Exploring Students' Learning of Integral Calculus using Revised Bloom's Taxonomy

By

Farzad Radmehr

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Abstract

Integral calculus is one of the topics involved in mathematical courses both at secondary and tertiary level with several applications in different disciplines. It is part of gateway mathematical courses at universities for many majors and important for the development of the science. Several studies had been undertaken for exploring students' learning of integral calculus, both at the secondary and tertiary level, using a variety of frameworks (e.g., Action-Process-Object-Schema (APOS) theory (Dubinsky, 1991). However, students' learning of integral calculus has not been explored in terms of metacognitive experiences and skills, and the number of studies which have explored metacognitive strategies in relation to the students' learning of integral calculus is limited. Therefore, this study used Revised Bloom's Taxonomy (RBT) (Anderson et al., 2001), Efklides's metacognition framework (Efklides, 2008), and an adaptation of VisA (Visualization and Accuracy) instrument (Jacobse & Harskamp, 2012) for exploring students' learning of integral calculus.

A multiple case study approach was used to explore students' learning of the integral-area relationships and the Fundamental Theorem of Calculus in relation to the RBT's factual, conceptual, and procedural knowledge, and the facets of metacognition including metacognitive knowledge, experiences, and skills. The study sample comprised of nine first year university and eight Year 13 students who participated in individual semi-structured interviews answering nine integral calculus questions and 24 questions related to the RBT's metacognitive knowledge. Integral calculus questions were designed to address different aspects of RBT's knowledge dimension and activate RBT-related cognitive processes. A think aloud protocol and VisA instrument were also used during answering integral calculus questions for gathering information about students' metacognitive experiences and skills. Ten undergraduate mathematics lecturers and five Year 13 mathematics teachers were also interviewed in relation to the teaching and learning of integral calculus to explore students' difficulties in the topic. The entire teaching of integral calculus in a first year university course and a Year 13 classroom were video recorded and observed to obtain a better understanding of the teaching and learning of integral calculus in the context of the study.

The study findings in terms of the RBT's factual knowledge show several students had difficulty with notational aspects of the Fundamental Theorem of Calculus (FTC) (e.g., Thompson, 1994) whereas this issue was not dominant for the definite integral. In relation to the RBT's conceptual and procedural knowledge for both topics, conceptual knowledge was less developed in students' minds in comparison to procedural knowledge (e.g., students had not developed a geometric interpretation of the FTC, whereas they were able to solve integral questions using the FTC). The obtained results were consistent with previous studies for these three types of knowledge. The study contributes to the current literature by sharing students' metacognitive knowledge, experiences and skills in relation to integral calculus. The findings highlight some student learning, monitoring, and problem-solving strategies in these topics. A comparison between University and Year 13 students' results showed students across this transition had different factual, conceptual, procedural, and metacognitive knowledge in these topics. For instance, University students in the sample use online resources more often than Year 13 students, are more interested in justifications behind the formulas, and have more accurate pre and post-judgments of their ability to solve integral questions. The information obtained using questions based on RBT and the metacognition framework indicates that these two together may be very useful for exploring students' mathematical learning in different topics.

Dedication

This thesis work is dedicated to my mother, Zohreh Ahmadnia, who has always loved me unconditionally. This work is also dedicated to my wife, Mozhgan Mohammadpour, who has been a constant source of support and encouragement during the challenges of this PhD study. I am truly thankful for having them in my life.

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Chapter One: Introduction and Overview

Chapter One is dedicated to describing the problem explored in the study and rationale for doing it (Section 1.1). It also includes the context of study, the researcher's background and the New Zealand context (Section 1.2). Key ideas of the main framework (Section 1.3), research questions, and focus of the study (Section 1.4) are also provided in the Chapter.

1.1 Introduction and rationale

Over the past two decades, there has been an increasing decline in the number of high school students taking intermediate and advanced mathematical courses (Kennedy, Lyons, & Quinn, 2014). For instance, in Australia, between 1992 to 2012, the number of students who were enrolled in the last year of schooling, Year 12, increased by 16 %, while the number of students who took intermediate and advanced mathematical courses decreased by 11 and 7 %, respectively. In addition, there has been an increasing research interest in the secondary-tertiary transition in mathematics due to reports of students opting not to study mathematics at University (e.g., Biza & Zachariades, 2010; Godfrey & Thomas, 2008; London Mathematical Society, 1995; Luk, 2005; Tall 1997; Thomas et al., 2010). The university mathematical courses (e.g., calculus, algebra) that can be chosen for investigating students' difficulties within the transition are varied based on the universities curriculum and policies. However, in first-year courses usually calculus is taught (e.g., Victoria University of Wellington, 2015) because of its importance as a tool for "viewing and analysing the physical world" (Anton, Bivens, & Davis, 2012, p. xxi). Among calculus topics, the secondary-tertiary transition has been studied for equations (Godfrey & Thomas, 2008) and derivatives (Biza & Zachariades, 2010) but to date, there is a lack of literature on transition in relation to integral calculus. This study used Revised Bloom's Taxonomy (RBT) (Anderson, et al., 2001) as a framework for investigating the secondary-tertiary transition in integral calculus in New Zealand. Rationale for undertaking the study is now described.

Students moving from the study of mathematics at secondary schools to university face social, cultural, cognitive, and didactical changes (Section 3.3) that can be challenging. Issues relating to the secondary-tertiary transition have been identified complex, and serious concerns about teaching and learning in these two levels have been reported (e.g., Clark & Lovric, 2008, 2009; Hourigan & O'Donoghue, 2007; Kajander & Lovric, 2005; London Mathematical Society, 1995; Luk, 2005; Tall 1997). For example, Hong, et al. (2009) reported there are differences between Year 13, the last year of schooling in New Zealand, and first-year tertiary calculus teaching from the points of view of teachers and lecturers (e.g., use of technology, access to resources, interaction with students, and purpose of teaching calculus). In New Zealand, research shows that there is not enough communication between teachers and lecturers of these two levels and misalignment between their curriculums have been found in some areas (e.g., sequences and series) (Thomas et al., 2010).

Calculus, taught internationally at secondary and tertiary levels, is within the focus of research in mathematics education at both levels (e.g., Biza & Zachariades, 2010; Green, 2010; Karaali, 2011). Calculus is chosen for this study for four reasons. First, it has applications in many sciences such as physics, engineering, and astronomy (Anton et al., 2012; Thomas, Weir, Hass, & Giordano, 2010). Second, its importance for developments of sciences is highlighted in the literature:

> Modern scientific thought has been formed from the concepts of calculus and is meaningless outside this context. When I speak of science, I do not restrict myself to other disciplines. In a very significant respect, mathematics itself came into being with the development of calculus (Bressoud, 1992, p. 615).

Third, at the secondary level, calculus can increase students' interest in doing Science, Technology, Engineering, and Mathematics (STEM) majors in university (McGivney-Burelle & Xue, 2013). Finally, at the tertiary level, calculus is a gateway to many advanced mathematical course work such as differential equations (Czocher, Tague, & Baker, 2013). Within the broad area of calculus, the rationale for choosing integral calculus for the study is now described.

Several studies reported students have difficulties with concepts within integral calculus (e.g., Jones, 2013; Kouropatov & Dreyfus, 2013; Kiat, 2005; Thomas & Hong, 1996). While the majority of students taking integral calculus are successful in applying basic procedures for finding antiderivatives, their understandings of the concepts are limited. For example, Thomas & Hong (1996) reported many students regard integral calculus “as a series of processes with associated algorithms and do not develop the grasp of concepts which would give them the necessary versatility of thought” (p. 577). Studies showed that students have difficulties with understanding the Fundamental Theorem of Calculus (FTC) (e.g., Thompson, 1994), integrating functions that are below the x-axis (e.g., Orton, 1983) or that have absolute value (e.g., $\int_{-1}^{1}|x+2|dx$) (Mundy, 1984), etc. In addition, Mahir (2009) found that students do not have enough conceptual understanding for solving integral calculus problems in: integrals, integral-area relation, the relationship between integrals as a function and the algebraic sum of areas, and the FTC.

The literature provides some useful information regarding students’ understanding of integral in terms of conceptual and procedural knowledge (e.g., Mahir, 2009). So far, however, there has been little discussion in the literature about metacognitive knowledge (Section 3.1.4) in relation to the teaching and learning of integral calculus.

Other reasons for choosing integral calculus for this study are:

- integral calculus is important for understanding a wide range of real-world problems, including a range of contexts in physics and engineering (e.g., Thompson & Silverman, 2008);
- it has been less frequently explored than limits, another fundamental calculus concept (Jones, 2013);
- many undergraduate and graduate courses in mathematics and engineering sciences rely heavily on parts of this topic (e.g., differential equations) (Czocher, Tague, & Baker, 2013); and
- personal experiences as a lecturer and tutor in relation to students’ difficulties motivated me to focus on this topic (Section 1.2.1).

It is common to use one or more frameworks to investigate the teaching and learning of mathematical concepts (e.g., Stewart, 2008). The following describes the rationales for choosing RBT as the main framework of the study.

For investigating the teaching, learning, and assessment of a mathematical concept, various frameworks of conceptual growth are proposed in the literature to explain how learning takes place in the mind (see Pegg & Tall, 2005; 2010), all of which have consequences for helping understand quality teaching (Bergsten, 2007). Two examples of frameworks commonly used in previous research (e.g., Jurdak & El Mouhayar, 2014;

Stewart, 2008) are Action-Process-Object-Schema (APOS) theory (Dubinsky, 1991) and the prestructural-unistructural-multistructural-relational-extended abstract levels in the Structure of the Observed Learning Outcome (SOLO) taxonomy (Biggs and Collis, 1982). These frameworks are one-dimensional, integrating cognitive processes and knowledge together. For example, SOLO taxonomy has five levels (Section 3.1.5), so that students' responses and learning can be classified into one of these levels. In APOS theory, mathematical concepts are considered to be learnt using four mental structures (i.e., action, processes, objects, and schemas) that are constructed using five reflective abstractions (Section 3.1.6).

In this study, a two-dimensional framework which separates the knowledge and cognitive processes is used. According to Anderson et al. (2001) educational objectives consist of verbs and nouns with the verbs describing the intended cognitive processes and the nouns describing the knowledge that needs to be acquired or constructed. The reason for choosing a two-dimensional framework is to analyse educational objectives (i.e., students' work, teaching activity, or curriculum documents) in more detail. A two-dimensional framework, often encapsulated in a table such as an RBT Table, enables separate and connected analysis of knowledge and cognitive process (Chapter Two) and can therefore provide more information about the transition between secondary and tertiary level than a one-dimensional framework. Others Reasons for choosing RBT for exploring the transition are:

- it is a framework that can be used for exploring teaching, learning, and assessment of a topic (Anderson, et al., 2001) but has been less in the focus of research and practices in mathematics education and has not been used for exploring the transition;
- it has metacognitive knowledge as part of the knowledge dimension;
- as the literature about metacognitive knowledge in relation to the teaching and learning of integral calculus is sparse, such research opens dialogue between lecturer, teachers, and researchers in relation to metacognitive knowledge in the context of teaching integral calculus;

- additionally, to date, the knowledge dimension of RBT has not been explored for integral calculus. By exploring the knowledge dimension of RBT, opportunities are created for designing educational objectives, teaching activities, and assessment based on RBT. This work can help lecturers, teachers, researchers, and curriculum designers involved in the transition to have a better understanding of RBT in the context of mathematics, particularly integral calculus, and to use or modify some of the examples in their practice and research. It provides an opportunity to think about improving the current teaching by improving students' awareness of metacognitive knowledge, and using it for learning and problem solving in mathematics. In addition, for evaluating the alignment between educational objectives, teaching activities, and assessment questions including achievement standards, these examples are useful for understanding where each material should be located in an RBT Table; and
- my previous experiences (Section 1.2.1) using this framework.

In the following argument, the rationale for choosing New Zealand as the context of the study is described.

New Zealand is chosen as the geographical context for the study for convenience and because integral calculus is taught at the secondary and tertiary levels, spanning the transition between these. Integral calculus has similar treatment in the *New Zealand Curriculum* (Ministry of Education, 2007a) to that of many countries such as the United States of America (Calculus course description: College Board, 2007) and Australia (State of Queensland: Mathematics B (Queensland Studies Authority, 2008); State of Victoria: Victorian Certificate of Education Study Design (Victorian Curriculum and Assessment Authority, 2013)). Calculus is commonly included in undergraduate university calculus courses internationally (e.g., Department of Mathematics and Statistics of the University of Melbourne, 2015). Therefore, the findings of the study are potentially highly relevant outside the New Zealand context as well as within New Zealand.

1.2 The context of the study

In this section, the researcher's background and the New Zealand context are described.

1.2.1 Researcher's background, stance, and perspective

My educational and research background are presented to show the motivation for undertaking the study and the experiences I bring to it. I hold a Bachelor degree in pure mathematics, a Master's and a PhD degree in mathematics with a focus in mathematics education. I have used quantitative research methods in mathematics education, especially, in relation to the psychology of learning mathematics. In particular, I explored how different affective factors (e.g., mathematics anxiety and attitude), working memory capacity, and different learning style affect students' mathematical problem solving. One of the reasons for undertaking the current study was to expand my knowledge about qualitative research methods in mathematics education.

I have worked as a teacher assistant in differential equations (ten years) in Iran and New Zealand and lectured pre-calculus, calculus, and multivariable calculus courses at the tertiary level (two years) in Iran. In New Zealand, I have tutored university calculus, multivariate calculus, and differential equation courses since 2014.

One of the reasons for choosing integral calculus as the focus of this study was that during marking students' work in differential equations, I frequently saw that students successfully found the appropriate procedure for solving first or second order differential equations; however, they faced more difficulties with the integration sections of the problems. Furthermore, I found that students have negative attitudes towards, and a lack of confidence in, integration work. From talking with students, I have come to the belief the difficulty is due to their perceptions of integration being more procedural than conceptual.

I have completed other research related to RBT including projects into the possible applications of RBT to different disciplines and within mathematics in general (Radmehr, Alamolhodaei, & Amani, 2012; Radmehr, Alamolhodaei, & Pezeshki, 2011), for analysing students' mathematical problem solving in the cognitive process (Fardin & Radmehr, 2013; Radmehr and Alamolhodaei, 2010), and the knowledge dimension of RBT (Fardin & Radmehr, 2013; Radmehr & Alamolhodaei, 2012). In addition, I have used RBT for exploring the relationship between mathematical problem-solving and psychological factors (i.e., mathematics anxiety, attitude, attention, working memory capacity, field-dependency, and multiple intelligences) (Fardin & Radmehr, 2013; Hajibaba, Radmehr, & Alamolhodaei, 2013; Rahbarnia, Hamedian, & Radmehr, 2014).

My previous studies focussed on analysing students' performance using quantitative approaches and I believe the applications of RBT also need to be explored using a qualitative approach. This reason along with reasons highlighted in Section 1.1 encouraged me to continue on working with RBT in this study. In the next section, the New Zealand schooling system in relation to teaching integral calculus is described.

1.2.2 The New Zealand context

In New Zealand, study at secondary school (also known as high school or College) for students in Years 9 to 13 is started at age 12 or 13, and compulsory up to age 16. A vast majority of secondary schools in New Zealand teach a programme of work that is based on *New Zealand Curriculum* (Ministry of Education, 2007b). The main secondary school qualification in New Zealand is the National Certificate of Educational Achievement (NCEA) (Ministry of Education, 2015a).

Integral Calculus is part of *New Zealand Curriculum* (Ministry of Education, 2007a) for secondary schools and also is taught in first and second-year courses at New Zealand universities. Typically students first learn about integral calculus at Year 12, age 15-16, by being introduced to the idea of anti-differentiation. Further topics in integral calculus are introduced at Year 13, age 16-17 (Ministry of Education, 2007a). The related achievement objectives in the *New Zealand Curriculum* in these years are:

> In a range of meaningful contexts, students will be engaged in thinking mathematically and statistically. They will solve problems and model situations that require them to:
>
> - apply differentiation and anti-differentiation techniques to polynomials [for Year 12] (Ministry of Education, 2007a, p. 22); and
> - choose and apply a variety of differentiation, integration, and anti-differentiation techniques to functions and relations, using both analytic and numerical methods [for Year 13] (Ministry of Education, 2007a, p. 22).

Integral calculus is part of the NCEA level two (New Zealand Qualifications Authority [NZQA], 2015) and three mathematics achievement standards (NZQA, 2013). Most students undertake their NCEA level two assessment in Year 12 and NCEA level three assessment in Year 13. The standard-based assessment uses achievement standards and achievement standards are *New Zealand Curriculum*-based describing:

> what a student needs to know, or what they must be able to achieve, in order to meet the standard… As students study new topics, their teachers will explain what will be assessed and how. Teachers ensure that students are prepared for assessment. If students pass the assessment, the standard is achieved (NZQA, 2015[1]).

In relation to integral calculus, the NCEA level two mathematics achievement standard focuses on antidifferentiating polynomials (NZQA, 2015). A further achievement standard related to integral calculus is set at NCEA level three (Figure 1.1). In both levels the assessment of integral calculus is external (i.e., tested nationally) rather than internally assessed within the schools.

AS91579 [3.7] Apply integration methods in solving problems

A.O.: M8-12 Form differential equations and interpret the solutions
M8-11a Choose and apply a variety of integration and anti-differentiation techniques to functions & relations using both analytical and numerical methods

Methods: -integrating power, polynomial, exponential (base e only), trigonometric, and rational functions
-reverse chain rule, trigonometric formulae
-rates of change problems
-areas under or between graphs of functions, by integration
-finding areas using numerical methods, eg the rectangle or trapezium rule
-differential equations of the forms $y' = f(x)$ or $y'' = f(x)$ for the above functions or situations where the variables are separable (eg $y' = ky$) in applications such as growth and decay, inflation, Newton's Law of Cooling and similar situations.

External 6 credits

Figure 1.1 Integral calculus in level three Mathematics Achievement Standards. Copyright 2013 by NZQA. Reprinted with permission.

The integral calculus topic has a credit value of six. "Credits are the currency of the NCEA qualification" (NZQA, 2016, p. 1). Each credit represents "ten hours of learning and assessment [including] teaching time, homework and assessment time" (NZQA, 2016, p. 1). A year-long course typically consists of 18 to 24 credits (NZQA, 2016), therefore, integral calculus is one of the main parts of the calculus course at Year 13. NZQA (2013) encourages teachers to teach "integration by parts" and "integration by substitution

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[1] No page number as the quote is copied from the NZQA website

techniques" and use resources available at Te Kete Ipurangi (TKI) senior secondary website (Ministry of Education, 2015b).

In New Zealand universities, integral calculus is taught in first and second Year calculus courses. For example, in Victoria University of Wellington, it is first taught to students in Calculus 1A (School of Mathematics and Statistics, 2015a), then further explored in Calculus 1B (School of Mathematics and Statistics, 2015b). Double, triple, and surface integrals are taught to students in Multivariable Calculus, at second year (School of Mathematics and Statistics, 2015c).

1.3 Revised Bloom's Taxonomy

In this section, RBT's structure and its key ideas are introduced to inform understanding the research questions and focus of the study. RBT will be described in more detail along with studies that used it as a framework in Chapter Two.

RBT (Anderson et al., 2001) was designed for considering potentially useful new approaches and theories of the late 20th century in relation to Bloom's Taxonomy (BT) (Bloom, Engelhart, Furst, Hill, & Krathwohl, 1956), such as metacognition (Flavell, 1979) and constructivism (Piaget, Elkind, & Tenzer, 1967). Another purpose for the revision of BT was to point out its value to educational researchers, as a tool that was "ahead of its time" (Anderson et al., 2001, p. xxi).

In RBT (Table 1.1) each cell is defined as an intersection of the knowledge and the cognitive process dimension. The knowledge dimension includes factual, conceptual, procedural, and metacognitive knowledge. Levels of the knowledge dimension are on a sequence from concrete in the 'factual knowledge' to abstract in the 'metacognitive knowledge' (Anderson et al., 2001). However, with regards to this continuum, sometimes there are overlaps between conceptual and procedural knowledge (Näsström, 2009) (Section 3.1.3).

Table 1.1

General RBT Table (*without inclusion of subtypes of the knowledge and the cognitive process dimension)*

The Knowledge Dimension	The Cognitive Process Dimension					
	1. Remembering	2. Understanding	3. Applying	4. Analysing	5. Evaluating	6. Creating
A. Factual knowledge						
B. Conceptual knowledge						
C. Procedural knowledge						
D. Metacognitive knowledge						

Note. Adapted from "*A taxonomy for learning, teaching, and assessing: A revision of Bloom's Taxonomy of Educational Objectives* (p. 28)," by L. W. Anderson, et al., 2001, New York: Longman. Copyright 2001 by the Addison Wesley Longman, Inc. Adapted with permission.

The cognitive process of RBT has six categories presented in verb format from a low cognitive complexity in '"remembering" to high cognitive complexity in "creating" (Anderson, 2005; Krathwohl, 2002). In order for RBT to be more applicable for lecturers and teachers, "strict hierarchy" was neglected to allow the categories to overlap one another (Krathwohl, 2002, p. 215). The hierarchical structure of RBT is not as strict as BT because some aspects of understanding (e.g., explaining) are more challenging than executing as a subcategory of applying (Krathwohl, 2002).

Educational objectives (e.g., learning objectives) can be classified using RBT by placing them in cell(s) in relation to the intersection of the column(s) suitable for categorising the verb(s) and the row(s) suitable for categorising the noun(s) (Krathwohl, 2002). For example, a learning objective based on RBT can be written using a verb to show the cognitive process and a noun to show the knowledge. "The learner will be able to [verb] [noun]" (Su & Osisek, 2011, p. 323). Amer (2006) describes the potential applications of the RBT Table including:

- “analyse the objectives of a unit or a syllabus” (p. 223);
- “help teachers not to confuse activities with objectives” (p. 225);
- “help teachers realise the relationship between assessment and teaching/learning activities” (p. 225); and
- “examine curriculum alignment” (p. 226).

These applications are described in detail in Chapter Two.

1.4 The study overview and focus

The focus of the study is how RBT can be used as a tool for exploring students’ mathematical learning in the context of integral calculus at Year 13 and first year university and what are the perspectives of teachers and lecturers towards students’ difficulties in this topic. The research questions that address the focus of the study are:

1. What examples of factual, conceptual, procedural, and metacognitive knowledge in integral calculus based on RBT can be found in Year 13 and first year university?

2. Using RBT as a lens, what are students' difficulties in solving integral questions in Year 13 and first year university?

3. What metacognitive *knowledge*, *experiences*, and *skills* do students hold about integral calculus in Year 13 and first year university?

4. What differences exist between student learning of integral calculus in Year 13 and first year university?

5. What are the perceptions of lecturers and teachers towards students’ difficulties in integral calculus?

For exploring the transition in integral calculus based on RBT, the RBT’s knowledge dimension must be contextualised for integral calculus. The factual, conceptual, procedural, and metacognitive knowledge for integral calculus needs to be defined in order to explore student learning of integral calculus based on RBT (to help answer the first research question).

To have a better understanding of the transition, Year 13 and first year university students were interviewed using questions designed based on RBT. The questions explored students' factual, conceptual, and procedural knowledge (to help answer the second and forth research question) as well as their metacognitive knowledge, experiences, and skill (to help answer the third and forth research question). In addition, to gain more insight about the transition, perceptions of teachers and lecturers about students' difficulties in integral calculus were explored (to help answer the fifth research question)

1.5 Chapter summary and thesis structure

This chapter explained the motivation, context, and research questions of the study. Chapter Two is dedicated to describing RBT and studies that have used it. Chapter Three reviews the literature relevant to the study. Chapter Four explains and justifies how the study was designed, including the research paradigm, the methodology of the study, explanations about the study sample, and the process of data collection. The data gathering instruments and how they have been designed and trialled, and how the data are analysed are described in Chapter Five.

Chapter Six describes the context of the study in terms of how integral calculus was taught in the University and the College settings and provides lecturers' and teachers' opinions about students' difficulties in integral calculus. Chapter Seven describes students' learning of integral calculus in relation to RBT's factual, conceptual, and procedural knowledge. Students' metacognitive knowledge, explored in relation to the questions designed based on the structure of RBT's metacognitive knowledge, is outlined in Chapter Eight. Chapter Nine provides explanations of students' metacognitive experiences and skills during solving integral questions.

Chapter Ten includes discussion of findings, conclusions of the study, discusses the limitations and implications of the study, and presents questions for further research.

Chapter Two: Revised Bloom's Taxonomy

In order to frame the study and what is known about the use of RBT in educational studies in general and in mathematics education in particular, this chapter is dedicated to the theoretical and empirical literature related to RBT. For this purpose, the major difference between Bloom's Taxonomy (BT) (Bloom et al., 1956) and RBT (Anderson et al., 2001) and reasons for designing RBT are provided (Section 2.1). Then, the subtypes of the knowledge and the subcategories of the cognitive process dimension are described in detail (Section 2.2). The rest of the chapter is dedicated to describing how RBT and its table have been used as described in the literature.

2.1 Bloom's Taxonomy and Revised Bloom's Taxonomy

In this section, BT, reasons for its revision, and its differences from RBT are provided to obtain a better understanding of RBT and how it can be used. BT (Bloom et al., 1956) has six categories including knowledge, comprehension, application, analysis, synthesis and evaluation. All the categories were divided into subcategories except application (Figure 2.1). BT is based on several assumptions such as "the categories were ordered from simple to complex and from concrete to abstract", and "mastery of each simpler category was prerequisite to mastery of the next more complex one", which presented a "cumulative hierarchy" (Krathwohl, 2002, pp. 212-213).

