the original system lifetime at the same point (since the system behaves as though it has not failed), i.e.

$$\begin{aligned} X_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_n} = \mathbf{0}, \boldsymbol{\gamma_n} = \mathbf{0}) &\overset{\mathrm{d}}{=} [X_1 - t \mid X_1 > t, Y_1 > u] \\ Y_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_n} = \mathbf{0}, \boldsymbol{\gamma_n} = \mathbf{0}) &\overset{\mathrm{d}}{=} [Y_1 - u \mid X_1 > t, Y_1 > u]\ . \end{aligned} \tag{8.13}$$

Sequence of perfect repairs. When all n repairs are perfect, the conditional residual time and usage at (t,u) are equal in distribution to those corresponding to the original system lifetime at the point $(t-t_n, u-u_n)$, where (t_n,u_n) is the last renewal point before (t,u), i.e.

$$\begin{aligned} X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n}=\mathbf{1},\boldsymbol{\gamma_n}=\mathbf{1}) &\overset{\mathrm{d}}{=} [X_1-(t-t_n)\mid X_1>t-t_n, Y_1>u-u_n] \\ Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n}=\mathbf{1},\boldsymbol{\gamma_n}=\mathbf{1}) &\overset{\mathrm{d}}{=} [Y_1-(u-u_n)\mid X_1>t-t_n, Y_1>u-u_n]\ . \end{aligned} \tag{8.14}$$

To derive the reliability functions corresponding to the conditional bivariate residual lifetime vector $\left(X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n}),\ Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n})\right)$, we use the conditional reliability function defined in (8.1). By setting, respectively, $y=0$ and $x=0$ in (8.1), we derive the conditional reliability vector

$$\varphi_{\bar{F}_{n+1}}(x,y;t,u,\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n}) := \left(\frac{\bar{F}(x+a_n(t,u),b_n(t,u))}{\bar{F}(a_n(t,u),b_n(t,u))},\ \frac{\bar{F}(a_n(t,u),y+b_n(t,u))}{\bar{F}(a_n(t,u),b_n(t,u))}\right)\ , \tag{8.15}$$

defined for $t>t_n$, $u>u_n$ and $x,y\geq 0$. As mentioned earlier, when $n=0$, then (8.15) reduces to (8.4), since $a_0(t,u)=t$ and $b_0(t,u)=u$.

In the previous section, we established that the conditional reliability function in (8.1) is increasing in each δ_i and each γ_i, for $i\in\{1,\ldots,n\}$ and $n\in\mathbb{N}_+$, when all other parameters remain fixed. This implies that, the components of the conditional reliability vector in (8.15) are also increasing in each δ_i and each γ_i (when other parameters are fixed), since they are derived from (8.1) by setting $y=0$ and $x=0$ respectively. This implies that, for $\delta'\geq\delta$ and $\gamma'\geq\gamma$, where $\delta,\delta',\gamma,\gamma'\in[0,1]$,

$$\begin{aligned} X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n=\delta',\gamma_n) &\geq_{\mathrm{ST}} X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n=\delta,\gamma_n)\ ; \\ X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n,\gamma_n=\gamma') &\geq_{\mathrm{ST}} X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n,\gamma_n=\gamma)\ , \end{aligned} \tag{8.16}$$

and similarly,

$$\begin{aligned} Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n=\delta',\gamma_n) &\geq_{\mathrm{ST}} Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n=\delta,\gamma_n)\ ; \\ Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n,\gamma_n=\gamma') &\geq_{\mathrm{ST}} Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n,\gamma_n=\gamma)\ , \end{aligned} \tag{8.17}$$

defined for $t>t_n$ and $u>u_n$, $n\in\mathbb{N}_+$. The orderings in (8.16) and (8.17) are true for any of the components of the n bivariate degrees of repair [I]. Therefore, the conditional residual time and usage both become stochastically larger when either component of any degree of

[I]These partial orderings are in effect univariate; see Section 3.1.3 for more on univariate partial orderings of distributions.

repair is increased, while all other parameters are fixed.

Given the inequalities in (8.16) and (8.17), using Definition 6.7 on page 134, the bivariate random variable

$$\begin{aligned}&\big(X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},(\delta_n,\gamma_n)=(\delta',\gamma')),\\ &\;Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},(\delta_n,\gamma_n)=(\delta',\gamma'))\big)\end{aligned} \tag{8.18}$$

is conditionally stochastically larger than the bivariate random variable

$$\begin{aligned}&\big(X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},(\delta_n,\gamma_n)=(\delta,\gamma)),\\ &\;Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},(\delta_n,\gamma_n)=(\delta,\gamma))\big)\end{aligned} \tag{8.19}$$

when $\delta' \geq \delta$ and $\gamma' \geq \gamma$– this is true for any of the n degrees of repair, when other parameters are fixed. The bivariate stochastic ordering discussed in Section 8.1 implies this conditional stochastic ordering.

8.2.2 General Repairs and Conditional Mean Residual Vectors

Using the conditional reliability vector we can derive the expected values of the conditional residual time and usage. The *conditional mean residual time* is derived as follows:

$$\begin{aligned}\mu^X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n}) &:= E[X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n})]\\ &= \int_0^\infty \mathrm{P}\{T_{n+1}-t>x \mid T_{n+1}>t, U_{n+1}>u, (T_1,U_1)=(t_1,u_1),\ldots,(T_n,U_n)=(t_n,u_n)\}\,dx\\ &= \int_0^\infty \frac{\mathrm{P}\{T_{n+1}>t+x, U_{n+1}>u \mid (T_1,U_1)=(t_1,u_1),\ldots,(T_n,U_n)=(t_n,u_n)\}}{\mathrm{P}\{T_{n+1}>t, U_{n+1}>u \mid (T_1,U_1)=(t_1,u_1),\ldots,(T_n,U_n)=(t_n,u_n)\}}\,dx\\ &= \int_0^\infty \frac{\bar F_{n+1}(t+x,u|\boldsymbol{t_n},\boldsymbol{u_n})}{\bar F_{n+1}(t,u|\boldsymbol{t_n},\boldsymbol{u_n})}\,dx = \int_0^\infty \frac{\bar F(a_n(t+x,u),\ b_n(t+x,u))}{\bar F(a_n(t,u),\ b_n(t,u))}\,dx\\ &= \int_0^\infty \frac{\bar F(x+a_n(t,u),\ b_n(t,u))}{\bar F(a_n(t,u),\ b_n(t,u))}\,dx\ ,\end{aligned} \tag{8.20}$$

and similarly, the *conditional mean residual usage* is given by

$$\begin{aligned}\mu^Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n}) &:= E[Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n})]\\ &= \int_0^\infty \frac{\bar F(a_n(t,u),\ y+b_n(t,u))}{\bar F(a_n(t,u),\ b_n(t,u))}\,dy\ .\end{aligned} \tag{8.21}$$

Let the corresponding conditional MRL vector be denoted by

$$\boldsymbol{\mu}_{\boldsymbol{n+1}}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n}) := \left(\mu^X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n}),\ \mu^Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n})\right) \ . \tag{8.22}$$

The relationships between the components of the conditional reliability vector in (8.15) and the corresponding components of the conditional MRL vector in (8.22) are the same as those of the components of the vectors $\varphi_{\bar{F}_1}(s,v;x,y)$ and $\boldsymbol{\mu_1}(x,y)$ of the original lifetime, given in (8.4) and (8.5); see Section 6.1.4. Note that, these relationships are equivalent to the univariate relationships discussed in Chapter 4.

Since the components of the conditional reliability vector are both increasing in each δ_i and each γ_i, for $i \in \{1,\ldots,n\}$, when other parameters are fixed, the corresponding components of the mean residual vector are also increasing in each δ_i and each γ_i, for $i \in \{1,\ldots,n\}$, when all other parameters are fixed. That is, stochastic ordering implies conditional mean residual (MR) ordering, i.e. for $\delta' > \delta$ and $\gamma' > \gamma$, the following inequalities hold:

$$\begin{aligned}
&X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n=\delta',\gamma_n) \geq_{\mathrm{MR}} X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n=\delta,\gamma_n) \ ; \\
&X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n,\gamma_n=\gamma') \geq_{\mathrm{MR}} X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n,\gamma_n=\gamma) \ ,
\end{aligned} \tag{8.23}$$

and similarly,

$$\begin{aligned}
&Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n=\delta',\gamma_n) \geq_{\mathrm{MR}} Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n=\delta,\gamma_n) \ ; \\
&Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n,\gamma_n=\gamma') \geq_{\mathrm{MR}} Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},\delta_n,\gamma_n=\gamma) \ .
\end{aligned} \tag{8.24}$$

The inequalities in (8.23) and (8.24) hold for either component of any of the n bivariate degrees of repair. Then, using Definition 6.9 on page 134, the bivariate random variable

$$\begin{gathered}
\big(X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},(\delta_n,\gamma_n)=(\delta',\gamma')), \\
Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},(\delta_n,\gamma_n)=(\delta',\gamma'))\big)
\end{gathered} \tag{8.25}$$

is conditionally stochastically larger in MR ordering than the bivariate random variable

$$\begin{gathered}
\big(X_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},(\delta_n,\gamma_n)=(\delta,\gamma)), \\
Y_{n+1}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_{n-1}},\boldsymbol{\gamma_{n-1}},(\delta_n,\gamma_n)=(\delta,\gamma))\big)
\end{gathered} \tag{8.26}$$

when $\delta' \geq \delta$ and $\gamma' \geq \gamma$, for $t > t_n$ and $u > u_n$– this is also true for any of the n degrees of repair, when other parameters are fixed. Therefore, the conditional stochastic ordering in

Section 8.2.1 implies the conditional stochastic ordering in terms of mean residual.

As the MRL vector in (8.5) completely determines the original bivariate distribution F (see Johnson [53]), the conditional MRL vector completely determines the distribution of the corresponding conditional bivariate lifetime given in (8.3).

8.2.3 General Repairs and Conditional Hazard Gradient Vectors

In the previous sections, we showed that as either component of any degree of repair increases, while all other parameters remain fixed, the components of the conditional reliability and MRL vectors increase. The monotonicity with the components of the corresponding hazard gradient is the opposite– i.e. as either component of any degree of repair increases, while all other parameters remain fixed, the components of the hazard gradient decrease.

Let the conditional hazard gradient vector, corresponding to the conditional MRL vector in (8.22), be denoted by

$$\mathbf{h_{n+1}}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n}) := \left(h_{X_{n+1}}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n}),\ h_{Y_{n+1}}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n})\right)\ , \tag{8.27}$$

which is defined for $t > t_n$ and $u > u_n$, where $n \in \mathbb{N}_+$. As in the univariate case, the hazard rate of the conditional mean residual time (usage) is the same as the hazard rate of the corresponding conditional inter-failure time (usage). The components of this conditional hazard gradient vector are such that

$$\begin{aligned} & h_{X_{n+1}}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n})\ dt \\ & \quad \approx \mathrm{P}\{T_{n+1} \le t+dt \mid T_{n+1} > t, U_{n+1} > u, (T_1,U_1) = (t_1,u_1),\ldots,(T_n,U_n) = (t_n,u_n)\} \\ & \quad = 1 - \mathrm{P}\{T_{n+1} > t+dt \mid T_{n+1} > t, U_{n+1} > u, (T_1,U_1) = (t_1,u_1),\ldots,(T_n,U_n) = (t_n,u_n)\} \\ & \quad = 1 - \frac{\bar{F}_{n+1}(t+dt,u|\boldsymbol{t_n},\boldsymbol{u_n})}{\bar{F}_{n+1}(t,u|\boldsymbol{t_n},\boldsymbol{u_n})} = 1 - \frac{\bar{F}(dt + a_n(t,u), b_n(t,u))}{\bar{F}(a_n(t,u),b_n(t,u))}\ , \end{aligned} \tag{8.28}$$

and similarly,

$$h_{Y_{n+1}}(t,u;\mathbf{t_n},\mathbf{u_n},\boldsymbol{\delta_n},\boldsymbol{\gamma_n})\ du \approx 1 - \frac{\bar{F}(a_n(t,u), du + b_n(t,u))}{\bar{F}(a_n(t,u),b_n(t,u))}\ . \tag{8.29}$$

Notice that, the conditional reliability functions appearing in (8.28) and (8.29) are respectively the components of the conditional reliability vector given in (8.15).

As shown in Section 8.2.1, the components of the conditional reliability vector are both increasing in each δ_i and each γ_i, $i \in \{1,\ldots,n\}$, when all other parameters of the function remain fixed. This implies that the corresponding components of the conditional hazard gradient vector are both decreasing in each δ_i and each γ_i, for $i \in \{1, \dots, n\}$ and $n \in \mathbb{N}_+$, when other parameters are fixed. This leads to the following ordering, in terms of hazard rate (HR), of the conditional residual times and usages: for $\delta' > \delta$ and $\gamma' > \gamma$,

$$
\begin{aligned}
&X_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-1}}, \delta_n = \delta', \gamma_n) \geq_{\text{HR}} X_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-1}}, \delta_n = \delta, \gamma_n) \ ; \\
&X_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-1}}, \delta_n, \gamma_n = \gamma') \geq_{\text{HR}} X_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-1}}, \delta_n, \gamma_n = \gamma) \ ,
\end{aligned}
\tag{8.30}
$$

which holds for any component of the n degrees of repair; and similarly,

$$
\begin{aligned}
&Y_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-1}}, \delta_n = \delta', \gamma_n) \geq_{\text{HR}} Y_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-1}}, \delta_n = \delta, \gamma_n) \ ; \\
&Y_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-1}}, \delta_n, \gamma_n = \gamma') \geq_{\text{HR}} Y_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-1}}, \delta_n, \gamma_n = \gamma) \ .
\end{aligned}
\tag{8.31}
$$

Therefore, using Definition 6.8 on page 134, the bivariate random variable

$$
\begin{gathered}
\big(X_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-1}}, (\delta_n, \gamma_n) = (\delta', \gamma')), \\
Y_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-1}}, (\delta_n, \gamma_n) = (\delta', \gamma'))\big)
\end{gathered}
\tag{8.32}
$$

is conditionally stochastically larger in HR ordering than the bivariate random variable

$$
\begin{gathered}
\big(X_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-1}}, (\delta_n, \gamma_n) = (\delta, \gamma)), \\
Y_{n+1}(t, u; \mathbf{t_n}, \mathbf{u_n}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-1}}, (\delta_n, \gamma_n) = (\delta, \gamma))\big)
\end{gathered}
\tag{8.33}
$$

when $\delta' \geq \delta$ and $\gamma' \geq \gamma$, for $t > t_n$ and $u > u_n$– this is also true for any of the n degrees of repair, when other parameters are fixed. Note that, the stochastic ordering with respect to the conditional hazard gradient vector implies the stochastic ordering with respect to both the conditional reliability and MRL vectors (from Sections 8.2.1 and 8.2.2).

As with the conditional MRL vector, the conditional hazard gradient vector completely determines the distribution of the corresponding conditional bivariate inter-failure lifetime, given in (8.3).

8.3 One-Dimensional Approach as Special Case of Two-Dimensional Approach

As discussed in Section 6.2.1, one approach to modeling consecutive failures of a system in two dimensions is to assume that, the cumulative usage at any time is a stochastic function of time. The stochastic nature of the relationship is often due to treating one or more parameters (e.g. usage rate) of the function as a random variable. The relationship between the cumulative usage and time is usually assumed to be linear; however, non-linear relationships can be considered; see for instance Yang and Nachlas [58] and Eliashberg et al [60]. This approach is known as the *one-dimensional approach* to modeling failures in two dimensions, since given the value of the stochastic parameters, the process reduces to a one-dimensional process; see Section 6.2.1.

In this section, we first derive the distribution functions of the consecutive failure points for a one-dimensional approach, where usage is modeled as a linear function of time. Then, we show that the distribution functions derived for the two-dimensional approach (proposed in Chapter 7), when usage is set as a linear function of time, reduce to the distribution functions derived for this one-dimensional approach. Therefore, the proposed two-dimensional modeling approach can be viewed as a generalization of this one-dimensional approach discussed by Blischke & Murthy [2].

8.3.1 One-Dimensional Approach

We begin by deriving general expressions for the distribution functions of the failure points using the above-mentioned one-dimensional approach. When the relationship between usage and time is assumed to be linear, the cumulative usage at time t is given by $M(t) = Rt$, where the usage rate R is modeled as a non-negative random variable. The cumulative usage function $M(.)$ is a non-decreasing function of time, and, at the start of the system lifetime, this cumulative usage is $M(0) = 0$.

Let T denote the time of the first failure. Then, the usage at first failure is $U := M(T) = RT$. When the usage rate is given, the cumulative usage at any time t is the deterministic function $m(t|R = r) = rt$, so that the usage at first failure is

$$[U|R = r] := m(T|R = r) = r\ [T|R = r]\ . \tag{8.34}$$

The notation $[.|R = r]$ is used to emphasize that the distributions of the variables are conditional on the usage rate $R = r$ (where this is obvious, we will simply use $U = rT$).

According to this one-dimensional approach, it is assumed that the usage rate for any particular system remains constant throughout the lifetime of the system, i.e. for all $t > 0$, $dm(t|R = r)/dt = r$. Therefore, given the usage rate $R = r$, the relationship between the usage and the time at system failure is

$$[U_n|R=r] := m(T_n|R=r) = r\ [T_n|R=r]\ , \tag{8.35}$$

for all $n \in \mathbb{N}_+$. That is, given the usage rate of the system, all failure points of the system lie on the same line. Therefore, when the usage rate is given, the failure process reduces to a point process in one dimension (here, time).

Distribution of the first failure point. The distribution of the first failure point (T, U) can be viewed as an average across the population of systems used at different rates [58], i.e.

$$\begin{aligned}
\mathrm{P}\{T \le t, U \le u\} &= E_R\big[\mathrm{P}\{T \le t, U \le u|R\}\big] \\
&= \int_0^\infty \mathrm{P}\{T \le t, U \le u|R=r\}\ dF_R(r) = \int_0^\infty \mathrm{P}\{T \le t, T \le \frac{u}{r}|R=r\}\ dF_R(r) \\
&= \int_0^{u/t} \mathrm{P}\{T \le t|R=r\}\ dF_R(r) + \int_{u/t}^\infty \mathrm{P}\{T \le \frac{u}{r}|R=r\}\ dF_R(r)\ ,
\end{aligned} \tag{8.36}$$

where $F_R(.)$ is the distribution function of the usage rate R. This joint distribution can also be derived as follows:

$$\begin{aligned}
\mathrm{P}\{T \le t, U \le u\} &= \int_0^t \mathrm{P}\{U \le u|T=s\}\ dF_T(s) = \int_0^t \mathrm{P}\{R \le \frac{u}{s}|T=s\}\ dF_T(s) \\
&= \int_0^t \left(\int_0^{u/s} f_{R|T}(r|s)\ dr\right) dF_T(s) = \int_0^t \int_0^{u/s} f_{T|R}(s|r)\ f_R(r)\ dr\ ds\ ,
\end{aligned} \tag{8.37}$$

where $F_T(.)$ is the marginal distribution of T, $f_R(.)$ is the density function of the usage rate R, $f_{T|R}(.|r)$ is the density function of the conditional variable $[T|R=r]$, and $f_{R|T}(.|s)$ is the density function of the usage rate R conditional on the time to first failure $T=s$. The representation in (8.37) has a straightforward interpretation: for any given $s \le t$, the point $(T,U) = (s,v)$ will be inside the rectangle $(0,t] \times (0,u]$, when $r \le u/s$; see Figure 8.2 for an illustration.

The expression in (8.36) can be easily simplified to arrive at the expression in (8.37), by changing the order of integration in both summands. However, to compare this one-dimensional approach to the proposed two-dimensional approach (from Chapter 7), we will mostly refer to the form in (8.36).

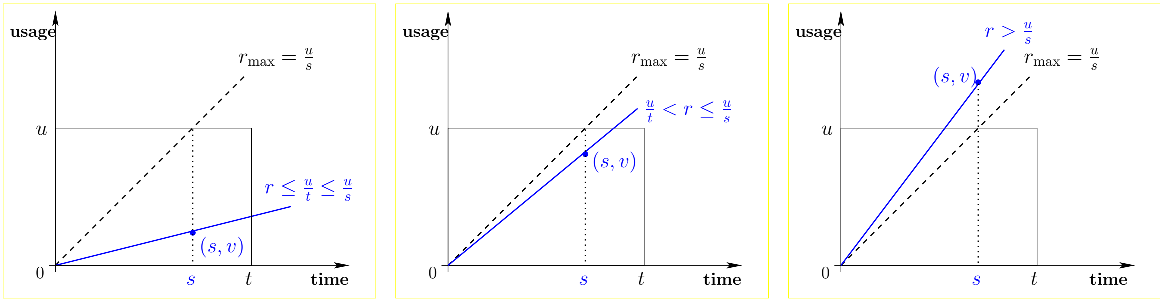


Figure 8.2: Illustrations of the first failure point $(T, U) = (s, v)$, given the usage rate $R = r$, where $s \in (0, t]$, and: $r \le \frac{u}{t}$ (left); $\frac{u}{t} < r \le \frac{u}{s}$ (middle); $r > \frac{u}{s}$ (right).

Distribution of the consecutive failure points. The joint distribution of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}) is derived as follows [cf. (8.36)]:

$$
\begin{aligned}
&\mathrm{P}\{T_{n+1} \le t, U_{n+1} \le u\} = \int_0^{u/t} \mathrm{P}\{T_{n+1} \le t | R = r\}\, dF_R(r) + \int_{u/t}^{\infty} \mathrm{P}\{T_{n+1} \le \frac{u}{r} | R = r\}\, dF_R(r) \\
&= \int_0^{u/t} \int_0^{t} \dots \int_0^{t_2} \mathrm{P}\{T_{n+1} \le t | T_1 = t_1, \dots, T_n = t_n, R = r\}\, f_{\boldsymbol{T_n}|R}(t_1, \dots, t_n | r)\, dt_1 \dots dt_n\, dF_R(r) \\
&+ \int_{u/t}^{\infty} \int_0^{u/r} \dots \int_0^{t_2} \mathrm{P}\{T_{n+1} \le \frac{u}{r} | T_1 = t_1, \dots, T_n = t_n, R = r\}\, f_{\boldsymbol{T_n}|R}(t_1, \dots, t_n | r)\, dt_1 \dots dt_n\, dF_R(r) \ ,
\end{aligned}
\tag{8.38}
$$

for $t, u \in \mathbb{R}_+$, where the conditional distribution $\mathrm{P}\{T_{n+1} \le t | R = r\}$, for $n \in \mathbb{N}_+$, also depends on previous failure points and the degrees of the corresponding repairs. The function $f_{\boldsymbol{T_n}|R}(t_1, \dots, t_n | r)$ is the density of the conditional random vector $[\boldsymbol{T_n} | R = r] := [(T_1, \dots, T_n) | R = r]$ at the point $(t_1, \dots, t_n)$, and can be derived as follows:

$$
f_{\boldsymbol{T_n}|R}(t_1, \dots, t_n | r) = f_{T_n|\boldsymbol{T_{n-1}},R}(t_n | t_1, \dots, t_{n-1}, r) \dots f_{T_2|T_1,R}(t_2 | t_1, r)\, f_{T_1|R}(t_1 | r) \ . \tag{8.39}
$$

In general, for $n \in \mathbb{N}_+$, the conditional density function is the derivative of the corresponding conditional distribution function, i.e.

$$
f_{T_{n+1}|\boldsymbol{T_n},R}(t | t_1, \dots, t_n, r) = \frac{\partial}{\partial t} \mathrm{P}\{T_{n+1} \le t | T_1 = t_1, \dots, T_n = t_n, R = r\} \ , \tag{8.40}
$$

defined for $t > t_n$.