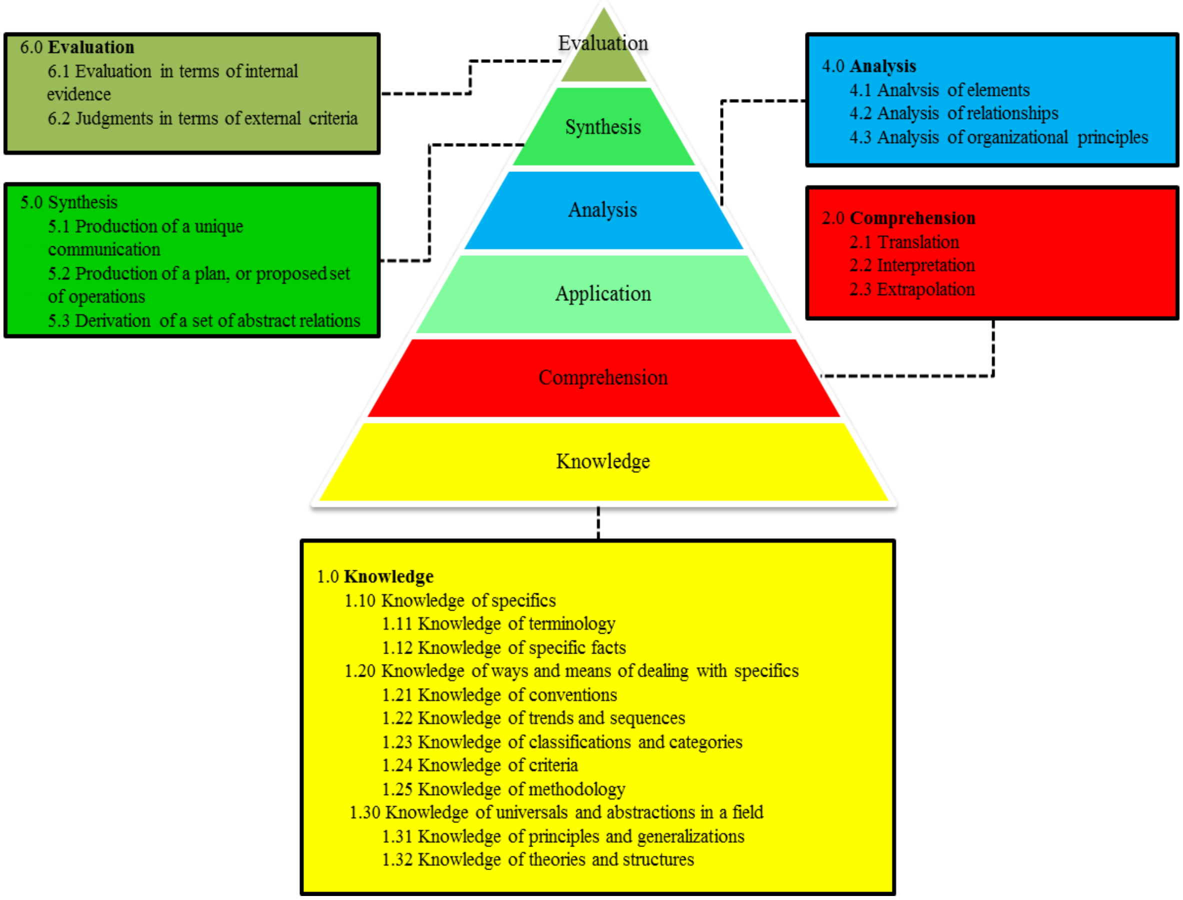
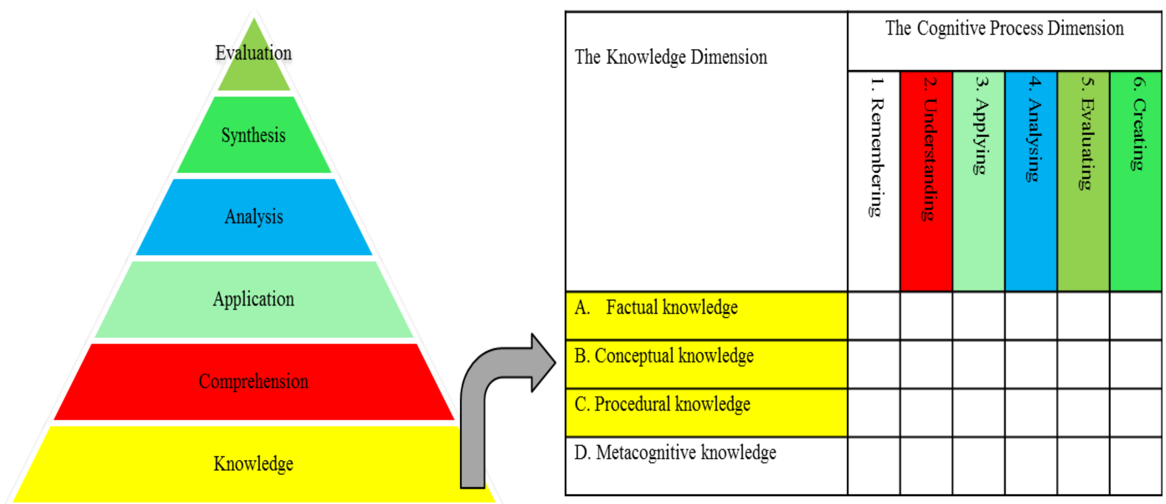


Figure 2.1 Structure of Bloom's Taxonomy

Adapted from Krathwohl (2002)

Researchers have identified several weaknesses and limitations of BT. For instance, some aspects of knowledge (e.g., conceptual knowledge about the Fundamental Theorem of Calculus) are more complex than certain demands of application. Moreover, synthesis is more challenging in comparison to evaluation (Kreitzer & Madaus, 1994), while evaluation is considered as the highest level in BT. In addition, the “cumulative hierarchy” (p. 212) of BT is not applicable in all situations (Krathwohl, 2002). For example, in mathematics, students could apply procedures for solving routine problems without mastering the concepts. Radmehr & Alamolhodaei (2010) showed that for the concept of limits and derivatives, students could apply procedures while they hold difficulties in understanding the related conceptual knowledge. Another weakness is related to classifying educational objectives. Educational objectives normally consist of some kinds of content (noun or noun phrase) and an explanation of how it should be acted on (verb or verb phrase) (Krathwohl, 2002) (Section 1.3). However, the knowledge category of BT consists of both noun and verb dimensions that brought “unidimensionality” to it (Krathwohl, 2002, p. 213). The noun was considered in subcategories of knowledge and the verb dimension was assumed in the definition of knowledge. In another words, knowledge is an unsuitable term to explain a level of thinking (Pickard, 2007).

Anderson and his colleagues (2001) amended BT to address its weaknesses by separating the noun and verb aspects and placed them in different dimensions. The noun aspect formed the knowledge dimension and the verb aspect shaped the cognitive process dimension. The first three levels of knowledge dimension consist of materials presented in BT in other terms as factual, conceptual and procedural knowledge (Figure 2.2; Table 2.1). Metacognitive knowledge was not included in BT. It was considered from the viewpoint of new learning theories in educational research regarding the importance of metacognition for learning (Section 3.1.4). Flavell’s definition of metacognition (1979) is used for the metacognitive level of RBT (Amer, 2006).



Bloom’s taxonomy **Revised Bloom’s taxonomy**

Figure 2.2 Relations between BT and RBT

Cognitive processes of RBT have the same number of categories as BT. Names of three categories were changed (i.e., remembering, understanding, and creating) and the positions of two were swapped (i.e., evaluating and creating) to address critique (e.g., Krietzer & Madaus, 1994). In addition, all the cognitive processes were revised to verb form in order to be more applicable for classifying educational objectives (Krathwohl, 2002).

Table 2.1

RBT Table with subtypes

	The Cognitive Process Dimension					
The Knowledge Dimension	1. **Remembering** 1.1 Recognising 1.2 Recalling	2. **Understanding** 2.1 Interpreting 2.2 Exemplifying 2.3 Classifying 2.4 Summarising 2.5 Inferring 2.6 Comparing 2.7 Explaining	3. **Applying** 3.1 Executing 3.2 Implementing	4. **Analysing** 4.1 Differentiating 4.2 Organising 4.3 Attributing	5. **Evaluating** 5.1 Checking 5.2 Critiquing	6. **Creating** 6.1 Generating 6.2 Planning 6.3 Producing
A. **Factual knowledge** Aa. Knowledge of terminology Ab. Knowledge of specific details and elements						
B. **Conceptual knowledge** Ba. Knowledge of classifications and categories Bb. Knowledge of principles and generalisations Bc. Knowledge of theories, models, and structures						
C. **Procedural knowledge** Ca. Knowledge of subject-specific skills and algorithms Cb. Knowledge of subject-specific techniques and methods Cc. Knowledge of criteria for determining when to use appropriate procedures						
D. **Metacognitive knowledge** Da. Strategic knowledge Db. Knowledge about cognitive tasks, including appropriate contextual and conditional knowledge Dc. Self-knowledge						

Note. RBT Table with subtypes of the knowledge and the cognitive process dimension. Adapted from "*A taxonomy for learning, teaching, and assessing: A revision of Bloom's Taxonomy of Educational Objectives* (p. 28)," by L. W. Anderson, et al., 2001, New York: Longman. Copyright 2001 by the Addison Wesley Longman, Inc. Adapted with permission.

2.2 RBT dimensions: Knowledge and cognitive process

In this section, the knowledge (Section 2.2.1) and cognitive process dimension (Section 2.2.2) of RBT are described to provide background to the analysis of the study's findings.

2.2.1 The knowledge dimension

This section is dedicated to the structure of the knowledge dimension of RBT. The knowledge dimension has 11 subtypes (Table 2.1). For each subtype, the definition is presented from the RBT handbook (Anderson et al., 2001), then, the definitions are again summarised in four tables, one for each type of knowledge. These tables were used for developing the interim RBT knowledge dimension for integral calculus (Section 5.1).

Factual knowledge

Factual knowledge is "the basic elements that students must know to be acquainted with a discipline or solve problems in it" (Anderson, et al., 2001, p. 46). It has two subtypes: knowledge of terminology, and knowledge of specific details and elements (Table 2.2).

Table 2.2

Structure of the factual knowledge

Subtype	An explanation of the subtype
Knowledge of terminology	"Knowledge of specific verbal and nonverbal labels and symbols" (Anderson, et al., 2001, p. 45).
Knowledge of specific details and elements	"Knowledge of events, locations, people, and dates" (p. 47). "Knowledge of sources of information" (p. 47).

Knowledge of terminology is referred to as "knowledge of specific verbal and nonverbal labels and symbols (e.g., words, numerals, signs, [and] pictures)" (Anderson, et al., 2001, p. 45). Knowledge of specific details and elements includes "knowledge of events, location, people, dates, sources of information, and the like" (Anderson, et al., 2001, p. 47).

Conceptual knowledge

Conceptual knowledge refers to the knowledge of "the interrelationships among the basic elements within a larger structure that enable them to function together" (Anderson, et al., 2001, p. 46). It has three subtypes (Table 2.3).

Table 2.3

Structure of the conceptual knowledge

Subtype	An explanation of the subtype
Knowledge of classifications and categories	Knowledge of "specific categories, classes, divisions, and arrangements that are used in different subject matter" (Anderson, et al., 2001, p. 49).
Knowledge of principles and generalisations	"Knowledge of particular abstraction that summarise observation of phenomena" (p. 51).
Knowledge of theories, models, and structures	"Knowledge of principles and generalisations together with their interrelationships that present a clear, rounded, and systematic view of a complex phenomenon, problem, or subject matter" (p. 51). "Knowledge of the different paradigms, epistemologies, theories, and models that different disciplines use to describe, understand, explain, and predict phenomena" (p. 52).

The first subtype, knowledge of classifications and categories, relates to "specific categories, classes, divisions, and arrangements that are used in different subject matter" (Anderson, et al., 2001, p. 49). It differs from the factual knowledge because it is more general and abstract. For example, knowing which numbers are integers and which are fractions can be considered as factual knowledge and categorised as rational numbers (Anderson et al., 2001). The second subtype, knowledge of principles and generalisations, refers to "knowledge of particular abstractions that summarise observations of phenomena" (Anderson, et al., 2001, p. 51). The last subtype, knowledge of theories, models, and structures includes "Knowledge of principles and generalisations together with their interrelationships that present a clear, rounded, and systematic view of a complex phenomenon, problem, or subject matter" (Anderson, et al., 2001, p. 51) and "knowledge of the different paradigms, epistemologies, theories, and models that different disciplines use to describe, understand, explain, and predict phenomena" (Anderson, et al., 2001, p. 52).

Procedural knowledge

Procedural knowledge is defined as knowledge of "how to do something; methods of inquiry, and criteria for using skills, algorithms, techniques, and methods" (Anderson, et al., 2001, p. 46). It has three subtypes (Table 2.4).

Table 2.4

Structure of the procedural knowledge

Subtype	An explanation of the subtype
Knowledge of subject specific skills and algorithm	"Knowledge of skills and algorithms that the process may be either fixed or more open but the end result is generally considered fixed" (Anderson, et al., 2001, p. 53).
Knowledge of subject specific techniques and methods	"Knowledge of how to think and attack a problem in a field rather than the results of such thought or problem solving" (p. 54).
Knowledge of criteria for determining when to use appropriate procedures	Knowledge of criteria that help to "make decisions about when and where using different types of subject-specific procedural knowledge" (p. 55).

Knowledge of subject specific skills and algorithm, the first subtype, refers to "knowledge of skills and algorithms that the process may be either fixed or more open but the end result is generally considered fixed" (Anderson, et al., 2001, p. 53). The second subtype, knowledge of subject specific techniques and methods, includes

> knowledge that is largely the result of consensus, agreement, or disciplinary norms rather than knowledge that is more directly an outcome of

> observation…knowledge [of] how experts in the field or discipline think and attack problems rather than the results of such thought or problem solving…The result is more open and not fixed (Anderson, et al., 2001, p. 54).

The last subtype, knowledge of criteria for determining when to use appropriate procedures, refers to knowledge of criteria that help to "make decisions about when and where using different types of subject-specific procedural knowledge" (Anderson, et al., 2001, p. 55).

Metacognitive knowledge

This section describes metacognitive knowledge in RBT and a broader discussion of metacognition is discussed in Chapter Three (Section 3.1.4). Metacognitive knowledge refers to "knowledge of cognition in general as well as awareness and knowledge of one's own cognition" (Anderson, et al., 2001, p. 46). It is different from using the metacognitive knowledge (i.e., monitoring, controlling, and regulating of cognition) which is related to the cognitive process dimension (Anderson et al., 2001). Metacognitive knowledge has three subtypes (Table 2.5).

Table 2.5

Structure of the metacognitive knowledge

Subtype	An explanation of the subtype
Strategic knowledge	"Knowledge of the **general strategies** for **learning**" (Anderson, et al., 2001, p. 56) (rehearsal, elaboration, and organisational). Knowing that planning, monitoring, and regulating cognition are useful.
	Knowledge of the **general strategies** for **thinking** and **problem solving**: "various heuristics students can use to solve problems, particularly problems that have no definitive solution method" (p. 57).
	Knowledge of "**general strategies** for **deductive** and **inductive thinking**, including evaluating the validity of different logical statements, avoiding circularity in argument, making appropriate inferences from different sources of data, and drawing on appropriate samples to make inferences" (p. 57).
Knowledge about cognitive tasks, including appropriate contextual and conditional knowledge	Knowledge about "different cognitive tasks can be more or less" (p. 57) challenging.
	Knowledge about when and how to use strategic knowledge.
	Knowledge of different cognitive tasks requires different strategic knowledge.
	Knowledge of "local situational and general social, conventional, and cultural norms for using different strategies" (p. 58).
Self-knowledge	Knowledge of "one's strengths and weaknesses in relation to cognition and learning" (p. 59) (e.g., self-awareness, self-efficacy, goal-orientation, and attitude).

Note. Some segments are bolded for adding emphasis to the original text.

Strategic knowledge, the first subtype, refers to the "knowledge of general strategies for learning, thinking, and problem solving" (Anderson, et al., 2001, p. 56). Learning strategies are grouped into three categories: rehearsal, elaboration, and organizational (Weinstein & Mayer, 1986). In addition to these general learning strategies, is the knowledge that planning, monitoring, and regulating cognition are useful for learning the topic and being a successful problem solver (Anderson, et al., 2001). General strategies for problem solving and thinking encompass "various heuristics students can use to solve problems, particularly… problems that have no definitive solution method" (Anderson, et al., 2001, p. 57). In addition to problem solving strategies, "there are general strategies for deductive and inductive thinking, including evaluating the validity of different logical statements, avoiding circularity in argument, making appropriate inferences from different sources of data, and drawing on appropriate samples to make inferences" (Anderson, et al., 2001, p. 57).

The second subtype, knowledge about cognitive tasks, including appropriate contextual and conditional knowledge, includes that

> different cognitive tasks can be more or less difficult, may make differential demands on the cognitive system, and may require different cognitive strategies… This knowledge reflects both what general strategies to use and how to use them. Conditional knowledge [as part of this subtype] refers to knowledge of the situations in which students may use metacognitive knowledge…Another important aspect of conditional knowledge is the local situational and general social, conventional, and cultural norms for using different strategies (Anderson, et al., 2001, pp. 57-58).

The last subtype, self-knowledge, relates to "one's strengths and weaknesses in relation to cognition and learning" (Anderson, et al., 2001, p. 59). It also includes individuals' beliefs about their motivations (e.g., self-awareness, self-efficacy, goal-orientation, and attitude). This aspect is also included in the taxonomy because these beliefs are social cognitive in nature and therefore related to the taxonomy (Anderson, et al. 2001).

2.2.2 The cognitive process dimension

This section presents the structure of the cognitive process dimension of RBT. The cognitive process has 19 subcategories (Table 2.6). They are described in the following subsections based on their definition in the RBT handbook (Anderson et al., 2001) and are summarised in Table 2.6.

Table 2.6

Subcategories of the cognitive process dimension

Cognitive process	Subcategory	Explanation
Remembering	Recognising	"retrieving relevant knowledge from long term memory in order to compare it with presented information" (Anderson, et al., 2001, p. 69)
	Recalling	"retrieving relevant knowledge from long term memory when given a prompt to do so" (p. 69)
Understanding	Interpreting	Converting "information from one representational form to another" (p. 70).
	Exemplifying	"Giving a specific example or instance of a general principle" (p. 70).
	Classifying	"Identifying something (e.g., particular instance or example) belongs to a certain category" (p. 72).
	Summarising	Suggesting "a single statement that represents presented information or abstracts a general theme" (p. 73).
	Inferring	"Finding a pattern within a series of examples or instances" (p. 73).
	Comparing	"Detecting similarities and differences between two or more objects, events, ideas, problems, or situation" (p. 75).
	Explaining	Constructing "a cause-and-effect model, including each major part in a system or each major event in the chain, and using the model to determine how a change in one part of the system or one "link" in the chain affects a change in another part" (p. 76).
Applying	Executing	Using constructed knowledge in an exercise, familiar task.
	Implementing	Referring to using knowledge in a problem, unfamiliar task.
Analysing	Differentiating	"Distinguishing the parts of a whole structure in terms of their relevance or importance" (p. 80).
	Organising	"Identifying the elements of a communication or situation and recognising how they fit together into a coherent structure" (p. 81).
	Attributing	Ascertaining "the point of view, biases, values, or intention underlying communications" (p. 82).
Evaluating	Checking	"Testing for internal inconsistencies or fallacies in an operation or act" (p.83).
	Critiquing	"Judging a product or operation based on externally imposed criteria and standards" (p. 84).
Creating	Generating	"Representing the problem and arriving at alternatives or hypotheses that meet certain criteria" (p. 86).
	Planning	"Devising a solution method that meets a problem's criteria, that is, developing a plan for solving the problem" (p. 87).
	Producing	"Carrying out a plan for solving a given problem that meets certain specifications" (p. 87).

Remembering

Remembering has two subcategories in RBT, recognising and recalling (Anderson et al., 2001). Recognising includes "retrieving relevant knowledge from long term memory in order to compare it with presented information" (Anderson et al., 2001, p. 69) whereas recalling is "retrieving relevant knowledge from long term memory when given a prompt to do so" (Anderson, et al., 2001, p. 69).

Understanding

Teaching activities that help students construct meaning from teaching materials are related to understanding (Anderson, et al., 2001). In RBT, understanding has seven subcategories including interpreting, exemplifying, classifying, summarising, inferring, comparing, and explaining. Interpreting refers to converting "information from one representational form to another" (Anderson, et al., 2001, p. 70). Exemplifying relates to giving "specific example or instance of a general principle" (Anderson, et al., 2001, p. 71). Classifying refers to identifying "something (e.g., particular instance or example) belongs to a certain category (e.g., concept or principle)" (Anderson, et al., 2001, p. 72). Summarising, the fourth subcategory, is related to suggesting "a single statement that represents presented information or abstracts a general theme" (Anderson, et al., 2001, p. 73). Inferring refers to "finding a pattern within a series of examples or instances" (Anderson, et al., 2001, p. 73). Comparing includes "detecting similarities and differences between two or more objects, events, ideas, problems, or situation" (Anderson, et al., 2001, p. 75). The last subcategory, explaining, refers to constructing

> a cause-and-effect model, including each major part in a system or each major event in the chain, and using the model to determine how a change in one part of the system or one "link" in the chain affects a change in another part (Anderson, et al., 2001, p. 76).

Applying

Applying has two subcategories in RBT, executing and implementing. Executing refers to using constructed knowledge in an exercise, familiar task, whereas implementing refers to using it in a problem, an unfamiliar task (Anderson, et al., 2001).

Analysing

Analysing "involves breaking material into its constituent parts and determining how the parts are related to one another and to an overall structure" (Anderson, et al., 2001, p. 79). Analysing is an extension of understanding and also can be considered as a prelude to evaluating and creating (Anderson, et al., 2001). In RBT, analysing has three subcategories including differentiating, organising, and attributing. Differentiating refers to "distinguishing the parts of a whole structure in terms of their relevance or importance" (Anderson, et al., 2001, p. 80). Organising relates to "identifying the elements of a communication or situation and recognising how they fit together into a coherent structure" (Anderson, et al., 2001, p. 81). The last subcategory, attributing, refers to ascertaining "the point of view, biases, values, or intention underlying communications" (Anderson, et al., 2001, p. 82).

Evaluating

Evaluating refers to making judgements based on some criteria (Anderson, et al., 2001). It has two subcategories in RBT, namely checking (i.e., "testing for internal inconsistencies or fallacies in an operation or act" (Anderson, et al., 2001, p. 83)), and critiquing (i.e., "judging a product or operation based on externally imposed criteria and standards" (Anderson, et al., 2001, p. 84)).

Creating

Creating is defined as "putting elements together to form a coherent or functional whole. "Objectives classified as *Create* have students make a new product by mentally reorganising some elements or parts into a pattern or structure not clearly present before" (Anderson, et al., 2001, p. 84). It has three subcategories in RBT including generating, planning, and producing. Generating relates to "representing the problem and arriving at alternatives or hypotheses that meet certain criteria" (Anderson, et al., 2001, p. 86). It differs from understanding in that the aim of generating is divergent, find various possibilities for solving the problem; whereas for understanding the goal is convergent, finding a single meaning for the problem through translating, exemplifying, etc. (Anderson, et al., 2001). Planning, the second subcategory, refers to "devising a solution method that meets a problem's criteria, that is, developing a plan for solving the problem" (Anderson, et al., 2001, p. 87). Producing, relates "carrying out a plan for solving a given problem that meets certain specifications" (Anderson, et al., 2001, p. 87).

2.3 Illustrating how RBT Tables are used

To clarify how educational materials are placed in the RBT Table, three examples are provided which were from Green (2010) in multivariable calculus and pre-calculus contexts. Undergraduate students, to match functions to graphs of simple functions such as $y = sin\, x$ or $y = e^x$, may only need to recall factual knowledge, as they work with these functions quite often. Therefore this question can be placed in the remembering factual knowledge cell (Example 1, Table 2.7).

Table 2.7

Sample examples in RBT Table

The Knowledge Dimension	The Cognitive Process Dimension					
	1. Remembering	2. Understanding	3. Applying	4. Analysing	5. Evaluating	6. Creating
A. Factual knowledge	Example 1			Example 3		
B. Conceptual knowledge		Example 3	Example 2 and 3			
C. Procedural knowledge					Example 3	
D. Metacognitive knowledge						

Note. Locating three examples in the RBT Table. Adapted from "Matching functions and graphs at multiple levels of Bloom's revised taxonomy," by K. H. Green, 2010, *PRIMUS, 20*(3), p. 217. Copyright 2010 by Taylor and Francis. Adapted with permission.

For matching more complex single variable functions to their graphs, such as $y = (x-1)^2 + 2$ or $y = (x+3)^2$, students may need to "apply conceptual knowledge" (Example 2, Table 2.7) (Green, 2010). However, matching functions to graphs of multivariable functions such as $z = \cos(xy)$ or $z = |x||y|$ involve different cells of RBT because:

- students are required to understand conceptual knowledge to find the symmetry or asymptotic properties of the functions and graphs;
- students need to analyse factual knowledge that is provided in graphs in order to find the behaviour of the functions;
- it is essential to evaluate procedural knowledge to find a strategy to solve the problem; and
- sometimes students need to apply procedural knowledge in order to calculate particular details about the functions like their value at the origin (Example 3, Table 2.7) (Green, 2010).

The next section describes how the RBT Table can be used for evaluating alignment in educational research.

2.4 RBT Table as an alignment framework

Alignment in educational settings consists of a picture of the relationship between standards, teaching, and assessment (Anderson, 2002). Evaluating alignment is important for three reasons (Anderson, 2005). Firstly, evaluating alignment provides information about the different effects of educational institutes (i.e., school, and university) on students' achievement. Second, poor alignment can cause neglect or underrating of the impact of teaching on students' learning. For instance, in the situation that instruction is not aligned with national assessment, the high quality of teaching cannot be recognised since it is not shown through the assessment. Finally, a high level of alignment is "central to the success of educational accountability programs" (Anderson, 2005, p. 111) since educational institutes are responsible for providing opportunities for their students to achieve the standards that have been considered for them (Anderson, 2005).

The RBT Tables can be used for evaluating alignment (Anderson, 2002, 2005; Näsström, 2009). The nouns and verbs included in the objectives/standards lead to finding suitable cell(s) for questions (Section 2.3). For analysing teaching materials and assessments, the procedure for finding suitable cell(s) is the same. Alignment can be evaluated through creating the three RBT Tables; one for curriculum (including educational objectives), another for teaching activities, and the third for assessment (including achievement standards). If similar cells are covered across the taxonomies, there is complete alignment. However, for example, if the same rows (levels of knowledge) but different columns (levels of cognitive process) or the same columns but different rows were covered in the taxonomies, partial alignment is said to occur. In checking alignment some cells may be found that are not addressed by any of curriculum, teaching, and assessment materials. These cells, which can be called "missed opportunities" (Anderson, 2005, p. 112), are suggested for consideration by the designer of curriculum and assessment

(Anderson, 2005). Moreover, lecturers and teachers could consider covering these cells to help enrich students' opportunities for learning.

2.5 Background to RBT as a framework

In this section, studies that used RBT are described in order to understand possible applications of RBT, and how the study should be designed. The use of RBT as a tool for analysing and informing teaching, learning, and assessment is well established in the literature across a wide range of disciplines. RBT has been used for evaluating alignment between curriculum, teaching, and assessment; planning teaching activities and lessons; designing assessment; and analysing students' performance. Such uses are collectively referred to as RBT's applications. In mathematics education, RBT has been used for analysing examination curricula, students' mathematical problem solving, and psychological factors affecting students' performance. Moreover, its potential for evaluating alignment has been addressed. A summary of these studies is described in the following sections.