The distribution function in (8.38) can also be simplified to resemble the form in (8.37), which may be a more intuitive representation. On Substituting for the conditional distribution function, we get

$$
\begin{aligned}
&\mathrm{P}\{T_{n+1} \le t, U_{n+1} \le u\} = \\
&= \int_0^{u/t}\int_0^{t}\left(\int_0^{t_{n+1}} \cdots \int_0^{t_2} f_{T_{n+1}|\boldsymbol{T_n},R}(t_{n+1}|t_1,\ldots,t_n,r)\; f_{\boldsymbol{T_n}|R}(t_1,\ldots,t_n|r)\; dt_1\ldots dt_n\right) dt_{n+1}\; dF_R(r) \\
&+ \int_0^{u/t}\int_0^{u/r}\left(\int_0^{t_{n+1}} \cdots \int_0^{t_2} f_{T_{n+1}|\boldsymbol{T_n},R}(t_{n+1}|t_1,\ldots,t_n,r)\; f_{\boldsymbol{T_n}|R}(t_1,\ldots,t_n|r)\; dt_1\ldots dt_n\right) dt_{n+1}\; dF_R(r)\ ,
\end{aligned}
\tag{8.41}
$$

which is defined only when the ordering of the failure points is as follows: $0 < t_1 < \cdots < t_n < t_{n+1} < \ldots$, for $n \in \mathbb{N}_+$. Then, on changing the order of integration of the two outer integrals (the integrals with respect to r and t_{n+1}), we get

$$
\begin{aligned}
&\mathrm{P}\{T_{n+1} \le t, U_{n+1} \le u\} = \\
&= \int_0^{t}\int_0^{u/t}\left(\int_0^{t_{n+1}} \cdots \int_0^{t_2} f_{T_{n+1}|\boldsymbol{T_n},R}(t_{n+1}|t_1,\ldots,t_n,r)\; f_{\boldsymbol{T_n}|R}(t_1,\ldots,t_n|r)\; dt_1\ldots dt_n\right) dF_R(r)\; dt_{n+1} \\
&+ \int_0^{t}\int_{u/t}^{u/t_{n+1}}\left(\int_0^{t_{n+1}} \cdots \int_0^{t_2} f_{T_{n+1}|\boldsymbol{T_n},R}(t_{n+1}|t_1,\ldots,t_n,r)\; f_{\boldsymbol{T_n}|R}(t_1,\ldots,t_n|r)\; dt_1\ldots dt_n\right) dF_R(r)\; dt_{n+1} \\
&= \int_0^{t}\int_0^{u/t_{n+1}}\left(\int_0^{t_{n+1}} \cdots \int_0^{t_2} f_{\boldsymbol{T_{n+1}}|R}(t_1,\ldots,t_n,t_{n+1}|r)\; dt_1\ldots dt_n\right) f_R(r)\; dr\; dt_{n+1} \\
&= \int_0^{t}\int_0^{u/t_{n+1}} f_{T_{n+1}|R}(t_{n+1}|r)\; f_R(r)\; dr\; dt_{n+1}\ .
\end{aligned}
\tag{8.42}
$$

The above expression can be interpreted as follows: the $(n+1)$-th failure point $(T_{n+1}, U_{n+1}) = (t_{n+1}, u_{n+1})$ will be within the rectangle $(0,t] \times (0,u]$, when $r \le u/t_{n+1}$, for each $t_{n+1} \le t$. Given that the points are ordered and on the same line, if the $(n+1)$-th point is in $(0,t] \times (0,u]$, then the n previous points are also within this rectangle; see Figure 8.3.

The exact expressions for the conditional density and distribution functions in the above equations depend on the chosen general repair model. In general, for $n \in \mathbb{N}_+$ and $t > t_n$, the conditional distribution function has the following form:

$$
\begin{aligned}
\mathrm{P}\{T_{n+1} \le t | T_1 = t_1, \ldots, T_n = t_n, R = r\} &= 1 - \mathrm{P}\{T_{n+1} > t | T_1 = t_1, \ldots, T_n = t_n, R = r\} \\
&= 1 - e^{-\int_{t_n}^{t} \tilde{\lambda}_r(s|h_s)\, ds}\ ,
\end{aligned}
\tag{8.43}
$$

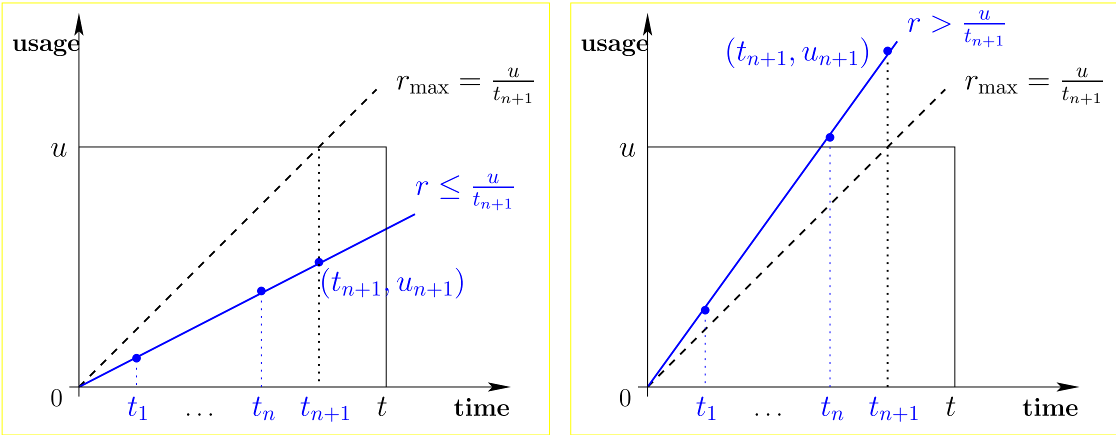


Figure 8.3: Illustrations of the $n+1$ failure points (t_i, u_i), $i \in \{1, \dots, n+1\}$, given the usage rate $R = r$, where the points are fixed such that $t_{n+1} \leq t$, and: $r \leq \frac{u}{t_{n+1}}$ (left); $r > \frac{u}{t_{n+1}}$ (right).

where $\tilde{\lambda}_r(t|h_t)$ is the conditional intensity function at time t, given the history $\mathcal{H}_t = h_t$, of the conditional failure process $\{N_X(t|r); t \in \mathbb{R}_+\}$ (see Section 3.1.1.2 for more on univariate intensity functions). Note that, the random variable $N_X(t|r)$ counts the number of failures of a system before time t, when the usage rate is $R = r$; see Section 6.2.1.

Given the conditional distribution function in (8.43), the corresponding conditional density function is of the form

$$f_{T_{n+1}|\boldsymbol{T_n},R}(t|t_1, \dots, t_n, r) = \tilde{\lambda}_r(t|h_t)\; e^{-\int_{t_n}^{t} \tilde{\lambda}_r(s|h_s)\, ds} \quad , \tag{8.44}$$

which is defined for $t > t_n$ and $n \in \mathbb{N}_+$.

An age reduction model for the one-dimensional approach . In the previous chapter, we proposed bivariate virtual age and usage functions to describe the effect of general repairs; see Section 7.2.1. In order to compare the one-dimensional approach to our two-dimensional modeling approach, we will use a similar univariate age reduction (or virtual age) model here. Let $A(t|r)$ denote the virtual age at time t, given the usage rate $R = r$. Then, this function is defined as follows:

$$A(t|r) = t - \sum_{i=1}^{N_X(t^-|r)} \delta_i \; A(T_i|r) \quad , \tag{8.45}$$

where, for simplicity, we have used the notation $A(T_i|r)$ for the virtual age at $[T_i|R = r]$. For $N_X(t^-|r) = n$ and given the associated failure points $T_i = t_i$, $i \in \{1, \dots, n\}$, we have the following realization of the virtual age function:

$$
\begin{aligned}
a_n(t|r) &= t - \sum_{i=1}^{n} \delta_i \; a_{i-1}(t_i|r) \\
&= t - \delta_1 \; t_1 - \delta_2 \; (t_2 - \delta_1 t_1) - \cdots - \delta_n \; (t_n - \delta_1 \; t_1 - \delta_2 \; (t_2 - \delta_1 t_1) - \dots) \; ,
\end{aligned} \tag{8.46}
$$

defined for $t > t_n$. Then, the corresponding conditional intensity function at time t, given this history, is given by $\tilde{\lambda}_r(t|h_t) = \lambda_0(a_n(t|r)|r)$, where $\lambda_0(.|r)$ is the baseline intensity function of the conditional stochastic process $\{N_X(t|r); t \in \mathbb{R}_+\}$. When all n repairs before time t are minimal, then $\tilde{\lambda}_r(t|h_t) = \lambda_0(t|r)$; and when all n repairs are perfect, then $\tilde{\lambda}_r(t|h_t) = \lambda_0(t - t_n|r)$. Note that, $a_n(.|r)$ is a decreasing function of each degree of repair. Therefore, when all repairs are imperfect, the function value is bounded between the two extremes $t - t_n$ and t, i.e. $t - t_n < a_n(t|r) < t$; see Section 3.2.3 for more on univariate age reduction models.

Substituting this conditional intensity function in (8.43), for $t > t_n$, we get the following conditional distribution function:

$$
\begin{aligned}
\mathrm{P}\{T_{n+1} \le t | T_1 = t_1, \dots, T_n = t_n, R = r\} &= 1 - e^{-\int_{t_n^+}^{t} \lambda_0(a_n(s|r)|r)\, ds} = 1 - e^{-\int_{a_n(t_n^+|r)}^{a_n(t|r)} \lambda_0(x|r)\, dx} \\
&= 1 - \mathrm{P}\{[T|R = r] > a_n(t|r) \mid [T|R = r] > a_n(t_n^+|r)\} \\
&\equiv 1 - \mathrm{P}\{T > a_n(t|r) \mid T > a_n(t_n^+|r), R = r\} \\
&= 1 - \frac{\mathrm{P}\{T > a_n(t|r) \mid R = r\}}{\mathrm{P}\{T > a_n(t_n^+|r) \mid R = r\}} \; .
\end{aligned} \tag{8.47}
$$

The sequence of expressions in (8.47) is due to the baseline intensity function being equal to the failure rate function of the original conditional lifetime $[T|R = r]$. Then, an exponent term of the form $\exp\{-\int_0^t \lambda_0(s|r)\; ds\}$ represents the probability that the original system (used at rate $R = r$) does not fail prior to time t. Note that, we have used t_n^+ in the lower limit of the first integral. This is because the virtual age function is defined as a left-continuous function, and therefore, the effect of all n previous repairs are taken into account only after the last failure, i.e. at t_n^+. The realization $a_n(t|r)$, which represents the virtual age function when n failures have occurred prior to t, is therefore defined only for $t \ge t_n^+$.

The above conditional distribution function can be interpreted as follows: the probability that the system fails for the $(n + 1)$-th time in the interval $(t_n, t]$ is equivalent to an identical system failing for the first time in the interval $(a_n(t_n^+|r), a_n(t|r)]$, given that both systems have been used at the same usage rate $R = r$.

Now, on substituting for the conditional distribution function in (8.38), we get the following (unconditional) distribution function for the $(n+1)$-th failure point (T_{n+1}, U_{n+1}):

$$
\begin{aligned}
&\mathrm{P}\{T_{n+1} \le t, U_{n+1} \le u\} \\
&= \int_0^{u/t}\int_0^t \cdots \int_0^{t_2} \left(1 - \frac{\mathrm{P}\{T > a_n(t|r) \mid R = r\}}{\mathrm{P}\{T > a_n(t_n^+|r) \mid R = r\}}\right) f_{\boldsymbol{T_n}|R}(t_1,\dots,t_n|r)\, dt_1 \dots dt_n\, dF_R(r) \\
&+ \int_{u/t}^{\infty}\int_0^{u/r} \cdots \int_0^{t_2} \left(1 - \frac{\mathrm{P}\{T > a_n(\frac{u}{r}|r) \mid R = r\}}{\mathrm{P}\{T > a_n(t_n^+|r) \mid R = r\}}\right) f_{\boldsymbol{T_n}|R}(t_1,\dots,t_n|r)\, dt_1 \dots dt_n\, dF_R(r) \;,
\end{aligned}
\tag{8.48}
$$

for $n \in \mathbb{N}_+$. These joint distributions can be used to find the distribution of the counts $\{N(t,u); t,u \in \mathbb{R}_+\}$; see Section 6.2.1.

8.3.2 Two-Dimensional Approach Reduced to One-Dimensional Approach

In the preceding section, we derived the distribution of failure points using the one-dimensional approach to modeling failures in two dimensions. In this section, we will show that the one-dimensional approach is a special case of our two-dimensional approach (proposed in Chapter 7).

The distribution function of the $(n+1)$-th failure point, for the proposed two-dimensional approach, is given by

$$
\begin{aligned}
F_{n+1}(t,u) =& \mathrm{P}\{T_{n+1} \le t, U_{n+1} \le u\} \\
=& \int_0^t\int_0^u \cdots \int_0^{t_2}\int_0^{u_2} F_{n+1}(t,u|\boldsymbol{t_n},\boldsymbol{u_n})\, f_n(\boldsymbol{t_n},\boldsymbol{u_n})\, du_1\, dt_1\; \dots\; du_n\, dt_n \\
=& \int_0^t\int_0^u \cdots \int_0^{t_2}\int_0^{u_2} \frac{V_F([a(t_n^+,u_n^+),a(t,u)] \times [b(t_n^+,u_n^+),b(t,u)])}{\bar{F}(a(t_n^+,u_n^+),b(t_n^+,u_n^+))}\, f_n(\boldsymbol{t_n},\boldsymbol{u_n})\, du_1\, dt_1 \\
& \dots\; du_n\, dt_n \;,
\end{aligned}
\tag{8.49}
$$

where $\boldsymbol{t_n} = (t_1,\dots,t_n)$ and $\boldsymbol{u_n} = (u_1,\dots,u_n)$; we derived this function in Section 7.3.1. Our goal is to prove that (8.48) is a spacial case of (8.49). Therefore, we will show that, in (8.49), when $U_i \overset{\text{set}}{=} r\; T_i$, for all $i \in \mathbb{N}_+$, then the distribution function in (8.49) will reduce to the distribution function in (8.48).

We split the proof into two parts: (i) we first individually derive the expressions for the bivariate virtual age and usage functions and the bivariate conditional distribution function; (ii) we then substitute these expressions in equation (8.49) and simplify to arrive at (8.48).

The virtual age and usage functions. According to the assumptions of the one-dimensional approach, for any given system, the usage rate between failures does not change. Then, when at least one failure of the system is observed, the usage rate can be computed, since $u_i/t_i = r$, for all $i \in \mathbb{N}_+$, where (t_i, u_i) is the realization of the i-th failure point of the system.

Let $a_n(t,u)$ and $b_n(t,u)$ denote respectively the realizations of the virtual age and virtual usage at time t and usage u, where n failures have occurred in the rectangle $(0,t] \times (0,u]$, at the points (t_i, u_i), $i \in \{1,\dots,n\}$, for $n \in \mathbb{N}_+$; see Section 7.2.1. Suppose we use the n-th point to compute the usage rate, i.e. $r = u_n/t_n$. Then, the following sets of information are equivalent:

$$\{(t_1,u_1),\dots,(t_n,u_n)\} \quad \Leftrightarrow \quad \{t_1,\dots,t_n, r = u_n/t_n\} \ . \tag{8.50}$$

The bivariate virtual age function at the point (t,u), when the number and points of previous failures are given, is given by

$$\begin{aligned} a_n(t,u) &= t - \sum_{i=1}^{n} \delta_i \, a_{i-1}(t_i,u_i) \\ &= t - \delta_1 \, t_1 - \delta_2 \, (t_2 - \delta_1 \, t_1) - \dots - \delta_n \, (t_n - \delta_1 \, t_1 - \delta_2 \, (t_2 - \delta_1 \, t_1) - \dots) \\ &= a_n(t|r) \ , \end{aligned} \tag{8.51}$$

for $t > t_n$ and $u > u_n$, where $r = u_n/t_n$, $n \in \mathbb{N}_+$. Similarly, the bivariate virtual usage function at the point (t,u), when we set $\gamma_i = \delta_i$, for all $i \in \mathbb{N}_+$, becomes

$$\begin{aligned} b_n(t,u) &= u - \sum_{i=1}^{n} \gamma_i \, b_{i-1}(t_i,u_i) = u - \sum_{i=1}^{n} \delta_i \, b_{i-1}(t_i,u_i) \\ &= u - \delta_1 \, u_1 - \delta_2 \, (u_2 - \delta_1 \, u_1) - \dots - \delta_n \, (u_n - \delta_1 \, u_1 - \delta_2 \, (u_2 - \delta_1 \, u_1) - \dots) \\ &= u - \delta_1 \, rt_1 - \delta_2 \, (rt_2 - \delta_1 \, rt_1) - \dots - \delta_n \, (rt_n - \delta_1 \, rt_1 - \delta_2 \, (rt_2 - \delta_1 \, rt_1) - \dots) \\ &= r \left[\frac{u}{r} - \delta_1 \, t_1 - \delta_2 \, (t_2 - \delta_1 \, t_1) - \dots - \delta_n \, (t_n - \delta_1 \, t_1 - \delta_2 \, (t_2 - \delta_1 \, t_1) - \dots) \right] \\ &= r \, a_n(\frac{u}{r}|r) \ , \end{aligned} \tag{8.52}$$

for $t > t_n$ and $u > u_n$, and $n \in \mathbb{N}_+$. Therefore, when u_i is set to $r \, t_i$, for all $i \in \mathbb{N}_+$, then both the virtual age and virtual usage functions can be expressed in terms of the conditional virtual age function in (8.46), when the number and points of previous failures are given.

The conditional distribution function. The bivariate conditional distribution function for the two-dimensional approach is defined in (7.16) on page 156, and is given by

$$\begin{aligned}
F_{n+1}(t,u|\boldsymbol{t_n},\boldsymbol{u_n}) &= \frac{V_F([a_n(t_n^+,u_n^+),a_n(t,u)]\times[b_n(t_n^+,u_n^+),b_n(t,u)])}{\bar{F}(a_n(t_n^+,u_n^+),b_n(t_n^+,u_n^+))} \\
&= \frac{F(a_n(t,u),b_n(t,u)) - F(a_n(t_n^+,u_n^+),b_n(t,u)) - F(a_n(t,u),b_n(t_n^+,u_n^+)) + F(a_n(t_n^+,u_n^+),b_n(t_n^+,u_n^+))}{\bar{F}(a_n(t_n^+,u_n^+),b_n(t_n^+,u_n^+))} \\
&= \frac{\bar{F}(a_n(t_n^+,u_n^+),b_n(t_n^+,u_n^+)) - \bar{F}(a_n(t_n^+,u_n^+),b_n(t,u)) - \bar{F}(a_n(t,u),b_n(t_n^+,u_n^+)) + \bar{F}(a_n(t,u),b_n(t,u))}{\bar{F}(a_n(t_n^+,u_n^+),b_n(t_n^+,u_n^+))} \\
&= 1 - \frac{\bar{F}(a_n(t_n^+,u_n^+),b_n(t,u)) + \bar{F}(a_n(t,u),b_n(t_n^+,u_n^+)) - \bar{F}(a_n(t,u),b_n(t,u))}{\bar{F}(a_n(t_n^+,u_n^+),b_n(t_n^+,u_n^+))}\ ;
\end{aligned} \tag{8.53}$$

see Section 7.2.2 for details. Notice that, we have expressed the conditional distribution function in terms of the original reliability function instead of the distribution function, so that the expression matches the form in (8.47).

Since $u_i = r\ t_i$, for all $i \in \mathbb{N}_+$, we can construct the usages u_i, $i \in \{1,\ldots,n-1\}$, when we have the failure times t_i, $i \in \{1,\ldots,n\}$ and the n-th usage u_n (equivalently, the usage rate $r = u_n/t_n$). Therefore, we will drop the redundant terms from the conditional distribution function $F_{n+1}(.,.|\boldsymbol{t_n},\boldsymbol{u_n})$ in (8.53). Now, given the above virtual age and usage functions, this conditional distribution function can be expressed in terms of the conditional virtual age function $a_n(.|r)$, given in (8.46), as follows:

$$\begin{aligned}
F_{n+1}(t,u|t_1,\ldots,t_n,u_n) &:= F_{n+1}(t,u|\boldsymbol{t_n},\boldsymbol{u_n}) \\
&= 1 - \frac{\bar{F}(a_n(t_n^+|r),r\ a_n(\frac{u}{r}|r)) + \bar{F}(a_n(t|r),r\ a_n(\frac{u_n^+}{r}|r)) - \bar{F}(a_n(t|r),r\ a_n(\frac{u}{r}|r))}{\bar{F}(a_n(t_n^+|r),r\ a_n(\frac{u_n^+}{r}|r))} \\
&= 1 - \frac{\bar{F}(a_n(t_n^+|r),r\ a_n(\frac{u}{r}|r)) + \bar{F}(a_n(t|r),r\ a_n(t_n^+|r)) - \bar{F}(a_n(t|r),r\ a_n(\frac{u}{r}|r))}{\bar{F}(a_n(t_n^+|r),r\ a_n(t_n^+|r))}\ ,
\end{aligned} \tag{8.54}$$

for $t > t_n$ and $u > u_n$, where $a_n(t_n^+,u_n^+) = a(t_n^+|r)$ is the virtual age immediately after the n-th repair, and $b_n(t_n^+,u_n^+) = r\ a_n\big(\frac{u_n^+}{r}|r\big)$ is the virtual usage immediately after the n-th failure at (t_n,u_n), for $n \in \mathbb{N}_+$. This virtual usage can be expressed as r times the conditional virtual age immediately after the n-th repair, since

$$a_n\big(\frac{u_n^+}{r}|r\big) = \lim_{\epsilon\to 0} a_n\big(\frac{u_n+\epsilon}{r}|r\big) = \lim_{\epsilon\to 0} a_n\big(t_n + \frac{\epsilon}{r}|r\big) = \lim_{\Delta\to 0} a_n\big(t_n+\Delta|r\big) = a_n\big(t_n^+|r\big)\ , \tag{8.55}$$

and therefore, in the last summand of (8.54), we have substituted $a_n\big(t_n^+|r\big)$ for $a_n\big(\frac{u_n^+}{r}|r\big)$.