2.5.1 Interpreting RBT for a topic

RBT cells are interpreted to a range of disciplines (e.g., music, nursing, and computer science education); Hanna (2007) provides examples for each cell of RBT in music education. For instance, an example of understanding conceptual knowledge in music education is "understand[ing], explain[ing], and discuss[ing] music concepts and music's relationships to other areas both within and outside of music" (Hanna, 2007, p. 10). Pickard (2007) did not cover all the cells of RBT for family and consumer sciences, but provided different examples for each level of the knowledge and cognitive process dimensions separately, without using the RBT Table. For instance, "forms of business ownership", "starch cookery principles", and "colour theory" (Pickard, 2007, p. 49) are examples of these respective subtypes of conceptual knowledge. Similarly, Thompson, Luxton-Reilly, Whalley, Hu, & Robbins (2008) provided examples for computer science education without using RBT Table. For instance, with reference to remembering, an example is "identifying a particular construct in a piece of code" (Thompson, et al., 2008,

p. 157). In mathematics, Green (2010) described the power of RBT in mathematics education by explaining three problems related to matching functions and graphs with multiple cells of RBT in pre-calculus and multivariate calculus (Section 2.3).

In terms of teaching activities, planning lessons, and assessment based on RBT, Ferguson (2002) designed an integrated class in history and English, planning the unit by specifying the objectives, determining the instructional activities, and designing formative and summative assessment all based on RBT. Such a procedure has been widely applied in nursing education (e.g., Su, Osisek, & Starnes, 2004; 2005; Su & Osisek, 2011). For instance, for understanding conceptual knowledge in the context of medical-surgical nursing lessons, an objective, a teaching activity and an assessment can be considered the following examples:

- objective: “Differentiate relevance among a set of clinical data related to clients with” myocardial infarction (Su et al., 2004, p. 118);
- teaching activity: “Classroom exercises—distinguish relationships among a set of data” (Su et al., 2004, p. 118); and
- assessment: “Multiple-choice test item—identify relevant information from a given set of data.” (Su et al., 2004, p. 118).

The next application of RBT described in this section is that of evaluating alignment.

2.5.2 Evaluating alignment based on RBT

Näsström, & Henriksson (2008) evaluated the alignment between standards and assessment for a syllabus in chemistry for upper secondary schools in Sweden. In addition, Näsström (2009) studied the efficiency of RBT for interpreting standards in mathematics through two panels (i.e., teachers and assessment experts) and revealed that RBT is an acceptable tool for this purpose; however, the expert panel was more consistent in their interpretation of standards in comparison to the teacher panel. Hanna (2007) analysed national standards for music education based on RBT, and in mathematics, Rizvi (2007) explained how to use RBT to analyse examination curricula of high school mathematics in Pakistan.

2.5.3 Analysing students' performance based on RBT

In computer science education, for the purpose of analysing students' performance based on RBT, Whalley et al., (2006) studied novice computer programmers' performance, and Alaoutinen & Smolander (2010) designed a survey questionnaire based on the cognitive process dimension of RBT. However, neither studies classified content in terms of the knowledge dimension.

In mathematics, Radmehr and Alamolhodaei (2010, 2012) quantitatively studied students' mathematical performance based on 24 cells of RBT in functions, limits, and derivatives. Moreover, Hajibaba, Radmehr, and Alamolhodaei (2013) investigated the effects of several psychological and affective factors on students' mathematical performances based on columns and rows of RBT. They found that the effects of attitude towards mathematics, anxiety, attention, field dependency, and working memory capacity on students' mathematical performance varied in terms of rows and columns of RBT. Also, Rahbarnia, Hamedian, and Radmehr (2014) investigated the relationships between different multiple intelligences and mathematical performance in different rows and columns of RBT. Consistent with Anderson (2005), the purpose of these studies were to show that researchers in the psychology of mathematics education would benefit from changing their typical questions to questions based on RBT in order to maximise the usefulness of information about the relationship between psychological factors and mathematical performance.

2.5.4 Strengths and weaknesses of previous studies into using RBT as a framework

In terms of defining the levels of knowledge of RBT, analysing students' performance, and providing assessment examples, educational objectives, and teaching activities based on RBT in different disciplines, three major concerns have been identified. First, previous studies have been done without documenting how these materials have been developed, refined, and validated (examples of these studies are: Hanna, 2007; Su, Osisek, & Starnes, 2004, 2005; Su, & Osisek, 2011). In these studies, researchers, based on their experiences in the discipline, provided materials using RBT. Second, in several studies, all the cells and subcategories of RBT are not covered, without providing justification as to whether the omitted categories are inapplicable for the discipline, or researchers cannot find the suitable materials for the content, or it is not in the interests of the researcher. An example in mathematics education is Green's (2010) study about functions. Finally, in several studies, the authors seem to have misconceptions about RBT. For instance, they have not considered RBT as a two dimensional taxonomy and therefore neglected one of the dimensions (e.g., Thompson et al., 2008).

In terms of the methods used in previous studies, they were varied (quantitative (e.g., Hajibaba et al., 2013; Radmehr & Alamolhodaei, 2010), qualitative (e.g., Hanna, 2007; Su, Osisek, & Starnes, 2004, 2005; Su, & Osisek, 2011), and mixed methods (e.g., Bümen, 2007; Näsström, & Henriksson, 2008; Näsström, 2009)). The quantitative studies used RBT for analysing students' performance; and the qualitative studies used RBT for providing educational objectives, assessment questions, and teaching activities, and the mixed methods employed RBT for evaluating alignment. The major methodological problems the researcher found in previous studies is that they have not refined and validated the materials based on RBT.

The design of this study seeks to address these concerns. The process of developing the interim RBT knowledge dimension is documented and resources used are provided (Section 5.1). In addition, the RBT knowledge dimension is not considered to be a refined or final version, instead, it is considered an interim knowledge dimension to be developed further in future studies. While all of the subcategories of the RBT cognitive process are not covered in the questions designed for exploring student learning (Section 4.3.1), questions are designed in such a way that each main category is being addressed (Section 5.2.1). To address both dimensions, the knowledge dimension was developed first; then, questions were designed to address different cognitive processes of these knowledge types.

2.6 Chapter summary

This chapter described BT's limitations and weaknesses that lead to the design of RBT. Then it described the RBT's structure to show how it can be used. It also included studies that used RBT as a framework, and discussed their limitations. All of these kinds of information were used in designing this study in order that the study findings could provide further understanding of the transition between school and university for students in relation to integral calculus.

Chapter Three: Literature Review

This review of the literature has the following structure. The first section discusses major theories and frameworks that influence the learning and teaching of mathematics. It is important to review these for understanding similarities and differences between RBT and these, to have a better understanding of how RBT can be used for informing the teaching and learning of mathematics. Section 3.2 is dedicated to the literature about teaching and learning of integral calculus, especially integral-area relationships and the FTC, to foreground the study findings. Section 3.3 reviews studies about the secondary-tertiary transition in mathematics, because these transition years are the focus of this study.

3.1 Major theories and frameworks that influence the teaching and learning of mathematics

Major theories and frameworks that influence the teaching and learning of mathematics are described in this section, including: cognitive and social constructivism, Skemp's theories of mathematical understanding; conceptual and procedural knowledge; metacognition; SOLO taxonomy (model); APOS theory; the notion of procept; Tall's theory of three worlds of mathematics; and Schoenfeld's framework for the analysis of mathematical problem-solving behaviour and decision making in teaching. These sections show how RBT fits with these theories, and discusses several aspects of the theories.

3.1.1 Cognitive and social constructivism

This section describes cognitive and social constructivism, two major theories that strongly influence the understanding of teaching and research in mathematics education. RBT's stand in relation to these theories is presented. Key ideas underlying these theories are described to understand the connection between them and RBT in order to explore whether RBT addresses them or not.

Constructivism and objectivism, previous methods used to explain the teaching of a variety of subjects including mathematics, provide different perspectives about teaching and learning (Biggs & Tang, 1997). To better understand constructivism, a description of objectivism is presented; then cognitive and social constructivism are described.

Objectivism considers knowledge existing as independent of the knower (Biggs & Tang, 1997). It is based on a positivist paradigm and sees understanding as "coming to know what already exists" (Biggs & Tang, 1997, p. 79). Teaching based on this theory (which has been described transmission teaching) is a matter of transmitting knowledge from the teacher's mind to the student's mind (Johnson, 2006), "learning of receiving and storing it accurately, and using it appropriately" (Biggs & Tang, 1997, p. 79). This teaching approach is considered to be teacher-centred with the teacher being the knowledge dispenser, and the final evaluator of learning (Johnson, 2006).

Constructivists have different points of view about knowledge and teaching. Based on this theory, meaningful learning occurs when students actively construct their knowledge (Biggs & Tang, 1997) rather than passively receive and store it. Construction of knowledge can happen through both personal and social activity, and relies on learners' prior knowledge, experiences, motivation, and orientation toward learning. The aim of constructivist style teaching is to help students' cognitive processes go in such a direction that teaching objectives can be achieved (Biggs & Tang, 1997; Powell & Kalina, 2009). To create opportunities so that meaningful learning occurs for students, teachers and lecturers are encouraged to find out what their students already know about the topic in order to present new information to them in such a way that students can create "personal meaning" (Powell & Kalina, 2009, p. 241) out of it.

Two types of constructivism considered in educational research are cognitive and social constructivism (Powell & Kalina, 2009). These are now described.

Cognitive constructivism

Cognitive constructivism relies on the work of Piaget (1953). Piaget's theory of cognitive development (1952) claimed that children construct knowledge through a process such that knowledge could not be given to them in a way that could immediately be understood and used. Piaget's cognitive theory has three main aspects, including schemas, adaptation processes, and stages of cognitive development (McLeod, 2015).

Piaget defined schemas as "a cohesive, repeatable action sequence possessing component actions that are tightly interconnected and governed by a core meaning", (Piaget, 1952, p. 7). It is a mental structure and can be considered as a way of organising knowledge (McLeod, 2015). Schemas can be considered as units of knowledge that are related to objects, actions, and abstract concepts in the world (McLeod, 2015).

In relation to the second aspect of his theory (i.e., adaptation processes), intellectual growth takes place through schema construction and is considered as a process of adaptation to the world (McLeod, 2015) through assimilation, accommodation, and equilibration. Assimilation refers to incorporating a new piece of knowledge into existing schemas, whereas, accommodation refers to changing the existing schema according to new knowledge (e.g., object, situation) because the current schema does not work or the new knowledge in conflict with the existing schema (McLeod, 2015, Swan, 2005). Equilibration is the force that brings about development, both when the new information can be assimilated or when it needs to be accommodated (McLeod, 2015). A final aspect of this theory, the stages of cognitive development (1952) (Table 3.1), are described as sensorimotor, preoperational, concrete, and formal operational (Wadsworth, 2004).

Table 3.1

Stages of Piaget's theory of cognitive development

Stage	Age range	Description of the stage
Sensorimotor	0-2	Using sense, physical activity, and later language for exploring environments.
Preoperational	2-7	Developing personal language skills but cannot understand thoughts of other people.
Concrete operational	7-11	Using personal logical reasoning to replace intuitive thought.
Formal operational	11 to adulthood	Using higher levels of thinking for problem-solving.

Piaget believed that this sequence is universal regardless of the learner's culture. This view was disputed by some scholars such as Vygotsky who believed social interaction is important for cognitive development (McLeod, 2015).

Social constructivism

Social constructivism is inspired by the works of Vygotsky (1962, 1978). Vygotsky's theories came after cognitive constructivism; highlighting the importance of social interactions in the construction of knowledge. Vygotsky believed students' social interactions with teachers and other students in classrooms are an integrated part of learning. Vygotsky claimed "cognitive skills and pattern of thinking are not primarily determined by innate factors,…, but rather are the products of activities practiced in the social institutions of the culture in which the individual lives" (Swan, 2005, p. 4). Based on his view, knowledge construction is a reciprocal process: first enacted socially, then internalised in the individual's mind. Internalised concepts then guide social interactions (Swan, 2005).

Vygotsky described a Zone of Proximal Development (ZPD) as a place where learning of a concept happens when students interact with teachers, peers or adults to construct knowledge (Powell & Kalina, 2009). Vygotsky was also interested in the role of language in learning and thinking, believing language and thought are closely related and language is important for forming thought (Swan, 2005).

Cognitive and social constructivism: A comparison

Despite the differences between cognitive and social constructivism, they have consistencies in relation to at least three things. Firstly, students construct knowledge based on their prior knowledge that is "relevant and meaningful" for them (Powell & Kalina, 2009, p. 241). Secondly, they place importance on inquiry learning, involving presenting a puzzling situation to students and asking them to solve the problem by collecting the required data, then evaluating the results (John Dewey, 1910 in Woolfolk, 2004). Thirdly, the lecturers and teachers are seen as facilitators helping students to construct their own knowledge (Powell & Kalina, 2009).

Cognitive constructivism theory in mathematics education is characterised by "students actively construct[ing] their mathematical ways of knowing as they strive to be effective by restoring coherence to the worlds of their personal experience" (Cobb, 1994, p. 13); whereas, social constructivism theory focuses on social and cultural aspects of mathematical activity and interactions in classrooms are examples of a "culturally organised practice of schooling" (Cobb, 1994, p. 15). These two theories have frequently been used in mathematics education (e.g., Averill, 2012; Bednarz & Janvier, 1988), and several differences between them are reported (Table 3.2); however, several scholars believe they are complementary and both should be considered in teaching and research (Cobb, 1994, Powell & Kalina, 2009). They believe social constructivism focuses on "the conditions for the possibility of learning" (Cobb, 1994, p. 13), whereas cognitive constructivism focuses "on what students learn and the processes by which they do so" (Cobb, 1994, p. 13).

Table 3.2

Difference between cognitive and social constructivist perspectives

Cognitive constructivist perspectives	Social constructivist perspectives
Inspired by the works of Piaget.	Inspired by the works of Vygotsky.
Ideas are constructed in the individual through a personal process.	Ideas are constructed through social interaction with others (e.g., teachers and students).
Student's thinking is analysed individually in terms of sensory-motor and conceptual processes.	Student's thinking is analysed through social actions.
Focus is on linking activity with "students' sensory-motor and conceptual activity" (Cobb, 1994, p.14).	Focus is on linking activity to "culturally organised practices" (Cobb, 1994, p.14).
Thinking precedes language and mathematical signs and symbols are tools that students use for expressing and communicating mathematical thinking.	Language precedes thinking, and mathematical signs and symbols are "carriers of established mathematical meaning or of a practice's intellectual heritage"(Cobb, 1994, p. 13).
Classroom interactions are considered as teachers' and students' efforts to adapt their individual activities.	Classroom interactions are examples of a "culturally organised practice of schooling" (Cobb, 1994, p. 15).
Focus is on "social and cultural basis of personal experience" (Cobb, 1994, p. 15).	Focus is on the "constitution of social and cultural processes by actively interpreting individuals" (Cobb, 1994, p. 15).

Note. Based on the difference reported by Cobb (1994) and Powell & Kalina (2009).

For improving teaching and learning drawing from these two theories, the literature suggested several practices such as:

- asking questions of students to explore whether they have difficulty with the concept and helping them to amend their misconception(s);
- students working on assignments while a teacher helps them; and
- encouraging students to talk about their culture and how the topic is related to that (Powell & Kalina, 2009).

Discussion

In relation to these theories, RBT is designed based on a constructivist approach to learning. RBT designers believe:

> students engage in active cognitive processing, such as paying attention to relevant incoming information, mentally organising incoming information into a coherent representation, and mentally integrating incoming information with the existing knowledge…The cognitive processes provide a means for describing the range of students' cognitive activities in constructivist learning; that is, …students can actively engage in the process of constructing meaning (Anderson, et al., 2001, p. 65).

In relation to the context of the study, the *New Zealand Curriculum* (Ministry of Education, 2007b) can be said to be strongly influenced by constructivism. Constructivist language can be found in the wording of its vision, values, key competencies, and principles. For instance, in terms of principles, students are considered to be at the centre of teaching, and learning and in terms of values, students are encouraged to value inquiry (Ministry of Education, 2007b), which is a component of constructivism. Thinking is a key competency in the *New Zealand Curriculum* which is strongly related to constructivism (e.g., developing understanding, constructing knowledge):

> Thinking is about using creative, critical, and metacognitive processes to make sense of information, experiences, and ideas. These processes can be applied to

purposes such as developing understanding, making decisions, shaping actions, or constructing knowledge. Intellectual curiosity is at the heart of this competency. Students who are competent thinkers and problem solvers actively seek, use, and create knowledge… (Ministry of Education, 2007b, p. 12).

The next section describes Skemps's theories about mathematical learning.

3.1.2 Skemp's theories of mathematical learning

Exploring what mathematical learning is, is important because this study explores student learning of a mathematical topic, integral calculus. To some extent, this section facilitates understanding the relationships between conceptual and procedural knowledge as two types of RBT knowledge. Skemp's theories of learning (1971, 1976, & 1979) are among major theories in relation to mathematical learning.

Similar to the Piaget definition of schema, Skemp (1971) considered the schema as the general psychological term for a mental structure which integrates existing knowledge and works as a tool for acquiring new knowledge (Skemp, 1971). Skemp (1971) highlighted "to *understand something means to assimilate it into an appropriate schema.* This explains the subjective nature of learning and also makes clear that this is not usually an all-or-nothing state" (p. 46).

Skemp (1976) discussed two types of understanding and learning: instrumental, knowing "rules without reasons" (p. 2); and relational, "knowing both what to do and why" (p. 2). For a group of students, instrumental understanding develops (in contrast to relational understanding) because it is easier to understand, requires less content, and the reward is quick and apparent (Skemp, 1976). However, relational understanding has its own advantages. In relational understanding, after content is learnt, it can be recalled more easily (Skemp, 1976). The content is often helpful for understanding other topics (Skemp, 1976). Relational understanding also acts as an internal reward because it "can be effective as a goal in itself" (Skemp, 1976, p. 10). If students are satisfied with relational understanding of their current topics, they may try to understand relationally new topics as well, "like a tree extending its roots" (Skemp, 1976, p. 10).

Skemp (1976) defined instrumental and relational mathematics in similar terms to his definitions of instrumental and relational understanding, and highlighted "I now believe that there are two effectively different subjects being taught under the same name, *'mathematics'…*what constitutes mathematics is not the subject matter, but a particular kind of knowledge about it" (pp. 6, 15). Characteristic of instrumental mathematical learning is learning several "fixed plans" (p. 14) that enable students to solve mathematical problems using a specific starting point. The plans tell students exactly what to do at each step. However, in relational mathematical learning, students construct a "conceptual structure (schema)" (p. 14) which helps them to create several plans that can be used for solving mathematical problems from any starting point. Comparing instrumental and relational understanding and learning with objectivism and constructivism (Section 3.1.1), objectivism is more linked to instrumental learning and constructivism is more linked to relational learning. The reason is both relational understanding and constructivism focus on creating plans rather than learning fixed plans.

Logical understanding, the third type of understanding described by Skemp in 1979 (Skemp, 1979) is:

> the ability to demonstrate that what has been stated follows of logical necessity, by a chain of inference, from (i) the given premises, together with (ii) suitably chosen items from what is accepted as established mathematical knowledge (axioms and theorems) (p. 47).

These three types of understanding are compared with RBT in the next section.

Discussion

A comparison between Skemp's understanding theories and RBT has not been considered in the literature to date; however to highlight the potential of RBT as a framework for analysing students' learning, a loose comparison is described here. In terms of the RBT knowledge dimension, instrumental understanding addresses procedural knowledge, relational understanding addresses both conceptual and procedural knowledge, and finally, logical understanding relates to metacognitive knowledge.

Instrumental understanding refers to knowing rules (without reasons) and RBT procedural knowledge is defined as knowledge of how to do something (Section 2.2.1), which could suggest that there is a close relationship between them. Relational understanding, knowing both what to do and why, relates to procedural knowledge because as part of this understanding the individual knows what to do. Relational understanding also encompass RBT conceptual knowledge because conceptual knowledge is knowledge of the interrelationships between elements (Section 2.2.1); knowing the relationships helps individuals to know why something works, which is part of relational understanding. In terms of logical understanding, strategic knowledge, a subtype of RBT metacognitive knowledge, relates to this type of understanding. One aspect of strategic knowledge is knowledge of general strategies for deductive and inductive thinking (Section 2.2.1) that relate to the definition of logical understanding.

Comparing Skemp's learning theories with the RBT Table shows that instrumental understanding relates to remembering and applying procedural knowledge (Table 3.3); whereas, relational understanding encompass these two as well as understanding procedural knowledge. It also includes remembering, understanding and applying conceptual knowledge. Logical understating relates to remembering, understanding, and applying metacognitive knowledge. In some instances, the higher cognitive processes of RBT are also related to relational and logical understanding.

Table 3.3

Comparing Skemp's learning theories with the RBT table

The Knowledge Dimension	The Cognitive Process Dimension					
	1. Remembering	2. Understanding	3. Applying	4. Analysing	5. Evaluating	6. Creating
A. Factual knowledge						
B. Conceptual knowledge	Relational understanding	Relational understanding	Relational understanding			
C. Procedural knowledge	Instrumental & relational understanding	Relational understanding	Instrumental & relational understanding			
D. Metacognitive knowledge	Logical understanding	Logical understanding	Logical understanding			

There are other definitions for mathematical understanding which are not in line with all parts of Skemp's theories for learning and understanding. For instance, Piaget (1978) believed understanding cannot be considered as how to do something. "Understanding brings out the reason for things…" (p. 222). Therefore, Piaget's (1978) notion of understanding is not in line with the idea of instrumental understanding as part of understanding. Skemp's views about instrumental and relational understanding are reflected as to how procedural and conceptual knowledge are defined (Stewart, 2008). The next section describes these two types of knowledge in the case of mathematics.

3.1.3 Conceptual and procedural knowledge

Both conceptual and procedural knowledge are two types of the RBT knowledge dimension (Section 2.2.1). In this section, conceptual and procedural knowledge, their relationship, and the tools used for measuring them, are described. Then, studies that have defined conceptual and procedural knowledge in integral calculus are reviewed. How conceptual and procedural knowledge are defined in mathematics and integral calculus are foregrounded here, as these two types of RBT knowledge will be interpreted in later analysis (Section 5.1).

Definitions of conceptual and procedural knowledge and their importance

The border between conceptual and procedural knowledge cannot be identified easily (Mahir, 2009) and sometimes they are not separable (Rittle-Johnson & Schneider, 2014). One of the first attempts to define these two terms in mathematics was done by Hiebert and Lefevre (1986). Based on their view, procedural knowledge refers to having "familiarities with the individual symbols of the system [and knowing] rules or procedures for solving mathematical problems" (p. 7) and conceptual knowledge is "knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete piece of information" (p. 3).

This conceptual knowledge definition is disputed by others (Baroody, Feil, & Johnson, 2007; Rittle-Johnson & Schneider, 2014), who claim it takes time for knowledge to be richly connected in the mind, and this happens for experts over time. Others consider conceptual knowledge as knowledge of concepts (Rittle-Johnson & Schneider, 2014), where concepts refers to "an abstract or generic idea generalized from particular instances" (Mish, 1999, p. 238).

Similarly to that of conceptual knowledge, Hiebert and Lefevre's procedural knowledge definition (1986) is argued by others (Canobi, 2009; Rittle-Johnson & Schneider, 2014). In more detail, familiarities with the symbols of the system are not considered in their definitions. For example, Rittle-Johnson & Schneider (2014) noted that procedural knowledge "is the ability to execute action sequences (i.e., procedures) to solve problems" (p. 1120).

Despite these differences in the definitions of conceptual and procedural knowledge, there is an agreement that to have a meaningful understanding of mathematics, a balance between providing opportunities for students to construct both conceptual and procedural knowledge should be considered (Gray & Tall, 1994; Mahir, 2009; Rittle-Johnson & Schneider, 2014; Stewart, 2008). Without having conceptual knowledge of a topic, procedural knowledge is limited. "Procedural knowledge must rest on conceptual base… procedural knowledge can be quite limited unless it is connected to a conceptual knowledge base" (Silver, 1986, p. 185). If conceptual and procedural knowledge are not linked in students' minds, "students may have a good intuitive feel for mathematics but not solve the problems, or they may generate answers, but not understand what they are doing" (Hiebert & Lefevre, 1986, p. 9).

Several types of teaching activities are suggested in the literature for improving both conceptual and procedural knowledge. Three examples are: encouraging students to compare alternative solution procedures; explaining the solution procedures for themselves, and exploring how they can solve problems before having instruction (Rittle-Johnson & Schneider, 2014). To achieve a better understanding of conceptual and procedural knowledge, the next section describes the relationships between these two types of knowledge.

Relationships between conceptual and procedural knowledge

Four perspectives on the relationship between conceptual and procedural knowledge are reported, including concept-first, procedure-first, inactivation, and iterative perspectives (Rittle-Johnson & Schneider, 2014) (These four perspectives are summarised in Table 3.4). However, the iterative perspective is currently the most accepted view, supported by several studies in different mathematical topics (See Rittle-Johnson & Schneider, 2014). The initial knowledge about the topic can be conceptual or procedural depending on prior knowledge within the topic. It is not important which one comes first, the important thing is that having one type of knowledge helps construct the other type (Rittle-Johnson & Schneider, 2014). For instance, conceptual knowledge assists in choosing suitable procedures during problem solving; and practising several procedures (procedural knowledge) may help achieve a better understanding of the concept (Rittle-Johnson & Schneider, 2014).

Table 3.4

Relationships between conceptual and procedural knowledge

	An explanation of the order
Concept-first	Individuals first construct conceptual knowledge, then build procedural knowledge from it by solving different problems.
Procedure-first	Individuals first construct procedural knowledge, then, using “abstraction processes”, build conceptual knowledge.
Inactivation	Individuals construct conceptual and procedural knowledge independently.
Iterative	Growth in Individuals’ conceptual and procedural knowledge is “bi-directional”; an increase in one will lead to “subsequent increase” in the other.

Note. These different perspectives are reported by Rittle-Johnson & Schneider (2014).

The next section describes the measures used for evaluating conceptual and procedural knowledge in the literature to inform how conceptual and procedural knowledge need to explored in the context of integral calculus in the study.

Measuring conceptual and procedural knowledge

There are differences between how conceptual and procedural knowledge can be measured. Conceptual knowledge can be measured using a variety of tools that can be categorised into two groups, implicit and explicit tools. A tool is classified into one or other of these, based on whether implicit or explicit knowledge of the concepts is required to be successful in the task. Examples of implicit tools are

- evaluating unfamiliar procedures, examples of concepts, and the quality of answers given by others;
- translating quantities between representational systems;
- comparing quantities; and
- sorting examples into categories (Rittle-Johnson & Schneider, 2014).

Examples of explicit tools are

- explaining judgement;
- generating or selecting definitions of a concept;
- explaining why procedures work; and
- drawing concept maps (Rittle-Johnson & Schneider, 2014).