Next, we need to show that this function reduces to the conditional distribution function from the one-dimensional approach, which is given in (8.47). Since $U = rT$ and given the usage rate $r = u_n/t_n$, the reliability function that appears in (8.54) is the reliability function of the point (T, U) given that $U = rT$. That is, for any $t > t_n$ and $u > u_n$, we have

$$\begin{aligned}
\bar{F}(a_n(t|r), r\, a_n(\frac{u}{r}|r)) &= \mathrm{P}\{T > a_n(t|r),\ U > r\, a_n(\frac{u}{r}|r) \mid R = r\} \\
&= \mathrm{P}\{T > a_n(t|r),\ T > a_n(\frac{u}{r}|r) \mid R = r\} \\
&= \mathrm{P}\{T > \max\left(a_n(t|r), a_n(\frac{u}{r}|r)\right) \mid R = r\} \\
&= \begin{cases} \mathrm{P}\{T > a_n(t|r) \mid R = r\} \ , & t > \frac{u}{r} \\ \mathrm{P}\{T > a_n(\frac{u}{r}|r) \mid R = r\} \ , & t \leq \frac{u}{r} \ , \end{cases}
\end{aligned} \tag{8.56}$$

since $t > (\leq)\ u/r$ is equivalent to $a_n(t|r) > (\leq)\ a_n(\frac{u}{r}|r)$, which follows from the definition of the virtual age function. Similarly,

$$\bar{F}(a_n(t_n^+|r), r\, a_n(t_n^+|r)) = \mathrm{P}\{T > a_n(t_n^+|r) \mid R = r\} \ ; \tag{8.57}$$

$$\bar{F}(a_n(t_n^+|r), r\, a_n(\frac{u}{r}|r)) = \begin{cases} \mathrm{P}\{T > a_n(t_n^+|r) \mid R = r\} \ , & t_n^+ > \frac{u}{r} \\ \mathrm{P}\{T > a_n(\frac{u}{r}|r) \mid R = r\} \ , & t_n^+ \leq \frac{u}{r} \ ; \end{cases} \tag{8.58}$$

and, for all $t \geq t_n^+$,

$$\begin{aligned}
\bar{F}(a_n(t|r), r\, a_n(t_n^+|r)) &= \begin{cases} \mathrm{P}\{T > a_n(t|r) \mid R = r\} \ , & t > t_n^+ \\ \mathrm{P}\{T > a_n(t_n^+|r) \mid R = r\} \ , & t \leq t_n^+ \end{cases} \\
&= \mathrm{P}\{T > a_n(t|r) \mid R = r\} \ .
\end{aligned} \tag{8.59}$$

For the one-dimensional approach, in deriving the unconditional distribution function in (8.48), we have considered the following two cases: (i) $r \leq u/t$, and (ii) $r > u/t$. We now need to determine the conditional distribution function in (8.54) for each of the two cases, by using the appropriate expressions from (8.56) - (8.59).

When $r \leq u/t$, which is equivalent to $t \leq u/r$, and in turn implies that $t_n^+ \leq u/r$ (since $t > t_n$), the conditional distribution function in (8.54) becomes

$$\begin{aligned}
&F_{n+1}(t, u|t_1, \ldots, t_n, u_n) \\
&= 1 - \frac{\mathrm{P}\{T > a_n(\frac{u}{r}|r) \mid R = r\} + \mathrm{P}\{T > a_n(t|r) \mid R = r\} - \mathrm{P}\{T > a_n(\frac{u}{r}|r) \mid R = r\}}{\mathrm{P}\{T > a_n(t_n^+|r) \mid R = r\}} \\
&= 1 - \frac{\mathrm{P}\{T > a_n(t|r) \mid R = r\}}{\mathrm{P}\{T > a_n(t_n^+|r) \mid R = r\}} \ ,
\end{aligned} \tag{8.60}$$

where $r = u_n/t_n$, for $n \in \mathbb{N}_+$.

When $r > u/t$, which is equivalent to $t > u/r$, the conditional distribution function in (8.54) becomes

$$\begin{aligned} &F_{n+1}(t,u|t_1,\ldots,t_n,u_n) \\ &= 1 - \frac{\mathrm{P}\{T > \max\left(a_n(t_n^+|r), a_n(\frac{u}{r}|r)\right) | R = r\} + \mathrm{P}\{T > a_n(t|r) | R = r\} - \mathrm{P}\{T > a_n(t|r) | R = r\}}{\mathrm{P}\{T > a_n(t_n^+|r) \mid R = r\}} \\ &= 1 - \frac{\mathrm{P}\{T > \max\left(a_n(t_n^+|r), a_n(\frac{u}{r}|r)\right) \mid R = r\}}{\mathrm{P}\{T > a_n(t_n^+|r) \mid R = r\}} \ . \end{aligned} \tag{8.61}$$

Having derived the expressions for the conditional distribution function, we now proceed to the second stage of the proof, which is substituting for this conditional distribution function in (8.49) and simplifying the expression to arrive at the one in (8.48).

The distribution function. In terms of the conditional distribution function, the (unconditional) distribution function in (8.49) can be expressed as follows:

$$\begin{aligned} &F_{n+1}(t,u) = \mathrm{P}\{T_{n+1} \le t, U_{n+1} \le u\} \\ &= \int_0^t \ldots \int_0^{t_2} \int_0^u F_{n+1}(t,u|t_1,\ldots,t_n,u_n) \left(\int_0^{u_n} \ldots \int_0^{u_2} f_n(\boldsymbol{t_n}, \boldsymbol{u_n})\, du_1 \ldots du_{n-1} \right) du_n\, dt_1 \ldots dt_n \\ &= \int_0^t \ldots \int_0^{t_2} \int_0^u F_{n+1}(t,u|t_1,\ldots,t_n,u_n) \left(\int_0^{u_n} \ldots \int_0^{u_2} f_{\boldsymbol{U_{n-1}}|\boldsymbol{T_n},U_n}(u_1,\ldots,u_{n-1}|t_1,\ldots,t_n,u_n)\, du_1 \ldots du_{n-1} \right) \\ &\qquad f_{U_n|\boldsymbol{T_n}}(u_n|t_1,\ldots,t_n)\, f_{\boldsymbol{T_n}}(t_1,\ldots,t_n)\, du_n\, dt_1 \ldots dt_n \\ &= \int_0^t \ldots \int_0^{t_2} \int_0^u F_{n+1}(t,u|t_1,\ldots,t_n,u_n)\, \mathrm{P}\{U_{n-1} \le u_n, \ldots, U_1 \le u_2 | T_1 = t_1, \ldots, T_n = t_n, U_n = u_n\} \\ &\qquad f_{U_n|\boldsymbol{T_n}}(u_n|t_1,\ldots,t_n)\, f_{\boldsymbol{T_n}}(t_1,\ldots,t_n)\, du_n\, dt_1 \ldots dt_n \ , \end{aligned} \tag{8.62}$$

where the function $f_{\boldsymbol{U_{n-1}}|\boldsymbol{T_n},U_n}$ denotes the density function of the first $n-1$ usages $\boldsymbol{U_{n-1}} = (U_1,\ldots,U_{n-1})$, given the values of the n failure points $\boldsymbol{T_n} = (T_1,\ldots,T_n)$ and the n-th usage U_n. We have used similar notations for the other two density functions.

Since, when $t_1,\ldots,t_n$, and u_n are given, the other $n-1$ usages are also given, and since the order $0 < u_1 < u_2 < \cdots < u_{n-1} < u_n < \ldots$ always holds, the probability function in (8.62) is unity, i.e.

$$\mathrm{P}\{U_{n-1} \le u_n, \ldots, U_1 \le u_2 | T_1 = t_1, \ldots, T_n = t_n, U_n = u_n\} = 1 \ . \tag{8.63}$$

We know that the density of the n-th usage U_n, given the n failure times is the derivative of the corresponding distribution function, i.e. for $u > u_n$ and $n \in \mathbb{N}_+$,

$$\begin{aligned} f_{U_n|\boldsymbol{T_n}}(u|t_1,\ldots,t_n) &= \frac{\partial}{\partial u}\,\mathrm{P}\{U_n \le u \mid T_1 = t_1,\ldots,T_n = t_n\} \\ &= \frac{\partial}{\partial u}\,\mathrm{P}\{R \le \frac{u}{t_n} \mid T_1 = t_1,\ldots,T_n = t_n\} \\ &= \frac{\partial}{\partial r}\,\mathrm{P}\{R \le r \mid T_1 = t_1,\ldots,T_n = t_n\}\;\frac{\partial}{\partial u}\frac{u}{t_n} \\ &= \frac{1}{t_n}\,f_{R|\boldsymbol{T_n}}(r|t_1,\ldots,t_n)\quad, \end{aligned} \tag{8.64}$$

where $f_{R|\boldsymbol{T_n}}(.|t_1,\ldots,t_n)$ is the density function of the usage rate R, given the failure times.

Using (8.63) and (8.64), the distribution function of (T_{n+1},U_{n+1}) in (8.62) can be further simplified to

$$\begin{aligned} F_{n+1}(t,u) &= \int_0^t \cdots \int_0^{t_2} \int_0^{u/t_n} F_{n+1}(t,u|t_1,\ldots,t_n,r)\; f_{R|\boldsymbol{T_n}}(r|t_1,\ldots,t_n)\; f_{\boldsymbol{T_n}}(t_1,\ldots,t_n)\; dr\; dt_1\ldots dt_n \\ &= \int_0^t \cdots \int_0^{t_2} \int_0^{u/t_n} F_{n+1}(t,u|t_1,\ldots,t_n,r)\; f_{\boldsymbol{T_n}|R}(t_1,\ldots,t_n|r)\; f_R(r)\; dr\; dt_1\ldots dt_n\quad, \end{aligned} \tag{8.65}$$

since $du_n = t_n\,dr$; $u_n \le u$ implies that $r \le u/t_n$; and

$$f_{R|\boldsymbol{T_n}}(r|t_1,\ldots,t_n)\; f_{\boldsymbol{T_n}}(t_1,\ldots,t_n) = f_{\boldsymbol{T_n}|R}(t_1,\ldots,t_n|r)\; f_R(r)\quad. \tag{8.66}$$

Note that, $F_{n+1}(t,u|t_1,\ldots,t_n,r) \equiv F_{n+1}(t,u|t_1,\ldots,t_n,u_n)$, since $u_i = r\,t_i$, for all $i \in \mathbb{N}_+$.

Continuing with the process of simplification, the distribution function in (8.65) becomes

$$\begin{aligned} F_{n+1}(t,u) = &\int_0^t \cdots \int_0^{t_2} \int_0^{u/t} \left(1 - \frac{\mathrm{P}\{T > a_n(t|r) \mid R = r\}}{\mathrm{P}\{T > a_n(t_n^+|r) \mid R = r\}}\right) f_{\boldsymbol{T_n}|R}(t_1,\ldots,t_n|r)\; f_R(r)\; dr\; dt_1\ldots dt_n \\ &+ \int_0^t \cdots \int_0^{t_2} \int_{u/t}^{u/t_n} \left(1 - \frac{\mathrm{P}\{T > \max\left(a_n(t_n^+|r), a_n(\frac{u}{r}|r)\right) \mid R = r\}}{\mathrm{P}\{T > a_n(t_n^+|r) \mid R = r\}}\right) f_{\boldsymbol{T_n}|R}(t_1,\ldots,t_n|r)\; f_R(r)\; dr \\ &\quad dt_1\ldots dt_n\quad, \end{aligned} \tag{8.67}$$

where we have split the inner integral (the integral with respect to r) into the two cases $r \le u/t$ and $u/t < r \le u/t_n$, and also substituted for the conditional distribution function using (8.60) and (8.61).

Now, by changing the order of integration between the innermost integral (the integral wrt r) and the outermost integral (the integral wrt t_n), we get

$$F_{n+1}(t,u) = \int_0^{u/t}\int_0^{t}\cdots\int_0^{t_2}\left(1-\frac{\mathrm{P}\{T>a_n(t|r)\mid R=r\}}{\mathrm{P}\{T>a_n(t_n^+|r)\mid R=r\}}\right) f_{\boldsymbol{T_n}|R}(t_1,\ldots,t_n|r)\; dt_1\ldots dt_n\; dF_R(r)$$
$$+\int_{u/t}^{\infty}\int_0^{u/r}\cdots\int_0^{t_2}\left(1-\frac{\mathrm{P}\{T>\max\left(a_n(t_n^+|r),a_n(\frac{u}{r}|r)\right)\mid R=r\}}{\mathrm{P}\{T>a_n(t_n^+|r)\mid R=r\}}\right) f_{\boldsymbol{T_n}|R}(t_1,\ldots,t_n|r)\; dt_1\ldots dt_n$$
$$dF_R(r)\;\;, \tag{8.68}$$

since (i) for $t_n \in (0,t]$, the possible range of the usage rate r is $\lim_{t_n\to 0}[u/t, u/t_n) = [u/t,\infty)$; and (ii) $u/t < r \le u/t_n$ implies that $t_n \le u/r < t$.

Since $t_n \le u/r$, the conditional probability in the second summand of (8.68) becomes

$$1-\frac{\mathrm{P}\{T>\max\left(a_n(t_n^+|r),a_n(\frac{u}{r}|r)\right)\mid R=r\}}{\mathrm{P}\{T>a_n(t_n^+|r)\mid R=r\}} = \begin{cases} 1-\frac{\mathrm{P}\{T>a_n(\frac{u}{r}|r)|R=r\}}{\mathrm{P}\{T>a_n(t_n^+|r)|R=r\}}\;, & t_n<\frac{u}{r} \\ 1-\frac{\mathrm{P}\{T>a_n(t_n^+|r)|R=r\}}{\mathrm{P}\{T>a_n(t_n^+|r)|R=r\}}=0\;,\; & t_n\ge\frac{u}{r}\;. \end{cases} \tag{8.69}$$

Finally, when (8.69) is inserted into (8.68), the distribution function becomes

$$\begin{aligned} F_{n+1}(t,u) &= \mathrm{P}\{T_{n+1}\le t, U_{n+1}\le u\} \\ &= \int_0^{u/t}\int_0^{t}\cdots\int_0^{t_2}\left(1-\frac{\mathrm{P}\{T>a_n(t|r)\mid R=r\}}{\mathrm{P}\{T>a_n(t_n^+|r)\mid R=r\}}\right) f_{\boldsymbol{T_n}|R}(t_1,\ldots,t_n|r)\; dt_1\ldots dt_n\; dF_R(r) \\ &+\int_{u/t}^{\infty}\int_0^{u/r}\cdots\int_0^{t_2}\left(1-\frac{\mathrm{P}\{T>a_n(\frac{u}{r}|r)\mid R=r\}}{\mathrm{P}\{T>a_n(t_n^+|r)\mid R=r\}}\right) f_{\boldsymbol{T_n}|R}(t_1,\ldots,t_n|r)\; dt_1\ldots dt_n\; dF_R(r)\;\;, \end{aligned} \tag{8.70}$$

which is exactly the distribution function in (8.48), derived for the one-dimensional approach. This completes our proof.

Therefore, our two-dimensional approach, under the assumption that the cumulative usage is a linear function of time, is equivalent to the one-dimensional approach to modeling failures in two dimensions (by Blischke & Murthy [2]).

Comments. The one- and two-dimensional approaches are two different descriptions of the physical space of the problem and require different sets of assumptions, which lead to two different models. From an application point of view, the one-dimensional approach is the simpler of the two, but requires that, for any given system in the population, cumulative usage be a deterministic, linear function of time. The proposed two-dimensional approach does not treat time and usage at failure as necessarily having a functional relationship, but models them simply as correlated bivariate random variables. In this sense, the two-dimensional approach is a generalization of the one-dimensional approach, so that when the cumulative usage is modeled as a deterministic, non-decreasing, linear function of time, and the distribution of the usage rate is known, then the two-dimensional model reduces to the one-dimensional model. In order to apply the models, it must first be determined which model's assumptions are met. For instance, given failure data on a sample of systems, if a linear trend is visible in the sequence of times and usages at failure of each system, then the one-dimensional approach is an appropriate choice, but if no such trend is present, then the two-dimensional approach is more suitable.

8.4 Chapter Conclusion

In this chapter, we discussed the properties of the two-dimensional general repair model proposed in Chapter 7. We showed that the conditional reliability of the system improves as each component of the bivariate degree of repair increases, when all remaining parameters are fixed. Here, the conditional bivariate reliability function is used as an indicator of the working condition of the system.

Next, we proved that, when the usage at failure is defined as a linear function of the corresponding failure time, then the proposed failure process in two dimensions reduces to the process constructed using the one-dimensional approach (which was discussed by Blischke & Murthy [2]).

In the following chapter, we illustrate the behavior of the failure (or general repair) process, through simulations of the process trajectories for various degrees of repair.

Chapter 9

Simulating the Failure Process with Applications in Warranty Cost Analysis

In this chapter, we suggest a procedure for simulating the failure or general repair process (proposed in Chapter 7) in two dimensions. We illustrate the effect of general repairs through simulations of the failure process for various degrees of repair. We also illustrate applications of the repair model in the context of two-dimensional warranty cost analysis.

This chapter is arranged as follows. In Section 9.1, we review the simulation methods that are used to generate observations from a bivariate distribution. In Section 9.2, we detail the steps involved in simulating the general repair process. In Section 9.3, we simulate the failure process to illustrate the properties of the general repair model and provide a numerical example of the application of the model in estimating warranty servicing costs using the simulation approach. In Section 9.4, we conclude with a chapter summary.

9.1 Generating Bivariate Observations

The literature on generating random variates from bivariate distributions is vast; see for instance Balakrishnan & Lai [67] and the references therein. In this section, we will briefly describe only those approaches used in this study for simulating the failure process in two dimensions.

9.1.1 Bivariate Simulation Approach: Conditioning Method

Let (X, Y) denote a bivariate lifetime random variable, i.e. $(X, Y) \in \mathbb{R}^2_+$. Let $f(.,.)$, $F(.,.)$ and $\bar{F}(.,.)$ denote respectively the joint density, distribution and reliability functions of X and Y. The relationships between the three bivariate functions is as follows:

$$F(x,y) = P\{X \leq x, Y \leq y\} = 1 - \bar{F}(x,0) - \bar{F}(0,y) + \bar{F}(x,y) \ , \tag{9.1}$$

$$\bar{F}(x,y) = P\{X > x, Y > y\} = 1 - F(x,\infty) - F(\infty,y) + F(x,y) \ , \tag{9.2}$$

$$f(x,y) = \frac{\partial^2}{\partial x \, \partial y} F(x,y) = \frac{\partial^2}{\partial x \, \partial y} \bar{F}(x,y) \ . \tag{9.3}$$

for $x, y \geq 0$.

In order to generate a point (x, y) from the distribution F, we use the conditioning approach, where:

(i) a Y variate is generated from its marginal distribution;

(ii) an X variate is then generated from its conditional distribution, given $Y = y$.

The order in which the variates are generated can be reversed, so that an X variate is generated first and a Y variate is then generated conditional on $X = x$ [67].

For the conditioning approach, we require the marginal and conditional distributions. Let $f_Y(.)$ and $f_{X|Y}(.|.)$ denote respectively the marginal density function of Y and the conditional density function of X given Y. Then, the corresponding reliability functions are given by

$$\bar{F}_Y(y) := \bar{F}(0,y) = P\{Y > y\} = \int_y^\infty f_Y(v) \, dv \ , \tag{9.4}$$

and

$$\bar{F}_{X|Y}(x|Y=y) := P\{X > x | Y = y\} = \int_x^\infty f_{X|Y}(s|y) \, ds = \int_x^\infty \frac{f(s,y)}{f_Y(y)} \, ds \ , \tag{9.5}$$

respectively (assuming the density functions exist). The corresponding marginal and conditional distribution functions are given by $F_Y(y) = 1 - \bar{F}_Y(y)$ and $F_{X|Y}(x|Y = y) = 1 - \bar{F}_{X|Y}(x|Y = y)$, respectively.

Note that, we have used the notation $\bar{F}_{X|Y}(.|Y = y)$ in order to distinguish this conditional reliability function from the conditional reliability function of X given $Y > y$, which we denote by $\bar{F}_{X|Y}(.|y)$. The reliability function of X given $Y > y$ can be derived from the joint bivariate reliability function: $\bar{F}(x,y) = \bar{F}_{X|Y}(x|y)\ \bar{F}_Y(y)$. For this simulation approach, the conditional distribution of interest is the one with reliability function given in (9.5).

The conditioning method of generating observations form a bivariate distribution reduces the problem to generating variates from two univariate distributions: the marginal distribution and the conditional distribution.

9.1.2 Univariate Simulation Approach: Inverse Transformation

The univariate simulation approach that we will use to simulate variates from the marginal and conditional distributions is the *inverse transformation method*.

Let $F(.)$ denote a continuous distribution function with support $\mathcal{S} \in \mathbb{R}$, and let U denote a random variable uniformly distributed on the interval $(0,1)$. Then, the random variable Z has distribution function $F(.)$, if $Z = F^{-1}(U)$, since

$$P\{Z \leq z\} = P\{F^{-1}(U) \leq z\} = P\{U \leq F(z)\} = F_U(F(z)) = F(z) \ , \tag{9.6}$$

where the distribution function of U is $F_U(u) = u$, for all $0 \leq u \leq 1$. The above expression follows from $F(.)$ being a monotone increasing function, so that the following events are equivalent [13]:

$$\{F^{-1}(U) \leq z\} \quad \Leftrightarrow \quad \{U \leq F(z)\} \ . \tag{9.7}$$

The inverse transformation method is also valid when the reliability function $\bar{F}(.)$ is used in place of the distribution function, i.e. when $Z = \bar{F}^{-1}(U)$. Since the reliability function $\bar{F}(.)$ is monotone decreasing, the following events are equivalent:

$$\{\bar{F}^{-1}(U) > z\} \quad \Leftrightarrow \quad \{U < \bar{F}(z)\} \ , \tag{9.8}$$

and therefore, $\bar{F}(.)$ is the reliability function of Z, if $Z = \bar{F}^{-1}(U)$:

$$P\{Z > z\} = P\{\bar{F}^{-1}(U) > z\} = P\{U < \bar{F}(z)\} = F_U(\bar{F}(z)) = \bar{F}(z) \ . \tag{9.9}$$

Algorithm 1: inverse transformation method. The algorithm for generating Z variates using the inverse transformation method is as follows:

(i) a U variate is generated from a uniform distribution over the interval $(0,1)$;

(ii) the transformation $F^{-1}(U)$ is used to compute the Z variate– one can equivalently use the transformation $Z = \bar{F}^{-1}(U)$.

When the inverse of the distribution function $F(.)$ can be expressed in closed form, then the solution to the equation $F(Z) = U$ is $Z = F^{-1}(U)$. When the inverse of the distribution function cannot be expressed in closed form, numerical methods can be used to find the approximate solution to the equation $F(Z) = U$; see for instance Abate et al. [72] and Devroye [73].

In our numerical illustrations, we come across situations where the inverse of the distribution (or reliability) function cannot be expressed in closed form. To generate variates from such distributions, we use the *secant method*, which is used when the distribution function $F(.)$ is not invertible, and therefore, the solution to $F(Z) = U$ is sought numerically [73].

Algorithm 2: secant method. The algorithm for finding the approximate solution $\tilde{Z}$ to $F(Z) = U$ using the secant method is as follows:

(i) a U variate is generated from a uniform distribution over the interval $(0, 1)$;

(ii) an interval $[a, b]$ to which the solution to $F(Z) = U$ belongs is found;

(iii) $\tilde{Z}$ is set to $a + \frac{U-F(a)}{F(b)-F(a)}\,(b-a)$;

(iv) if $F(\tilde{Z}) \leq U$, then a is assigned the $\tilde{Z}$ value, otherwise b is assigned the $\tilde{Z}$ value;

(v) steps (iii) and (iv) are repeated until $b - a \leq \Delta$ (where $\Delta > 0$ is some preassigned tolerance).

With this approach, the approximate solution is found by continually narrowing the interval $[a, b]$, which contains the exact solution, until an acceptable width is reached. Suppose that $Z^{\star}$ is the exact solution to $F(Z) = U$. When the tolerance Δ is small (close to zero), then the approximate solution $\tilde{Z}$ derived from this method converges to the true solution $Z^{\star}$, if $F(Z) = U$ has a unique solution (which is the case for a monotonically increasing $F(.)$ function). Note that, when we use $\frac{1}{2}$ in place of the proportion $\frac{U-F(a)}{F(b)-F(a)}$, then the secant method becomes what is known as the *bisection method*, which also converges when the tolerance is set to a value close to zero; see Devroye [73]. When the reliability function $\bar{F}(.)$ is used instead of the distribution function $F(.)$, then steps (ii) - (iv) in Algorithm 2 need to be replaced by the following steps:

(ii) an interval $[a, b]$ to which the solution to $\bar{F}(Z) = U$ belongs is found;

(iii) $\tilde{Z}$ is set to $a + \frac{\bar{F}(a)-U}{\bar{F}(a)-\bar{F}(b)}\ (b-a)$;

(iv) if $\bar{F}(\tilde{Z}) > U$, then a is assigned the $\tilde{Z}$ value, otherwise b is assigned the $\tilde{Z}$ value.

The secant method assumes that the distribution function $F(.)$ is available in closed form. When the distribution function is available only as the integral of the density function, we use numerical approximation to compute the values of the distribution function.

9.1.3 Simulation Check: Empirical Distribution Functions

In order to test the simulation approach, at each stage, we will plot the empirical distribution function of the generated data and compare this to the corresponding theoretical distribution function.

The univariate empirical distribution function, denoted by $\hat{F}(.)$, is computed as follows:

$$\hat{F}(u) = \frac{\sum_{i=1}^{m} \mathbb{I}_{\{y_i \le u\}}}{m} \ , \tag{9.10}$$

for $u \ge 0$, where m is the number of Y variates generated, $\{y_1, \dots, y_m\}$ is the sequence of generated variates, and $\mathbb{I}_.$ is an indicator function defined as follows:

$$\mathbb{I}_{\{y_i \le u\}} = \begin{cases} 1 \ , & y_i \le u \\ 0 \ , & \text{otherwise} \ . \end{cases} \tag{9.11}$$

To plot the univariate empirical distribution function, we compute the function at equidistant points $\{u_1, \dots, u_q\}$ in an interval $[0, L]$ (which includes the range of the generated variates). The chosen distance L/q between these points will depend on the number and range of the generated variates. Empirical distribution functions are usually plotted as step-functions. To get a good approximation for the distribution of the generated variates, it is important that the number of variates generated is large and the distance L/q is small.

The bivariate empirical distribution function, denoted by $\hat{F}(.,.)$, is computed as follows:

$$\hat{F}(u,v) = \frac{\sum_{i=1}^{m} \mathbb{I}_{\{x_i \le u \ \& \ y_i \le v\}}}{m} \ , \tag{9.12}$$

for $u, v \ge 0$, where m is the number of generated data points, $\{(x_1, y_1), \dots, (x_m, y_m)\}$ is the sequence of generated data points, and $\mathbb{I}_{.}$ is an indicator function defined as follows:

$$\mathbb{I}_{\{x_i \le u \;\&\; y_i \le v\}} = \begin{cases} 1 \;, & x_i \le u \text{ and } y_i \le v \\ 0 \;, & \text{otherwise} \;. \end{cases} \tag{9.13}$$

To plot the bivariate empirical distribution function, we compute the function at grid points $\{(u_i, v_j); i \in \{1, \ldots, q\}, j \in \{1, \ldots, p\}\}$ in the region $[0, L] \times [0, K]$ (which includes the generated data points). The area of the grid cells is $\frac{L}{q}\frac{K}{p}$. For a closer approximation of the distribution of generated data points, we generate a large number of points and choose small grid cells.

9.2 Simulating the Failure or General Repair Process

In this section, we describe the steps involved in simulating the failure process. After the first failure of the system, the distribution of any given failure point depends on the degree (effectiveness) of all preceding general repairs; see Section 7.2. Therefore, we begin with describing the process of generating the point of first failure (Section 9.2.1); then, we describe the process of generating a failure point following general repair(s) (Section 9.2.2); and finally, we will outline the steps involved in generating the consecutive failure points, where each failure is followed by a general repair (Section 9.2.3).

9.2.1 Generating the Original Bivariate Lifetime

Let the point (X, Y) denote the first failure of the system, where X is the time at first failure and Y is the usage at first failure. The point (X, Y) is the original bivariate lifetime of the system. The reliability function of the original lifetime (X, Y), that will be used in the numerical illustrations, is of the following form:

$$\begin{aligned} \bar{F}(x, y) &= e^{-R_X(x)}\, e^{-R_Y(y)}\, e^{-\,\theta\, R_X(x)\, R_Y(y)} \\ &= \bar{F}_X(x)\, \bar{F}_Y(y)\, e^{-\,\theta\, R_X(x)\, R_Y(y)} \;, \end{aligned} \tag{9.14}$$

for $x, y \ge 0$, where $\theta \in [0, 1]$. The functions $R_X(.)$ and $R_Y(.)$ are the marginal cumulative failure rate functions with respect to time and usage respectively, and $\bar{F}_X(.)$ and $\bar{F}_Y(.)$ are the corresponding marginal reliability functions; see Section 7.4 for details.

The chosen marginal distributions are both IFR Weibull distributions, i.e. the marginal cumulative failure rate functions are defined as follows:

$$R_X(x) = \left(\frac{x}{\alpha_1}\right)^{\beta_1} \quad ; \qquad R_Y(y) = \left(\frac{y}{\alpha_2}\right)^{\beta_2} \ , \tag{9.15}$$

for $x, y \geq 0$, where $\alpha_1, \alpha_2 > 0$ and $\beta_1, \beta_2 > 1$.

The joint density function corresponding to the bivariate reliability function in (9.14) is given by

$$f(x,y) = f_X(x) f_Y(y)\ e^{-\,\theta\, R_X(x)\, R_Y(y)} \left\{ 1 + \theta\ \left(R_X(x) + R_Y(y) + \theta\ R_X(x)\ R_Y(y) - 1\right) \right\} \ , \tag{9.16}$$

for $x, y \geq 0$, where $f_X(.)$ and $f_Y(.)$ are the marginal density functions of X and Y, respectively.

In order to generate a bivariate observation from this distribution, we will first generate a Y variate from its marginal distribution, and then we will generate an X variate from its conditional distribution given the Y variate; see Section 9.1.1.

9.2.1.1 Generating from the Marginal Distribution of Y

The marginal distribution of the usage Y at first failure is Weibull, which has an easily invertible distribution/reliability function, and therefore, we use the inverse transformation method to generate a Y variate; see Algorithm 1 in Section 9.1.2. Given a U variate and the Weibull marginal, we solve for Y in the following equation:

$$e^{-\left(\frac{Y}{\alpha_2}\right)^{\beta_2}} = U \ , \tag{9.17}$$

to get the transformation of interest. Note that, we have used the reliability function to derive the transformation:

$$\begin{aligned} e^{-\left(\frac{Y}{\alpha_2}\right)^{\beta_2}} = U \qquad & \Leftrightarrow \qquad \left(\frac{Y}{\alpha_2}\right)^{\beta_2} = -\ln U \\ & \Leftrightarrow \qquad \left(\frac{Y}{\alpha_2}\right) = \left(-\ln U\right)^{1/\beta_2} \\ & \Leftrightarrow \qquad Y = \alpha_2\ \left(-\ln U\right)^{1/\beta_2} \ . \end{aligned} \tag{9.18}$$

Note that, since U is in the interval $(0, 1)$, $-\ln U$ is in the interval $(0, \infty)$, and therefore Y is non-negative.

The density function of the above Weibull distribution, denote by $f_Y(.)$, is given by

$$
\begin{aligned}
f_Y(y) &= -\frac{d}{dy}\,\bar{F}_Y(y) \\
&= \frac{\beta_2}{\alpha_2}\left(\frac{y}{\alpha_2}\right)^{\beta_2-1} e^{-\left(\frac{y}{\alpha_2}\right)^{\beta_2}} .
\end{aligned}
\tag{9.19}
$$

Example 1: We generate 1000 variates from the marginal distribution of Y, with parameter values set to: $\alpha_2 = 1.75$ and $\beta_2 = 1.5$. In Figure 9.1, we have plotted the empirical distribution function and histogram of the generated data, together with the theoretical distribution and density functions. The empirical distribution function $\hat{F}(.)$ is computed at all $u \in \{0, 0.05, 0.1, \ldots, 8\}$. $\square$

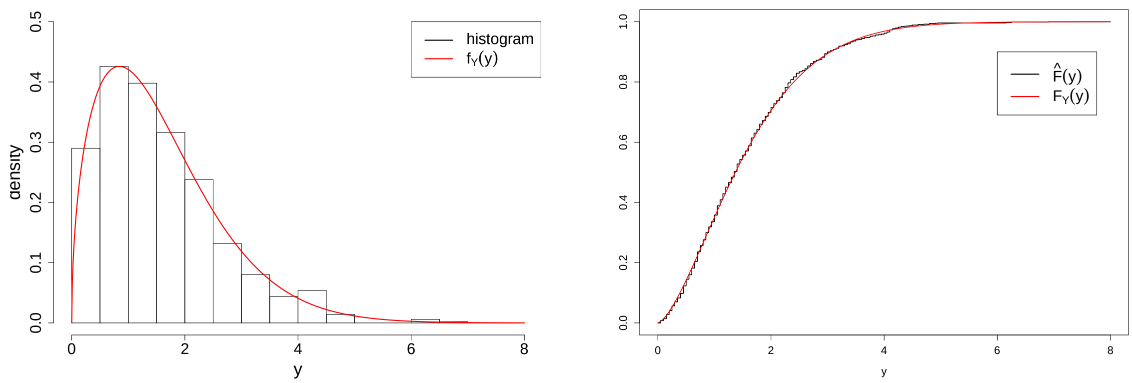


Figure 9.1: Plots of the histogram (left) and the empirical distribution function (right) of 1000 variates generated from the marginal distribution of Y. The theoretical density and distribution functions are superimposed.

9.2.1.2 Generating from the Conditional Distribution of X given Y = y

To generate an X variate given $Y = y$, we need the associated conditional distribution. For $x, y \geq 0$, the conditional density $f_{X|Y}(.|.)$ at (x, y) is the ratio $f(x, y)/f_Y(y)$, where $f(.,.)$ is the joint density function in (9.16) and $f_Y(.)$ is the marginal density function in (9.19). Therefore, for $x, y \geq 0$,

$$
f_{X|Y}(x|y) = f_X(x)\, e^{-\,\theta\, R_X(x)\, R_Y(y)} \left\{ 1 + \theta\, \left(R_X(x) + R_Y(y) + \theta\, R_X(x)\, R_Y(y) - 1 \right) \right\} \tag{9.20}
$$

where $f_X(.)$ is the marginal density function of X, which is also Weibull, and is given by

$$
f_X(x) = \frac{\beta_1}{\alpha_1}\left(\frac{x}{\alpha_1}\right)^{\beta_1-1} e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}} . \tag{9.21}
$$

We can derive the corresponding conditional distribution function as follows:

$$
\begin{aligned}
F_{X|Y}(x|Y=y) &= \int_0^x f_{X|Y}(s|y)\, ds \\
&= \int_0^x f_X(s)\, e^{-\,\theta\, R_X(s)\, R_Y(y)} \left\{ 1 + \theta\, \left(R_X(s) + R_Y(y) + \theta\, R_X(s)\, R_Y(y) - 1 \right) \right\} ds \ .
\end{aligned} \tag{9.22}
$$

The conditional reliability function is simply $\bar{F}_{X|Y}(x|Y=y) = 1 - F_{X|Y}(x|Y=y)$.

This conditional distribution function cannot be expressed in closed form, and therefore, numerical integration is used to approximate the definite integral in (9.22). Also, the inverse of this function cannot be derived in closed form, and therefore, we use the secant method for simulating the X variates; see Algorithm 2 in Section 9.1.2.

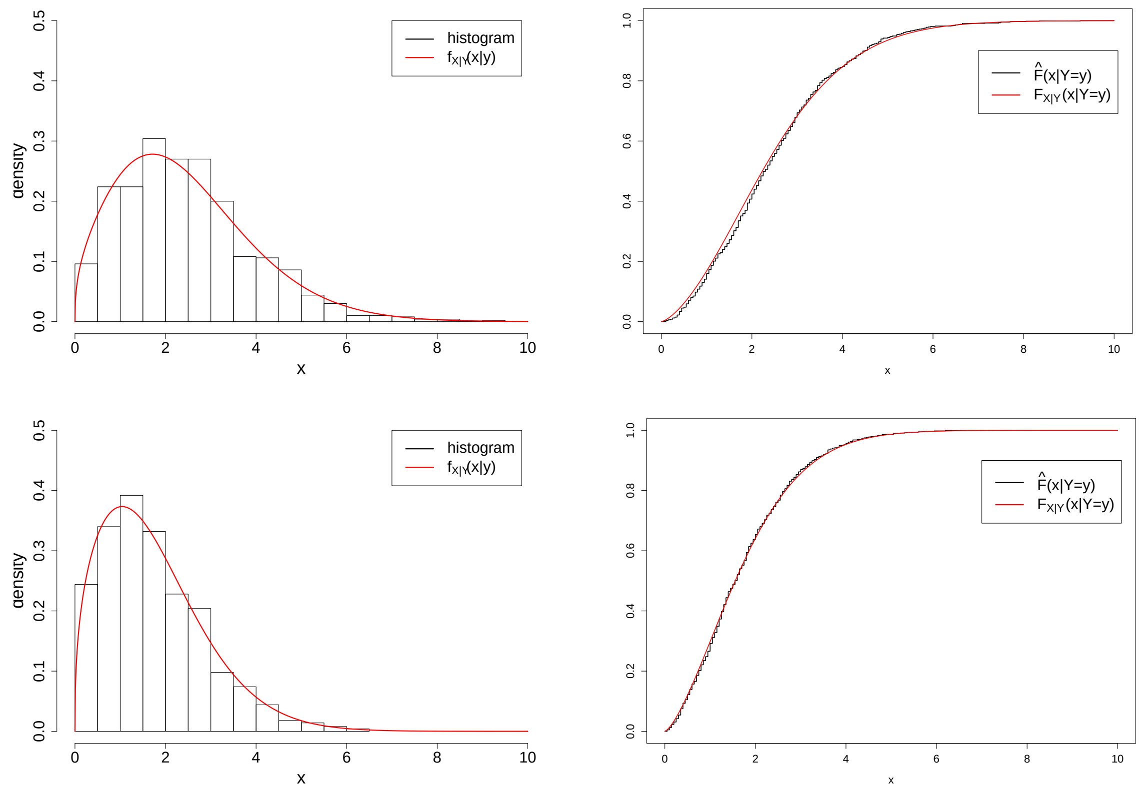


Figure 9.2: Plots of the histogram (left) and the empirical distribution function (right) of 1000 variates generated from the conditional distribution of X given $Y = y$, for $y = 0.5$ (top row) and $y = 1.5$ (bottom row). The theoretical density and distribution functions are superimposed.

Example 2: For each $y \in \{0.5, 1.5\}$, we generate 1000 variates from the conditional distribution of X given $Y = y$, with parameter values set to: $\alpha_1 = 2$, $\beta_1 = 1.4$, $\alpha_2 = 1.75$, $\beta_2 = 1.5$, and $\theta = 0.7$. In Figure 9.2, we have plotted the empirical distribution function and histogram of the generated X variates, along with the theoretical distribution and density functions given in (9.20) and (9.22). For the stopping rule $b - a \leq \Delta$, the tolerance is set to $\Delta = 0.0001$. The empirical distribution function $\hat{F}(.)$ is computed at all $u \in \{0, 0.05, 0.1, \ldots, 10\}$.

Notice that, in Figure 9.2, when the system usage Y at failure increases from 0.5 (top row) to 1.5 (bottom row), the distribution of the generated points is more skewed to the right. If the usage at failure is high, then it is likely that the system fails earlier in time. □

9.2.1.3 Generating Bivariate Observations $(\mathbf{X}, \mathbf{Y})$

In the preceding sections, we demonstrated how to generate variates from the marginal distribution of Y and the conditional distribution of X given $Y = y$. To generate points from the bivariate distribution, for each generated $Y = y$, we generate one X variate.

The bivariate distribution function of the point (X, Y) of the first failure of the system (i.e. the original lifetime) is given by

$$F(x, y) = 1 - \bar{F}_X(x) - \bar{F}_Y(y) + \bar{F}(x, y) \ , \tag{9.23}$$

for $x, y \geq 0$, where $\bar{F}(.,.)$ is the bivariate reliability function in (9.14), and $\bar{F}_X(.)$ and $\bar{F}_Y(.)$ are the marginal reliability functions of X and Y respectively, which are both IFR Weibull.

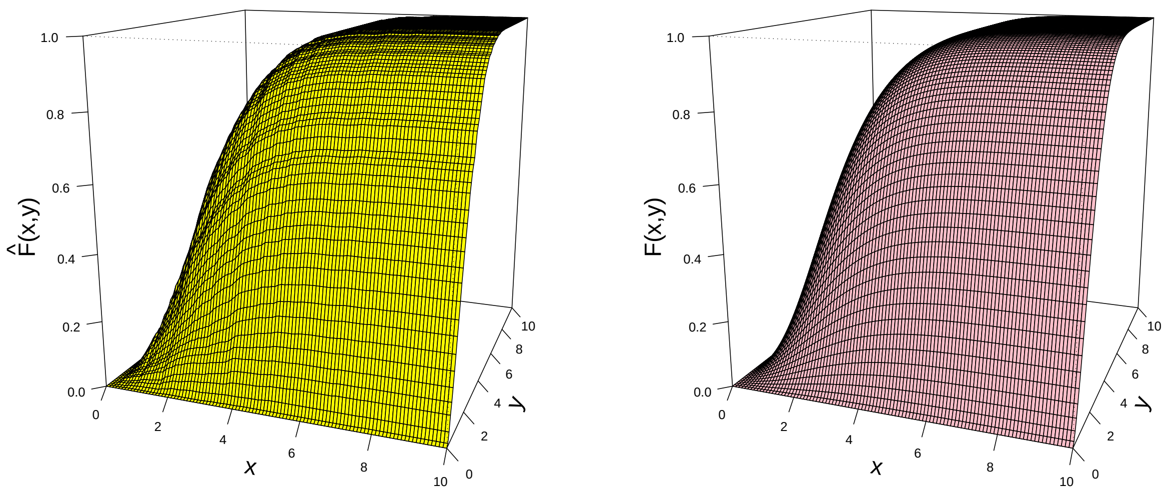


Figure 9.3: Plots of the empirical distribution function (left) of 1000 generated (X, Y) points and the theoretical distribution function (right).

Example 3: We generate 1000 points from the bivariate distribution of (X, Y), using the conditioning method. The parameter values are set to: $\alpha_1 = 2$, $\beta_1 = 1.4$, $\alpha_2 = 1.75$, $\beta_2 = 1.5$ and $\theta = 0.7$. In Figure 9.3, we have plotted the empirical distribution function of the generated data points (left), along with the theoretical bivariate distribution function $F(.,.)$ in (9.23) (right).

In Figure 9.3, both the empirical (left) and theoretical (right) distribution functions are computed at grid points $(u,v) \in \{0, 0.1, 0.2, \dots, 10\} \times \{0, 0.1, 0.2, \dots, 10\}$. The smooth appearance of the empirical distribution (left plot) is because the function is not plotted as a step function, but instead linearly interpolated between its values at the grid points.

Notice that, in Figure 9.3, the (empirical) distribution of the generated data is close to the theoretical distribution from which the data was generated. $\square$

9.2.2 Generating the Bivariate Inter-failure Lifetimes

Let (T_n, U_n) denote the n-th failure point and (X_{n+1}, Y_{n+1}) denote the $(n+1)$-th bivariate inter-failure lifetime, for $n \in \mathbb{N}_+$. Then, $(X_1, Y_1) = (T_1, U_1)$ is the original lifetime (or point of first failure) of the system. For $n \in \mathbb{N}_+$, T_n is the time and U_n is the usage at the n-th failure, X_{n+1} is the time between the n-th and $(n+1)$-th failures, and Y_{n+1} is the usage accumulated between the n-th and $(n+1)$-th failures. The relationship between the failure points and inter-failure lifetimes is as follows:

$$(T_n, U_n) = \left(X_1 + \cdots + X_n,\ Y_1 + \cdots + Y_n\right) \ , \tag{9.24}$$

for $n \in \mathbb{N}_+$. Note that, since the components of the inter-failure lifetimes are positive, the failure points are ordered such that:

$$\begin{aligned} &0 < T_1 < T_2 < \cdots < T_n < \dots \\ &0 < U_1 < U_2 < \cdots < U_n < \dots \ . \end{aligned} \tag{9.25}$$

In Section 9.2.1, we discussed the procedure for generating the point $(T_1, U_1) = (X_1, Y_1)$ ($\equiv (X, Y)$) of first failure. In this section, we discuss the steps involved in generating the consecutive failure points $\{(T_{n+1}, U_{n+1}); n \in \mathbb{N}_+\}$, where each failure is immediately followed by an instantaneous general repair. As before, we use the conditioning method for generating the bivariate observations; see Section 9.1.1.

To generate the failure points, we first generate the bivariate inter-failure lifetimes, and then compute the failure points using the relationship in (9.24). The distributions that we require in order to generate the bivariate lifetime (X_{n+1}, Y_{n+1}), for $n \in \mathbb{N}_+$, are:

(i) the marginal distribution of Y_{n+1}, given all previous failure points $(t_1, u_1), \dots, (t_n, u_n)$;

(ii) the conditional distribution of X_{n+1}, given $Y_{n+1} = y_{n+1}$ and all previous failure points $(t_1, u_1), \ldots, (t_n, u_n)$.

Note that, these marginal and conditional distributions depend on all previous failures and the degrees of the corresponding general repairs.

To derive these distributions, we need the joint distribution of X_{n+1} and Y_{n+1}, for $n \in \mathbb{N}_+$, given all previous failure points. We derived and discussed this joint distribution in Section 7.3.2. Here, we use this joint reliability function to derive the marginal and conditional distributions of interest.

Given the previous n failure points, the conditional reliability function of (X_{n+1}, Y_{n+1}) at the point (x, y) is equal to the conditional reliability function of (T_{n+1}, U_{n+1}) at the point (t, u), where $x = t - t_n^+ \geq 0$ and $y = u - u_n^+ \geq 0$, for $n \in \mathbb{N}_+$. The point (t_n^+, u_n^+) is used to denote the earliest possible instance of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}), where $T_{n+1} > T_n = t_n$ and $U_{n+1} > U_n = u_n$ [cf. (9.25)]. Then, for $x, y \geq 0$, the conditional reliability function of the bivariate inter-failure lifetime (X_{n+1}, Y_{n+1}), given the n previous failure points, is given by

$$\begin{aligned}
\bar{G}_{n+1}(x, y | \boldsymbol{t_n}, \boldsymbol{u_n}) &= \bar{F}_{n+1}(t_n^+ + x, u_n^+ + y \mid \boldsymbol{t_n}, \boldsymbol{u_n}) \\
&= \frac{\bar{F}(a_n(t_n^+ + x, u_n^+ + y), b_n(t_n^+ + x, u_n^+ + y))}{\bar{F}(a_n(t_n^+, u_n^+), b_n(t_n^+, u_n^+))} \\
&= \frac{\bar{F}(x + a_n(t_n^+, u_n^+), y + b_n(t_n^+, u_n^+))}{\bar{F}(a_n(t_n^+, u_n^+), b_n(t_n^+, u_n^+))} \ ,
\end{aligned} \tag{9.26}$$

where $\boldsymbol{t_n} = (t_1, \ldots, t_n)$, $\boldsymbol{u_n} = (u_1, \ldots, u_n)$, and $\bar{F}_{n+1}(.,. \mid \boldsymbol{t_n}, \boldsymbol{u_n})$ denotes the conditional reliability function of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}) given all previous failures; see Sections 7.2.2 and 7.3.2 for details. The function $\bar{F}(.,.)$ is the original reliability function given in (9.14). The quantities $a_n(t_n^+, u_n^+)$ and $b_n(t_n^+, u_n^+)$ denote respectively the virtual age and usage immediately following the n-th repair. The last expression in (9.26) follows from the definitions of the virtual age and usage functions: for $x = t - t_n^+ \geq 0$ and $y = u - u_n^+ \geq 0$, the virtual age at the point (t, u) is

$$\begin{aligned}
a_n(t, u) &= t - \sum_{i=1}^{n} \delta_i \, a_{i-1}(t_i, u_i) \\
&= t - t_n^+ + t_n^+ - \sum_{i=1}^{n} \delta_i \, a_{i-1}(t_i, u_i) \\
&= x + a_n(t_n^+, u_n^+) \ ,
\end{aligned} \tag{9.27}$$

and the virtual usage at this point is

$$\begin{aligned} b_n(t,u) &= u - \sum_{i=1}^{n} \gamma_i \, b_{i-1}(t_i, u_i) \\ &= u - u_n^+ + u_n^+ - \sum_{i=1}^{n} \gamma_i \, b_{i-1}(t_i, u_i) = y + b_n(t_n^+, u_n^+) \ , \end{aligned} \tag{9.28}$$

where (δ_i, γ_i) denotes the degree of the i-th repair, $i \in \{1, \ldots, n\}$; see Section 7.2.1. It must be noted that, when the number of failures before the point (t, u) is given, the virtual age function depends only on the failure times and the virtual usage function depends only on the failure usages; see Sections 7.2.1 and 8.3.2.