The main feature of measuring conceptual knowledge is that the task should be relatively unfamiliar to participants in order for them to find out the answer by using their conceptual knowledge, not using a known procedure for solving it (Rittle-Johnson & Schneider, 2014).

Methods of measuring procedural knowledge are not as varied as for conceptual knowledge and are easier to assess. Most of the time participants are asked to solve a number of familiar problems and their work is evaluated in terms of accuracy of the answer or procedures (Gray & Tall, 1994; Rittle-Johnson & Schneider, 2014). Familiar tasks are used because participants have solved questions related to them, therefore, they have constructed suitable procedures for solving them (Rittle-Johnson & Schneider, 2014). The next section describes how conceptual and procedural knowledge were explored in literature in the context of integral calculus.

Conceptual and procedural knowledge in integral calculus

Mahir (2009) evaluated undergraduate students' conceptual and procedural knowledge in integral calculus. His definition of conceptual and procedural knowledge is inspired by Hiebert and Lefevre's (1986) definitions.

> Conceptual knowledge is knowledge which is connected to the other pieces of knowledge, and the holder of the knowledge also recognises the connection. The connections between the pieces of knowledge are as important as the pieces themselves. Procedural knowledge consists of [the] formal language of mathematics, and of rules, algorithms and procedures used to solve mathematical tasks (Mahir, 2009, pp. 201-202).

Mahir (2009), based on his understanding of conceptual and procedural knowledge, defined them in the context of integral calculus. He defined conceptual knowledge as knowing "the definite integral of a function is the limit of Riemann sums, the integral-area relation and the Fundamental Theorem of Calculus" (p. 202); and procedural knowledge as knowing "the integral techniques used to find the primitive of a function" (p. 202). Mahir used a questionnaire with five questions. Two questions measured procedural knowledge, ($e.g., \int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$), two measured both conceptual and procedural knowledge depending on the method used for solving the question (e.g., $\int_{\sqrt{2}}^{2\sqrt{2}} \sqrt{6} - \sqrt{8-x^2}\,dx$), and one measured conceptual knowledge (question three of students interview, Section 5.2.1). For evaluating whether students have conceptual or procedural knowledge, the methods students solved the questions with were considered. If the question was solved using integral-area relationship, Mahir (2009) claimed the presence of conceptual knowledge, and if the question was solved using integral techniques, then, the presence of procedural knowledge was claimed.

Kiat's (2005) study is also useful for exploring conceptual and procedural knowledge in the context of integral calculus. In this study, when exploring students' difficulties in solving integral calculus problems, three types of errors were defined including conceptual, procedural, and technical errors. He described conceptual error as "failure to grasp the concepts in the problem or errors that rise from failure to appreciate the relationships involved in the problem" (p. 41) and procedural errors as errors that "arise from failure to carry out manipulations or algorithms despite having understood the concepts behind the problem." (p. 41). Instances of conceptual error in integral calculus from Kiat's (2005) study are:

- students do not split the integral when a function crosses the axis of integration. For example, not splitting the integral at $x = 4$ for $\int_0^5 x(x-4)dx$; and
- students make a mistake in setting up integral limits.

Two examples of procedural error reported by Kiat (2005) are: not adding the constant c when finding the antiderivative of an indefinite integral; and confusing differentiation with integration.

The last study reviewed here is Thomas and Hong's (1996) study. For exploring students' conceptual understanding of integrals, they used questions in which an integrand is not explicitly given in the question. For example, "If $\int_1^3 f(t)dt = 8.6$, then write down the value of $\int_2^4 f(t-1)dt$" (p. 575). Questions like this caused difficulties for several students who have a process-oriented view to integral calculus (Thomas and Hong, 1996).

Discussion

The definitions highlighted by Rittle-Johnson & Schneider (2014) are in line with RBT's definitions for conceptual and procedural knowledge. However, Hiebert and Lefevre's conceptual and procedural definition (1986) encompasses not only RBT's conceptual and procedural definition, but also RBT's factual knowledge definition. In more detail, familiarity with the individual symbols of the system that was considered in Hiebert and Lefevre's definition (1986) of procedural knowledge is related to the first subtype of factual knowledge (i.e., knowledge of terminology) that includes knowledge of specific verbal and nonverbal labels and symbols (Section 2.2.1).

Now that an insight into the nature of conceptual and procedural knowledge has been presented, the next section explores the nature of metacognitive knowledge.

3.1.4 Metacognition

Metacognition (Flavell, 1979; Schneider, & Lockl, 2002) is traditionally defined as any knowledge or cognitive activity that individuals have about any aspect of cognitive activities (Flavell, Miller, and Miller, 1993). It refers to "meta-level knowledge and mental action used to steer cognitive processes" (Jacobse & Harskamp, 2012, p. 133).

One important addition to BT was the inclusion of metacognitive knowledge in RBT. Metacognitive knowledge is the last type of RBT's knowledge dimension (Section 2.2.1) which is part of the broader notion, metacognition (Flavell, 1979; Schneider, &

Lockl, 2002). In this section, after highlighting the importance of metacognition, metacognitive knowledge from Flavell's model of cognitive processing (1979) is presented due to its importance in designing RBT. Then, three general facets (the terminology that is used for the components of metacognition) of metacognition presented in several studies (e.g., Efklides, 2006, 2008; Kim, Park, Moore, & Varma, 2013; Schneider, & Artelt, 2010) are explained and compared with the RBT's metacognitive knowledge. This has been done in order to link the current research about metacognition to RBT. Then, instruments for measuring metacognition are explored as a background to the tools that are used for gathering data in this study.

Metacognitive activities are necessary for successfully solving mathematical problems (Lester, 1982; Silver, 1982; Verschaffel, 1999). Skills in building relevant toolkit (e.g., separating relevant from irrelevant information) and using a variety of heuristics are components of successful problem solving that is related to metacognition (Lester, 1982). Metacognition is instrumental in building an appropriate representation of the given problem and checking the outcome of problem solving (Garofalo, & Lester, 1985; Verschaffel, 1999). Metacognition is also related to the decisions that a problem solver makes, relating to their personal beliefs and values, between different cognitive strategies to find the solution during mathematical problem solving (Silver, 1982). Beliefs and values about learning, problem solving, and mathematical problem solving are important in the encoding and retrieval of mathematical content knowledge (Silver, 1982). Knowing the definition of metacognition and its importance for mathematical problem solving, the next section describes Flavell's definition of metacognition through his model of cognitive processing (1979).

Flavell's model of cognitive processing

Based on Flavell's model of cognitive monitoring (1979), monitoring of cognitive tasks happens through the interactions of metacognitive knowledge, metacognitive experiences, goals, tasks, and strategies. Flavell (1979) highlighted that metacognitive knowledge "consists primarily of knowledge or beliefs about what factors act and interact in what ways to affect the course and outcome of cognitive enterprises" (p. 907). The major categories of these factors are persons, tasks, and strategies.

The persons' category refers to an individual's beliefs about himself and others as cognitive processors, and includes intra-individual (Flavell, 1979) (e.g., you prefer and are more confident in finding area with respect to the x-axis rather than the y-axis when both ways can be used for finding the area using integral calculus), inter-individual differences (e.g., between your friends, Sara knows integral calculus better than others), and universals of cognition (e.g., knowing one might not understand the lecture notes if one has not attended the lectures).

The tasks' category relate to how the cognitive enterprise should be managed and how you are likely to be successful in achieving the goal(s) based on the information provided in the given situation (Flavell, 1979) (e.g., your knowledge about how you want to check if you have found the correct answer for a definite integral problem).

The third category, 'strategies', refers to knowledge about strategies that are effective for achieving goals in different conditions (Flavell, 1979) (e.g., knowing in the questions related to areas, volumes, and surface areas, sketching the related graphs of functions and considering a cross section will help to determine which method/way should be used).

In practical situations, metacognitive knowledge combines two or all of these factors (Flavell, 1979). For instance, Sara may believe that (unlike John) (person factor) first she should use integration with respect to the y-axis (rather than integration with respect to the x-axis) (strategy factor) in solving area problems when the given functions are presented as $x = f(y)$ (task factor).

Garofalo, & Lester (1985) describes Flavell's metacognitive knowledge definition (1979) in the context of mathematics. The person factor involves self-assessment of capabilities and limitations in relation to different topics in mathematics. It also includes personal beliefs regarding "the nature of mathematical ability, the relationship of performance in mathematics to performance in other areas, and the effects of affective variables such as motivation, anxiety, and perseverance" (Garofalo, & Lester, 1985, p. 167). Mathematical task knowledge includes personal beliefs "about the subject of mathematics as well as beliefs about the nature of mathematics tasks" (p. 167). "An awareness of the effects of task features such as content, context, structure, and syntax on task difficulty" (p. 167) are also included in this type of knowledge. The third factor, mathematical strategy knowledge, includes "knowledge of algorithms and heuristics, but it also includes a person's awareness of strategies to aid in comprehending problem statements, organising information or data, planning solution attempts, executing plans, and checking results" (p. 168).

The next section describes the facets of metacognition, including metacognitive knowledge, experiences, and skills.

Metacognition and its facets

A number of researchers have tried to describe metacognition and its facets (e.g., Efklides, 2006, 2008; Flavell, 1979; Kim et al. 2013; Schneider, & Lockl, 2002; Tarricone, 2011). However, there is fuzziness in the concept and structure of metacognition (Akturk, & Sahin, 2011; Tarricone, 2011) due to different models and meanings presented in the literature, and it is not always easy to distinguish between cognition and metacognition (Garofalo, & Lester, 1985). However, metacognition is accepted as a model of cognition that works at a meta-level to cognition by monitoring and controlling cognitive tasks (Efklides, 2006). Three main facets of metacognition recognised in current research (e.g., Efklides, 2006, 2008; Kim et al., 2013; Schneider, & Artelt, 2010; Tarricone, 2011) are: metacognitive knowledge or knowledge of cognition; metacognitive skills or regulation of cognition; and metacognitive experiences or concurrent metacognition. These three facets are described in the following paragraphs.

Metacognitive knowledge (or knowledge of cognition)

Metacognitive knowledge is a declarative knowledge about cognition (Efklides, 2006). It refers to individuals' explicit or implicit knowledge (i.e., ideas, beliefs, theories) about persons (how individuals perform and feel about different tasks), tasks (its categories, features, relationships, and the way they work), goals (the goals individuals follow within different tasks and situations), and strategies (including different strategies and when, why, and how they should be used) (Efklides, 2006, 2008).

Metacognitive knowledge derives from a person's long-term memory about him/herself or others. It also encompasses knowledge about the various cognitive functions (e.g., thinking, and memory) in terms of what they are and how they work (Efklides, 2006, 2008). In addition, it includes epistemic cognition, knowledge of the criteria of validity of knowledge that involve knowledge about limits of knowing, the certainty of knowing, and criteria of knowing (Kitchner, 1983). Finally, there is some evidence (e.g., Kuhn, 2000) that theory of mind (Bartsch, & Wellman, 1995), beliefs that individuals have about people's minds, including their own, can be considered as knowledge of cognition.

Self-monitoring and monitoring others' cognitive activities, communicating with other people, and awareness about personal metacognitive experiences help individuals to constantly develop, update, and revise their metacognitive knowledge (Efklides, 2006, 2008; Flavell, 1979; Kim, et al., 2013).

Metacognitive skills (or regulation of cognition)

The second facet of metacognition, metacognitive skills, is a procedural knowledge (Efklides, 2006) that refers to activities that help individuals to control their cognitive activities such as learning (Schraw, 1998).

These activities are performed deliberately and consist of activities such as task orientating, planning, monitoring, regulating, and evaluating. Task orientating relates to understanding the task requirements, planning corresponds to steps that need to be done to achieve the goal(s) or doing the task(s), monitoring refers to monitoring activities during implementing strategies, and evaluating and regulating relate to checking the outcome of cognitive processing and modifying it when it fails (Efklides, 2006, 2008; Garofalo, & Lester, 1985; Schraw, 1998; Veenman, & Elshout, 1999).

Metacognitive experience (or concurrent metacognition)

Metacognitive experience as the last facet of metacognition is "what the person is aware of and what she or he feels when coming across a task and processing the information related on it" (Efklides, 2008, p. 279).

Unlike metacognitive knowledge and skills, metacognitive experience is present in working memory (Efklides, 2006). It encompasses feelings of knowing, familiarity, difficulty, confidence, and satisfaction. Metacognitive experiences also include judgment of learning, estimation about effort, and time that is needed and spent on the task as well as estimating the correctness of solution. Another aspect of metacognitive experiences is online task specific knowledge, that is, task information, ideas, and thoughts that a person knows about the task which they are engaged with. Metacognitive knowledge that individuals retrieve from memory to perform the task is also part of online task specific knowledge (Efklides, 2001, 2006, 2008; Schneider, & Lockl, 2002).

Having an understanding of these metacognition facets, the next section describes the instruments that have been used for measuring metacognition.

Instruments for measuring metacognition

Two main types of metacognition measure are reported in the literature (e.g., Jacobse, & Harskamp, 2012; Schneider, & Artelt, 2010), offline and online measures. Offline measures are those that assess metacognitive knowledge without concurrent problem solving assessment, whereas online measures assess metacognitive skills and experiences during problem solving activity (Schneider, & Artelt, 2010). Both interviews (e.g., Kreutzer, Leonard, & Flavell, 1975) and questionnaires (e.g., the Motivated Strategies for Learning Questionnaire (Pintrich, & De Groot 1990), and the Metacognitive Awareness Inventory (Schraw, & Dennison, 1994)) have been used for measuring individuals' metacognitive knowledge.

Questionnaires generally include statements about metacognitive monitoring and regulation and individuals need to rate to which degree the statements apply to them.

Measuring metacognitive knowledge using a self-report questionnaire has the advantage of being easily administered. However, the results may be not accurate because of the social desirability factor (McNamara, 2011; Jacobse, & Harskamp, 2012; Schneider, & Artelt, 2010); "the basic human tendency to present oneself in the best possible light" (Fisher, 1993, p. 303), and the issue of memory distortion when recalling what has been done during a cognitive task (Jacobse, & Harskamp, 2012; McNamara, 2011).

One proven online effective method of measuring metacognition is a think-aloud protocol (Ericsson, & Simon, 1993). For using this protocol, a person's thinking, which is verbalised during working on a task, is collected. Then, it is transcribed and coded based on a scheme (e.g., Kim et al., 2013); or without transcribing, each recorded video file is coded based on a scheme (e.g., Jacobse, & Harskamp, 2012). Measuring metacognition using a think-aloud protocol is time-consuming, but it provides more reliable information than questionnaires because it is collected while the learner executes the task and it is less affected by the social desirability factor and memory distortions (Jacobse, & Harskamp, 2012; Veenman, 2011).

Knowing the three facets of metacognition (metacognitive knowledge, experiences, and skills), and instruments measuring it, it is worth comparing facets of metacognition with the RBT's metacognitive knowledge.

Discussion

Anderson, et al. (2001) acknowledge two facets of metacognition, knowledge of cognition (which relates to RBT's metacognitive knowledge), and the monitoring, control, and regulation of cognition (which relates to metacognitive skills of Efklides's (2006, 2008) metacognition framework). RBT's metacognitive knowledge as stated above is a representation of Flavell's (1979) model:

> In Flavell's (1979) classical article on metacognition, he suggested that metacognition included knowledge of strategy, task, and person variables. We have represented this general framework in our categories by including …

> strategic knowledge, …, knowledge about cognitive task, [and] self-knowledge (Anderson et al., 2001, p. 56).

Metacognitive experience as a facet of metacognition is not explicitly mentioned in the RBT handbook. However, considering the fact that metacognitive experiences are present in the working memory, it seems it is related to the cognitive process dimension of RBT. Evaluating as a cognitive process of RBT seems to be especially related to the judgement, a component of metacognitive experiences. In addition, comparing Table 2.5 and Table 3.4 shows that RBT's metacognitive knowledge covers most aspects of the metacognitive knowledge facet of Efklides's framework (2006, 2008).

3.1.5 SOLO taxonomy (or model)

In this section, the SOLO taxonomy is described, then compared with RBT. Knowing the structure of the SOLO taxonomy could help us to have a better understanding of how learning happens for students and also comparing it with RBT could help us to have a better understanding of RBT.

The SOLO taxonomy (also called the SOLO model (Pegg & Tall, 2010)) is known both as a local framework of conceptual growth, describing how learning happens for individuals when they construct a concept in mind; and as a global framework of knowledge growth, describing how knowledge develops over a long period of time (Pegg & Tall, 2010).

If SOLO is used as a local framework, Biggs and Collis (1982) claimed that understanding of a concept is constructed through a cycle of stages that can be described as prestructural, unistructural, multistructural, relational, and extended abstract. Students' responses to a task can then be categorised into one of these levels (Table 3.5).

Table 3.5

SOLO as a local framework

SOLO Levels	A description of the level
Prestructural	Response represents the use of no relevant aspect.
Unistructural	Response represents the use of one relevant aspect.
Multistructural	Response represents the use of several disjoint aspects.
Relational	Response represents the use of all aspects related into an integrated whole.
Extended abstract.	Comprehensive use of all relevant aspects together with related hypothetical constructs and abstract principles.

Note. Adapted from "Van Hiele levels and the SOLO taxonomy," by M. Jurdak, 1991, *Mathematical Education in Science and Technology, 22*(1), 57-60. Copyright 1991 by Taylor and Francis. Adapted with permission.

In terms of global framework, Biggs and Collis (1982) proposed that knowledge develops by successive modes of operation including sensori-motor, ikonic, concrete symbolic, formal, and post formal (Table 3.6). Their model is similar to Piaget's theory of cognitive development.

Table 3.6

SOLO as a global framework

SOLO Modes	Age range	A description of the mode
Sensori-motor	Soon after birth	A person reacts to the physical environment. For the very young child it is the mode in which motor skills are acquired.
Ikonic	From 2 years	A person internalises actions in the form of images. It is in this mode that the young child develops words and images that can stand for objects and events.
Concrete symbolic	From 6-7 years	A person thinks through use of a symbol system such as written language and number systems.
Formal	From 15 to 16 years	A person considers more abstract concepts. This can be described as working in terms of "principles" and "theories". Students are no longer restricted to a concrete referent.
Post formal	Possibly at around 22 years	A person is able to question or challenge the fundamental structure of theories or disciplines.

Note. Adapted from "The fundamental cycle of concept construction underlying various theoretical frameworks," by J. Pegg, & D. Tall, 2010, In B. Sriraman & L. English (Eds.), *Theories of Mathematics Education: Seeking new frontiers* (pp. 173-192). Berlin Heidelberg: Springer.

In the following, the SOLO taxonomy is compared with RBT to show the potential of RBT.

Discussion

Several studies highlighted differences between BT and SOLO (e.g., Hattie & Purdie, 1998); however, no study has been found which compared SOLO and RBT. RBT as a taxonomy can be used for analysing students' learning, and can be compared with the local aspect of SOLO. The aim of this study is not to do so in depth, however, it seems RBT provides more information when is used for exploring student learning because it is a two-dimensional taxonomy with 24 cells with the inclusion of metacognitive knowledge.

On the other hand, local aspect of the SOLO taxonomy integrates both knowledge and cognitive processes, therefore, student learning can be explored in only one dimension (within five levels). Having familiarity with SOLO taxonomy, the next section is dedicated to APOS theory, another framework which is frequently used in mathematics education.

3.1.6 APOS theory

APOS theory is introduced by Dubinsky (Dubinsky, 1991) as a constructivist learning theory for describing how mathematical concepts are constructed in the mind. It is a theory with roots in the works of Piaget that can be used for designing teaching activities and analysing students' mathematical problem solving and learning (Arnon et al., 2014). APOS is frequently used for research and curriculum development in mathematics education, especially at secondary and tertiary levels (e.g., Bayazit, 2010; Dubinsky & Wilson, 2013).

Based on APOS theory, construction of mathematical knowledge is not linear, but how a mathematical concept is constructed in the mind is assumed in a hierarchical manner (Arnon et al., 2014). Mathematical concepts cannot be understood directly, therefore, mental structures are necessary for making sense of concepts (Piaget & Garcia, 1989). According to APOS theory, mathematical concepts are learnt using four mental structures (i.e., action, processes, objects, and schemas) that are constructed by five reflective abstractions (mental mechanisms) including interiorisation, encapsulation, coordination, reversal, de-encapsulation, and thematisation (Figure 3.1) (Arnon et al., 2014).

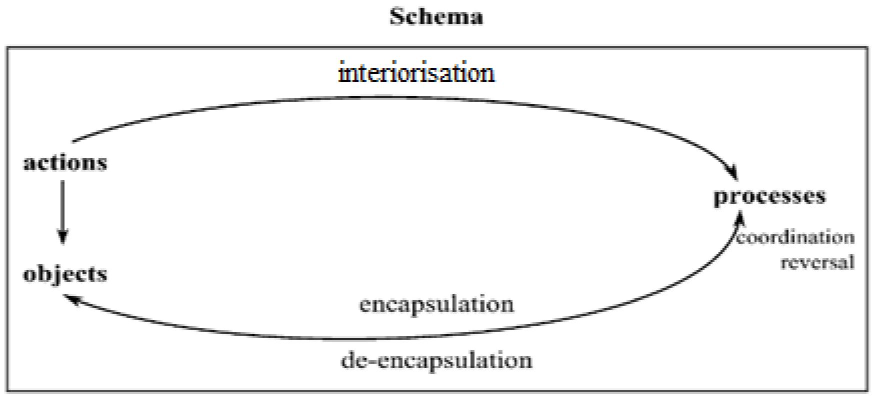


Figure 3.1 Mental structures and mechanisms of APOS theory.

Adapted from "*APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education*," by L. Arnon, J. Cottrill, E. Dubinsky, A. Oktac, S. Roa Fuentes, M. Trigueros, K. Weller, 2014. New York, USA: Springer.

Learning a mathematical concept starts with manipulating previously learnt objects (mental and physical) in order to shape actions. Through interiorisation, actions form processes. Then, objects form by encapsulation. Objects can be de-encapsulated to the processes they were originated from. Finally, schemas are the place where actions, processes, and objects can be organised (Asiala et al. 1996).

Genetic decomposition is a key part of APOS theory. It is a hypothetical model describing "the mental structures and mechanisms that a student might need to construct in order to learn a specific mathematical concept" (Arnon et al., 2014, p. 27). It can help in understanding students' difficulties in learning a concept and also be used for designing teaching activities (Arnon et al., 2014).

Discussion

The same argument that was mentioned for the SOLO taxonomy can be made for APOS theory. APOS also integrates knowledge and cognitive processes, and does not address metacognitive knowledge explicitly. Therefore, in terms of analysing students' learning, more information is able to be provided by RBT in comparison to APOS. The nature of these lenses differ; however, a comparison of these frameworks shows that RBT has potential that should be revisited by scholars in mathematics education for exploring students' learning.

Another lens which is linked to APOS theory is Garry and Tall's (1994) notion of procept (Tall, 1999). The next section is dedicated to describing this notion.

3.1.7 The notion of procept

Gray and Tall (1994) introduced another lens for exploring how mathematical ideas are developed in the mind. They provided a lens which integrates concept and process and highlights the importance of symbols in mathematical learning to represent processes or objects. For example, a symbol, $\int_1^x cost\ dt$, evokes the process of integration and also the concept of integral calculus. The cognitive combination of these three (i.e., symbol, process, and concept) can be considered as a procept in the context of integral calculus (Thomas, & Hong, 1996). Gray and Tall (1994) defined an elementary procept as the amalgam of three components including a process which creates a mathematical object, a symbol representing the process or the object. For instance, the symbol 2 + 3 evokes the process of addition or the concept of sum (Gray & Tall, 1994).

Then, they extended this notion to highlight the flexibility of mathematical thinking. "A procept consists of a collection of elementary procepts that have the same object" (Gray & Tall, 1994, p. 121). For instance, the procept 6, includes "the process of counting 6 and a collection of other representations such as 3 + 3,…,8 - 2, and so on" (p. 121) indicating "the flexible way in which 6 may be decomposed and recomposed using different processes" (p. 121).

As a sequence of defining procept, proceptual thinking is also defined by Gray & Tall (1994) as opposed to procedural thinking. "Proceptual thinking is characterised by the ability to compress stages in symbol manipulation to the point where symbols are viewed as objects that can be decomposed and recomposed in flexible ways" (p. 132).

Discussion

Comparing procept with RBT, especially the knowledge dimension, it seems that procept addresses the first three types of RBT knowledge dimension. Symbols address the first subcategory of factual knowledge, knowledge of terminology (Section 2.2.1), process relates to procedural knowledge, and object addresses the conceptual knowledge. However, procept, like SOLO taxonomy and APOS theory, does not address metacognitive knowledge explicitly.

3.1.8 Tall's theory of three worlds of mathematics

Tall (2004, 2006, and 2008) proposed the theory of the three worlds of mathematics for describing how humans learn mathematics in the long term; relevant to this study that explores student mathematical learning in the context of integral calculus. Tall introduced two terminologies, 'set-befores' and 'met-befores', for explaining his theory. He claimed that a human learns mathematics through three attributes located in a person's genes 'set-befores' birth, including:

- "recognition of patterns, similarities, and differences;
- repetition of sequences of actions until they become automatic;
- language to describe and refine the way we think about things" (Tall, 2008, p. 6).

He also claimed personal development relies on a person's interpretations of new situations using experiences that are 'met-befores' (Tall, 2008). Tall (2008) defining a met-before as "a current mental facility based on specific prior experiences of the individual" (p. 6). A met-before can be consistent or inconsistent with new situations. For example, for a finite set, when some elements are taken, the cardinality of the set is reduced. However, for infinite sets (e.g., natural number), removing odd numbers does not affect the cardinality of the set. The effect of negative met-befores is sometimes neglected and causes major difficulties for some students (Tall, 2008).

Tall's theory of three worlds of mathematics (2004, 2006, and 2008) refers to three interrelated blended sequences of development of mathematical thinking that are constructed by the three set-befores (i.e., recognition, repetition, and language). These sequences worlds are conceptual-embodied, proceptual-symbolic, and axiomatic-formal worlds.

The first world is “based on perception of and reflection on properties of objects, initially seen and sensed in the real world but then imagined in the mind” (Tall, 2008, p.7). The second world, the proceptual-symbolic world, is developed on the conceptual-embodied world using action. It is symbolised as thinkable concepts “that function both as processes to do and concepts to think about (procepts)” (Tall, 2008, p.7). The third world, the axiomatic-formal world, which is based on formal definitions and proofs, “reverses the sequence of construction of meaning from definitions based on known objects to formal concepts based on set-theoretic definitions” (Tall, 2008, p.7).