For a given sequence of failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$, the virtual age and usage immediately after the n-th repair are constant with respect to x and y. To make the expressions appearing in the remainder of this chapter more concise, we will henceforth use a_n and b_n to denote the virtual age and usage immediately following the n-th failure: i.e. $a_n(t_n^+, u_n^+) =: a_n$ and $b_n(t_n^+, u_n^+) =: b_n$, for $n \in \mathbb{N}_+$. With this new notation, the reliability function in (9.26) can be expressed as follows:

$$\bar{G}_{n+1}(x, y | \boldsymbol{t_n}, \boldsymbol{u_n}) = \frac{\bar{F}(x + a_n, y + b_n)}{\bar{F}(a_n, b_n)} \ . \tag{9.29}$$

Then, the joint density function corresponding to this conditional reliability function is given by

$$\begin{aligned} g_{n+1}(x, y | \boldsymbol{t_n}, \boldsymbol{u_n}) &:= \frac{\partial^2}{\partial x \, \partial y} \, \bar{G}_{n+1}(x, y | \boldsymbol{x_n}, \boldsymbol{y_n}) \\ &= \frac{\frac{\partial^2}{\partial x \, \partial y} \, \bar{F}(x + a_n, y + b_n)}{\bar{F}(a_n, b_n)} = \frac{f(x + a_n, y + b_n)}{\bar{F}(a_n, b_n)} \ , \end{aligned} \tag{9.30}$$

since the denominator is a constant with respect to both x and y, and the derivatives of the virtual age $a_n = a_n(t_n^+, u_n^+)$ and virtual usage $b_n = b_n(t_n^+, u_n^+)$, with respect to x and y, are 0. The function $f(.,.)$ is the original bivariate density function given in (9.16).

Next, using the above joint reliability and density functions, we derive the marginal distribution of Y_{n+1} and the conditional distribution of X_{n+1}, given $Y_{n+1} = y_{n+1}$, and given the previous n failures, for $n \in \mathbb{N}_+$.

9.2.2.1 Generating Observations from the Marginal Distribution of $\mathbf{Y_{n+1}}$

Let $\bar{G}_{Y_{n+1}}(.|\boldsymbol{t_n}, \boldsymbol{u_n})$ denote the marginal reliability function of the $(n+1)$-th inter-failure usage Y_{n+1}, given all previous failure points (or equivalently all previous bivariate inter-failure lifetimes), for $n \in \mathbb{N}_+$. This reliability function can be derived from the joint reliability function in (9.29), by simply setting $x = 0$. That is, for $n \in \mathbb{N}_+$,

$$
\begin{aligned}
\bar{G}_{Y_{n+1}}(y|\boldsymbol{t_n}, \boldsymbol{u_n}) &:= \bar{G}_{n+1}(0, y|\boldsymbol{t_n}, \boldsymbol{u_n}) \\
&= \frac{\bar{F}(a_n, y + b_n)}{\bar{F}(a_n, b_n)} \\
&= \frac{e^{-R_X(a_n)}\ e^{-R_Y(y+b_n)}\ e^{-\ \theta\ R_X(a_n)\ R_Y(y+b_n)}}{e^{-R_X(a_n)}\ e^{-R_Y(b_n)}\ e^{-\ \theta\ R_X(a_n)\ R_Y(b_n)}} \\
&= \frac{e^{-R_Y(y+b_n)\ \left[1+\theta\ R_X(a_n)\right]}}{e^{-R_Y(b_n)\ \left[1+\theta\ R_X(a_n)\right]}} \ ,
\end{aligned}
\tag{9.31}
$$

where $R_X(.)$ and $R_Y(.)$ are the cumulative failure rate functions of the original lifetime variables X and Y respectively, given in (9.15). Note that, except for $R_Y(y + b_n)$ in the numerator of the last expression in (9.31), all other quantities are constant with respect to y.

To further simplify expressions, we introduce the constants (with respect to y) k_1 and k_2, where

$$
k_1 := 1 + \theta\ R_X(a_n) \ , \tag{9.32}
$$

and

$$
k_2 := e^{-R_Y(b_n)\ \left[1+\theta\ R_X(a_n)\right]} = e^{-k_1\ R_Y(b_n)} \ . \tag{9.33}
$$

Then, using k_1 and k_2, the marginal reliability function in (9.31) can be written as follows:

$$
\begin{aligned}
\bar{G}_{Y_{n+1}}(y|\boldsymbol{t_n}, \boldsymbol{u_n}) &= \frac{e^{-k_1\ R_Y(y+b_n)}}{k_2} \\
&= \frac{e^{-k_1\ \left(\frac{y+b_n}{\alpha_2}\right)^{\beta_2}}}{k_2} \ .
\end{aligned}
\tag{9.34}
$$

This function is easily invertible, and therefore, we use the inverse transformation method to generate Y_{n+1} variates. The solution Y to the equation $\bar{G}_{Y_{n+1}}(Y|\boldsymbol{t_n}, \boldsymbol{u_n}) = U$, is derived as follows:

$$
\begin{aligned}
\frac{e^{-k_1\ \left(\frac{Y+b_n}{\alpha_2}\right)^{\beta_2}}}{k_2} = U \qquad &\Leftrightarrow \qquad e^{-k_1\ \left(\frac{Y+b_n}{\alpha_2}\right)^{\beta_2}} = k_2\ U \\
&\Leftrightarrow \qquad k_1\ \left(\frac{Y+b_n}{\alpha_2}\right)^{\beta_2} = -\ln(k_2\ U) \\
&\Leftrightarrow \qquad \frac{Y+b_n}{\alpha_2} = \left(\frac{-\ln(k_2\ U)}{k_1}\right)^{1/\beta_2} \\
&\Leftrightarrow \qquad Y = \alpha_2\ \left(\frac{-\ln(k_2\ U)}{k_1}\right)^{1/\beta_2} -\ b_n \ .
\end{aligned}
\tag{9.35}
$$

Since the virtual age a_n and the virtual usage b_n depend on all n previous failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$ and the degrees $\{(\delta_1, \gamma_1), \ldots, (\delta_n, \gamma_n)\}$ of the corresponding repairs, we require these values to generate the Y_{n+1} variates. Once we compute the values of a_n and b_n, we compute the values of k_1 and k_2, and then use the transformation

$$Y_{n+1} = \alpha_2 \left(\frac{-\ln(k_2\, U)}{k_1} \right)^{1/\beta_2} - \; b_n \tag{9.36}$$

to generate the Y_{n+1} variates; see Algorithm 1 on page 209.

Note that, Y_{n+1} in (9.36) is positive, since $\alpha_2 \left(\frac{-\ln(k_2\, U)}{k_1} \right)^{1/\beta_2} \geq b_n$; the proof is as follows. We know that, the U variate is uniform on $(0, 1)$, and therefore, the value of $-\ln U = |\ln U|$ (which is a monotone function of U) is in the interval $(0, \infty)$. Then, using the definitions of k_2 in (9.33) and $R_Y(.)$ in (9.15), we have

$$\begin{aligned}
\alpha_2 \left(\frac{-\ln(k_2\, U)}{k_1} \right)^{1/\beta_2} &= \alpha_2 \left(\frac{-\ln k_2 + |\ln U|)}{k_1} \right)^{1/\beta_2} \\
&= \alpha_2 \left(\frac{k_1\, R_Y(b_n) + |\ln U|)}{k_1} \right)^{1/\beta_2} \\
&= \alpha_2 \left(R_Y(b_n) + \frac{|\ln U|)}{k_1} \right)^{1/\beta_2} \\
&= \alpha_2 \left(\left(\frac{b_n}{\alpha_2} \right)^{\beta_2} + \frac{|\ln U|)}{k_1} \right)^{1/\beta_2} \quad ,
\end{aligned} \tag{9.37}$$

which decreases to b_n when $|\ln U|$ approaches 0 (or equivalently, when U approaches 1).

When all repairs are perfect (i.e. have degree $(1, 1)$), then the distributions of the inter-failure usages Y_{n+1}, $n \in \mathbb{N}$, are identical to the original marginal distribution with reliability function $\bar{F}_Y(y) = \exp\{-(y/\alpha_2)^{\beta_2}\}$. When all repairs are perfect, $a_n = b_n = 0$ and $k_1 = k_2 = 1$, and therefore, the transformation in (9.36) reduces to the transformation in (9.18).

The marginal density function of the inter-failure usage Y_{n+1}, given all previous failure points, is given by

$$\begin{aligned}
g_{Y_{n+1}}(y|\boldsymbol{t_n}, \boldsymbol{u_n}) := -\frac{\partial}{\partial y}\, \bar{G}_{Y_{n+1}}(y|\boldsymbol{t_n}, \boldsymbol{u_n}) &= -\frac{\frac{\partial}{\partial y}\, \bar{F}(a_n, y + b_n)}{\bar{F}(a_n, b_n)} \\
&= \frac{e^{-R_X(a_n)}\, \left[1 + \theta\, R_X(a_n)\right]\, r_Y(y + b_n)\, e^{-R_Y(y+b_n)\, \left[1+\theta\, R_X(a_n)\right]}}{e^{-R_X(a_n)}\, e^{-R_Y(b_n)\, \left[1+\theta\, R_X(a_n)\right]}} \\
&= \frac{k_1\, r_Y(y + b_n)\, e^{-k_1\, R_Y(y+b_n)}}{k_2} \quad ,
\end{aligned} \tag{9.38}$$

for $n \in \mathbb{N}_+$, where k_1 and k_2 are defined in (9.32) and (9.33), and $r_Y(.)$ is the marginal failure rate function, which is the derivative of the corresponding cumulative failure rate function, i.e. for $u \geq 0$,

$$r_Y(u) = \frac{d}{du} R_Y(u) = \left(\frac{\beta_2}{\alpha_2}\right) \left(\frac{u}{\alpha_2}\right)^{\beta_2 - 1} . \tag{9.39}$$

Example 4: We generate 1000 observations from the marginal distribution of the second inter-failure usage Y_2, given the point of first failure $(t_1, u_1) = (x_1, y_1) = (1.1, 2.3)$. In Figure 9.4, we have plotted the histogram and empirical distribution function of the Y_2 variates, generated using the transformation in (9.36). In Figure 9.4, in the top row, the degree of the first repair is set to $(\delta_1, \gamma_1) = (0.3, 0.5)$, and in the bottom row, the degree of the first repair is set to $(\delta_1, \gamma_1) = (1, 1)$. The other parameter values are set to: $\alpha_1 = 2$, $\beta_1 = 1.4$, $\alpha_2 = 1.75$, $\beta_2 = 1.5$, and $\theta = 0.7$. The empirical distribution functions are computed at all $u \in \{0, 0.02, 0.04, \ldots, 8\}$.

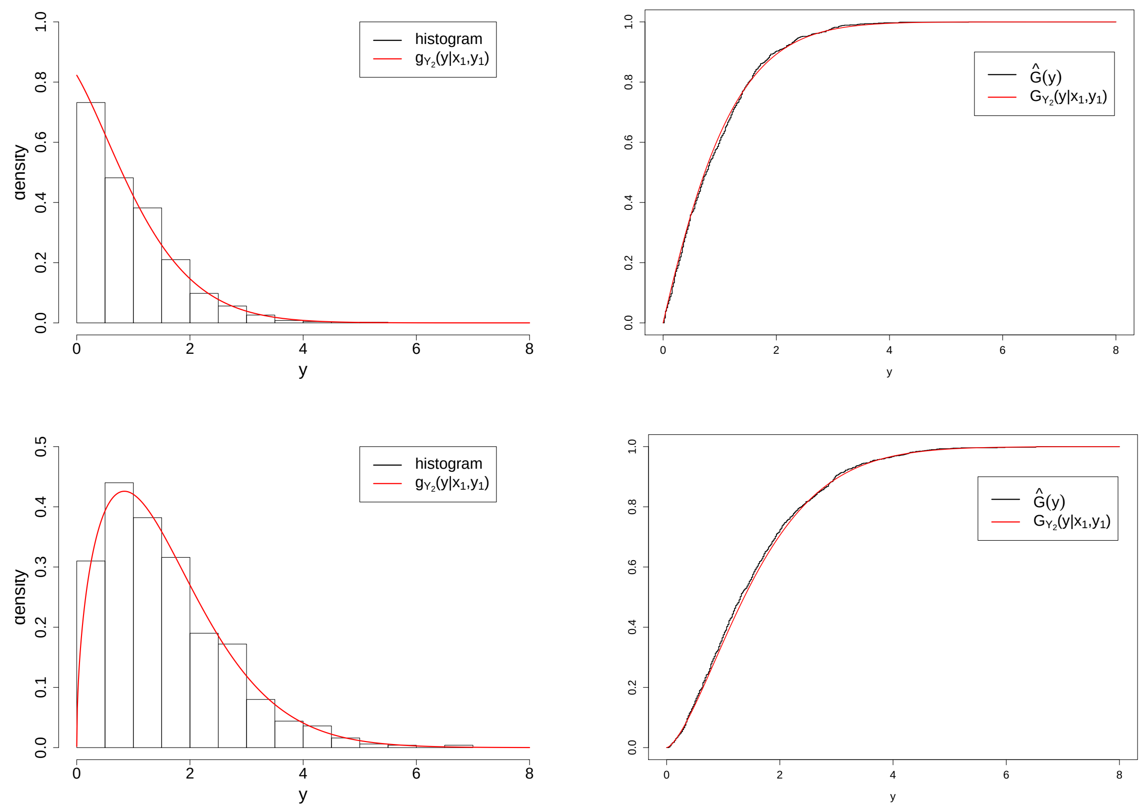


Figure 9.4: Plots of the histogram (left) and empirical distribution function (right) of 1000 variates generated from the marginal distribution of Y_2, given the previous failure point $(t_1, u_1) = (1.1, 2.3)$. The degree of repair is: $(\delta_1, \gamma_1) = (0.3, 0.5)$ (top row); $(\delta_1, \gamma_1) = (1, 1)$ (bottom row). The respective theoretical density and distribution functions are superimposed.

In Figure 9.4, notice that, when the degree of the repair following the first failure is increased from $(0.3, 0.5)$ (top row) to $(1, 1)$ (bottom row), the distribution function approaches 1 at a lower rate. As the degree of repair increases, the reliability of the system further improves, which results in the second failure point being less likely to happen close to the first. This is observable in the density functions in the two rows: in the bottom row, longer inter-failure usages are more likely.

It must be noted that, when the degree of repair is $(\delta_1, \gamma_1) = (1, 1)$ (bottom row in Figure 9.4), the system is in effect replaced by a new system following the first failure. Therefore, $a_1 = a_1(t_1^+, u_1^+) = 0$, $b_1 = b_1(t_1^+, u_1^+) = 0$, and $k_1 = k_2 = 1$, which reduces the distribution of Y_2 to the distribution of $Y_1 \equiv Y$, which is the usage at first failure; see Section 9.2.1.1. Notice that, the density and distribution functions in Figure 9.4 (bottom row) and Figure 9.1 are identical. □

9.2.2.2 Generating Observations from the Conditional Distribution of $\mathbf{X_{n+1}}$ given $\mathbf{Y_{n+1} = y}$

For $n \in \mathbb{N}_+$, the conditional distribution of the $(n+1)$-th inter-failure time X_{n+1}, given $Y_{n+1} = y$ and all previous failure points, can be derived from the corresponding joint and marginal distributions discussed earlier. The conditional density function for this distribution is derived as follows:

$$
\begin{aligned}
g_{X_{n+1}|Y_{n+1}}(x|y; \boldsymbol{t_n}, \boldsymbol{u_n}) &:= \frac{g_{n+1}(x, y|\boldsymbol{t_n}, \boldsymbol{u_n})}{g_{Y_{n+1}}(y|\boldsymbol{t_n}, \boldsymbol{u_n})} \\
&= \frac{f(x + a_n, y + b_n) / \bar{F}(a_n, b_n)}{-\frac{\partial}{\partial y}\, \bar{F}(a_n, y + b_n) / \bar{F}(a_n, b_n)} \\
&= \frac{f(x + a_n, y + b_n)}{-\frac{\partial}{\partial y}\, \bar{F}(a_n, y + b_n)} \\
&= \frac{f(x + a_n, y + b_n)}{e^{-R_X(a_n)} \left[1 + \theta\, R_X(a_n)\right] r_Y(y + b_n)\, e^{-R_Y(y + b_n)\left[1 + \theta\, R_X(a_n)\right]}} \\
&= \frac{f(x + a_n, y + b_n)}{k_1\, r_Y(y + b_n)\, \bar{F}(a_n, y + b_n)}\,,
\end{aligned} \tag{9.40}
$$

where $g_{n+1}(.,.|\boldsymbol{t_n}, \boldsymbol{u_n})$ denotes the density function of (X_{n+1}, Y_{n+1}) given in (9.30) and the function $g_{Y_{n+1}}(.|\boldsymbol{t_n}, \boldsymbol{u_n})$ denotes the marginal density function of Y_{n+1} given in (9.38), both conditional on all previous failure points. The expressions for the original joint reliability and density functions, $\bar{F}(.,.)$ and $f(.,.)$, are given in (9.14) and (9.16) respectively. Note that, as before, $a_n = a_n(t_n^+, u_n^+)$, $b_n = b_n(t_n^+, u_n^+)$, and $k_1 = 1 + \theta\, R_X(a_n)$.

The distribution function corresponding to the density function in (9.40) is given by

$$
\begin{aligned}
G_{X_{n+1}|Y_{n+1}}(x|Y_{n+1}=y;\boldsymbol{t_n},\boldsymbol{u_n}) &= \int_0^x g_{X_{n+1}|Y_{n+1}}(s|y;\boldsymbol{t_n},\boldsymbol{u_n})\,ds \\
&= \int_0^x \frac{f(s+a_n,y+b_n)}{k_1\,r_Y(y+b_n)\,\bar{F}(a_n,y+b_n)}\,ds\ ,
\end{aligned}
\tag{9.41}
$$

and the corresponding reliability function is given by

$$
\bar{G}_{X_{n+1}|Y_{n+1}}(x|Y_{n+1}=y;\boldsymbol{t_n},\boldsymbol{u_n}) := 1 - G_{X_{n+1}|Y_{n+1}}(x|Y_{n+1}=y;\boldsymbol{t_n},\boldsymbol{u_n})\ . \tag{9.42}
$$

The conditional distribution function cannot be expressed in closed form– the above integral is computed numerically. To generate variates from this distribution, we will use the secant method discussed in Section 9.1.2; see Algorithm 2 on page 210.

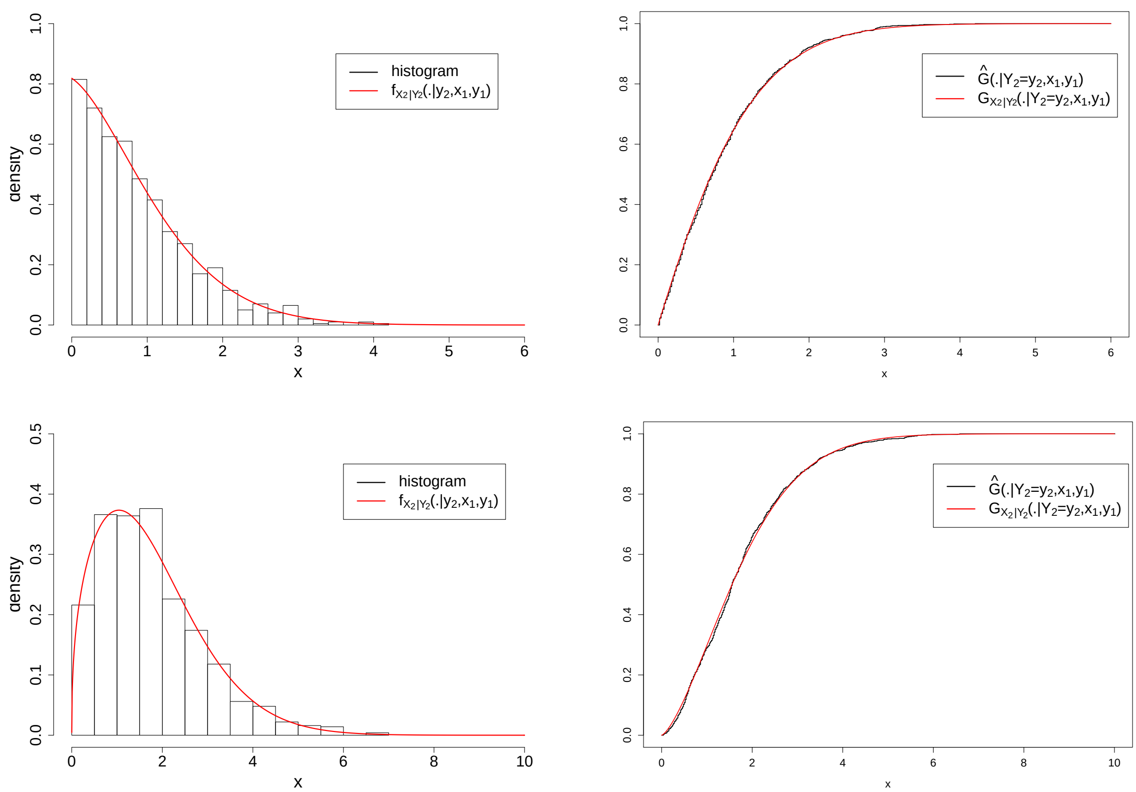


Figure 9.5: Plots of the histogram (left) and empirical distribution function (right) of 1000 variates generated from the conditional distribution of the 2-nd inter-failure time X_2, given $Y_2 = 1.5$ and first failure point $(x_1 = y_1) = (1.1, 2.3)$. The degree of repair is: $(\delta_1, \gamma_1) = (0.3, 0.5)$ (top row); $(\delta_1, \gamma_1) = (1, 1)$ (bottom row). The respective theoretical distribution functions are superimposed.

Example 5: We generate 1000 variates from the conditional distribution of X_2, given $Y_2 =$

1.5 and first failure point $(t_1, u_1) = (x_1, y_1) = (1.1, 2.3)$. In Figure 9.5, we have plotted the histogram and empirical distribution function, along with the theoretical density and distribution functions given in (9.40) and (9.41). The degree of the repair following the first failure is set to: $(\delta_1, \gamma_1) = (0.3, 0.5)$ in the top row and $(\delta_1, \gamma_1) = (1, 1)$ in the bottom row. The chosen parameter values are: $\alpha_1 = 2$, $\beta_1 = 1.4$, $\alpha_2 = 1.75$, $\beta_2 = 1.5$, and $\theta = 0.7$. The empirical distribution functions are computed at equidistant points, set 0.02 units apart.