Discussion

If the three ‘set-befores’ descried in Tall’s theory being compared with RBT (which has not been done in the literature), the first set-before seems to relate to remembering knowledge, because recognition relates to the first subcategory of remembering, recognising. Patterns, similarities and differences can be related to any type of knowledge, therefore the first set-before addresses the first column of RBT Table (Table 2.1). In terms of the second set-before, repetition seems to be related to executing, the first subcategory of applying. Actions seems to relate to procedural knowledge, therefore the second set-before, repetition of sequences of actions until they become automatic, might address applying procedural knowledge cell. Regarding the third set-before, as mentioned earlier, language relates to knowledge of terminology, the first subtype of factual knowledge. Describe might be related to the second RBT’s cognitive process, understanding, particularly, the last subcategory, explaining. However, refine can encompass higher cognitive processes, especially evaluating. Therefore, the last set-before possibly addresses four cells including understanding, applying, analysing, and evaluating factual knowledge cells. Therefore, all set-befores can be located in the RBT Table, showing the potential of

RBT for use in mathematics education, and how it is in line with other theories in mathematics education.

In terms of the three words, considering their definition and RBT's structure, the first and the third world, conceptual-embodied world and axiomatic-formal world, seems to be related to conceptual knowledge, whereas, the second world, the proceptual-symbolic world, seems to address the RBT factual, conceptual, and procedural knowledge. However, this theory does not explicitly mention how metacognitive knowledge can be developed over time.

3.1.9 Schoenfeld's frameworks of the analysis of mathematical problem-solving behaviour and decision making in teaching

Schoenfeld (e.g., 1985, 1987, 1992, 2010) has made a major contribution to how student mathematical problem solving can be studied, claiming four categories are "necessary and sufficient for understanding problem-solving success or failure" (Schoenfeld, 2010, p. 4). The four categories for exploring include:

- what mathematical knowledge do students know?
- what problem-solving strategies do students have for solving unfamiliar problems?
- how do students monitor and regulate resources (e.g., time) for problem-solving? and
- what beliefs do student have about mathematics, context, themselves, etc (Schoenfeld, 2010).

Schoenfeld (2010) argued mathematical knowledge is essential for problem-solving. There are occasions where the knowledge is the "make-or-break factor" (p. 4). In unfamiliar problems, using heuristic strategies sometimes make the solution reachable. How much time and effort should be spent on a problem is also a key variable for being a successful problem solver. Personal beliefs sometimes lead students to avoid or attack problems, and also influence their success in problem solving (Schoenfeld, 2010).

Another major contribution of Schoenfeld's research is his framework (2010) for decision making in teaching. Schoenfeld (2014) highlighted:

> people's moment-by-moment decision making in teaching…can be modeled as a function of their resources (especially their knowledge, but also the tools at their disposal), orientations (a generalization of beliefs, including values and preferences), and goals (which are often chosen on the basis of orientations and available resources) (p. 406).

Discussion

Schoenfeld's frameworks seem to be in line with RBT. In relation to the framework for analysing problem solving, the first category fits with the RBT's factual, conceptual, and procedural knowledge. The second addresses strategic knowledge, the first subcategory of RBT's metacognitive knowledge. The third category relates to metacognitive experiences and skills, and the last relates to self-knowledge, the third subcategory of RBT's metacognitive knowledge.

Regarding Schoenfeld's framework (2010) for decision making in teaching, resources related to the RBT's knowledge dimension, especially the factual, conceptual, and procedural knowledge, and orientation and goal address the third RBT subtype of metacognitive knowledge, self-knowledge.

After presenting major theories and frameworks that influence teaching and learning of mathematics and how they are addressed in RBT, the next section is dedicated to describing studies that focus on the teaching and learning of integral calculus.

3.2 Teaching and learning of integral calculus

This section reviews the literature in relation to the teaching and learning of integral calculus in two streams. The first comprises the definite integral, the Riemann integral, and the area under curve(s), and the second focusses on the FTC. The first stream is chosen because it is being taught at both Secondary and tertiary level. The FTC is chosen because it is an important theorem that connects the definite and indefinite integral (Section 3.2.1) and there is a lack of research about it.

3.2.1 The definite integral, Riemann sums, and area under curve(s)

In this section, first the importance of Riemann sums in integral calculus is presented. Then, students' difficulties with this topic reported in the literature are described to be used for justifying the study's findings. Teaching activities that are suggested in the literature for teaching integral calculus are also described in this section.

The concept of Riemann sum, $\sum_{i=1}^{n} f(c_i)\Delta x_i$, is an important part of integral calculus for at least three reasons. Firstly, for functions where the antiderivative cannot be expressed in terms of elementary functions (e.g.,$\int_2^5 \frac{1}{lnx} dx$), numerical integration methods should be used for finding the integral, Riemann sums is one of the approaches that can be used. Secondly, structures of other numerical methods (e.g., trapezoid rule) are based on Riemann sums; therefore, if students understand Riemann sums, their understanding facilitates learning of other methods. Thirdly, understanding Riemann sums helps students with solving definite integral problems by knowing what to integrate and how to set up the bounds of integral (Sealey, 2006, 2014).

A number of important concepts are involved in Riemann sums and definite integral, $\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$, including series, functions, limits, rate of change, and multiplication (Sealey, 2006). Students' difficulties with understanding the definite integral as the limit of a sum are highlighted in the literature (Orton, 1983; Grundmeier, Hansen, & Sousa, 2006). For instance, Orton (1983) found that even the best student in his sample had difficulty with solving problems related to understanding the definite integral as the limit of a sum.

To achieve a better insight about how students might develop the concept of Riemann integral, a framework for characterising students' understanding of Riemann sums and the definite integral is proposed (Sealey, 2014). The framework has a pre-layer, and four layers (Table 3.7). Using the framework, Sealey (2014) found that layer 1, the product of $f(x)$ and Δx, is the most complex part of problem-solving for students. "Difficulties in this layer are not necessarily related to the operation of multiplication and performing calculations, but are typically related to understanding how the product is formed and understanding how to use each factor within the product" (Sealey, 2014, p. 238).

Table 3.7

Riemann integral framework

Layer	Symbolic representation
Pre-layer	$\left[\frac{1}{c}.f(x_i)\right]$ and or $[c.\Delta x]$
Layer 1: Product	$\left[\frac{1}{c}.f(x_i)\right].[c.\Delta x]$
Layer 2: Summation	$\sum_{i=1}^{n} f(x_i)\Delta x$
Layer 3: Limit	$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$
Layer 4: Function	$f(b) = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$

Several studies have explored the relationship between definite integral and area under curve(s). These studies reported many students are able to do routine procedures for finding area using integral techniques, however, their understanding about why such a procedure should be performed is limited (e.g., Artigue, 1991, Thomas & Hong, 1996). Students' difficulties are reported when the function is below the x-axis (Orton, 1983) or absolute values are involved in the integrand (Mundy, 1984) (Section 1.1). In addition, Kiat (2005) found that if the graph of the integrand is not given to students, students can fail to set up integrals correctly for finding the area, and that shows their understanding of definite integral is procedural and that they cannot make the connection between definite integral and area. In detail, Kiat (2005) reported 80% of students failed to realise that, for finding the area enclosed between $y = x(x-4)$ and the x-axis from $x = 0$ to $x = 5$, the integral should be split at $x = 4$. Difficulties in setting up the right integral are also reported when the graphs of the curves are given in the questions. Kiat (2005) found that 55% of students could not set up the correct integrals for finding the shaded area in a question where one of the curves is above and one below the x-axis (Kiat, 2005).

Mahir (2009) explored conceptual and procedural knowledge of undergraduate students in integral calculus (Section 3.1.3, measuring conceptual and procedural knowledge in integral calculus). He found that students developed a satisfactory level of procedural knowledge in this topic. In detail, 92 % and 74% of students answered correctly two questions that can be solved using procedural knowledge in integral calculus. However, this study reported a lack of students' conceptual knowledge in integral calculus. In detail, in three questions that can be solved using conceptual knowledge in integral calculus, only 8%, 16%, and 24% of students were able to use their conceptual knowledge for solving the questions. In addition, Rasslan and Tall (2002) reported students' difficulties in understanding the definite integral as an area under curve for piecewise defined functions and improper integrals.

In terms of symbols and notations of integral calculus, "dx" in $\int f(x)dx$ causes conflicts and contradictions for a number of students. The reason is, some students are told not to cancel dx in $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ when solving questions related to chain rule as it has no separate meaning. However, in integral calculus dx has a different meaning, showing integration should be done with respect to x (Tall, 1992).

In relation to the definition of definite integral, Grundmeier, Hansen, and Sousa (2006) reported students had great difficulty with learning the symbolic definition of a definite integral. Within the sample of the study consisting of 52 students, only one student was able to provide the symbolic definition of the definite integral and only 35% were able to provide a correct verbal definition. However, the study findings showed that students' knowledge of the definition of a definite integral did not affect students' ability to do routine integral problems. In the sample of the study, more than 60% of students were able to evaluate the definite integral of a trigonometric function. Such findings are supported by others (e.g., Rasslan, & Tall, 2002) that a majority of students are not able to write meaningfully about the definition of definite integral. One reason is related to the learning and teaching approach of some teachers and lecturers that focuses on the procedural aspect of calculus (Bezuidenhout, 2001).

Teaching integral calculus

Researchers have provided suggestions for teaching integral calculus based on their study findings. For instance, Orton (1983) suggested the focus of integral calculus teaching should be on determining the enclosed area as a limit of a sum rather than integration techniques for solving different types of integrals. Providing diagrams and graphs as much as possible is also recommended in order that students have a better understanding of the relationship between definite integral and area (Orton, 1983). Integrating technology to teach integral calculus might also be helpful for focusing on conceptual ideas (Thomas & Hong, 1996; Hong & Thomas, 2015). Kiat (2005) suggested asking students to compare differentiation and integration techniques because he found that many students have confusion about these two processes. He also highlighted that many students make technical errors during solving integral problems; therefore, he suggested remedial lessons and revision worksheets related to prerequisite knowledge for integral calculus for preparing students for this topic. Another suggestion about teaching integral calculus, especially at secondary school, is related to teaching limits. Limit is not in the focus of secondary school curriculum in several countries (e.g., Singapore (Kiat, 2005)) whereas it is necessary for understanding the definition of the definite integral; therefore, it is suggested that more attention be paid to this topic in order for students to have a better understanding of integral calculus (Kiat, 2005).

3.2.2 The Fundamental Theorem of Calculus

FTC is an important part of integral calculus because it connects the definite and indefinite integral together and provides an efficient method for evaluating definite integrals using anti-derivatives (Anton et al., 2012). It expresses the relationship between the accumulation of a quantity and the rate-of-change of the accumulation (Thompson, 1994). It is recognised as one of the intellectual hallmarks in the development of calculus (Carlson, Persson, & Smith, 2003).

The FTC has two parts. The first part says, "if f is continuous on [a,b] and F is any antiderivative of f on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$ (Anton et al., 2012, p. 363)". The second part states

> If f is continuous on an interval, then f has an antiderivative on that interval. In particular, if a is any point in the interval, then the function F defined by $F(x) = \int_a^x f(t)dt$ is an antiderivative of f; that is, $F'(x) = f(x)$ for each x in the interval, or in an alternative notation $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$" (Anton et al., 2012, p. 370).

For understanding the FTC, encapsulating both differentiation and integration seem to be required (Thomas & Hong, 1996). Literature shows that a number of students could apply the FTC for finding the definite integral; however, they do not know why the FTC provides the results (Orton, 1983).

Student difficulties with the FTC are reported to be related to students' understanding of function (e.g., Thompson, 1994), limits (e.g., Thompson & Silverman, 2008), rate of change (Thompson, 1994), and the notational aspect of the accumulation function (Thompson & Silverman, 2008). Several studies reported that both undergraduate and high school students have difficulties with understanding limits (e.g., Tall, 1992; Tall, Vinner, 1981), which might prevent students from understanding the FTC (Thompson & Silverman, 2008). From the notational aspect, the role of t in $\int_a^x f(t)dt$ is confusing for a number of students (Thompson & Silverman, 2008).

The concept of accumulation function in the FTC, represented by $F(x) = \int_a^x f(t)dt$, involves several parts that make it hard for some students to understand it (Thompson & Silverman, 2008). First, students need to understand $f(t)$ is a number depending on the value of t. Second, they need to have a covariational understanding (Carlson et al., 2003; Carlson, Larsen, & Jacobs, 2001) of the relationship between t and f, which means understanding that as the value of t changes from $[a, x]$, the value of $f(t)$ varies accordingly. The third step is understanding the bounded area accumulating, as t and $f(t)$ vary, and these values are changing in tandem (Thompson & Silverman, 2008).

Teaching the Fundamental Theorem of Calculus

The literature suggested some changes in the focus of teaching integral calculus. Thompson & Silverman (2008) highlighted that the focus should be on the accumulation function rather than the traditional emphasis on finding a number representing the area enclosed by curves over an interval: "This paper presents a call for increased emphasis on the FTC as explicating an inherent relationship between accumulation of quantities in bits and the rate at which an incremental bit accumulates" (p. 51). To make this happen, the idea of covariation should also be considered in teaching calculus (Thompson & Silverman, 2008). In addition, to achieve a better understanding of the role of the accumulation function in the FTC, constructing, representing, and understanding Riemann sums should also be in focus (Thompson & Silverman, 2008).

A teaching experiment was conducted to help first year students to understand the accumulation function and the FTC (Carlson et al., 2003). In this design, the culmination of the course was understanding the FTC. The course started with reviewing function and from the start of the course, the concept of covariation of quantities was the focus and that reasoning was used for teaching other topics (e.g., rate of change, limits). Their study findings showed a high success rate in terms of understanding the concept of accumulation function and the FTC.

Discussion

Previous studies in calculus that used RBT as a lens examined matching functions and graphs (Green, 2010) and functions, limits, and derivatives (Radmehr & Alamolhodaei, 2010, 2012). Therefore, analysing students' performance in integral calculus based on RBT is potentially useful and could reveal further information about students' difficulties in this topic. In addition, previous studies that explored students' mathematical problem solving based on RBT were quantitative while the current study used a mixed approach (both quantitative and qualitative). In addition, there is a lack of research about students' metacognitive knowledge, experiences, and skills in the context of integral calculus that makes this study novel.

3.3 The secondary-tertiary transition in mathematics

Integral calculus is part of the upper secondary school and undergraduate university curriculum; therefore, transition effects between these two levels should be considered in this study. The secondary-tertiary transition is considered as a "rite of passage" (Clark & Lovric, 2008, 2009), with three phases including separation from high school, liminal (transition from high school to university), and incorporation into university. Separation happens during the time students are still in secondary school and expect to be going to university. The liminal phase includes to "the end of high school, the time between high school and university, and the start of first year at a university" (Clark & Lovric, 2009, p. 756). The final phase, incorporation, includes the first year at university (Clark & Lovric, 2009).

The secondary-tertiary transition in mathematics is a complex issue and serious concerns about teaching and learning in these two levels have been reported (e.g., Hourigan & Donoghue, 2007; Kajander & Lovric, 2005; London Mathematical Society, 1995; Luk, 2005; Tall, 1997). These concerns have been classified as epistemological and cognitive, sociological and cultural, and didactical (De Guzmán, Hodgson, Robert, and Villani, 1998). These classifications were derived from different discussions during the International Congress on Mathematical Education (ICME) conferences and from administering questionnaires to undergraduate students about their difficulties with university mathematics (De Guzmán, Hodgson, Robert, and Villani, 1998). Those most pertinent to this study are the cognitive and didactical aspects because RBT is particularly powerful for examining cognitive and didactical aspects of teaching and learning.

Regarding the cognitive aspect (which is related to concepts in the discipline and how individuals approach it) of the secondary-tertiary transition, students may face difficulties in university since mathematical concepts are presented in more depth and problems require more technical and conceptual understanding in comparison to secondary level teaching (De Guzmán et al., 1998). Moreover, facing formal deductive approaches like proving theorems in the first year in university is another problem for students at this transition (London Mathematical Society, 1995; Tall 1997).

In relation to the didactical aspect (which is related to teaching methods and the performance of lecturers and teachers) of the transition, there are concerns such as lack of innovation in teaching methods, feedback procedures, and designing assessment in universities. In addition, lecturers in the study were found not to have strong pedagogical and didactical abilities and paid less attention to course design than teachers (De Guzmán et al., 1998).

In New Zealand, a comprehensive two-year study about the transition from secondary to tertiary mathematics took place (Hong et al., 2009; Thomas et al., 2010). The results of the study are similar to previous research in some aspects. For instance, not all of the students were affected in the same manner through the transition, and there were several differences between teaching mathematics in schools and universities in term of emphasis and style. However, other aspects of this research may be specific to New Zealand. For example, Thomas et al. (2010) reported that there was a misalignment between the curriculum at secondary and tertiary level in some areas such as the omission of series and vectors from NCEA. This led to problems for students whose studies were based on NCEA in that when they enter tertiary education they have not been introduced to these concepts. However, the study claimed that students who studied under the Cambridge Curriculum (The Association of Cambridge Schools in New Zealand, 2011) made the transition more easily. In addition, communication between teachers and lecturers is important for understanding issues involved in each side of the transition, but it was not focused on by universities and schools up to the time they conducted their study (Hong et al., 2009).

Discussion

In terms of calculus, the cognitive secondary-tertiary transition has been studied for functions (Godfrey & Thomas, 2008) and derivatives (Biza & Zachariades, 2010) but there is a lack of research in this regard for integral calculus. Therefore, exploring students' learning of integral calculus based on RBT in Year 13 and University level will provide new insight into cognitive aspects of the secondary-tertiary transition in integral calculus.

3.4 Chapter summary

This chapter discussed the potential of RBT by comparing it with learning theories and frameworks that are frequently used in mathematics education. In addition, it describes the literature about the teaching and learning of integral calculus and the secondary-tertiary transition in mathematics to show the potentials of RBT for providing further insight into student learning in these topics, and also for being used to interpret the study findings.

Chapter Four: Study Design

In this chapter, the research paradigm and the methodology of the study are described and justified. Explanations about the study sample and the process of data collection are also presented. The main data gathering instruments and how they have been designed are described in Chapter Five. The research questions are restated here and the approach taken to answering them is presented.

1. What examples of factual, conceptual, procedural, and metacognitive knowledge in integral calculus based on RBT can be found in Year 13 and first year university?

2. Using RBT as a lens, what are students' difficulties in solving integral questions in Year 13 and first year university?

3. What metacognitive *knowledge*, *experiences*, and *skills* do students hold about integral calculus in Year 13 and first year university?

4. What differences exist between student learning of integral calculus in Year 13 and first year university?

5. What are the perceptions of lecturers and teachers towards students' difficulties in integral calculus?

4.1 Research paradigm

A Research paradigm can be defined as "a general philosophical orientation about the world and the nature of research that a researcher brings to a study" (Creswell, 2014, p. 6) that influences the way knowledge is studied, interpreted, and obtained (Mackenzie, & Knipe, 2006). The research paradigm addresses ontological, epistemological, and methodological matters (Punch, 2013). It describes the nature of reality (ontology), the relationship between the researcher and the reality (epistemology), and indicates methods suitable for studying the reality (methodology) (Punch, 2013).

In a post-positivist world there is an increasing variety of paradigms (e.g., post-positivism, social-constructivism, feminism, social realism, transformative, and pragmatism) (Blaikie, 2009;

Creswell, 2014; Teddlie & Tashakkori, 2009), therefore a pragmatic approach has been chosen for the study. Such an approach allows researchers to use both quantitative and qualitative approaches for answering the research question(s). Pragmatism considers that knowledge can be obtained in a social, historical, and political context. Pragmatists are focusing on research questions, and design data collection and analysis, in a way that provides insights into the problem without any dependency on alternative paradigms (Creswell, 2014; Mackenzie, & Knipe, 2006; Morgan, 2007).

In relation to investigating the students' learning of integral calculus using RBT, both qualitative (i.e., interpretive) and quantitative (i.e., post-positivism) approaches are used. A qualitative approach was used because it relies on understanding a phenomenon from the participants' point of view in the context in which it happens (Stake, 2006), and is found on a belief that knowledge is subjective and socially embedded (Cohen, Manion, & Morrison, 2011). Considering the fact that some aspects of metacognitive knowledge (e.g., self-knowledge) are subjective, using a qualitative approach is necessary to explore students' metacognitive knowledge in integral calculus. It is also productive for exploring how students perform on tasks whose design is based on RBT at a micro level. Quantitative approaches are useful for comparing students' results from university and Year 13 at a macro level. They are also useful for answering the fourth research question, by using statistical inference tests to explore whether there is any significant difference between the performance and knowledge of students at these respective levels. Table 4.1 provides rationales for rejecting the other main paradigms (i.e., post-positivism, interpretive, and transformative) (Creswell, 2014) for this study.

Table 4.1

Rationale for rejecting the main paradigms other than pragmatism for this study

Paradigm	The main reason for rejecting the paradigm for the study
Post-positivism	The study was not only committed to comparing students' performance in integral calculus questions based on RBT in general. The performance of each student was analysed in depth (micro focus).
Interpretive	The study not only relied on qualitative analysis, but a comparison of students' performances in general between university and Year 13 students was also considered (macro focus).
Transformative	The research questions were not related to social justice and marginalised people and did not include an action agenda for changing the lives of participants.

The next section describes the research methodology undertaken based on the chosen paradigm.

4.2 Research methodology

Having the pragmatic paradigm in mind, a mixed method study, predominantly qualitative, was designed within the case study approach (Yin, 2012, 2014) as a research methodology. In this section, the case study approach and the rationales for using it are justified.

Case study is a research methodology which "investigates a contemporary phenomenon (the "Case") in its real-world context, especially when the boundaries between phenomenon and context may not be clearly evident" (Yin, 2014, p. 2). Case study methodology is suitable when the research questions are descriptive, explanatory, or exploratory and the goal of the study is to emphasise the study of a phenomenon within its real-world context (Baxter & Jack, 2008; Yin, 2012). Case study is chosen because of the nature of the research questions and the aim of the study. Three main steps of case study design are: defining the case; selecting one of four types of case study design (i.e., single holistic, single embedded, multiple holistic, and multiple embedded design); and deciding whether to use theory to inform methodological steps such as case selection, data collection, data analysis, etc. (Yazan, 2015).

A single case study is suggested for five situations (Yin, 2014), including: critically testing a theory; explanation of the conditions and circumstances of an everyday situation; an extreme or unusual case; a longitudinal case, and when the researcher has access to a case that was previously inaccessible to research (Yin, 2014). Multiple case studies are used when there are two or more cases can be selected for the study, considering the time and expense involved, they are "believed to be literal replications, such as a set of cases with exemplary outcomes in relation to some evaluation questions" (Yin, 2014, p. 62). If there is more than one level of analysis in a case, then an embedded Case study should be used instead of a holistic approach (Yin, 2014). In this study, a case was defined as an educational institution in New Zealand (i.e., University or College) in which integral calculus was taught in 2014. The two selected Cases consist of a sample of students and lecturers/teachers who were interviewed and the teaching of integral calculus was observed and recorded within the samples (Figure 4.1). Information about study participants is presented in method section (Section 4.3.2).

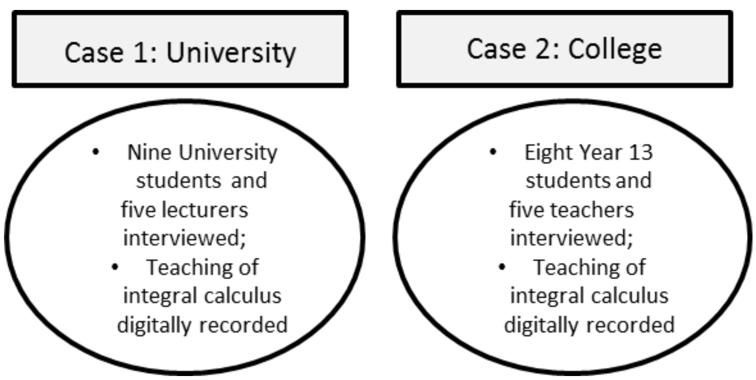


Figure 4.1 Structure of the study Cases

In terms of case study design, embedded multiple-case studies were conducted, with the goal of understanding how students learn integral calculus and what are their difficulties in learning this topic at Year 13 and first year university at their respective institutions. The reason for having such design was that university and Year 13 students might have different learning experiences of integral calculus, and lecturers/teachers might have different opinions toward students' difficulties in learning integral calculus.

In terms of analysis, the study is a comparative case study (Kaarbo & Beasley, 1999) in the sense that a comparison is conducted between students' learning of integral calculus at University

(Case 1) and College (Case 2). In addition to these two Cases, five lecturers from another University were interviewed to explore students' difficulties in learning integral calculus, however, as no students were interviewed from the second University, these five lecturers were not considered as a Case.

As this research is exploratory, emerging theory is intended to be derived from an analysis of the data collected rather than existing beforehand. As such, grounded theory (Charmaz, 2006) has been chosen as an approach to the data analysis because of its usefulness for constructing explanations of complex phenomena, students' learning of integral calculus, and its being frequently used in mathematics education research (e.g., Frejd, 2013; Johnson, 2015; Roh, 2010; Triantafillou, & Potari, 2010). For example, Roh (2010) categorised students based on their way of using a counting process by abstracting interviewees' responses, gestures, language expressions, and diagrams (Roh, 2010). Similarly, in this study, students' responses to the interview questions were used for generating theory about the ways students learn integral calculus.

Grounded theory provides a guideline for identifying, refining, and integrating categories within a set of data, and shows researchers how to link them to create a theory (Willig, 2013). When using grounded theory for data analysis, the goal of a researcher is to generate theory from data (Willig, 2013). Using grounded theory for data analysis influenced the research method selected for the study, of using a semi-structured one-to-one interview with students to elicit data about students' learning of integral calculus.

Key aspects of grounded theory data analysis consist of different processes, including memo-writing, coding, constant comparative analysis, and theoretical saturation (Charmaz, 2006; Willig, 2013). Memo-writing was done throughout the process of data collection and analysis to keep a written record of theory development. By using coding, including initial and focused coding, (Charmaz, 2006) categories that emerge from data were identified. Constant comparative analysis, defined as "inductive processes of comparing data with data, data with category, category with category, and category with concept" (Charmaz, 2006, p. 187) were done to find out similarities and differences between emerging categories and theory. Finally, data analysis continued until theoretical saturation was reached, and coding was continued until no new categories could be recognised. Due to the importance of coding in qualitative research, more explanation about how coding was done in the study is presented below.

The first step of moving from data to generating theory is coding (Charmaz, 2006). Coding refers to "categorising segments of data with a short name that simultaneously summarises and accounts for each piece of data" (p. 43). Coding shows how the data was selected, separated, and sorted. Two main phases of grounded theory coding are initial and focused coding (Charmaz, 2006). In the initial phase, words, lines, and segments of data are labelled and the codes are considered "provisional, comparative, and grounded in data" (p. 48). After constructing direction through initial coding, focused coding should be done, which is "using the most significant/and or frequent earlier codes to sift through the large amount of data" (p. 57).

In relation to inductive or deductive approaches, the coding in relation to student learning was considered inductive, and defined as moving from data to theory, rather than testing a theory with a data set, which is defined as a deductive approach (Blackstone, 2012). However, considering the study findings (Chapter Seven to Nine) at a macro level, the results show the degree of the effectiveness of using the combination of RBT and Efklides's metacognition framework for exploring students learning. This aspect of the analysis related to deductive approach towards reasoning. An example of how coding was done at a micro level through the data is presented below.

The following (Figure 4.2) shows three responses from students in relation to their orientation toward taking the calculus course (Section 8.4.2). All three responses were identified as being from students who like calculus regardless of the fact that this course was useful for their further study. In the figure, the clouds are the initial codes for these segments and the focused code that was chosen for them is *like*.

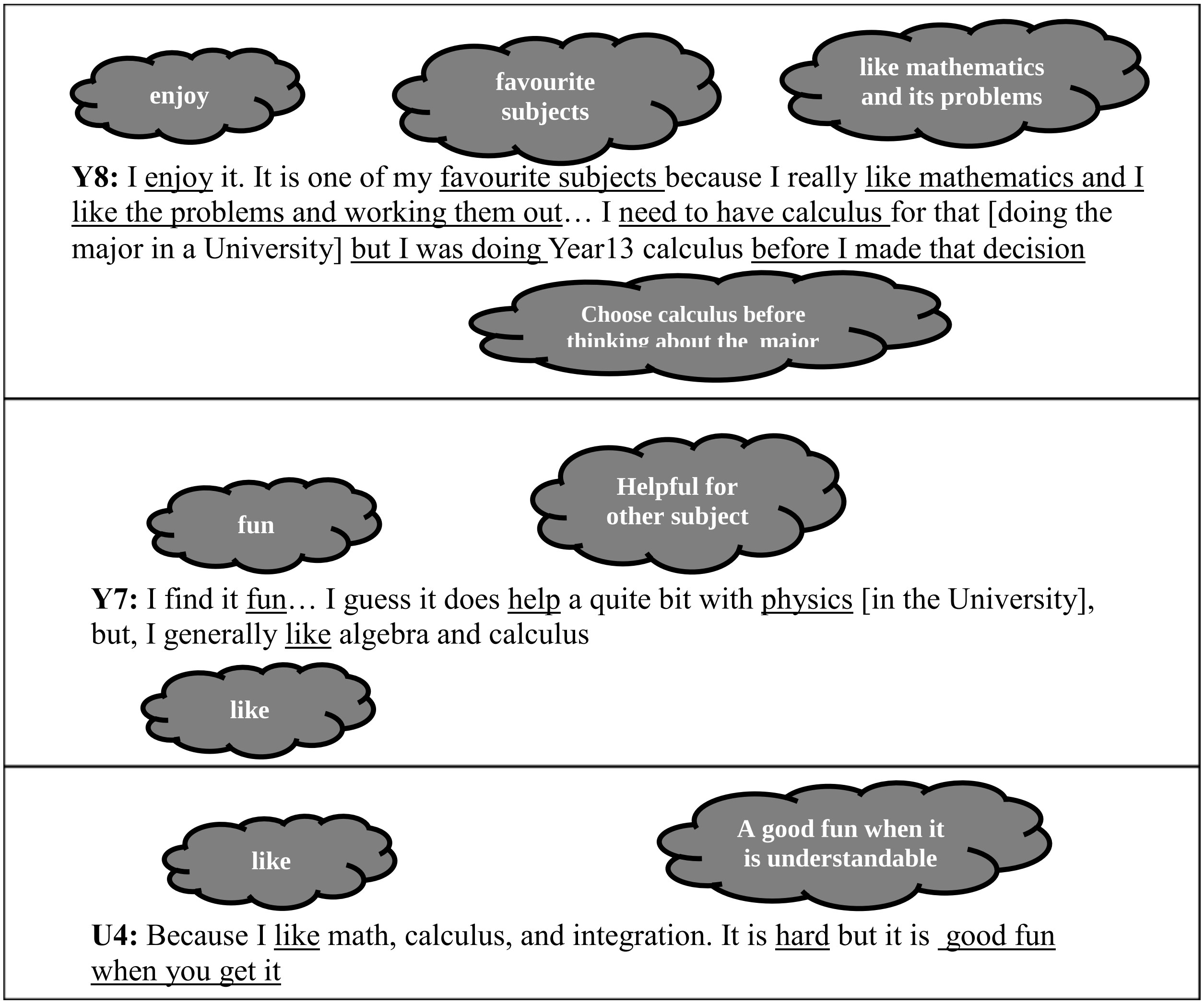


Figure 4.2 Example of how the coding was done for the data

In detail, first, the segments were read and the initial codes were assigned. Then, by comparing the students' responses, and considering the fact that all of them, used the word *like* for expressing their reason for taking the calculus course, the focused code for these segments was chosen as like.

4.3 Method

In the following section the method of the study is described, including describing how the data were collected, how participants were recruited, and how the reliability, validity, and ethical considerations were considered in the study.

The data collection had four Stages (Table 4.2). In Stage One, Tool development, the instruments of the study were designed (Chapter Five) and trialled. Then, undergraduate mathematics lectures and Year 13 teachers (Stage Two: Lecturers/teachers interview), University and College students (Stage Three: Students interview) were interviewed in 2013-2014 academic year using the instruments. The entire teaching of integral calculus in a course in a University in New Zealand and a College in Wellington were observed and digitally recorded (Stage Four: Video recording and observing).

Table 4.2

Stages of data collection

Stages	Data collection	Participants	Purpose
Stage One: Tool development (qualitative)	Documents: (The *New Zealand Curriculum*, NCEA level 3 mathematics achievement standards, Year 13 and university Calculus text books and revision books, summative assessment in school and university, Senior subject guide on TKI website, past exam on integral calculus).	The researcher: designing the instruments	To design the interviews for next stages of the study.
Stage Two: Lecturers/teachers interview (mixed methods, mainly qualitative)	Interviews: semi-structured-audio recording.	The researcher: interviewing mathematics undergraduate lecturers (10 participants) and Year 13 mathematics teachers (5 participants)	To explore lecturers and teachers perspective about students' learning of integral calculus
Stage Three: Students interview (mixed methods, mainly qualitative)	Interviews: semi-structured- audio & video recording.	The researcher: interviewing University students (9 students) and College students (8 students)	To explore students' learning of integral calculus.
Stage Four: Video recording and observing (mixed methods, mainly qualitative)	Video recording classrooms, lectures, and tutorials.	The researcher: observing and memo-writing, first year undergraduate calculus students and their lecturers plus two tutors, and Year 13 calculus students and their mathematics teacher (one regular class of a College).	To have a better understanding of how the topics are taught at the courses.

4.3.1 General structure of the interviews

In this section, how the interviews were conducted and trailed is described. The details of how the students' interview questions were designed are provided in Section 5.2. One-to-one semi-structured interviews (DiCicco-Bloom & Crabtree, 2006) are the main source of data collection in this study. The interviews were semi-structured in a sense that the wording of the questions was developed in advance, but further questions (probes) were used depending on how clear and complete the interviewer judged the interviewee's response to be. Interview research "is distinctive in its reliance on direct, usually immediate, interaction between the researcher and participant" (Salmons, 2015, p. 1). In qualitative research, interviewing is one of the most popular methods for generating data (King, & Horrocks, 2010) and is frequently used in grounded theory research in mathematics education (e.g., Johnson, 2015). In interviews, questions can be explained more thoroughly and questions without any responses are minimised (Snowball & Willis, 2011). However, collecting data using interviews is more time and energy consuming than using questionnaires and gives less time for participants to think about the questions in some instances (Snowball & Willis, 2011).

For trialing the interview questions, they were discussed with the PhD supervisors in several sessions and their comments were considered. For instance, the number of questions and how they should be asked was changed during these consultations. In some occasions, the supervisors considered themselves as teachers and have tried to help the researcher to understand how the question should be reworded. In addition, instead of using one question for each cell of RBT, questions were designed in a way that several cells and types of knowledge were assessed in each question. Then, a PhD student in mathematics who had experiences of teaching calculus answered all questions (lecturer, teacher, and student interview) and her feedback was also considered for the final version. The first University and College student who participated in the interview were used to trail the tools. However, as no major problem was found with the interview questions (i.e., the questions were understandable for students and they had no difficulty with interpreting them), these students' responses were considered part of the study data. In the following sections, the interview questions are introduced.

Lecturers' and teachers' interview

Interviews were begun with general and open questioning about the way lecturers and teachers teach integral calculus in their classes, in order to put interviewees at ease and not feel as if their knowledge is being tested in interviews (Levenson, 2012). The opening question was: *Could you please describe a typical day of teaching integral calculus in your class/lecture in relation to finding the area enclosed between curves and the Fundamental Theorem of Calculus?* The interviewer did not critique or probe further their response to this question, instead, after their response, started asking the interview questions. Then, the interview questions became more direct and specific. The interviewees were asked 24 questions during a one to three hour interview, however, only one is presented in the thesis as it being central to the focus of the study: *What are students' main difficulties toward finding the area enclosed between curves? How about the Fundamental Theorem of Calculus?*

Student interviews

RBT cells represent different types of intended student learning (Anderson, 2005, Chapter Two). Analysing students' learning in relation to all 24 RBT cells required a number of questions and consequently could take several hours of students' time, so, this was not undertaken. When working with 24 cells is not practical, Anderson (2005) suggests students' learning can be analysed in relation to RBT's rows or columns:

> Organising the current research in terms of the cells of the Taxonomy Table (or, alternatively, the row or columns if the cells prove to be too specific) may provide a level of understanding of the research which has not been possible to this point in time (p. 110).

Student learning of the integral-area relationship and the Fundamental Theorem of Calculus (FTC) were explored using 23 questions (e.g., Appendix 1[2]). The first nine questions comprised integral-area and FTC questions to explore factual, conceptual, procedural knowledge,

[2] In Appendix 1, the University version is placed. In the Year 13 version, in the second part, after the fundamental theorem of calculus, the word "definite integral " were placed for students who do not know the name of this theorem. In addition, in the first question of part two, instead of lecturer/tutor/instructor, the word "teacher" was used.

metacognitive skills, and metacognitive experiences. The remaining 14 explored students' metacognitive knowledge (Figure 4.3).

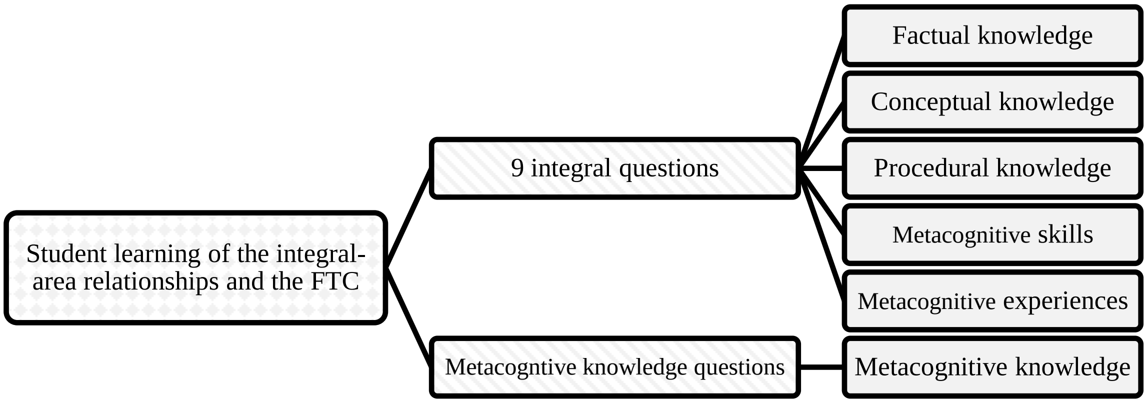


Figure 4.3 Structure of interview questions with students

4.3.2 Study participants

The following section describes information about the study participants and how they were selected. It includes teachers, lecturers, and students who were interviewed. Their classes were observed and video recorded. Reasons for the methods of recruiting participants are given.

Interview participants

In the following paragraph, information about the undergraduate mathematics lecturers and Year 13 mathematics teachers who were interviewed is presented first, and then information about the students is provided.

Lecturers and teachers

For the selection of lecturers and teachers, theoretical sampling was used, meaning the sample is "suitable for illuminating and extending relationships and logic among constructs" (Eisenhardt, & Graebner, 2007, p. 27). The criteria for being chosen for the interview were: having the experience of teaching calculus for at least three years; and being interested in knowing more about the learning and teaching of integral calculus. These criteria were chosen to provide a higher probability that participants would have an established understanding of students' difficulties in the context of integral calculus and be aware of restrictions in the teaching and learning of integral calculus. The researcher cannot provide further information about his relationship with participants in the Cases as it may identify the Cases in New Zealand. However, the researcher can confirm he was not a lecturer or teacher in these two Cases.

Ten undergraduate mathematics lecturers from two universities in New Zealand and five Year 13 mathematics teachers from a College in Wellington, New Zealand, were interviewed. The reason for having at least five participants from each group of lecturers and teachers was that for building theory from data, four to ten participants are reported to be enough for driving theory from data (Eisenhardt, 1989). These two Universities are among the top five universities in New Zealand (QS World University Rankings, 2014) and regularly offer calculus courses that include integral calculus to first year university students. The College is one of 11 Colleges in Wellington (Te Kete Ipurangi, 2014) which offered calculus courses to Year 12 and Year 13 students. Lecturers and Year 13 mathematics teachers in the sample have had a diversity of teaching experiences in calculus, ranging from 4 to 33 years for lecturers and 8 to 29 years for teachers (Table 4.3). The lecturers and teachers pseudonyms were given in each institute in order of length of teaching experiences, to assist in exposing any connections between their responses about students' difficulties and their respective lengths of teaching experience.

Table 4.3

Participants' information: lecturers and teachers

	Qualifications	Years of teaching calculus
Undergraduate mathematics lecturers		
University1		
L11	B.Sc.(Hons.), & PhD in mathematics	4
L12	B.Sc. in mathematics/computer science, honours in mathematics, M.Sc., & PhD in mathematics	5
L13	B.Sc. in Physics, M.Sc., & PhD in mathematics	30
L14	B.Sc., M.Sc., & PhD in mathematics	30
L15	B.Sc.(Hons.), & PhD in Physics	33
University 2		
L21	B.Sc. in mathematics, M.A. & PhD in mathematics education	4
L22	B.Sc. in mathematics, & education, M.Sc. & PhD in mathematics	7
L23	B.Sc., M.Sc., & PhD in mathematics, & Dip. T.*	11
L24	B.Sc. & M.A. in statistics	17
L25	B.Sc.(Hons.) in Mathematics, M.Sc., & PhD in mathematics education	More than 30
Year 13 mathematics teachers		
T1	B. A. in engineering, & Dip. T.	8
T2	B.Sc. in mathematics, & Dip. T.	12
T3	M.Sc. in Military Science	20
T4	B.Sc.(Hons.) in physics, & Dip. T.	20
T5	B. A. in Sociology, & Dip. T.	29

*Note.** Dip. T. = Graduate diploma for teaching.

This research is a comparative case study (Section 4.2), therefore, the three important criteria (Kaarbo & Beasley, 1999) for selecting cases in comparative case studies were considered including " [1] selecting comparable cases, [2] selecting cases that vary on the dependent variable, and [3] selecting cases across subgroups of the population to address alternative explanations" (Kaarbo & Beasley, 1999, p. 380). Having approximately the same number of students (i.e., nine University and eight Year 13 students) in Case 1 and 2, makes these two Cases comparable in terms of exploring student learning. Having students with different calculus backgrounds (next Section) increased the chances that the second and third criteria were being met in this study.

Students

Theoretical sampling (Eisenhardt & Graebner, 2007) was also considered for choosing students for interviews in the College. For this purpose, in the College that was chosen using convenience sampling, students with different calculus performance were chosen from a scholarship and a regular class (Section 4.3.3). The reason for having students with different calculus performance is to have a better understanding of student learning of integral calculus, their difficulties, and meeting the criteria for selecting cases in a comparative case study. Eight students (Table 4.4) from the College were interviewed, including two of the best students from the scholarship class, and two students from each group of high, medium, and low calculus performance from the regular class (see Section 4.3.3 for details about participant recruitment). Having two students from each group was done to avoid having a single student representing a group of students (e.g., scholarship, high, medium, and low). Students' codes are based on the level of their calculus background (Section 4.3.3) to help illuminate any connections between students' points of view about learning integral calculus and their calculus background.

Table 4.4

Participants' information: College students who participated in the interviews

	Gender	Calculus background
Y1	Male	Low
Y2	Male	Low
Y3	Male	Medium
Y4	Male	Medium
Y5	Male	High
Y6	Male	High
Y7	Male	Scholarship student
Y8	Male	Scholarship student

The method of choosing University students for the interview was different because the lecturers did not know their students' calculus background. Therefore, all students enrolled in the course were invited by email to participate and nine students volunteered (Table 4.5). While interviewing University students, different performances were observed, indicating that students with different calculus backgrounds were included.

Table 4.5

Participants' information: University students who participated in the interviews

	Gender
U1	Male
U2	Male
U3	Male
U4	Male
U5	Female
U6	Female
U7	Male
U8	Male
U9	Male

Digital recording and classroom observation participants

The entire lectures and two tutorial classes related to the teaching of integral calculus were recorded in Case 1. In Case 2, the entire teaching of integral calculus in one regular classroom was recorded. The camera and sound recorder were placed to capture the board and the lecturer/teacher/tutor. Memo-writing during the video recording of the classes was conducted using a form (Appendix 2) to help analyse the recording data and be a backup if the video and sound file were lost or corrupted. The form has three sections for memo writing, including a section on the learning objectives of the session, a section for teaching activities that were used in the classroom, and a section for the interactions between the instructor and students. In addition, a small RBT table was inserted in each section for locating the learning objectives, teaching activities, and interaction in the RBT table. However, those Tables were not included in the analysis of this study because an evaluation of alignment was not conducted in the study due to the quantity of data from other sources.

4.3.3 Participant recruitment

In this section, the methods of selecting study participants are described.

Case 1

Regarding the interviews with lecturers in Case 1, the information sheet (e.g., Appendix 3) and the consent form (e.g., Appendix 4) were delivered to the head of the Mathematics Department, and after his agreement, the researcher met the lecturers face to face, asking about their interest in being interviewed. In Case 1 all were involved with teaching calculus in the academic year 2013-2014, and met the criteria of the study.

For choosing students for interview, an email including the consent form and information sheet about the interview was sent to each student enrolled in the course. All students who volunteered were interviewed.

Regarding video recording of the lectures and tutorials, the head of the Mathematics Department discussed the project to the lecturer and tutors of the course, then, the researcher sought permission to record and observe their classes. In the first session of their lectures and tutorials, before starting integral calculus topic, the lecturer and tutors introduced the project to the students, and informed them that if they were not happy with the video recording of the teaching they should inform the researcher or the lecturer of the course and ask to be excluded from the analysis. In addition, the information sheets and consent form were placed on the website for the course. It was neither practical nor necessary to obtain students' written consent for videoing because of the large number of students enrolled in the course and the video recorder being placed in a way that it captured only the board and the lecturer. In the tutorials, the consent form and information sheets were delivered to the students and each was asked for their permission to be included in the video recording, because of smaller number of students in the classrooms. All students agreed to be video recorded and signed the consent form.

University 2

In University 2, the research project was described to one of the lecturers in the Mathematics Department. He discussed the project within the department, and based on the minimum criteria of the study, the five lecturers who volunteered were introduced to the researcher.

Case 2

In Case 2, the information sheets and the consent form were sent to the head of the Mathematics Department asking for permission to collect data. The researcher asked him to send the consent form and information sheet to the principal and seek his permission to do the research. With the agreement of the principal, the information sheet and the consent form were delivered to

Year 13 mathematics teachers. All of the Year 13 mathematics teachers of the College agreed to be interviewed and met the criteria of the study.

Regarding the interviewing of students, the head of the Mathematics Department and the teacher talked to the scholarship and regular students, respectively, describing the project. The scholarship students were chosen by the head of the Mathematics Department and were happy to be interviewed. The researcher asked the head of the Mathematics Department to choose the best students from that class. High, medium, and low calculus performance students were chosen by the calculus teacher of the regular class. Following the initial agreement of students, their email addresses were passed to the researcher. The researcher sent the consent form and the information sheet to the students. With students' agreement and after they have signed the consent form, the interviews were conducted during the College hours.

The researcher asked the head of the Mathematics Department to choose a class with regular students so that the results of the study would be more applicable to the actual situation of teaching integral calculus in Year 13. Regarding the observation and video recording of the regular classroom, the head of the Mathematics Department talked to its teacher about the study, and asked his permission. With the agreement of the teacher, the consent form and information sheets were delivered to him. The teacher then talked to the students of his classroom, delivered consent forms and information sheets to them, and asked their permission. All students agreed to be video recorded and signed the consent form. This happened in case students' voices were captured by the recorder when they asked questions.

4.3.4 Reliability

Evaluating the quality of research is important for its findings to be utilised in practice (Noble & Smith, 2015). Reliability is one of the factors used for evaluating the quality of a research. In quantitative studies, reliability refers to techniques that show that if the study is repeated in the same context using the same methods, similar results will be obtained (Shenton, 2004). In qualitative research, dependability is closely related to the notion of reliability in quantitative research (Golafshani, 2003) and can be addressed by providing details of how the study was conducted (Shenton, 2004). The researcher's background, the study context, rationale and decisions regarding participant selection and design of the instruments are all described to show the degree of reliability that can be assigned to the study.

4.3.5 Validity

Validity in quantitative research refers to “the precision in which the findings accurately reflect the data” (Noble & Smith, 2015, p. 34). In qualitative studies, validity refers to using different procedures for evaluating the findings (Creswell, 2009). Validity has two main streams: internal, which refers to how truthful the findings are; and external, which refers to how applicable the findings are for other settings (Decrop, 1999). Several criteria are suggested (e.g., Creswell, 2009; Golafshani, 2003; Tracy, 2010) for ensuring the validity of the findings, such as considering multivocality, triangulation, whether a reasonable amount of time has been spent on the site of research, possible bias of the researcher, and the provision of a thick description about the study.

Multivocal research relates to including “multiple and varied voices in the qualitative report and analysis” (Tracy, 2010, p. 844). In the study, having students with different calculus backgrounds could ensure the provision for different voices about how students learn integral calculus. In addition, having lecturers’ and teachers’ opinions about students’ difficulties could help towards a better understanding of how students learn integral calculus and what are their difficulties in the topic.

Triangulation is related to use of different sources of data, methods, theories, and researchers to increase the validity of the findings (Denzin, 2006; Shenton, 2004). It is based on the “premise that no single method ever adequately solves the problem of rival explanations. Because each method reveals different aspects of empirical reality, multiple methods of data collection and analysis provide more grist for the research mill” (Patton, 1999, p. 1192).

The study used data triangulation by interviewing students with different calculus backgrounds, interviewing lecturers/teachers from different institutions with various teaching experiences, recording and observing the teaching of integral calculus from different institutions to provide a better understanding of student learning. Theory triangulation has been done by using not only RBT for exploring student learning of integral calculus, but also using facets of metacognition to have a better understanding of student learning. In relation to method triangulation, three different methods were used for collecting data from students, including conduction a semi-structured interview, using a think-aloud protocol, and the VisA instrument (Figure 4.4).

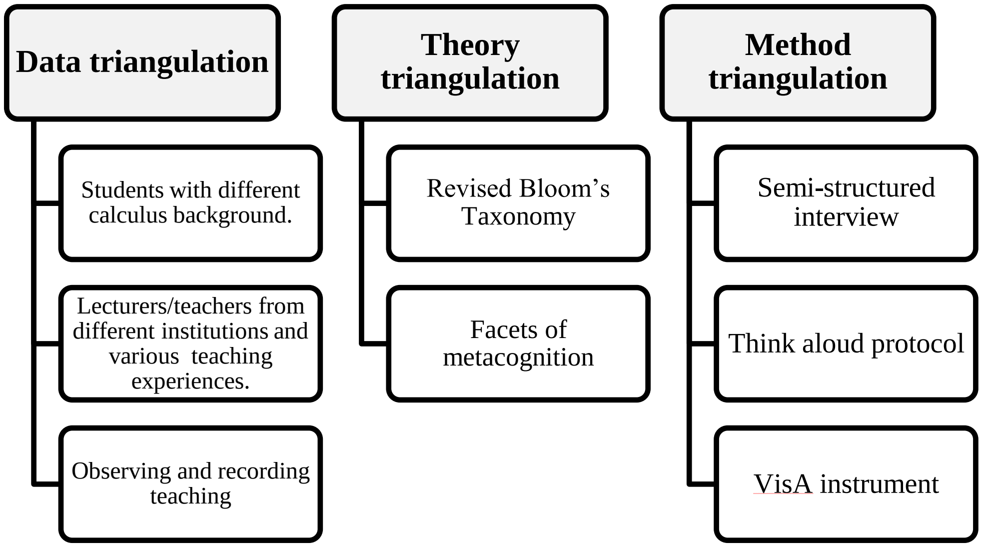


Figure 4.4 Summary of the triangulations considered in the study

Concerning the researcher being on the site for a reasonable time, the researcher spent more than three months in Cases 1 and 2, video recording and observing teaching, and interviewing lecturers, teachers, and students.

In relation to the bias of the researcher, the description of the researcher's background (Section 1.2.1) describes the position from which the researcher approached the study, enabling identification of any possible bias in the study. Provision of excerpts of students' verbal thoughts and workings for each category of analysis, helps readers to identify the researcher's interpretation within each category.

Regarding thick description, the study design provides the information about participants, their context, and how they have been recruited. Details in the results chapters are also provided to help readers reach their own conclusions about the results and understand how the study fits with other settings (i.e., external validity) (Tracy, 2010).

4.3.6 Ethical considerations

An ethical norm should be considered in research to promote the purpose of the study (e.g., obtaining knowledge and truth) and values that are essential to collaborative work (e.g., trust and mutual respect). This would also make the researcher accountable to society, and may bring public support for the study (Resnik, 2011). The study followed the ethical guidelines of the Human Ethics Committee of Victoria University of Wellington (VUW) and New Zealand Association for Research in Education's (NZARE) ethical guidelines (NZARE, 1998). Ethics approval from the Human Ethics committee of VUW was obtained (Approval number: 20851). Several ethical issues were considered during the study as suggested by the literature (e.g., Creswell, 2014; Johnson & Christensen, 2012; NZARE, 1998; Resnik, 2011) such as:

- the purpose of the study, and how the data was to be collected and would be used were explained to the participants before data collection, using the study information sheets (e.g., Appendix 3);

- the participants were informed in the information sheets that they could withdraw from the study without giving any reasons and no pressure was imposed on them to sign the consent form (e.g., Appendix 4);

- the participants and the sites of the research were treated with respect. For example, the researcher tried not to make noise during data collection (e.g., observing and video recording the classrooms);

- the confidentiality of the participants was considered by using pseudonyms;

- the researcher introduced himself to participants as a Ph.D. student;

- the research findings were intended to be reported objectively, frankly, and without any prejudice; and

- summary of research findings was passed to those participants who showed an interest in knowing about the results in the consent forms.