In Figure 9.5, as observed with the marginal distribution of Y_2 in Figure 9.4, when the degree (δ_1, γ_1) of the first repair increases from $(0.3, 0.5)$ (top row) to $(1, 1)$ (bottom row), the next failure time $T_2 = X_2 + x_1$ (for a given Y_2) is less likely to be close to the first failure time $t_1 = x_1$ (i.e. the second inter-failure time is more likely to be farther from zero).

Also, when the degree of the first repair is $(1, 1)$ (i.e. perfect repair), the conditional distribution of X_2 given $Y_2 = y$ is equal to the conditional distribution of $X_1 = X$ given $Y_1 = Y = y$, discussed in Section 9.2.1.2. Notice that, the theoretical distributions in Figure 9.5 (bottom row) and Figure 9.2 (bottom row) are identical. □

9.2.2.3 Generating Bivariate Observations $(\mathbf{X_{n+1}}, \mathbf{Y_{n+1}})$

In the preceding sections, for a sequence of previous failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$, we demonstrated how to generate variates from the marginal distribution of Y_{n+1} and the conditional distribution of X_{n+1}, given $Y_{n+1} = y$, both conditional on previous failure points, for $n \in \mathbb{N}_+$. To generate points from the bivariate distribution of (X_{n+1}, Y_{n+1}), for each generated $Y_{n+1} = y$, we generate one X_{n+1} variate.

The distribution function of the bivariate inter-failure lifetime (X_{n+1}, Y_{n+1}), given all previous failure points, is given by

$$G_{n+1}(x, y|\boldsymbol{t_n}, \boldsymbol{u_n}) := 1 - \bar{G}_{n+1}(x, 0|\boldsymbol{t_n}, \boldsymbol{u_n}) - \bar{G}_{n+1}(0, y|\boldsymbol{t_n}, \boldsymbol{u_n}) + \bar{G}_{n+1}(x, y|\boldsymbol{t_n}, \boldsymbol{u_n}) \ , \quad (9.43)$$

for $x, y \geq 0$ and $n \in \mathbb{N}_+$, where $\bar{G}(., .|\boldsymbol{t_n}, \boldsymbol{u_n})$ is the corresponding bivariate reliability function given in (9.26).

Example 6: We generate 1000 observations from the bivariate distribution of (X_2, Y_2), given the point of first failure $(t_1, u_1) = (1.1, 2.3)$. The chosen degree of the first repair is $(\delta_1, \gamma_1) = (0.4, 0.6)$. The parameter values are set to: $\alpha_1 = 2$, $\beta_1 = 1.4$, $\alpha_2 = 1.75$, $\beta_2 = 1.5$, and $\theta = 0.7$. In Figure 9.6, we have plotted the empirical distribution function of the generated bivariate observations (left), along with the theoretical bivariate distribution function $G_2(., .|x_1, y_1)$ in

(9.43) (right). Both the empirical and theoretical distribution functions are computed at grid points $(u, v) \in \{0, 0.05, 0.1, \ldots, 10\} \times \{0, 0.05, 0.1, \ldots, 10\}$.

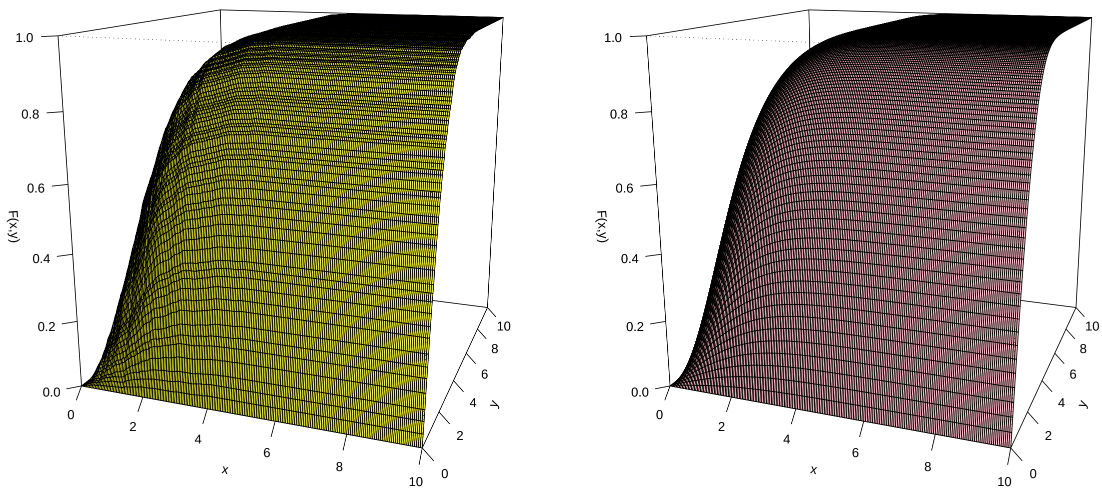


Figure 9.6: The empirical distribution function (left) of 1000 points generated from the bivariate distribution (right) of the inter-failure lifetime (X_2, Y_2), given the previous failure point.

In Figure 9.6, the smooth appearance of the empirical distribution (left plot) is because the function is not plotted as a step function, but instead linearly interpolated between its values at the grid points. Notice that, the distribution of the generated data points (left) is close to the theoretical distribution (right) from which the data points were generated. □

9.2.3 Generating Consecutive Failure Points (Failure Process)

In the previous sections, we discussed the procedures for generating variates from the marginal and conditional distributions of the inter-failure lifetimes $\{(X_n, Y_n); n \in \mathbb{N}_+\}$. Using these procedures, we can now describe the steps involved in generating the consecutive failure points $\{(T_n, U_n); n \in \mathbb{N}_+\}$ of the failure process in two dimensions. As discussed earlier, generating the failure points is equivalent to generating the bivariate inter-failure lifetimes, since the time and usage at failure are partial sums of inter-failure times and usages respectively; see (9.24).

Algorithm 3: generating the failure process. The algorithm to generate a sequence of failure points in a region $(0, w_t] \times (0, w_u] \subset \mathbb{R}^2_+$ is as follows:

(i) generate the first point $(t_1, u_1) = (x_1, u_1)$ (see Section 9.2.1);

(ii) set $n = 1$;

(iii) select the degree (δ_n, γ_n) of repair for the n-th failure, and compute the virtual age $a_n = a_n(t_n^+, u_n^+)$ and the virtual usage $b_n = b_n(t_n^+, u_n^+)$ immediately following the n-th repair (refer to (9.27) and (9.28) on page 218);

(iv) generate the $(n+1)$-th bivariate inter-failure lifetime (x_{n+1}, y_{n+1})– the distributions of the inter-failure lifetimes depend on the computed virtual age a_n and virtual usage b_n (see Section 9.2.2);

(v) compute the $(n+1)$-th failure point $(t_{n+1}, u_{n+1}) = \big(t_n + x_{n+1},\ u_n + y_{n+1}\big)$;

(vi) if $t_{n+1} > w_t$ or $u_{n+1} > w_u$, stop– the number of failures in $(0, w_t] \times (0, w_u]$ is then n; otherwise, set $n = n + 1$, and return to step (iii); see Figure 9.7.

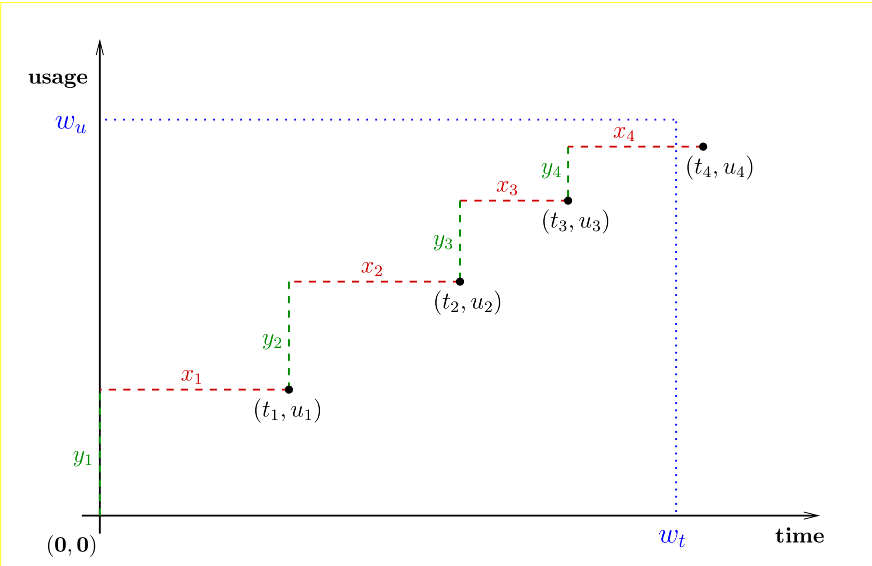


Figure 9.7: Illustration of 4 consecutive failure points $\{(t_1, u_1), \ldots, (t_4, u_4)\}$, along with the inter-failure times $\{x_1, \ldots, x_4\}$ and usages $\{y_1, \ldots, y_4\}$, where the 4-th failure occurs outside the region $(0, w_t] \times (0, w_u]$. The simulation run terminates after the 4-th point.

Example 7: We generate multiple sample trajectories of the failure process in two dimensions, using the above simulation procedures. In Figure 9.8, we have plotted the trajectories for the following parameter values: $\alpha_1 = 2$, $\beta_1 = 1.4$, $\alpha_2 = 1.75$, $\beta_2 = 1.5$, and $\theta = 0.7$. Following each failure of the system, a general repair is performed, which affects the distribution of the succeeding failure points. For this example, we have set the degrees of repair to $(\delta_n, \gamma_n) = (0.5, 0.5)$, for all $n \in \mathbb{N}_+$.

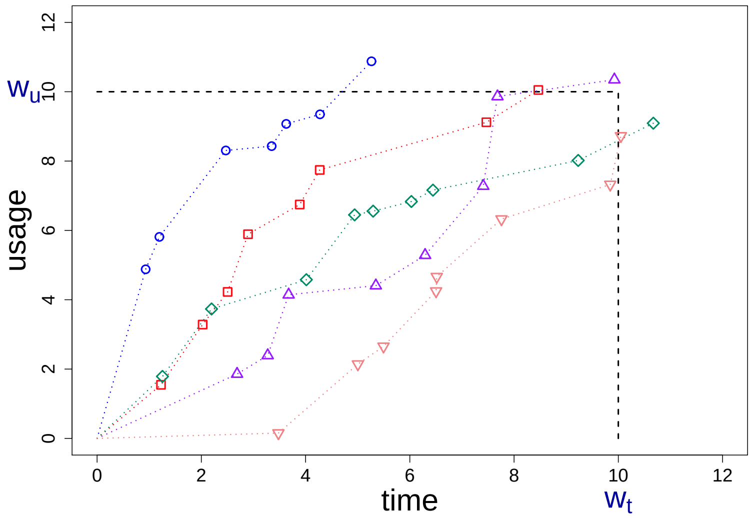


Figure 9.8: Plot of simulated trajectories of the failure process over the region $(0, 10] \times (0, 10]$. The numbers of points in each trajectory that are within the region range from 6 to 8.

In Figure 9.8, we have also plotted the first point outside the region of interest, which is the point at which a simulation run terminates. To distinguish the individual trajectories, we have plotted lines connecting the failure points from each run. □

9.3 Numerical Illustrations

In this section, we provide a numerical example to illustrate applications of the proposed repair model and simulation approach in the context of two-dimensional warranty cost analysis. We use the simulation approach to also illustrate the effect of the bivariate degrees of repair on the (expected) number of failures.

Two-dimensional warranties are characterized by a region $\mathcal{W} \subset \mathbb{R}_+^2$, with the axes representing the variables of the warranty policy (here, time and usage). We consider the rectangular warranty coverage $\mathcal{W} = (0, w_t] \times (0, w_u]$, where w_t and w_u denote the warranty limits in terms of time and usage, respectively. With this type of warranty policy, the system remains under warranty until either limit is exceeded. Also, the warranty considered is a free-repair (or free-replacement) warranty, i.e. the manufacturer agrees to repair (or replace) the failed system at no charge to the consumer; see Section 2.3. Most automobile manufacturers offer this type of warranty, with usage quantified by the distance traveled: for example, the warranty coverage ends when 5 years or $60,000$ miles is exceeded.

9.3.1 Estimating Expected Number of Failures (Claims)

Failures of a system can lead to claims under warranty, which can result in additional costs to the manufacturer (warrantor). Therefore, failure modeling is an important aspect of warranty cost analysis. Often the process of modeling the number of claims made under warranty is simplified to be equivalent to modeling the number of failures of the system under warranty. The necessary assumptions are: (a) every failure of the system results in an immediate and valid warranty claim (with processing time being negligible, i.e. equal to zero); and (b) every claim is followed immediately by an instantaneous repair (or replacement); see Section 2.4. Then, the expected number of failures of a system covered by a warranty policy can serve as an estimate of the number of repairs (or replacements) performed under warranty.

Let $N(w_t, w_u)$ denote the number of failures in the warranty region $(0, w_t] \times (0, w_u]$. To estimate the expected number of failures, denoted by $E[N(w_t, w_u)]$, we simulate the failure process multiple times, using the simulation approach described earlier, and then compute the sample average:

$$\widehat{E}[N(w_t, w_u)] = \frac{1}{m} \sum_{i=1}^{m} n_i(w_t, w_u) \ , \tag{9.44}$$

where m is the number of simulations (i.e. trajectories of the failure process), and $n_i(w_t, w_u)$ is the number of failures for the i-th simulation run, where $i \in \{1, \ldots, m\}$ and $m \in \mathbb{N}_+$. The sample variance of the simulated numbers is given by

$$\widehat{\text{var}}[N(w_t, w_u)] = \frac{1}{m-1} \sum_{i=1}^{m} \left(n_i(w_t, w_u) - \widehat{E}[N(w_t, w_u)] \right)^2 \ . \tag{9.45}$$

The sample relative standard deviation (or estimated coefficient of variation), which we denote by $\hat{c}_v$, can then be computed as follows:

$$\hat{c}_v = \frac{\sqrt{\widehat{\text{var}}[N(w_t, w_u)]}}{\widehat{E}[N(w_t, w_u)]} \ . \tag{9.46}$$

The purpose of this simulation is to illustrate the effect of the degrees of repair on the expected number of failures. According to the general repair model, following a minimal repair, the system is simply restored to an operational state and its working condition does not change. The working condition of a system following a non-minimal repair is equivalent to the working condition of an identical system at a younger age and lower usage– the virtual age and usage of the system immediately following the repair depend on the degree of the repair. As the degree of repair increases (from $(0, 0)$ for a minimal repair to $(1, 1)$ for a perfect repair), so does the reliability of the system, and therefore, it is reasonable to expect fewer failures for higher degrees of repair; see Section 8.1.

As before, let (δ_n, γ_n) denote the bivariate degree of the general repair following the n-th failure of the system, for $n \in \mathbb{N}_+$. For simplicity, we set $(\delta_n, \gamma_n) = (\delta, \gamma)$, for all $n \in \mathbb{N}_+$, where $\delta, \gamma \in [0, 1]$. That is, the general repairs for any given failure process are all of the same effectiveness (or degree).

For this illustration, we choose the following values for the components of the degree of repair: $\mathcal{D} = \{0.0, 0.1, 0.2, \ldots, 1.0\}$. We set the warranty coverage limits to $w_t = 5$ (years) and $w_u = 6$ ($\times 10,000$ miles). Then, for each $(\delta, \gamma) \in \mathcal{D}^2$, we simulate the process $m = 10,000$ times and compute the average number of failures. The averages, along with the corresponding sample relative standard deviations, are tabulated in Table 9.1.

Table 9.1: Estimates $\widehat{E}[N(5, 6;\ \delta, \gamma)]$ of the expected number of failures under warranty, along with the corresponding relative standard deviations $\hat{c}_v$, for various repair degrees $(\delta, \gamma) \in \mathcal{D}^2$

		γ										
		0.0	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
δ	**0.0**	6.198	5.130	4.527	4.161	3.831	3.610	3.451	3.300	3.142	3.041	2.921
		0.5898	0.5276	0.5037	0.4834	0.4838	0.4759	0.4740	0.4774	0.4844	0.4834	0.4840
	0.1	5.511	4.708	4.237	3.918	3.639	3.438	3.248	3.141	3.013	2.906	2.805
		0.5428	0.5042	0.4862	0.4675	0.4634	0.4623	0.4634	0.4633	0.4618	0.4634	0.4695
	0.2	4.984	4.404	4.004	3.697	3.443	3.256	3.151	3.003	2.889	2.787	2.670
		0.5234	0.4884	0.4732	0.4606	0.4596	0.4589	0.4506	0.4591	0.4549	0.4532	0.4570
	0.3	4.635	4.158	3.787	3.509	3.327	3.152	2.995	2.873	2.757	2.682	2.575
		0.5105	0.4847	0.4690	0.4586	0.4489	0.4482	0.4428	0.4494	0.4514	0.4491	0.4493
	0.4	4.371	3.949	3.590	3.392	3.181	3.017	2.890	2.783	2.672	2.574	2.493
		0.5152	0.4796	0.4672	0.4477	0.4478	0.4441	0.4390	0.4368	0.4414	0.4504	0.4444
	0.5	4.170	3.757	3.497	3.262	3.075	2.929	2.776	2.693	2.565	2.493	2.409
		0.5094	0.4740	0.4602	0.4483	0.4451	0.4443	0.4394	0.4375	0.4461	0.4437	0.4472
	0.6	3.966	3.612	3.326	3.126	2.948	2.799	2.693	2.604	2.511	2.415	2.352
		0.5087	0.4773	0.4605	0.4512	0.4470	0.4428	0.4349	0.4350	0.4359	0.4409	0.4302
	0.7	3.852	3.449	3.254	3.030	2.838	2.728	2.619	2.519	2.424	2.333	2.271
		0.5037	0.4920	0.4571	0.4530	0.4497	0.4426	0.4411	0.4375	0.4351	0.4374	0.4337
	0.8	3.630	3.361	3.115	2.938	2.787	2.612	2.541	2.456	2.363	2.281	2.203
		0.5196	0.4873	0.4709	0.4526	0.4486	0.4478	0.4346	0.4335	0.4325	0.4337	0.4420
	0.9	3.459	3.208	2.975	2.814	2.689	2.554	2.441	2.356	2.282	2.219	2.138
		0.5284	0.4969	0.4739	0.4638	0.4494	0.4493	0.4462	0.4405	0.4398	0.4319	0.4387
	1.0	3.330	3.057	2.844	2.677	2.560	2.439	2.353	2.274	2.210	2.136	2.080
		0.5276	0.5003	0.4870	0.4733	0.4617	0.4616	0.4459	0.4471	0.4405	0.4391	0.4397

Each cell of Table 9.1 corresponds to simulation runs of the failure (or general repair) process $\{(T_n, U_n); n \in \mathbb{N}_+\}$ characterized by the corresponding degree of repair, $(\delta, \gamma) \in \mathcal{D}^2$. In each cell, the first number is the average number of failures, $\widehat{E}[N(5,6;\ \delta,\gamma)]$, over the $m = 10,000$ simulations of the failure process; and the second number (smaller font size) is the sample relative standard deviation $\hat{c}_v$. Here, we use the notation $\widehat{E}[N(w_t, w_u;\ \delta,\gamma)]$ to specify that the estimates are for a process where all repairs are of a given degree (δ,γ). To improve the readability of the table, the diagonal cells (where the components of the degree of repair are both equal) are colored.

Notice that, as the bivariate degree (δ,γ) of the general repairs increases (along the diagonal), the estimated expected number of failures decreases, from 6.090 for all-minimal repairs to 2.077 for all-perfect repairs. This is due to the increased improvement in the reliability of the system following repairs of higher degrees. As discussed in Section 8.1, at any point, the conditional reliability of the system following an imperfect repair is bounded between the conditional reliabilities following minimal and perfect repairs. This improved reliability also occurs when either component δ or γ of the bivariate degree of repair is fixed and the other is increased. The trend is observed in Table 9.1, where the average number of failures in each column (row) decreases along the rows (columns).

We have plotted the estimates from Table 9.1 in Figure 9.9, where the effects of the components δ and γ of the degrees of repair are more discernible. Notice that, the average number of failures decreases as δ and γ, for $(\delta,\gamma) \in \mathcal{D}^2$, increase.

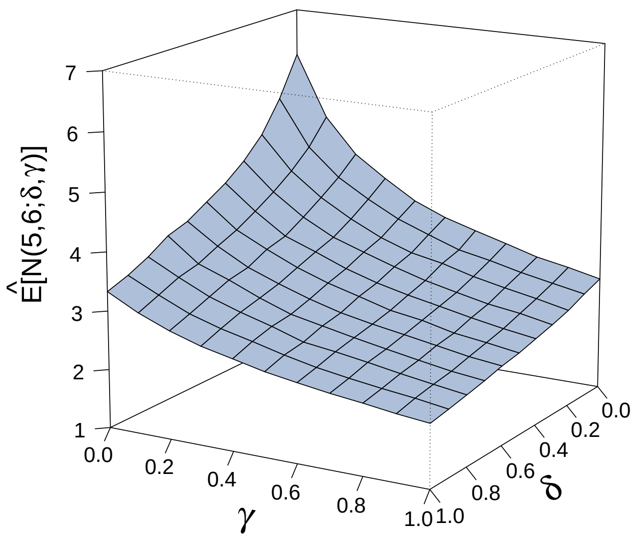


Figure 9.9: Plot of the estimates $\widehat{E}[N(5,6;\delta,\gamma)]$ from Table 9.1 against $\delta \in \mathcal{D}$ and $\gamma \in \mathcal{D}$ along the x- and y-axes. The interpolated values of $\widehat{E}[N(5,6;\delta,\gamma)]$ are represented by the gray surface.

The estimates in Table 9.1 are each computed from 10,000 simulations of the failure process. Additional simulation runs may be necessary to get more accurate estimates. For this illustration, we chose $m = 10,000$ based on the elementary test in Table 9.2, where we computed the estimates $\widehat{E}[N(5,6;\ \delta,\gamma)]$ for $\delta = \gamma \in \mathcal{D}$ and for an increasing sequence of numbers of simulations and stopped at 10,000 runs, since the difference between the estimates was small. Each column in Table 9.2, corresponds to a varying number m of simulations of a failure process, where all repairs are of degree (δ,γ), such that $\delta = \gamma \in \mathcal{D}$. The estimates in the last column of Table 9.2 are the estimates in the diagonal cells of Table 9.1.

Table 9.2: Estimates $\widehat{E}[N(5,6;\ \delta,\gamma)]$ of the expected number of failures under warranty, along with sample relative standard deviations $\hat{c}_v$, for repair degree components $\delta = \gamma \in \mathcal{D}$ and an increasing number m of simulations

		number of simulations: m									
		1000	**2000**	**3000**	**4000**	**5000**	**6000**	**7000**	**8000**	**9000**	**10000**
(δ,γ)	**(0.0,0.0)**	6.044	6.088	6.127	6.136	6.155	6.177	6.181	6.176	6.221	6.198
		0.6084	0.5997	0.5936	0.5895	0.5864	0.5860	0.5894	0.5910	0.5872	0.5898
	(0.1,0.1)	4.647	4.742	4.752	4.705	4.716	4.705	4.705	4.702	4.706	4.708
		0.5209	0.4994	0.4976	0.5011	0.5012	0.5045	0.5045	0.5051	0.5056	0.5042
	(0.2,0.2)	4.017	3.962	3.970	3.997	3.998	3.997	4.000	4.002	4.011	4.004
		0.4697	0.4795	0.4771	0.4757	0.4754	0.4737	0.4707	0.4718	0.4713	0.4732
	(0.3,0.3)	3.543	3.534	3.535	3.507	3.526	3.518	3.511	3.511	3.517	3.509
		0.4465	0.4452	0.4432	0.4512	0.4482	0.4519	0.4548	0.4560	0.4576	0.4586
	(0.4,0.4)	3.206	3.196	3.182	3.193	3.185	3.188	3.192	3.185	3.184	3.181
		0.4353	0.4377	0.4477	0.4436	0.4436	0.4444	0.4439	0.4456	0.4476	0.4478
	(0.5,0.5)	2.896	2.904	2.897	2.931	2.928	2.927	2.921	2.920	2.917	2.929
		0.4631	0.4556	0.4504	0.4433	0.4423	0.4420	0.4444	0.4452	0.4475	0.4443
	(0.6,0.6)	2.663	2.685	2.684	2.681	2.673	2.668	2.671	2.674	2.685	2.693
		0.4351	0.4352	0.4335	0.4358	0.4378	0.4387	0.4393	0.4384	0.4360	0.4349
	(0.7,0.7)	2.565	2.521	2.534	2.538	2.544	2.538	2.530	2.525	2.522	2.519
		0.4269	0.4365	0.4332	0.4299	0.4310	0.4340	0.4356	0.4359	0.4376	0.4375
	(0.8,0.8)	2.382	2.415	2.400	2.385	2.374	2.362	2.363	2.361	2.365	2.363
		0.4275	0.4223	0.4281	0.4306	0.4336	0.4359	0.4343	0.4327	0.4329	0.4325
	(0.9,0.9)	2.210	2.204	2.216	2.219	2.223	2.221	2.221	2.217	2.213	2.219
		0.4242	0.4375	0.4354	0.4341	0.4340	0.4345	0.4349	0.4328	0.4330	0.4319
	(1.0,1.0)	2.074	2.057	2.075	2.066	2.069	2.076	2.076	2.077	2.078	2.080
		0.4491	0.4448	0.4467	0.4441	0.4442	0.4458	0.4430	0.4419	0.4416	0.4397

It must be noted that, for the bivariate case, the ordering of the effectiveness of the general repairs, unlike the univariate case, is not complete. For instance, whether an imperfect repair of degree $(0.3, 0.5)$ is more (or less) effective than an imperfect repair of degree $(0.5, 0.3)$ depends on the parameters of the original bivariate distribution; see Section 8.1.

9.3.2 Estimating Expected Total Warranty Servicing Costs

In the context of warranty cost analysis, the expected cost of repairs under warranty is used as an estimate of the total cost of servicing the warranty. This expected cost is a function of the number of failures occurring while the system is under warranty. In the previous section, we illustrated estimating the expected number of failures under warranty using the suggested simulation approach. In this section, we illustrate estimating the expected total warranty servicing costs for pre-assigned degrees of repair.

Let C_i denote the cost of the general repair following the i-th failure of the system, for $i \in \mathbb{N}_+$– this cost can be stochastic or fixed. Then, the total warranty servicing cost, for the rectangular warranty coverage $(0, w_t] \times (0, w_u]$, is given by

$$C(w_t, w_u) = \sum_{i=1}^{N(w_t, w_u)} C_i \ , \tag{9.47}$$

where $N(w_t, w_u)$ denotes the number of failures under warranty. When the repair costs are stochastic, the expected total warranty servicing cost is derived as follows:

$$\begin{aligned} E[C(w_t, w_u)] &= E\left[E\left[\sum_{i=1}^{N(w_t, w_u)} C_i \middle| N(w_t, w_u) \right] \right] \\ &= \sum_{n=1}^{\infty} E\left[\sum_{i=1}^{n} C_i \right] \ \mathrm{P}\{N(w_t, w_u) = n\} \ , \end{aligned} \tag{9.48}$$

where the inner expectation is with respect to the random repair costs and the outer expectation is with respect to the number of failures $N(w_t, w_u)$. The expected cost is zero for $n = 0$. When the repair costs are fixed, then the inner expectation in (9.48) can be dropped, i.e. $E\big[\sum_{i=1}^{n} C_i\big] = \sum_{i=1}^{n} C_i$.

For this illustration, we assume that the cost of a general repair is a deterministic function of its bivariate degree of repair, so that when the degree of the general repair is given, its cost is fixed (i.e. not stochastic). Let c_{min} and c_{per} denote the cost of a minimal repair (degree $(0, 0)$) and a perfect repair (degree $(1, 1)$), respectively. The minimum cost of a general repair is set to c_{min} and the maximum cost is set to c_{per}. These costs are constants and represent aggregates of the various costs (e.g. claims processing, transport, servicing, etc.) of a repair/replacement under warranty; see Blischke & Murthy [2].

We suggest the following example cost function for the cost of a general repair with degree (δ, γ), for $\delta, \gamma \in [0, 1]$:

$$c(\delta, \gamma) = c_{\min} + c_{\text{TD}}\, \delta + c_{\text{UD}}\, \gamma \ , \tag{9.49}$$

where $c_{\text{TD}}, c_{\text{UD}} \geq 0$. Then, when $(\delta, \gamma) = (0, 0)$, we have

$$c(0, 0) = c_{\min} \ , \tag{9.50}$$

and when $(\delta, \gamma) = (1, 1)$, we have

$$c(1, 1) = c_{\min} + (c_{\text{TD}} + c_{\text{UD}}) = c_{\text{per}} \ . \tag{9.51}$$

Therefore, the cost of a perfect repair is $c_{\text{TD}} + c_{\text{UD}}$ units in addition to the cost of a minimal repair. The costs c_{TD} and c_{UD} can be viewed as the costs of removing all time-related damage and usage-related damage, respectively. When we set $c_{\text{TD}} = c_{\text{UD}} = 0$, then the cost of a general repair is independent of its degree, i.e. constant for all repairs. When $c_{\text{TD}} < (>)\ c_{\text{UD}}$, the cost of removing time-related damage is less (higher) than the cost of removing usage-related damage.

The example cost function in (9.49) is set up such that the cost of an imperfect repair is bounded between the costs of minimal and perfect repairs, i.e. for $(\delta, \gamma) \in [0, 1] \times [0, 1] \setminus \{(0, 0), (1, 1)\}$,

$$0 \ < \ c_{\min} = c(0, 0) \ \leq \ c(\delta, \gamma) \ \leq \ c_{\text{per}} = c(1, 1) \ < \ \infty \ . \tag{9.52}$$

Since $c_{\text{TD}} \geq 0$ and $c_{\text{UD}} \geq 0$, the cost function is increasing in each component of the degree of repair (δ, γ), when the other is fixed.

Given the example cost function and the bivariate degrees of repair, the expected total warranty servicing cost in (9.48) becomes

$$\begin{aligned} E[C(w_t, w_u)] &= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} c(\delta_i, \gamma_i) \right) \ \mathrm{P}\{N(w_t, w_u) = n\} \\ &= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \left(c_{\min} + c_{\text{TD}}\, \delta_i + c_{\text{UD}}\, \gamma_i \right) \right) \ \mathrm{P}\{N(w_t, w_u) = n\} \ . \end{aligned} \tag{9.53}$$

This expected cost, for pre-assigned degrees of repair, can be estimated by: (i) simulating the associated failure process using the simulation approach described in Section 9.2.3; (ii) computing the sum of the costs of the repairs for each trajectory of the process (i.e. total cost per run); and (iii) computing the sample average over the total costs.

When we have a constant cost for all general repairs, then the expected total warranty servicing cost is simply the product of the cost of an individual general repair and the expected number of failures under warranty, i.e.

$$E[C(w_t, w_u)] = c_{\text{rep}}\ E[N(w_t, w_u)]\ , \tag{9.54}$$

for a constant cost of repair $c_{\text{rep}} > 0$. Consider, for instance, the numerical example in Section 9.3.1, where we set $(\delta_i, \gamma_i) = (\delta, \gamma)$, for all $i \in \mathbb{N}_+$. For any given $(\delta, \gamma) \in [0,1]^2$, the expected cost in (9.53) reduces to

$$\begin{aligned} E[C(w_t, w_u;\ \delta, \gamma)] &= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} (c_{\min} + c_{\text{TD}}\ \delta + c_{\text{UD}}\ \gamma) \right)\ \mathrm{P}\{N(w_t, w_u) = n\} \\ &= \left(c_{\min} + c_{\text{TD}}\ \delta + c_{\text{UD}}\ \gamma\right) \sum_{n=1}^{\infty} n\ \mathrm{P}\{N(w_t, w_u) = n\} \\ &= \left(c_{\min} + c_{\text{TD}}\ \delta + c_{\text{UD}}\ \gamma\right)\ E[N(w_t, w_u;\ \delta, \gamma)]\ . \end{aligned} \tag{9.55}$$

To estimate this expected total warranty servicing cost, we can use the estimated expected number of failures:

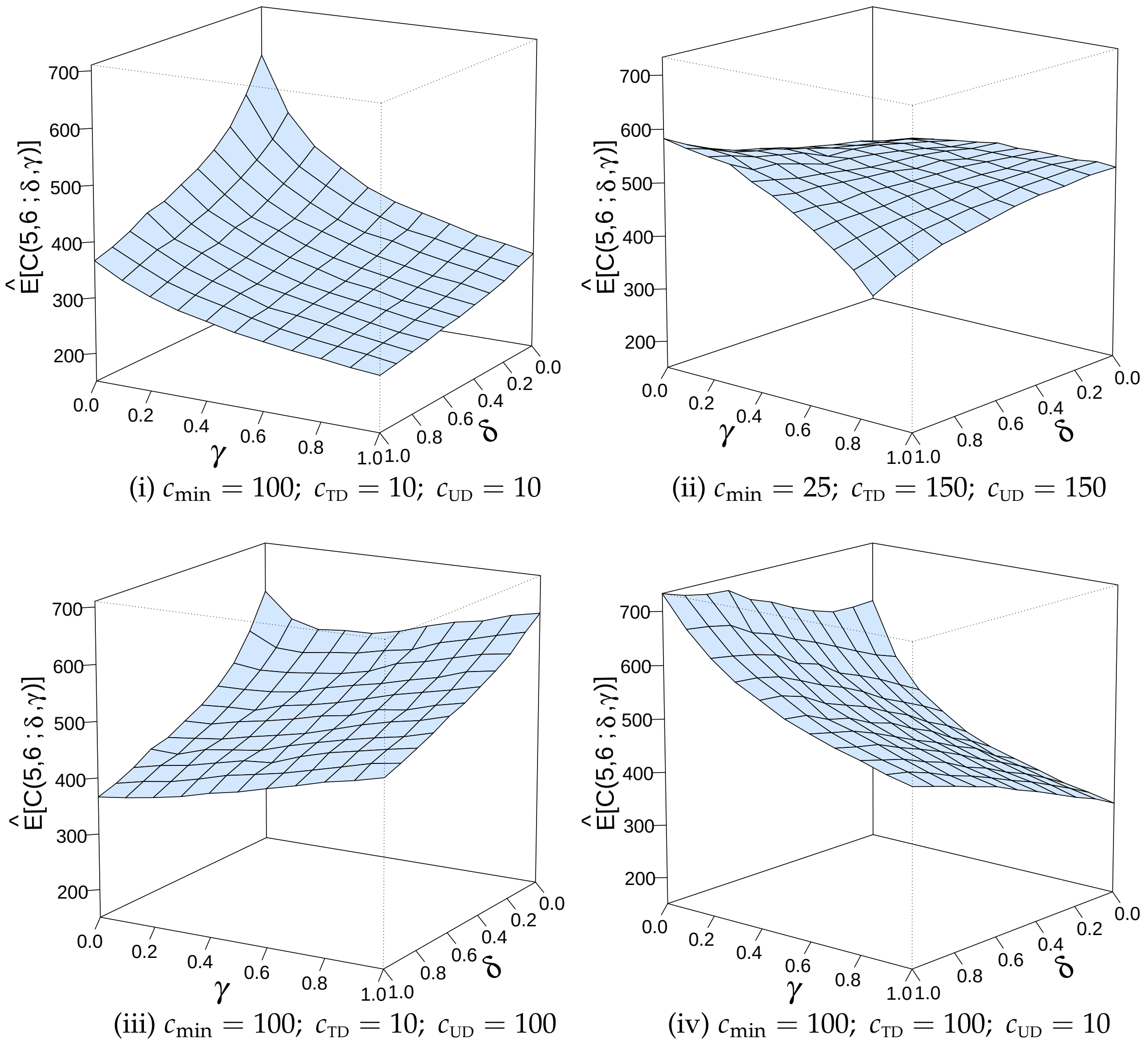
$$\widehat{E}[C(w_t, w_u;\ \delta, \gamma)] = \left(c_{\min} + c_{\text{TD}}\ \delta + c_{\text{UD}}\ \gamma\right)\ \widehat{E}[N(w_t, w_u;\ \delta, \gamma)]\ . \tag{9.56}$$

Table 9.3: Estimates $\widehat{E}[C(5, 6;\ \delta, \gamma)]$ of the expected total warranty servicing cost computed using the estimates of the expected number of failures from Table 9.1, where $c_{\min} = 100$ (\$) and $c_{\text{TD}} = c_{\text{UD}} = 10$ (\$)

		γ										
		0.0	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
δ	**0.0**	619.85	518.18	461.79	428.61	398.39	379.00	365.79	353.10	339.39	331.43	321.33
	0.1	556.58	480.25	436.37	407.42	382.13	364.44	347.49	339.24	328.38	319.71	311.31
	0.2	508.36	453.59	416.39	388.23	364.96	348.35	340.26	327.28	317.81	309.31	299.03
	0.3	477.43	432.47	397.68	372.00	355.99	340.42	326.43	316.05	306.06	300.35	290.99
	0.4	454.61	414.66	380.56	362.97	343.58	328.83	317.85	308.88	299.23	290.87	284.20
	0.5	437.84	398.25	374.18	352.27	335.21	322.18	308.14	301.60	289.86	284.15	277.08
	0.6	420.38	386.50	359.24	340.74	324.27	310.72	301.63	294.21	286.22	277.76	272.83
	0.7	412.20	372.51	354.71	333.31	315.07	305.58	296.00	287.21	278.74	270.58	265.68
	0.8	392.05	366.39	342.62	326.12	312.12	295.16	289.65	282.46	274.10	266.83	259.93
	0.9	377.04	352.87	330.27	315.22	303.88	291.18	280.67	273.25	267.01	261.79	254.43
	1.0	366.30	339.29	318.49	302.51	291.80	280.51	272.98	266.01	260.82	254.21	249.59

As an illustration, in Table 9.3, we present the estimates (averages) of the total warranty servicing costs corresponding to Table 9.1, for cost parameters $c_{\min} = 100$ and $c_{\text{TD}} = c_{\text{UD}} = 10$ (which makes the cost of a minimal repair \$100 and the cost of a perfect repair \$120). Each cell in Table 9.3 is $\widehat{E}[C(5,6;\ \delta,\gamma)] = (100 + 10\,\delta + 10\gamma)\ \widehat{E}[N(5,6;\ \delta,\gamma)]$, where $\widehat{E}[N(5,6;\ \delta,\gamma)]$, for $(\delta,\gamma) \in \mathcal{D}^2$, where $\mathcal{D} = \{0.0, 0.1, 0.2, \ldots, 1.0\}$, are tabulated in Table 9.1.

Notice that, in Table 9.3, the ordering of the estimates is the same as the ordering of the estimates of the expected number of failures in Table 9.1. This is because the value of the minimum cost $c_{\min} = 100$ is large in comparison to the value of the additional cost $c_{\text{TD}} + c_{\text{UD}} = 20$. When $c_{\min}$ is relatively large, the estimated expected numbers of failure are in effect being multiplied by a positive constant, and therefore, the ordering is preserved.



(i) $c_{\min} = 100;\ c_{\text{TD}} = 10;\ c_{\text{UD}} = 10$ (ii) $c_{\min} = 25;\ c_{\text{TD}} = 150;\ c_{\text{UD}} = 150$

(iii) $c_{\min} = 100;\ c_{\text{TD}} = 10;\ c_{\text{UD}} = 100$ (iv) $c_{\min} = 100;\ c_{\text{TD}} = 100;\ c_{\text{UD}} = 10$

Figure 9.10: The estimated expected warranty servicing costs $\widehat{E}[C(5,6;\ \delta',\gamma')]$, plotted with actual estimates at grid points $(\delta,\gamma) \in \mathcal{D}^2$, and interpolated values between grid points. The cost parameters are displayed under each plot.

To illustrate some possible trends, in Figure 9.10, we have plotted the estimates for various values of the cost parameters. Note that, the values of $\widehat{E}[C(5, 6;\ \delta', \gamma')]$, as a function of $(\delta', \gamma') \in [0, 1]^2$, are interpolated between the grid points $(\delta, \gamma) \in \mathcal{D}^2$.

9.4 Chapter Conclusion

In this chapter, we developed a simulation procedure to simulate the failure (or general repair) process and estimate the expected number of failures from the simulated trajectories, for various values of the bivariate degrees of repair. We observed that, as the degrees of the general repairs increase, the estimated expected number of failures decreases. As shown in the previous chapter, the reliability of the system further improves as the value of either component of any degree of repair increases (when all other parameters are fixed), which leads to a decrease in the expected number of failures.

We also discussed an application of the proposed general repair process in the context of warranty cost analysis, where we used the simulation approach to estimate the expected cost of servicing a free-replacement warranty having a rectangular coverage.

Part IV

Conclusion

Chapter 10

Conclusion

It is well-known that general repairs, along with returning a failed system to a operational state, often improve the working condition of the system, and thus, increase its reliability which affects the rate of future failures of the system. General repair models provide a unified framework for the realistic modeling of consecutive failures of repairable engineered systems. These models have broad applications in many fields, such as reliability modeling and warranty cost analysis. The overall goal of this study was

> *to advance the state of the art in modeling consecutive failures of a repairable system, by modeling the effect of general repairs on the working condition of the system.*

In this study, two specific types of repairable system were considered:

(I) systems whose working condition initially improves with age or usage, and whose lifetimes are modeled as univariate random variables having a bathtub-shaped failure rate function;

(II) systems whose working condition deteriorates with age and usage, and whose lifetimes are modeled as bivariate random variables having a decreasing conditional reliability function.

The main contributions of the thesis were arranged in two parts, with each part corresponding to one of the two types of system outlined above. For each case, the modeling procedure involved the following steps:

(i) **The system:** we began with describing a characterization of the system in terms of its working condition and the variables affecting it;

(ii) **The lifetime distribution:** we described the class of lifetime distributions used to model the lifetime of the original system (i.e. before its first failure);

(iii) **The general repair model:** we proposed a model to reflect the effect of repairs on the working condition of the system following the repairs; the model involved defining the effect of each repair in terms of either the failure rate function or the conditional reliability function of succeeding lifetimes (i.e. lifetimes between consecutive failures);

(iv) **The model properties:** we examined the effect of the parameters (here, degrees of repairs) of the proposed general repair model on the reliability of the system;

(v) **The failure (or general repair) process:** we described the associated failure process and derived the distributions of the consecutive failure points and inter-failure lifetimes by applying the proposed general repair model.

In each of the two cases, to simplify the modeling process, the following assumptions were made:

(a) each failure of the system is followed by an immediate repair, i.e. the time between the failure and undertaking the repair is set equal to zero;

(b) all repairs are instantaneous, i.e. the downtime of the system is set equal to zero.

Under these assumptions, consecutive failure points of the system are the only points at which repairs are performed. Therefore, the point process associated with the sequence of failures is the same as the process associated with the sequence of consecutive repair points.

Having developed the models, we then demonstrated the applications of each of the two models in the context of warranty cost analysis, where failure modeling is essential to the accurate estimation of the expected number of warranty claims and the associated costs.

10.1 Remarks on the One-Dimensional Repair Model

The first problem that we considered in this study was *modeling the effect of repairs (and hence, consecutive failures) of a system whose working condition is initially improving with age (or with usage).* The lifetime distribution used to model the time to first failure of such systems is a univariate distribution with a bathtub-shaped failure rate function.

Most of the existing literature on modeling repairs performed on this type of system assume that all initial failures of the system are rectified by minimal repair, and only when the system begins to deteriorate, general repairs of higher degree are performed following failures. Here, we proposed a model to describe the effect of general, non-minimal repairs performed while the working condition of the system is still improving.

Modeling approach. We modeled the effect of general repairs as changes in the conditional intensity function of the corresponding failure process, such that, following each non-minimal repair, the conditional reliabilities associated with the succeeding lifetimes increased. A distinguishing characteristic of the proposed model is that perfect repairs– which are assumed to be the most effective of general repairs– were not modeled simply as a replacement of the failed system with a new and identical system. Since the system is initially improving, a replacement is not always the most effective rectification action.

Model properties. The degree of a general repair reflects the effectiveness of the repair and a repair that is more effective is expected to result in greater reliability improvement. Illustrations of the proposed general repair model showed that the conditional reliability and mean residual lifetime functions are both increasing in each degree of repair, when all other parameters of the functions are fixed. As expected, since the reliability improvement increased as the degree of any given repair increased, the expected number of consecutive failures of the system, where each failure was rectified by a general repair, decreased.

Research contributions. The following are the main contributions of the first part of this research.

(1.1) We developed a new model to describe the effect of general repairs on the working condition of systems whose lifetimes can be modeled with a distribution having a bathtub-shaped failure rate function (e.g. systems that initially improve with age/usage before beginning to wear out).

(1.2) We formalized a definition of perfect repair, which for systems that are initially improving is not equivalent to a replacement. By removing the initial decreasing failure rate phase, the model reduces to a renewal process, where perfect repair is equivalent to replacement.

(1.3) We investigated the properties of the proposed general repair model: the conditional reliability and mean residual lifetime functions following repairs are both increasing in each degree of repair (when all other parameters are fixed).

(1.4) We derived the distributions of the consecutive failure points and inter-failure lifetimes of the associated failure process.

(1.5) We suggested warranty servicing strategies, and derived the expected costs using the proposed general repair model.

10.2 Remarks on the Two-Dimensional Repair Model

The second problem that we considered in this study was *modeling the effect of repairs (and hence, consecutive failures) of a system whose working condition deteriorates with both age and usage*. The bivariate lifetime distribution used to model the point of first failure of such systems is a distribution having the bivariate increasing failure rate property.

Failures of a system can be attributed to changes in more than one measure of its working condition– here, we chose age and usage as the two variables of interest. In the literature, general (imperfect) repair models for systems whose lifetime is modeled with a bivariate distribution generally involve reducing the failure process to a one-dimensional process by, for instance, assuming a relationship between age and usage or by defining a composite scale. Then, univariate repair models are used to describe the effect of repairs. Here, we proposed a new approach to model the effect of general repairs performed on a system whose lifetime is modeled as a bivariate random variable.

Modeling approach. We modeled the effect of repairs as changes in the bivariate conditional reliability function, such that, following a general repair, the system is at least as reliable as a system that has not failed (or a minimally repaired system). Specifically, the effect of a general repair is modeled as a possible decrease in the virtual age and the virtual usage of the system, which is equivalent to replacing the failed system with an identical system at a younger age and with lower usage. The proposed general repair process is a generalization of the renewal (replacement) process in two dimensions and it also includes the minimal repair process in two dimensions as a special case. Therefore, when all repairs are perfect, the bivariate inter-failure lifetimes are independent and identically distributed. When repairs are minimal or imperfect, the bivariate inter-failure lifetimes are neither independent nor identically distributed– the distribution of any of these bivariate lifetimes depends on all failure points and the degrees of the general repairs before it.