4.3.7 Structure of data analysis

Lecturers/teachers' points of view about students' difficulties in integral calculus are presented in Chapter Six, which also includes contextual information, obtained from observing and video recording the teaching of integral calculus in Cases 1 and 2.

Students' responses to integral questions are explored in Chapter Seven. Students' responses to metacognitive knowledge questions are presented in Chapter Eight. When it was possible, they were compared to student performance in solving integral-area problems. This was done to explore whether they applied their metacognitive knowledge during problem-solving and to find out how students' metacognitive knowledge could be related to their factual, conceptual, and procedural knowledge. Students' metacognitive experiences and skills are described and compared to students' problem-solving in Chapter Nine.

4.4 Chapter summary

In this chapter, the research paradigm, the methodology, and the method of the study are explained and justified. The detailed structure of main data gathering instruments, the students' interview, and how the interim RBT knowledge dimension for integral calculus was developed, are described in the next chapter.

Chapter Five: The Development of Interim Knowledge Dimension and Associated Data Gathering Tool

In this chapter, firstly the way in which RBT knowledge dimension has been contextualised for integral calculus within this study is explained. Secondly the detailed structure of students' interviews are presented and justified. It is important to acknowledge that the cognitive processes activated in students' minds during problem solving depends on students' prior knowledge and experiences. Prior knowledge is found to be one of the important prerequisites for learning (Gurlitt & Renkl, 2010) and affects mathematical performance (Weinert and Helmke, 1998). One question can activate recall of a type of knowledge for one student, while for another who has not seen similar questions, creativity can be activated as the student needs to construct a type of knowledge. Even within each cognitive process, different subcategories may be activated by students, depending on their prior knowledge and experience. For instance, in terms of applying a type of knowledge, if students have previously solved a similar question to those answered during interviews, executing (Section 2.2.2) a type of knowledge was involved, however, if they have not solved similar questions before, implementing (Section 2.2.2) a type of knowledge might be activated for solving the problem. Considering the subjectivity in the cognitive process activated in students' minds, student learning was analysed in terms of RBT's types of knowledge. For doing so, the RBT's knowledge dimension for integral calculus was developed, as described in the following sections.

5.1 The interim RBT knowledge dimension for integral calculus

This section is dedicated to demonstrating the process and care taken in contextualising RBT's knowledge dimension for integral calculus. It is important to first define the knowledge dimension for integral calculus (Section 1.1) in order to be able to explore student learning of integral calculus based on RBT. When designing the RBT knowledge dimension for integral calculus, the following resources (Figure 5.1) were used:

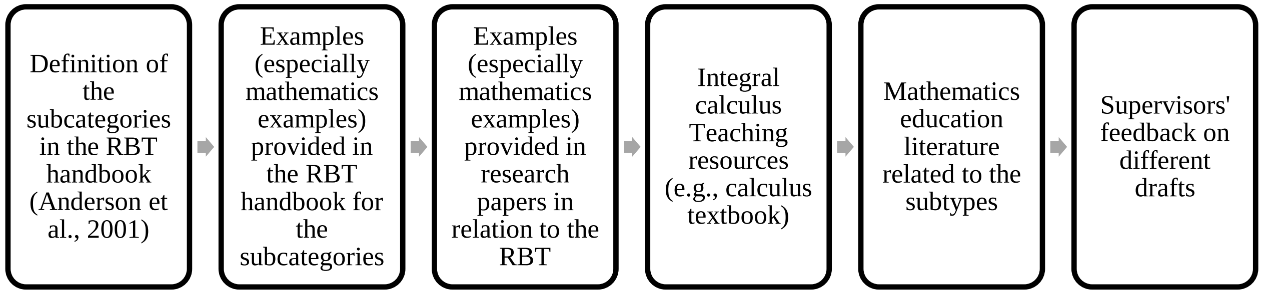


Figure 5.1 Resources used for designing the RBT knowledge dimension for integral calculus

- first, the definitions of the RBT's knowledge dimension subtypes were obtained from the RBT handbook (Anderson et al., 2001) (Section 2.2.1);
- the examples provided in the RBT handbook were considered as the second important resources because they were chosen by the designers of RBT, who were likely to have a clear understanding of what constituted each subtype. Examples in the handbook were provided from various disciplines including mathematics. When, for a subtype, examples from mathematics and other disciplines were provided, more attention was given to the mathematical examples;
- the third source was research papers that defined RBT in particular disciplines (Section 2.5.1). Such papers were given a high status because the examples were peer-reviewed by academics before publishing as a research paper. However, they were not considered as valid as the first and the second resources because they were not defined by the designers of RBT;
- integral calculus teaching resources (Section 5.2.1) were considered for choosing examples that related to the focus of the study and made the examples useful for lecturer and teachers teaching the topics;
- the next resource used were the mathematics education literature about the RBT types of knowledge (e.g., conceptual and procedural knowledge) and examples provided in them were used for developing the RBT examples (e.g., Mahir, 2009); and
- finally, PhD supervisors provided feedback on the interim knowledge dimension and their comments were used.

The following sections provide a detailed explanation of how the 11 subtypes of RBT's knowledge dimension for integral calculus were developed using the process above.

5.1.1 Factual knowledge: Knowledge of terminology

For the knowledge of terminology, the examples provided in the RBT handbook (Anderson et al., 2001) comprise knowledge of "the alphabet, scientific terms (e.g., labels of parts of a cell), the standard representational symbols on maps and charts" (p. 47). No mathematical examples were found in the handbook or other resources for knowledge of terminology. Considering the definition of the subtype (Section 2.2.1) and these examples, in the context of integral calculus, the knowledge of terminology can be considered to include the definition of such terms as integration, anti-differentiation, antiderivative, along with the meaning of commonly used symbols such as the integral sign, integrand, information that lecturers/teachers teach students to acquaint them with this topic. The reason is these are the specific verbal and nonverbal symbols used in integral calculus for introducing this topic. For instance, the definition of antiderivative is the first definition in the Anton et al. (2012, p. 322) calculus textbook for the section on the indefinite integral.

5.1.2 Factual knowledge: Knowledge of specific details and elements

This subtype comprises knowledge of events, locations, people, dates, and sources of information (Section 2.2.1). For this subtype, again, no mathematical examples were found in the RBT handbook. Some examples in the handbook (Anderson et al., 2001) are "knowledge of major facts about particular cultures and societies,…, [and] knowledge of the more significant names, places, and events in the news" (p. 48).

In the context of integral calculus, a first suitable example might be the history of the development of integral calculus (e.g., knowing that Newton and Leibniz are those who realised that the Fundamental Theorem of Calculus is an efficient tool for computing areas and integrals without using the limits of sums (Anton et al., 2012)). The history of integral calculus is frequently highlighted in calculus textbooks (e.g., Anton et al., 2012; Stewart, 2008). In addition, the importance of the history of mathematics in the teaching and learning of mathematics is highlighted in several studies (e.g., Clark, & Thoo, 2014; Huntley, & Flores, 2010). Sharing the history of mathematics with students humanises mathematics. Students can be helped to understand mathematics as a human endeavour (e.g., Clark, & Thoo, 2014) and they can gain a new appreciation of mathematics (e.g., Huntley, & Flores, 2010). The history of mathematics is also reported as a motivating and exciting factor for learning mathematics (e.g., Huntley, & Flores, 2010; Mayfield, 2014). Providing the history of mathematics is found to be helpful for understanding mathematical concepts at a deeper level (e.g., Huntley, & Flores, 2010) and remembering them for a longer time (e.g., Mayfield, 2014).

The second example of this subtype might be knowledge about the context. Context has different meanings in educational settings (Van den Heuvel-Panhuizen, 2005; Harvey & Averill, 2012). It is related to both the learning-environment context, and task context. Learning-environment context refers to different situations where learning takes place and interpersonal aspect of learning, whereas, task context are "words and pictures that help students to understand the task, or concerning the situation or event in which the task is situated" (Van den Heuvel-Panhuizen, 2005, p. 2).

In terms of integral calculus, the aspect that is relevant to the knowledge dimension is knowing the importance of realistic context problems/questions for student learning and knowing different realistic context problems/questions in relation to integral calculus. Realistic context problems are referred to as problems where the context is imaginable for learners, and does not necessary come from the real world (e.g., could be from the fantasy world of fairy tales) (Van den Heuvel-Panhuizen, 2005). An example of a realistic context problem in integrals is

> A modern sport stadium is being planned, with a curved concrete roof. Each of the end walls is 25 meters long. Its height, y meters, is given by $y = 5\sqrt{x} - x + 4$, where x meters is the horizontal distance from the beginning of the wall. Draw a sketch showing the shapes of one of these walls and use a definite integral to find its area (Barnes, 1993, p. 39).

An example of the knowledge of sources of information in the handbook (Anderson et al., 2001) is "particular books, writings, and other sources of information on specific topics and problems" (pp. 47-48). For the knowledge of sources and information, in integral calculus, two different aspects of knowledge might be involved. The first is calculus textbooks, websites, etc. that provide information for those who want to learn about integral calculus, and the second is data that is presented in integral problems. For example, in the area problem of finding the area enclosed between $x = y^2$ and $y = x - 2$, the two functions are the specific details and information needed for finding the enclosed area. Figure 5.2 summarises factual knowledge in the context of integral calculus.

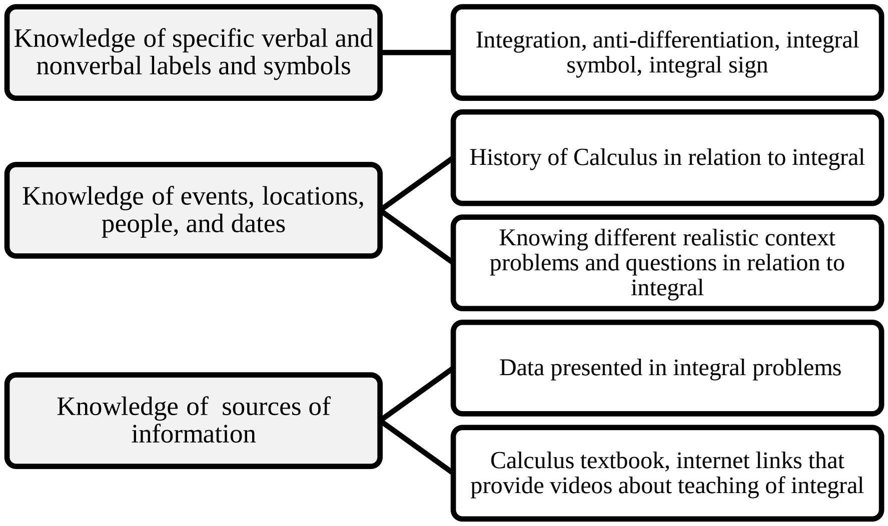


Figure 5.2 The factual knowledge in the context of integral calculus

The next section describes the first subtype of RBT conceptual knowledge for integral calculus.

5.1.3 Conceptual knowledge: Knowledge of classifications and categories

Knowledge of classification and categories is important to not constrain student learning by "misclassification of information into inappropriate categories" (Anderson et al., 2001, p. 50). For instance, if a student cannot classify an integral into one of the types of indefinite, definite, and improper integral, they cannot identify the appropriate procedures for finding the integral. No mathematical examples were provided in the handbook for this subtype. Examples from other disciplines are knowledge of "the parts of sentences (e.g., nouns, verbs, adjectives), [and] knowledge of different kinds of psychological problems" (p. 50). Conceptual knowledge in mathematics and integral calculus are defined in the mathematics education literature (Section 3.1.3), however, it seems those definitions might fit better with the second subtype of RBT's conceptual knowledge as they do not address the different classifications and categories (Section 3.1.3).

Considering the definition (Section 2.2.1) and the examples, classifications and categories in the context of integral (Figure 5.3) include knowing that there are different types of integral (i.e., indefinite, definite, and improper integrals), different integrands (e.g., rational function, and exponential), different methods of finding areas (e.g., with respect to the x and y-axis) and volumes (i.e., slices by disk, washers, and cylindrical shell). Students need to distinguish these classifications to be able to apply suitable procedural knowledge.

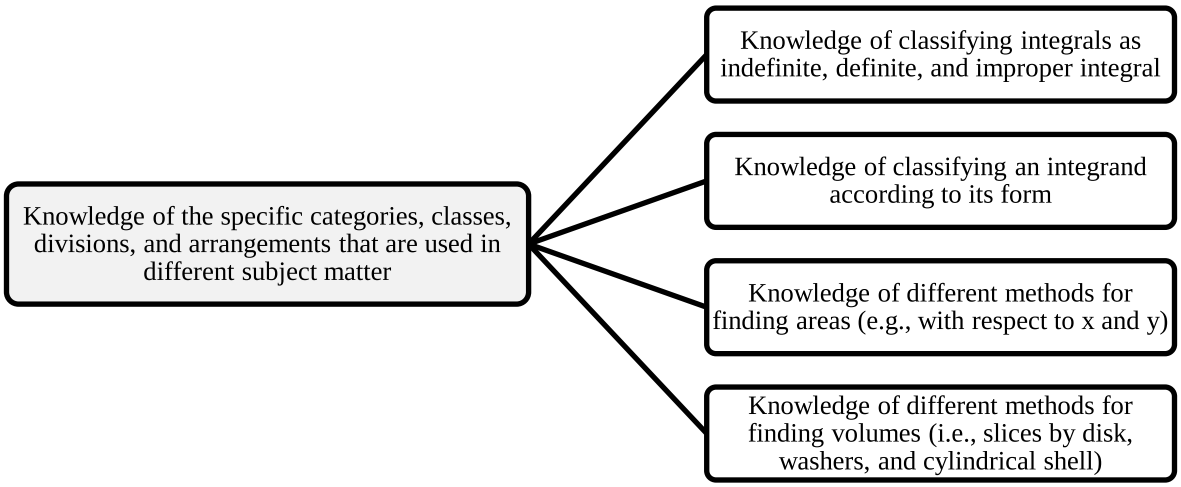


Figure 5.3 First subtype of the conceptual knowledge in the context of integral calculus

The second subtype of RBT's conceptual knowledge (Section 2.2.1) is described in the next section.

5.1.4 Conceptual knowledge: Knowledge of principles and generalisations

Knowledge of principles and generalisations, abstractions that summarise observation of phenomena (Section 2.2.1), is important in any discipline for solving problems and studying phenomena (Anderson et al., 2001). Mathematical examples of this subtype in the RBT handbook are "knowledge of the principles that govern rudimentary arithmetic operations (e.g., the commutative principle, the associative principle)" (p. 51) and "Pythagorean theorem" (p. 46). Mahir (2009) defined conceptual knowledge for integral calculus as knowing "the definite integral of a function is the limit of Riemann sums, the integral-area relation, and the Fundamental Theorem of Calculus" (p. 202) (Section 3.1.3).

Considering the definition of this subtype (Section 2.2.1) and the examples, the subtype in the context of integral calculus are 1) principles used for finding the antiderivatives of different types of functions (e.g.,$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ (n \neq -1)$) which derive from the relationships between derivative and antiderivatives, 2) knowledge about the relationship between the definite integral and Riemann sums (e.g.,$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_i)\Delta x$), and 3) the FTC, the overarching theorem in integral calculus that relates definite and indefinite integrals together. The second and third aspects are the same as claimed by Mahir (2009).

In relation to the first aspect, If $\int x^3 dx = \frac{x^4}{4} + c$ is considered as one observation, and $\int x^{\frac{2}{3}} dx = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + c$ as another observation, and the integral of different polynomial functions as other observations, the abstraction that summarise these observations is $\int x^n dx = \frac{x^{n+1}}{n+1} + C, (n \neq -1)$. Therefore, the knowledge of principles and generalisations that are defined as particular abstractions that summarise observation of phenomena consists of all formulas in integral calculus topics. In addition to that, as conceptual knowledge is rich in relationships (Section 3.1.3), for this subtype, these formulas should be considered with the rationale that produce them. For instance, $(\frac{x^{n+1}}{n+1} + c)' = x^n$ is also part of conceptual knowledge about $\int x^n dx = \frac{x^{n+1}}{n+1} + C$. However, the way that the formulas are used is related to knowledge of subject specific skills and algorithms, the first subtype of procedural knowledge.

The integral-area and integral-volume relationship are also part of this subtype because they can be stated as a formula, and as an abstraction that summarises which principles should be used for solving area/volume problems. This subtype in the context of integral calculus is summarised below.

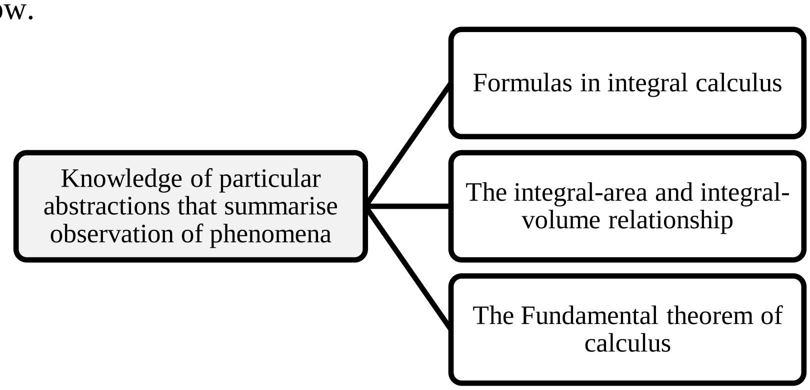


Figure 5.4 Second subtype of the conceptual knowledge in the context of integral

The next section describes the third subtype of RBT conceptual knowledge for integral calculus.

5.1.5 Conceptual knowledge: Knowledge of theories, models, and structures

This subtype is different from the previous subtype in the sense that it focuses "on a set of principles and generalisations related in some way to form a theory, model, or structure" (Anderson et al., 2001, p. 52); whereas, the second subtype of conceptual knowledge does not need to be related to other parts (Anderson et al., 2001). No mathematical examples were found for this subtype in the handbook. Some examples from other disciplines are "knowledge of a relatively complete formulation of the theory of evolution, knowledge of genetic models (e.g., DNA), … [and knowledge about] behavioral, cognitive, and social constructivist theories in psychology" (p. 52). Considering the definition of the subtype, the whole set of theorems (e.g., the FTC, the mean value theorem for integral), proofs, and formulas in relation to indefinite, definite, and improper integrals that form integral calculus topic can be considered as fitting within this subtype.

5.1.6 Procedural knowledge: Knowledge of subject-specific skills and algorithms

Subject-specific skills and algorithms are those processes where the end result is to fix whether the processes are open or fixed (Section 2.2.1). Mathematical examples of this subtype can be found in the resources: "Knowledge of algorithms used with mathematical exercises. The procedure of multiplying fractions in arithmetic,…, knowledge of various algorithms for solving quadratic equations" (Anderson et al., p. 53). In mathematics education literature (Section 3.1.3), Mahir (2009) defined procedural knowledge as knowledge of integral techniques for finding the antiderivative of functions. This definition seems broad encompassing all three subtypes of procedural knowledge identified in RBT. Sections 5.1.6 to 5.1.8 provides the detailed definition of procedural knowledge in integral calculus for these subtypes.

Considering the definition of the subtype (Section 2.2.1) and the examples, knowing how to solve one specific type of integral calculus problem is related to the first subtype of procedural knowledge. The main skills and algorithms in the context of integral calculus are those related to how to find an anti-derivative of different types of integrands and knowing how to find areas, volumes, etc. (i.e., applications of definite integrals) (Figure 5.5). For example, knowing how to find the anti-derivative of different types of trigonometric functions such as odd and even powers of $\cos x$ is related to this subtype.

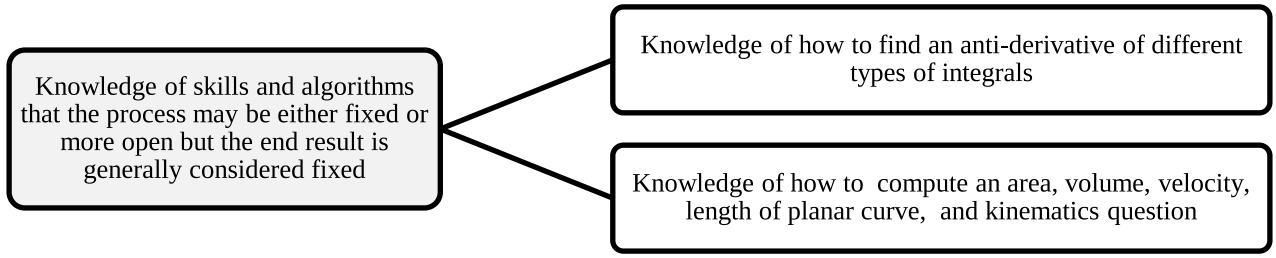


Figure 5.5 The first subtype of procedural knowledge in the context of integral calculus

The following section describes the second subtype of procedural knowledge for integral calculus.

5.1.7 Procedural knowledge: Knowledge of subject-specific techniques and methods

In contrast to the first subtype where the knowledge is specifically related to knowing one approach, the second subtype is broader. It is the "knowledge of how of think and attack a problem in a field" (Anderson et al., 2001, p. 53, Section 2.2.1). No mathematical examples were found in the resources for this subtype. Examples from other disciplines are "knowledge of relevant research methods,…, knowledge of techniques used by scientists in seeking solutions to problems" (Anderson, et al., p. 54). Considering the definition (Section 2.2.1) and the examples, research methods in mathematics can be considered for this subtype; those method that used/or are being used for proving mathematical statements such as direct proof, proof by mathematical induction, and proof by contradiction. Theorems and proofs which are the results of research in mathematics are related to conceptual knowledge; but, the methods that are used and are being used for creating theorems and proof, are related to this subtype because these are the ways that mathematicians attack and solve problems in mathematics.

At a lower level of abstraction for attacking integral problems, a technique that is used is sketching an integrand and a cross-section in area/volume problems to decide which method should be used for solving the problems (e.g., Anton et al., 2012) (Figure 5.6).

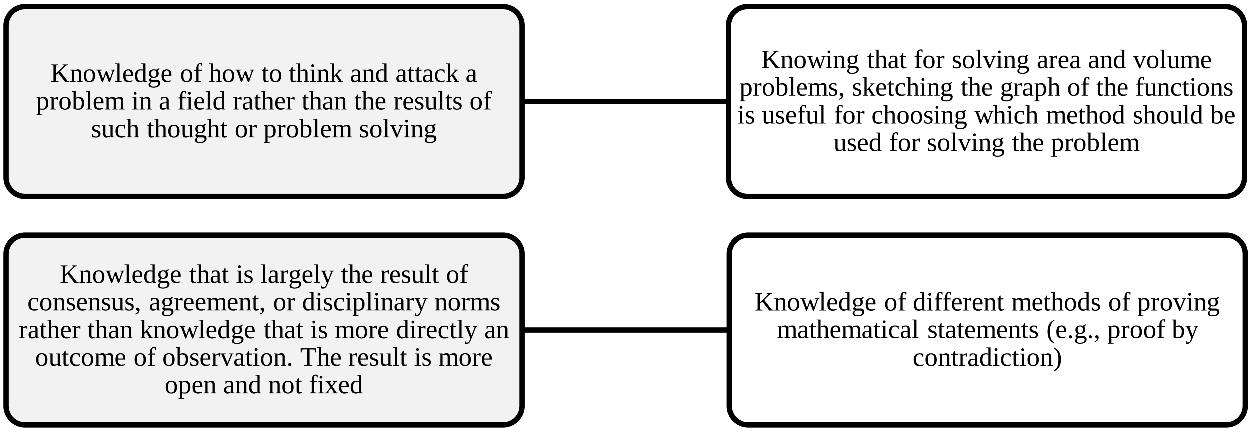


Figure 5.6 The second subtype of the procedural knowledge in the context of integral calculus

The next section describes the last subtype of RBT's procedural knowledge for integral calculus.

5.1.8 Procedural knowledge: Knowledge of criteria for determining when to use appropriate procedures

This subtype refers to knowledge that helps decision making about *when* and *where* to use different subject specific knowledge (Section 2.2.1). For this subtype, a mathematical example is provided in the handbook (Anderson et al., 2001): "knowledge of the criteria for determining which method to use in solving algebraic equations" (p. 55). Considering the definition of the subtype (Section 2.2.1) and the examples, the subtype in integral calculus might be knowing which method should be used in volume/area/indefinite integral problems when the method is not indicated in the question (Figure 5.7). For example, to find the enclosed area between $x = y^2$ and $y = x - 2$, it is easier to set up the integral using integration with respect to the y-axis, as the enclosed area can be represented by $\int_{-1}^{2}(y + 2 - y^2)dy$. However, for finding the enclosed area using integration with respect to the x-axis, the area needs to be split and several integrals need to be set up. Knowing that it is easier to find the definite integral using the FTC rather than Riemann sums, when the antiderivative can be found in terms of elementary functions, is another example of this subtype.

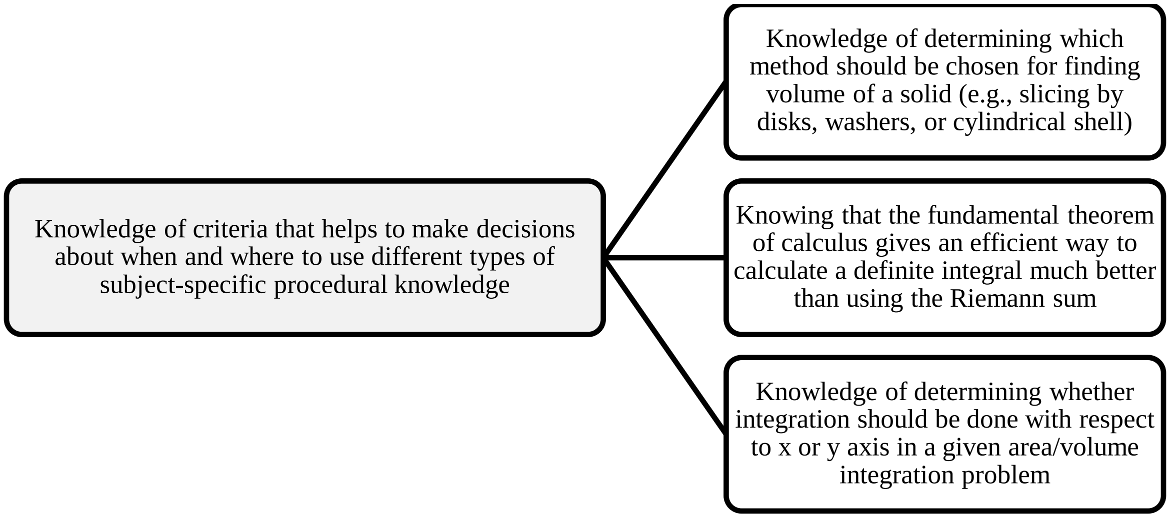


Figure 5.7 The third subtype of the procedural knowledge in the context of integral calculus

The first subtype of metacognitive knowledge in the context of integral calculus is described in the next section.