Model properties. The bivariate increasing failure rate property implies that the corresponding components of the hazard gradient vector are both increasing. Following each general non-minimal repair, the reliability of the system is increased, and hence, the components of the hazard gradient vector associated with the succeeding bivariate inter-failure lifetime decrease. The effectiveness of a general repair is modeled with a bivariate degree of repair, where the first component represents the proportion of decrease in the virtual age and the second component represents the proportion of decrease in the virtual usage. The improvement in system reliability following a failure is proportional to the bivariate degree of repair, for subsets of the degrees of repair for which there is complete ordering (for instance, when both components of the degree of repair are equal). For subsets of the degrees of repair where there is no complete ordering, the effectiveness of a repair depends among others on the parameters of the chosen lifetime distribution. Simulations of the failure (or general repair) process showed that, for a given region, the expected numbers of failures corresponding to an increasing sequence of bivariate degrees of repair are decreasing.

Research contributions. The following are the main contributions of the second part of this research.

(2.1) We developed a new model to describe the effect of general repairs on the working condition of systems whose lifetimes can be modeled with a bivariate distribution having the bivariate increasing failure rate property (e.g. systems that deteriorate with age and usage).

(2.2) The model generalized to two dimensions the one-dimensional virtual age models proposed by Kijima [27]. It is also a generalization of both the renewal process in two dimensions proposed by Hunter [61] and the minimal repair process in two dimensions proposed by Baik et al. [64].

(2.3) We investigated the properties of the proposed general repair model with respect to the suggested bivariate degrees of repair: the bivariate reliability function and the components of the associated conditional reliability and mean residual vectors are all increasing functions of either component of any degree of repair; and the components of the associated hazard gradient vector are both decreasing functions of either component of any degree of repair (when all other parameters are fixed).

(2.4) We derived the distributions of the bivariate failure points and inter-failure lifetimes associated with the failure process in two dimensions.

(2.5) We suggested an approach to simulate trajectories of the failure process in two dimensions, using the conditional distributions of the bivariate inter-failure lifetimes. The simulated trajectories can be used to estimate the expected number of failures and expected warranty servicing costs for various strategies.

10.3 Possible Future Research

(i) In modeling the consecutive failures of the system, we assumed that the general repairs are instantaneous (i.e. the time to repair the system was set to zero). This assumption is reasonable in the context of reliability analysis, if it can be assumed that the system does not deteriorate during downtime (while being repaired). In the context of cost analysis however, there may be penalties associated with the downtime of the system. For instance, while the system is being repaired, a temporary replacement system may be required which results in additional costs. In this case, setting the duration of the downtime to zero does not provide one with accurate cost estimates– modeling system downtime following failures as non-zero random variables may be more appropriate. One possible research direction is to develop models for non-zero repair times.

(ii) A second open question, which relates to systems with lifetime distributions having bathtub-shaped failure rate function, is to determine when a repair is more beneficial than the replacement of the system (this may be based on cost analysis as well as system reliability analysis). According to the proposed repair model, for a certain period of time immediately following a general repair, the repaired system performs "better" (in terms of system reliability) than a replaced (new) system. If the system becomes obsolete before this period ends, then a general repair may be more beneficial than a replacement (assuming that the cost of the repair is less than the cost of the replacement); however, if the system is in use after this period, a replacement may be more appropriate.

(iii) For one-dimensional repair models, there is complete ordering of the (univariate) degrees of repair, i.e. a repair of higher degree is more effective than a repair of lower degree. For the two-dimensional model however, where the degrees of repair are bivariate, the ordering of the degrees of repair in terms of their effectiveness (improvement is system reliability) is partial. A possible study can be conducted to determine some form of complete ordering of the bivariate degrees, so that there is a one-to-one correspondence between the degrees of repair and their effectiveness in improving system reliability.

(iv) In this study, the degrees of the repairs were assumed to be pre-assigned (known)– they can also be modeled as bivariate random variables with some dependence structure or as parameters of the lifetime distributions. Possible research can include investigating methods to model and estimate the degrees of repair in the proposed general repair models using failure data.

(v) With both general repair models, the system considered was in effect a single-component system. Most real-world systems are multi-component with components that may or may not be dependent. A possible study can include extending the proposed models to multi-component systems.

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List of Acronyms

BCR	bivariate conditional reliability
BDFR	bivariate decreasing failure rate
BDMRL	bivariate decreasing mean residual life
BFR	bathtub-shaped failure rate
BIFR	bivariate increasing failure rate
BIMRL	bivariate increasing mean residual life
BNBU	bivariate new-better-than-used
BNBUE	bivariate new-better-than-used in expectation
BNWU	bivariate new-worse-than-used
BNWUE	bivariate new-worse-than-used in expectation
CFR	constant failure rate
CW	combination warranty
DFR	decreasing failure rate
DMRL	decreasing mean residual life
FR	failure rate (ordering of distributions)
FRW	free-replacement (or free-repair) warranty
IFR	increasing failure rate
IMRL	increasing mean residual life
MR	mean residual (ordering of distributions)
MRL	mean residual lifetime
MRT	mean residual time
MRU	mean residual usage
NBU	new-better-than-used
NBUE	new-better-than-used in expectation
NWU	new-worse-than-used
NWUE	new-worse-than-used in expectation

PRW	partial rebate warranty
PrW	pro-rata warranty
RW	rebate warranty
ST	stochastic (ordering of distributions)
UFR	U-shaped failure rate

Notation

$A(.)$	age modification function: $A(t)$ is the modified age at time t
$A(t\|r)$	conditional virtual age at time t, given usage rate $R = r$
$A(t, u)$	virtual age at time t and usage u
$A(T_n^+)$	modified age immediately after the repair following the n-th failure of the system at time T_n
$A(T_n^+\|r)$	conditional virtual age immediately after the repair following the n-th failure of the system at time T_n (given system usage rate $R = r$)
$A(T_n^+, U_n^+)$	virtual age immediately after the repair following the n-th failure of the system at time T_n and usage U_n
A_2'	second change-point of the modified baseline intensity function $\lambda_1(.)$
$a(t, u)$	realization of the virtual age $A(t, u)$
$a(t_n^+)$	realization of the modified age $A(T_n^+)$ immediately after the n-th repair
$a(t_n^+\|r)$	realization of the conditional virtual age $A(T_n^+\|r)$ immediately after the n-th repair
$a(t_n^+, u_n^+)$	realization of the virtual age $A(T_n^+, U_n^+)$ immediately following the n-th repair
a_1	first change-point of a BFR function
a_2	second change-point of a BFR function
a_2'	realization of the change-point A_2'
$a_n(t)$	realization of the age modification function $A(.)$ at time t, given n failures before t
$a_n(t\|r)$	realization of the conditional virtual age $A(t\|r)$, given n failures before time t
$a_n(t, u)$	realization of the virtual age $A(t, u)$, given n failures before time t and usage u

$B(t,u)$	virtual usage at time t and usage u
$B(T_n^+,U_n^+)$	virtual usage immediately after the repair following the n-th failure of the system at time T_n and usage U_n
$b(t,u)$	realization of the virtual usage $B(t,u)$
$b(t_n^+,u_n^+)$	realization of the virtual usage $B(T_n^+,U_n^+)$ immediately following the n-th repair
$b_n(t,u)$	realization of the virtual usage $B(t,u)$, given n failures before time t and usage u
$C(w)$	total cost of servicing a warranty having warranty period $(0,w]$
$C(w_t,w_u)$	total cost of servicing a warranty having warranty region $(0,w_t] \times (0,w_u]$
$c(t;\delta_t)$	cost of a general repair performed at time t, having degree δ_t
$c(\delta,\gamma)$	cost of a general repair with bivariate degree of repair (δ,γ)
c_{min}	cost of a minimal repair
c_{per}	cost of a perfect repair
$F^{**}F_n$	convolution of F with F_n
$F(.)$	distribution function of the lifetime $X \equiv T$ of the original system
$F(.,.)$	distribution function of the bivariate lifetime $(X,Y) \equiv (T,U)$ of the original system
$F_n(.)$	distribution function of the n-th failure time T_n
$F_n(.,.)$	distribution function of the n-th failure point (T_n,U_n)
$F_{n+1}(.\|t_1,\ldots,t_n)$	distribution function of the $(n+1)$-th failure time T_{n+1}, given the n previous failure times $(t_1,\ldots,t_n)$
$F_{n+1}(.,.\|\boldsymbol{t_n},\boldsymbol{u_n})$	distribution function of the $(n+1)$-th failure point (T_{n+1},U_{n+1}), given the n previous failure points (times and usages) $\{(t_1,u_1),\ldots,(t_n,u_n)\}$
$F_t(t+x)$	conditional distribution function of the original system at time $t+x$, given the system is operational at time t
$F_Z(.)$	marginal distribution function of random variable Z
$F_{Z\|V}(.\|v)$	conditional distribution function of random variable Z, given random variable $V > v$
$F_{Z\|V}(.\|V=v)$	conditional distribution function of random variable Z, given random variable $V = v$
$\bar{F}(.)$	reliability function of the lifetime $X \equiv T$ of the original system

$\bar{F}(.,.)$	reliability function of the bivariate lifetime $(X, Y) \equiv (T, U)$ of the original system
$\bar{F}_n(.)$	reliability function of the n-th failure time T_n
$\bar{F}_n(.,.)$	reliability function of the n-th failure point (T_n, U_n)
$\bar{F}_{n+1}(.\|t_1, \ldots, t_n)$	reliability function of the $(n+1)$-th failure time T_{n+1}, given the n previous failure times $(t_1, \ldots, t_n)$
$\bar{F}_{n+1}(.,.\|\boldsymbol{t_n}, \boldsymbol{u_n})$	reliability function of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}), given the n previous failure points (times and usages) $\{(t_1, u_1), \ldots, (t_n, u_n)\}$
$\bar{F}_t(t+x)$	conditional reliability function of the original system at time $t+x$, given the system is operational at time t
$\bar{F}_t(t+x\|\mathcal{H}_t)$	conditional reliability function of the original system at time $t+x$, given the system is operational at time t, conditional on the history $\mathcal{H}_t$ of the failure process (i.e. taking into account the effect of repairs performed before time t)
$\bar{F}_Z(.)$	marginal reliability function of random variable Z
$\bar{F}_{Z\|V}(.\|v)$	conditional reliability function of random variable Z, given random variable $V > v$
$\bar{F}_{Z\|V}(.\|V = v)$	conditional reliability function of random variable Z, given random variable $V = v$
$f(.)$	density function of the lifetime $X \equiv T$ of the original system
$f(.,.)$	density function of the bivariate lifetime $(X, Y) \equiv (T, U)$ of the original system
$f_n(.)$	density function of the n-th failure time T_n
$f_n(.,.)$	density function of the n-th failure point (T_n, U_n)
$f_n(t_1, \ldots, t_n)$	joint density function of the first n failure times $\{T_1, \ldots, T_n\}$, at the point $(t_1, \ldots, t_n)$
$f_n(\boldsymbol{t_n}, \boldsymbol{u_n})$	joint density function of the first n failure points $\{(T_1, U_1), \ldots, (T_n, U_n)\}$, at $(t_1, u_1, \ldots, t_n, u_n)$
$f_{n+1}(.\|t_1, \ldots, t_n)$	conditional density function of the $(n+1)$-th failure time T_{n+1}, given the n previous failure times $(t_1, \ldots, t_n)$
$f_{n+1}(.,.\|\boldsymbol{t_n}, \boldsymbol{u_n})$	conditional density function of the $(n+1)$-th failure point (T_{n+1}, U_{n+1}), given the n previous failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$
$f_Z(.)$	marginal density function of random variable Z

$f_{Z\|V}(.\|v)$	conditional density function of random variable Z, given random variable $V > v$
$f_{Z\|V}(.\|V = v)$	conditional density function of random variable Z, given random variable $V = v$
$G_n(.)$	distribution function of the n-th inter-failure lifetime X_n
$G_n(.,.)$	distribution function of the n-th bivariate inter-failure lifetime (X_n, Y_n)
$G_{n+1}(.,.\|\boldsymbol{t_n}, \boldsymbol{u_n})$	conditional distribution function of the $(n+1)$-th bivariate inter-failure lifetime (X_{n+1}, Y_{n+1}), given the n previous failure points (times and usages) $\{(t_1, u_1), \ldots, (t_n, u_n)\}$
$G_{Y_{n+1}}(.\|\boldsymbol{t_n}, \boldsymbol{u_n})$	conditional distribution function of the $(n+1)$-th inter-failure usage Y_{n+1}, given the n previous failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$
$G_{X_{n+1}\|Y_{n+1}}(.\|y, \boldsymbol{t_n}, \boldsymbol{u_n})$	conditional distribution function of the $(n+1)$-th inter-failure time X_{n+1}, given the n previous failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$ and given the $(n+1)$-th inter-failure usage $Y_{n+1} = y$
$\bar{G}_n(.)$	reliability function of the n-th inter-failure lifetime X_n
$\bar{G}_n(.,.)$	reliability function of the n-th bivariate inter-failure lifetime (X_n, Y_n)
$\bar{G}_{n+1}(.,.\|\boldsymbol{t_n}, \boldsymbol{u_n})$	conditional reliability function of the $(n+1)$-th bivariate inter-failure lifetime (X_{n+1}, Y_{n+1}), given the n previous failure points (times and usages) $\{(t_1, u_1), \ldots, (t_n, u_n)\}$
$\bar{G}_{Y_{n+1}}(.\|\boldsymbol{t_n}, \boldsymbol{u_n})$	conditional reliability function of the $(n+1)$-th inter-failure usage Y_{n+1}, given the n previous failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$
$\bar{G}_{X_{n+1}\|Y_{n+1}}(.\|y, \boldsymbol{t_n}, \boldsymbol{u_n})$	conditional reliability function of the $(n+1)$-th inter-failure time X_{n+1}, given the n previous failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$ and given the $(n+1)$-th inter-failure usage $Y_{n+1} = y$
$g_n(.)$	density function of the n-th inter-failure lifetime X_n
$g_n(.,.)$	density function of the n-th bivariate inter-failure lifetime (X_n, Y_n)
$g_{n+1}(.,.\|\boldsymbol{t_n}, \boldsymbol{u_n})$	conditional density function of the $(n+1)$-th bivariate inter-failure lifetime (X_{n+1}, Y_{n+1}), given the n previous failure points (times and usages) $\{(t_1, u_1), \ldots, (t_n, u_n)\}$
$g_{Y_{n+1}}(.\|\boldsymbol{t_n}, \boldsymbol{u_n})$	conditional density function of the $(n+1)$-th inter-failure usage Y_{n+1}, given the n previous failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$
$g_{X_{n+1}\|Y_{n+1}}(.\|y, \boldsymbol{t_n}, \boldsymbol{u_n})$	conditional density function of the $(n+1)$-th inter-failure time X_{n+1}, given the n previous failure points $\{(t_1, u_1), \ldots, (t_n, u_n)\}$ and given

	the $(n+1)$-th inter-failure usage $Y_{n+1} = y$
$H(.,.)$	cumulative bivariate hazard rate function: $H(t,u) = -\ln \bar{F}(t,u)$
$\mathcal{H}_t$	history of the failure process in one dimension, available at time t
$\mathcal{H}_{t,u}$	history of the failure process in two dimensions, available at time t and usage u
h_t	realization of the failure process history $\mathcal{H}_t$
$h_{t;n}$	realization of the failure process history $\mathcal{H}_t$, given n failures (repairs) before time t
$h_{t,u}$	realization of the failure process history $\mathcal{H}_{t,u}$
$\boldsymbol{h}(t,u)$	hazard gradient vector at time t and usage u: $\boldsymbol{h}(t,u) = \nabla H(t,u)$
$\boldsymbol{h_{n+1}}(t,u;\boldsymbol{t_n},\boldsymbol{u_n})$	conditional hazard gradient vector at time t and usage u, given the first n system failures (defined for $t > t_n$ and $u > u_n$)
$h_T(t,u)$	time component of the hazard gradient vector $\boldsymbol{h}(t,u)$: $h_T(t,u) = \partial H(t,u)/\partial t$ (also denoted by $h_X(t,u)$)
$h_{X_{n+1}}(t,u;\boldsymbol{t_n},\boldsymbol{u_n})$	time component of the hazard gradient vector $\boldsymbol{h_{n+1}}(t,u;\boldsymbol{t_n},\boldsymbol{u_n})$
$h_U(t,u)$	usage component of the hazard gradient vector $\boldsymbol{h}(t,u)$: $h_U(t,u) = \partial H(t,u)/\partial u$ (also denoted by $h_Y(t,u)$)
$h_{Y_{n+1}}(t,u;\boldsymbol{t_n},\boldsymbol{u_n})$	usage component of the hazard gradient vector $\boldsymbol{h_{n+1}}(t,u;\boldsymbol{t_n},\boldsymbol{u_n})$
$\mathbb{I}_{\mathcal{B}}$	indicator function of event $\mathcal{B}$ (which is 1 when $\mathcal{B}$ occurs, and is 0 otherwise)
$\mathbb{N}$	set of natural numbers $\{0,1,\dots\}$
$\mathbb{N}_+$	set of positive integers $\{1,2,\dots\}$
$N(t)$	number of system failures in the interval $(0,t]$ (before time t)
$N(t,u)$	number of system failures in the region $(0,t] \times (0,u]$ (before time t and usage u)
$N(\mathcal{A})$	number of system failures in the region $\mathcal{A}$ (which is usually of the form $\mathcal{A} = (x_1,x_2] \times (y_1,y_2]$)
$N_X(t)$	number of system failures before time t (marginal count)
$N_Y(u)$	number of system failures before usage u (marginal count)
$N_X(t\|r)$	number of failures in the interval $(0,t]$, of a system used at rate $R = r$
$\mathbb{R}$	real line $(-\infty,\infty)$
$\mathbb{R}_+$	set of non-negative real numbers $[0,\infty)$
$R(.)$	cumulative univariate failure rate function: $R(t) = -\ln \bar{F}(t)$

$R_Z(.)$	marginal cumulative failure rate function of random variable Z
$r(.)$	univariate failure rate function: $r(t) = dR(t)/dt = f(t)/\bar{F}(t)$
$r(.,.)$	bivariate failure rate function of Basu [45]: $r(t,u) = f(t,u)/\bar{F}(t,u)$
$r_Z(.)$	marginal failure rate function of random variable Z: $r_Z(t) = dR_Z(t)/dt$
$r_T(.\|U > u)$	failure rate function of time T to first failure, given usage at first failure is $U > u$: $r_T(t\|U > u) = h_T(t,u)$
$r_U(.\|T > t)$	failure rate function of usage U at first failure, given time to first failure is $T > t$: $r_U(u\|T > t) = h_U(t,u)$
T	time to first failure of the original system (also denoted by X)
T_n	time of the n-th failure of the system
$\boldsymbol{T_n}$	vector of the first n failure times: $\boldsymbol{T_n} = (T_1, \ldots, T_n)$
t_n	realization of the time T_n of the n-th system failure
$\boldsymbol{t_n}$	realization of the vector $\boldsymbol{T_n}$ of the first n failure times: $\boldsymbol{t_n} = (t_1, \ldots, t_n)$
U	usage at first failure of the original system (also denoted by Y)
U_n	usage at the n-th failure of the system
$\boldsymbol{U_n}$	vector of the first n failure usages: $\boldsymbol{U_n} = (U_1, \ldots, U_n)$
u_n	realization of the usage U_n at the n-th system failure
$\boldsymbol{u_n}$	realization of the vector $\boldsymbol{U_n}$ of the first n failure usages: $\boldsymbol{u_n} = (u_1, \ldots, u_n)$
$V_F(\mathcal{B})$	F-volume of the set $\mathcal{B} = [x_1, x_2] \times [y_1, y_2]$, which is defined as follows: $V_F(\mathcal{B}) = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1)$
w	end (limit) of the one-dimensional warranty period
w_t	time limit of the two-dimensional, rectangular warranty region
w_u	usage limit of the two-dimensional, rectangular warranty region
X	time to first failure of the original system (also denoted by T)
X_t	residual lifetime of the system at time t: $X_t = [X - t\|X > t]$
X_{n+1}	time between the n-th and $(n+1)$-th failures of the system: $X_{n+1} = T_{n+1} - T_n$, where $X_1 = T_1$
Y	usage at first failure of the original system (also denoted by U)
Y_{n+1}	usage accumulated between the n-th and $(n+1)$-th failures of the system: $Y_{n+1} = U_{n+1} - U_n$, where $Y_1 = U_1$
γ_n	usage component of the n-th bivariate degree (δ_n, γ_n) of repair, for the repair process in two dimensions
δ_n	time component of the n-th bivariate degree (δ_n, γ_n) of repair, for

	the repair process in two dimensions; or degree of the n-th general repair, for the repair process in one dimension
$\Lambda(.)$	cumulative intensity function of the failure process in one dimension: $\Lambda(t) = E[N(t)]$
$\Lambda(.,.)$	cumulative intensity function of the failure process in two dimensions: $\Lambda(t,u) = E[N(t,u)]$
$\lambda_0(.)$	baseline intensity function of the failure process in one dimension (equal to the failure rate function $r(.)$ of the original lifetime)
$\lambda_0(.\|r)$	baseline intensity function of the conditional failure process in one dimension, given the system usage rate $R = r$
$\lambda_0(.,.)$	baseline intensity function of the failure process in two dimensions
$\lambda_1(.)$	modified baseline intensity function (which is a BFR function with change-points a_1 and A_2')
$\tilde{\lambda}(t\|\mathcal{H}_t)$	intensity function of the failure process in one dimension, conditional on the history $\mathcal{H}_t$ of the process at time t
$\tilde{\lambda}(t,u\|\mathcal{H}_{t,u})$	intensity function of the failure process in two dimensions, conditional on the history $\mathcal{H}_{t,u}$ of the process at point (t,u)
$\mu(t)$	mean residual lifetime function at time t: $\mu(t) = E[X_t]$
$\mu(t\|\mathcal{H}_t)$	mean residual lifetime function at time t, conditional on the history $\mathcal{H}_t$ of the failure process in one dimension
$\boldsymbol{\mu}(t,u)$	mean residual vector at time t and usage u
$\boldsymbol{\mu_{n+1}}(t,u;\boldsymbol{t_n},\boldsymbol{u_n})$	conditional mean residual vector at time t and usage u, given the first n failures of the system (defined for $t > t_n$ and $u > u_n$)
$\mu_T(t,u)$	time component of the mean residual vector $\boldsymbol{\mu}(t,u)$ (also denoted by $\mu_X(t,u)$)
$\mu_{X_{n+1}}(t,u;\boldsymbol{t_n},\boldsymbol{u_n})$	time component of the mean residual vector $\boldsymbol{\mu_{n+1}}(t,u;\boldsymbol{t_n},\boldsymbol{u_n})$
$\mu_U(t,u)$	usage component of the mean residual vector $\boldsymbol{\mu}(t,u)$ (also denoted by $\mu_Y(t,u)$)
$\mu_{Y_{n+1}}(t,u;\boldsymbol{t_n},\boldsymbol{u_n})$	usage component of the mean residual vector $\boldsymbol{\mu_{n+1}}(t,u;\boldsymbol{t_n},\boldsymbol{u_n})$
T_{a_1}	endpoint of the DFR phase of the univariate conditional intensity function $\tilde{\lambda}(t\|\mathcal{H}_t) = \lambda_1(A(t))$
τ_{a_1}	realization of the endpoint T_{a_1}
$\boldsymbol{\phi_{\bar{F}}}(s,v;t,u)$	conditional reliability vector at point $(t+s,u+v)$, given that the

	system is in an operational state at time t and usage u
$\phi_{\bar{F}_{n+1}}(s, v; t, u, \boldsymbol{t_n}, \boldsymbol{u_n})$	conditional reliability vector at point $(t+s, u+v)$, given that the system is in an operational state at time t and usage u, and given n failures (repairs) before point (t, u)