5.1.9 Metacognitive knowledge: Strategic knowledge

Strategic knowledge comprises the general strategies for learning, thinking, and problem solving (Section 2.2.1). In RBT, strategic knowledge is considered to include strategies that can be used across different tasks and disciplines (Anderson, et al., 2001); therefore, the examples provided are general rather than subject-specific. Some examples are:

> knowledge of various mnemonic strategies for memory (e.g., the use of acronyms such as Roy G Biv for the colours of the spectrum,…, knowledge of various organisational strategies such as outlining or diagramming,…, knowledge of various elaboration strategies such as parapharasing and summarising,…, knowledge of comprehension-monitoring strategies such as self-testing or self-questioning (p. 57).

In terms of the learning strategies in the context of integral calculus (Figure 5.8), knowing that making a plan for solving different integral calculus problems or knowing summarising techniques such as making concept maps for classifying concepts, procedures, and formulas are useful for learning integral calculus, might be included. If students have problems with memorising formulas or procedures, while encouraging students to solve several integral calculus examples to help them to know how the procedures work, knowing how mnemonic strategies can be used to help with memorising and recalling procedures such as order of integration by parts (e.g., LIATE

(Logarithmic, Inverse trig, Algebraic, Trig, and Exponential functions)), relates to this subtype. Knowing the importance of prior knowledge and how it affects problem solving is another aspect of strategic knowledge. Students could evaluate their prior knowledge and try to amend their weaknesses in previous materials that are related to integration such as derivatives, limits, and functions (e.g., Thompson & Silverman, 2008). A final aspect of the learning strategy is knowing useful approaches for dealing with material that is hard to understand, such as asking questions of others (e.g., peers, teacher/lecturer/tutor), looking at different references for different approaches to topic, seeking videos on the internet that explain the topic, and practising more questions and re-reading material.

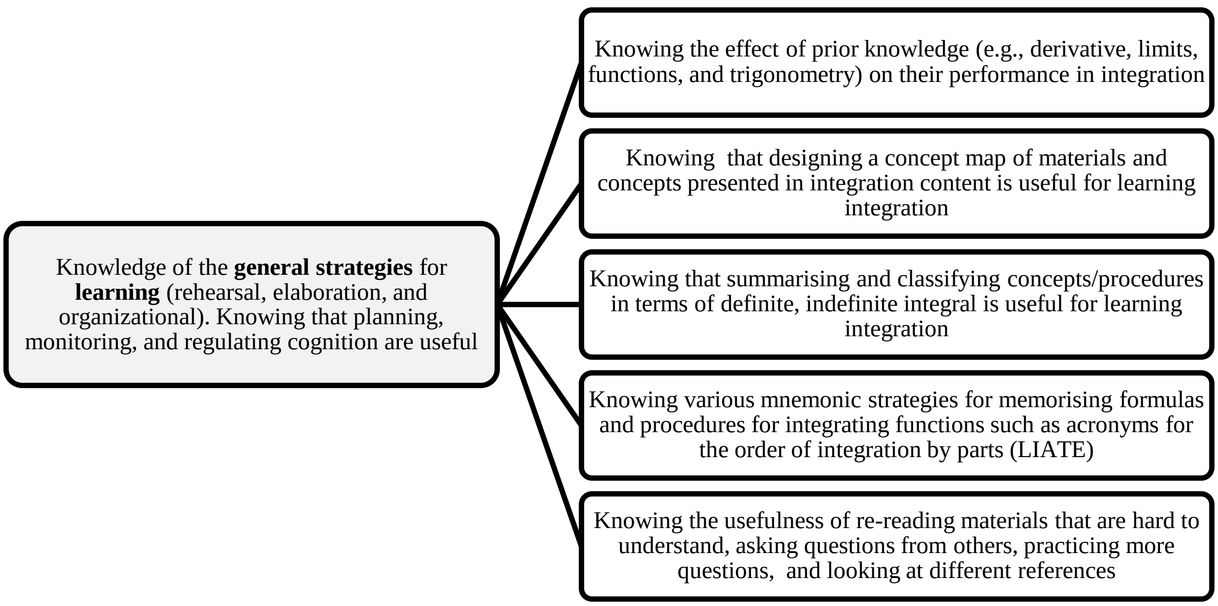


Figure 5.8 General strategies for learning in the context of integral calculus

In terms of knowledge of monitoring strategies, three strategies might be considered in the context of integral calculus (Figure 5.9). In the definite integral, a strategy can be knowing that sketching the integrand in the given interval and finding (or approximating) the net area is useful to check whether the answer makes sense. For indefinite integrals, knowing that the answer can be checked by differentiating the anti-derivative might be related to this subtype. In numerical integration, knowing that the results of numerical integration can be checked using the Fundamental Theorem of Calculus can be a consideration (if the antiderivative of the integrand can be shown with elementary functions).

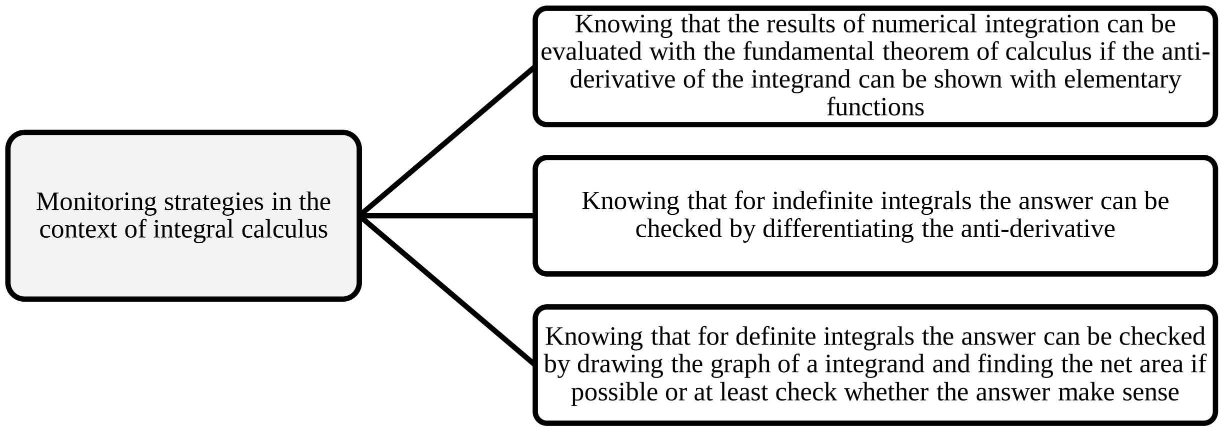


Figure 5.9 Examples of monitoring strategies in the context of integral calculus

Concerning knowledge of general problem solving strategies, several are highlighted in the literature. For instance, Schoenfeld (1987) noted "never use any difficult techniques before checking to see whether simple techniques do the job" (p. 191) in the exam situation. Using several approaches briefly to find out which approach could solve the problem is a general strategy used by mathematicians (Schoenfeld, 1987).

In terms of general problem solving strategies for integral calculus, for indefinite integrals, one strategy can be to look to simplify the integrand if possible - such as through an obvious substitution, then classify the integrand according to its form, and if the substitution does not work, try another substitution or another method. Knowing some problems might be attempted a variety of ways, that if one chosen strategy does not work, going back, "relooking" at the problem from a different perspective might give a clue to another strategy (for instance, another substitution) could be useful. For the definite integral in area/volume problems, having a plan that consists of the following might be useful: A) Sketch the integrand, B) Sketch a cross section, C) decide to integrate with respect to the x or y axis, choosing the method (i.e., slices by disk, washers, and cylindrical shell) based on the cross section (Thomas, Weir, & Hass, 2010).

In terms of the fourth aspect of this subtype, knowledge of general strategies for deductive and inductive thinking (Table 2.5, Section 2.2.1) in the context of integral calculus, being familiar with deductive, abductive, and inductive reasoning in mathematics might be included. The result of having such knowledge is being able to realise whether a proof of a theorem in integral calculus is correctly constructed or not. In addition, if students are asked to prove a theorem, familiarity with these types of reasoning can help them. However, students who do not have this aspect of strategic knowledge, when a theorem is given to them, may only provide an example that meets the conditions of the theorem as a proof, claiming “evidence is proof” (Chazan, 1993).

5.1.10 Metacognitive knowledge: Knowledge about cognitive tasks, including appropriate contextual and conditional knowledge

Knowledge about cognitive tasks is also part of metacognitive knowledge (Flavell, 1979, Anderson et al., 2001). Some examples of this subtype are

> knowledge that recall tasks (i.e., short-answer items) generally make more demands on the individual’s memory system than recognition tasks (i.e., multiple-choice items),…, knowledge that a primary source may be more difficult to understand than a general textbook or popular textbook,…, knowledge that general problem-solving heuristics may be most useful when the individual lacks relevant subject- or task-specific knowledge (Anderson, et al., 2001, pp. 58-59)

This subtype has four aspects (Table 2.5). Considering the definition and the examples, for this subtype (Figure 5.10), in relation to knowledge about different cognitive tasks that can be more or less challenging, the first aspect, knowing that some integral calculus problems are harder than others can be considered. In addition, knowing that useful references (e.g., videos on YouTube) may be easier to follow than others (e.g., lecture notes) and this may differ from one person to another, can be considered.

In terms of knowledge about when and why to use strategic knowledge, the second aspect, and different cognitive tasks require different strategic knowledge, the third aspect (Table 2.5), an example can be knowing that maybe different preparation should be done for taking a multiple choice exam in comparison to a constructed response exam. An example of knowledge that different cognitive tasks require different strategic knowledge is knowing that materials that are presented in university can be more challenging than materials presented in senior secondary school in the sense that the rationale behind the formulas and proofs might be asked in the University section, and university students may face difficulties if they want to take the same approach to learning as they had previously.

A few examples of the fourth aspect, knowledge of local situational and general social, conventional, and cultural norms for using different strategies, are knowing that lecturers may ask similar questions in their exams over different academic years, looking at previous integral calculus exams may give better idea of what will appear on integration exams, and different calculus lecturers may have different approaches to teaching and this can lead to having different types of questions in exams.

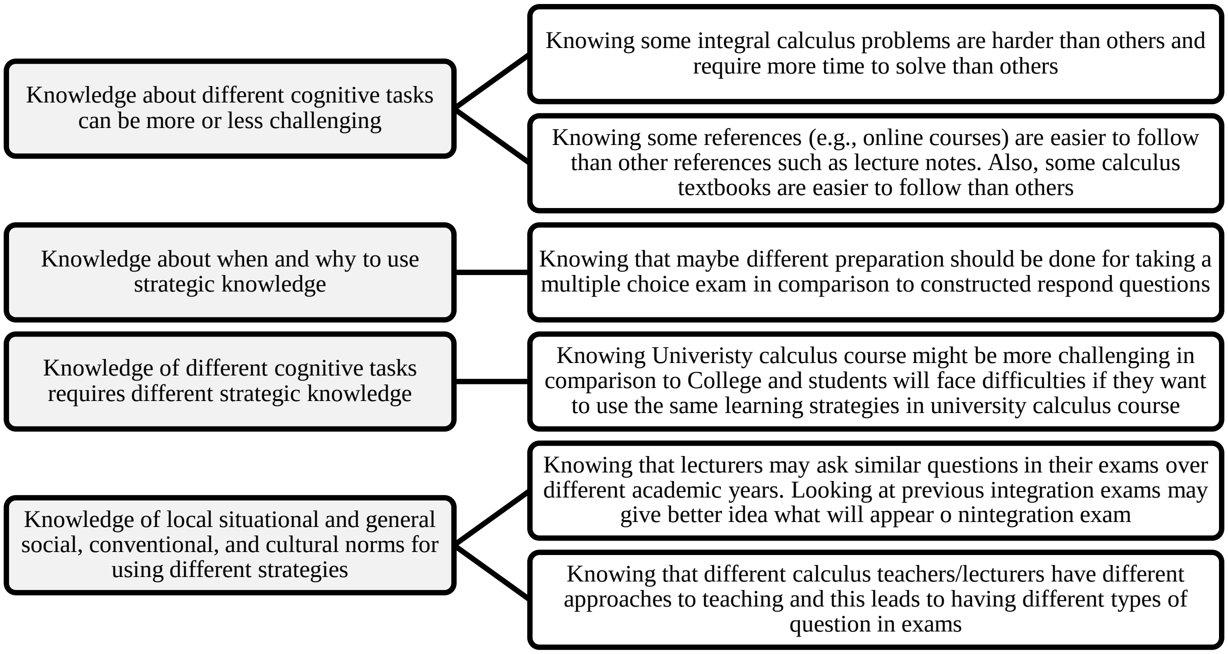


Figure 5.10 The second subtype of metacognitive knowledge in the context of integral calculus

The next section describes the last subtype of RBT's knowledge dimension.

5.1.11 Metacognitive knowledge: Self-knowledge

Self-knowledge is an important part of student learning. If students are not aware that some aspects of their factual, conceptual, or procedural knowledge are not well-structured, it is unlikely they will make an effort to learn them (Anderson et al., 2001). In addition, if students' perceptions of their weaknesses and strengths are not accurate, these perceptions can inhibit their success when solving problems. Several examples are presented in the RBT handbook (Anderson et al., 2001) for this subtype:

> Knowledge that one is knowledgeable in some areas, but not in others… knowledge of one's goal for performing a task,…, knowledge of one's personal interest in a task,…, knowledge that one tends to rely on one type of "cognitive tool" (strategy) in a certain situation (p. 60).

Considering the definition (Section 2.2.1) and the examples, self-knowledge in the context of integral calculus (Figure 5.11) might include awareness of one's own weaknesses and strengths in relation to solving the different types of integral calculus problems and doing different types of exams (i.e., multiple choice, constructed response). Another consideration might be awareness about personal goal orientation (i.e., mastery, performance, or avoidance) (Keys, Conley, Duncan, & Domina, 2012) and attitude toward learning integral calculus and its effects on learning and problem solving.

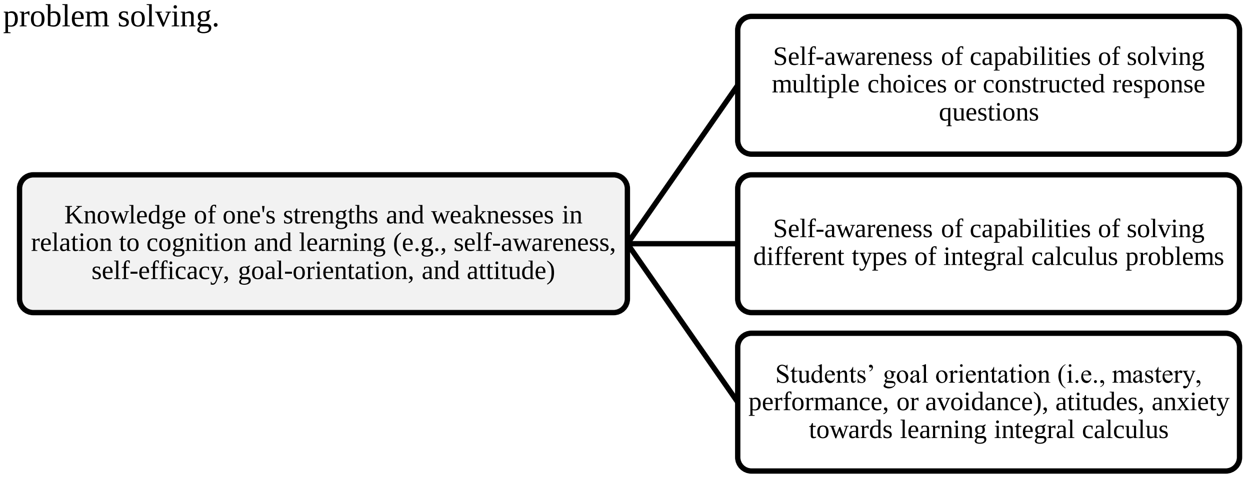


Figure 5.11 The third subtype of metacognitive knowledge in the context of integral calculus

Having the interim RBT knowledge dimension, the next step was using it to design the instruments of the study.

5.2 Detailed structure of students' interviews

In this section, the students' interview questions are presented and how they were designed is described. First, the integral questions are provided (Section 5.2.1), then, metacognitive knowledge questions are presented (Section 5.2.2). Section 5.3.3 describes how metacognitive experiences and skills were measured in the interviews.

5.2.1 Integral calculus questions asked in the interviews

The integral questions were chosen in such a way that several cognitive processes might be activated during the solving of each question. The questions do not cover all 19 RBT subcategories of the cognitive process dimension (Section 2.2.2); however, at least one of each of the six cognitive processes of RBT is addressed. For designing and choosing integral calculus questions for the interviews, the following were considered:

- questions used in previous research in integral calculus (e.g., Jones, 2013; Kiat, 2005; Mahir, 2009; Orton, 1983; Sealey, 2006; Thomas, & Hong, 1996);
- the materials used for teaching integral calculus in most secondary schools in New Zealand (e.g., the *New Zealand Curriculum* (Ministry of Education, 2007b); NCEA level 3 mathematics achievement standards (New Zealand Qualifications Authority, 2013); Delta mathematics (Barton & Laird, 2002);
- the materials used for teaching integral calculus in some Universities in New Zealand (e.g., Calculus: Early transcendentals (Anton, Bivens, & Davis, 2012)) and outside of New Zealand (e.g., Calculus: Early transcendentals (Stewart, 2008); Thomas' Calculus Early Transcendentals (Thomas et al., 2010); and
- observation of all taught classes on integral calculus for both Cases.

How the nine questions can activate different RBT's cognitive processes are described first, then how the questions address different types of RBT's knowledge are presented.

Q1. Please calculate the area enclosed between the curve $x = y^2$ and $y = x - 2$ in two ways. Which way is better to use? Why?

The first part of Q1 (i.e., calculate the area enclosed between curves) is a typical question of the integral calculus topic that explores whether students have the knowledge to use the definite integral for finding areas enclosed between curves. It is typical, as similar questions were used for teaching and assessment in the Cases of the study (Section 6.1). While the two functions are not sophisticated, the curves cross the x −axis and the lower curve changes if the integral is set up with respect to the x-axis. Therefore, the question could challenge some students. In addition, solving the question in two ways, and evaluating which way is better to use, is not a standard question format according to the consulted resources and observed teaching (Section 6.1). This part of the item is intended to activate a higher cognitive process in students' minds.

For Q1, remembering might be activated to recall the ways in which areas enclosed between curves can be calculated using integral calculus. Then the methods need to be applied for solving the question. Finally, critiquing, a subcategory of evaluating, might be included to figure out which method is better to use for finding the enclosed area in Q1.

Q2. What do you understand by A = $\int_a^b [f(x) - g(x)]dx$ and B = $\int_c^d [w(y) - v(y)]dy$? Can you justify how these formulas are derived? Can you justify when each one is used?

Similar questions to Q2 were asked in the literature about exploring student learning of integral calculus. For example, Rasslan & Tall (2002) in their survey asked students, *In your opinion, what is* $\int_a^b f(x)dx$ *(the definite integral of the function* f *on the interval* $[a, b]$*)* or Rösken & Rolka (2007) explored students' concept definition of integral calculus using the following questions:

"What do you understand by $\int_a^b f(x)dx$*?*

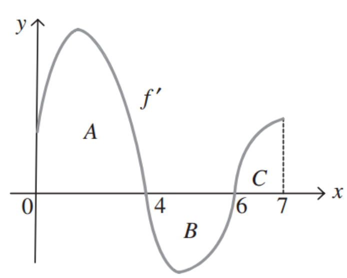
Give a geometric definition of the integral and an illustration.

Give an analytic definition of the integral and an illustration" (p. 188).

Students in both Cases were taught Riemann sums and Riemann integral (Section 6.1). However, this happened for students in Case 2 at the end of teaching integral calculus, and examples were not solved for students using Riemann sums and Riemann integral. Therefore, students in Case 2 might have more difficulty with Q2 in comparison to students in Case 1.

In relation to Q2, remembering may be activated to recognise the symbol of integral, integrand, lower and upper bound, and dx in A= $\int_a^b [f(x) - g(x)]dx$ and B= $\int_c^d [w(y) - v(y)]dy$ and for recalling $\int_a^b [f(x) - g(x)]dx$ and $\int_c^d [w(y) - v(y)]dy$ can be used for finding enclosed area between curves. Understanding might be involved for explaining how each of A = $\int_a^b [f(x) - g(x)]dx$ and B = $\int_c^d [w(y) - v(y)]dy$ is derived and being used.

Q3. "The graph of f'(x), the derivative of $f(x)$, is sketched below. The area of the regions, $A, B,$ and C are 20, 8, and, 5 square units, respectively. Given that $f(0) = -5$, find the value of $f(6)$" (Mahir, 2009, p. 203).



This question is chosen from a Mahir (2009) study focusing on exploring students' conceptual and procedural knowledge in integral calculus. This question was designed for exploring students' conceptual knowledge (Mahir, 2009). Students in both Cases might have difficulty with this question as the function, $f(x)$, is not given in the question explicitly (Thomas and Hong, 1996). In addition, this question might be an atypical question for students in the Cases as they have not seen questions like this in the teaching and assessment of integral calculus (Section 6.1). This question was intended to activate analysing to distinguish which areas (i.e., A, B, and C) should be used for finding $f(6)$. Other cognitive processes that might be activated include remembering how the area under the graph of $f'(x)$ is linked to $f(x)$ through the FTC, and then executing might be involved for solving the question.

Q4. Are these examples solved correctly? Please justify your answer.

Ex.1: Find, if possible, the area between the curve $y = x^2 - 4x$ and the x-axis from $x = 0$ to $x = 5$.

$\int_0^5 (x^2 - 4x)dx = [\frac{x^3}{3} - \frac{4x^2}{2}]_{x=0}^{x=5} = \left[\frac{5^3}{3} - \frac{4(5)^2}{2}\right] - \left[\frac{(0)^3}{3} - \frac{4(0)^2}{2}\right] = \frac{-25}{3}$.

Ex.2: Find, if possible, the area enclosed between the curve $y = \frac{1}{x^2}$ and the x-axis from $x = -1$ to $x = 1$.

$$\int_{-1}^{1} \frac{1}{x^2} dx = \int_{-1}^{1} x^{-2} dx = [\frac{(x)^{-1}}{(-1)} = \frac{-1}{x}]_{x=-1}^{x=1} = \frac{-1}{1} - \frac{(-1)}{(-1)} = -2.$$

The first example in Q4 is based on an item from Kiat (2005) that looked at students' difficulties in solving integral calculus problems:

"Find the area between the curve y = x(x – 4) and the x-axis from x = 0 to x = 5" (p. 58). Students may find an incorrect area for this question if they do not sketch the graph of the curve. In addition, students who only focus on integral techniques and do not pay enough attention to the integral-area relationship may also make mistakes in answering this question.

The dominant cognitive process that might be activated in Q4 is checking, a subcategory of evaluating whether the examples are solved correctly, because as mentioned (Section 2.2.2), checking refers to testing for internal inconsistencies or fallacies in an operation or act. Similarly to previous questions, remembering and applying might also be activated for remembering how the area can be calculated and then be executed for the question.

Christou, Mousoulides, Pittalis, Pitta-Pantazi, & Sriraman (2005) proposed a taxonomy for

Q5. Please can you pose a problem about the area enclosed between a curve and a line with any two arbitrary bounds that will give an answer of 1 (i.e., the enclosed area will be equal to one)?

problem posing processes that can be useful for desiging problem posing questions. Based on this taxonomy, problem posing questions can be classified in one of the following types: a) editing, b) selecting, c) comprehending, and d) translating quantitative information. Q5 is classified as selecting quantitative information because it requires students to pose a problem that is appropriate to the given answer (Christou et al., 2005). This task is more difficult than editing as students need to focus on relationships between the given information (Christou et al., 2005).

This question might be challenging for students in both Cases as they have not seen problem posing questions in integral calculus topic in their classes (Section 6.1). Problem posing activity in Q5 is related to creating. Students need to create a mathematical question that meets the given conditions in Q5. Other cognitive processes like remembering and applying might also be involved.

Q6. Find the derivative of the following functions.

- $O(x) = \int_1^x \frac{1-t}{t^2-2t-9} dt$
- $\mathrm{G}(x) = \int_0^{x^2} r^2\sqrt{1+r^3}\, dr$
- $D(x) = \int_{2x-5}^{4x+4} t^3 dt$

Q6 is designed to explore whether students are able to use the second part of the FTC to find the derivative of the definite integral. This question is similar to tutorial and assignment questions students practised in Case 1 (Section 6.1.1). The integrals were designed in such a way that the antiderivatives can be found using basic integral techniques; therefore, students who are not completely familiar with the FTC, Case 2, also have this chance to solve the question.

For solving Q6, students might need to remember how to use the second part of the FTC for finding the derivatives. Applying might also be activated for executing procedures for finding the answers.

Q7. What do you understand by $F(x) = \int f(x)dx$?

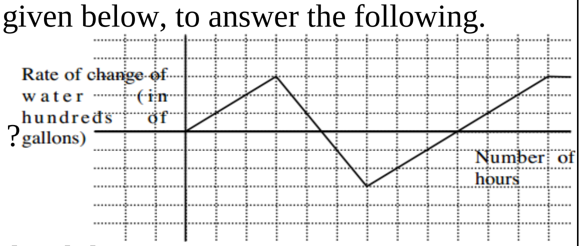
What do you understand by $F(x) = \int_a^x f(t)dt$?

What do you understand by $\int_a^b f(x)dx = F(b) - F(a)$? When do you use this formula? Can you justify how it is derived?

What do you understand by $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$? When do you use this formula? Can you justify how it is derived?

The design of Q7 is similar to Q2 and the wording of the question is inspired by the literature (Rösken & Rolka, 2007). Students in Case 1 have been taught both parts of the FTC; therefore, they might have better performance in this question in comparison to students in Case 2. Students in Case 2 have not been taught the second part of the FTC; therefore, they might not able to answer this question completely (Section 6.1). The reason for asking this question of these students is to know what parts of the FTC are more challenging for them. Q7, similarly to Q2, might activate remembering and understanding. Remembering for recognising symbols in the questions and linking them to the FTC, recalling the statement of the FTC, etc. Understanding might also be activated for being able to explain how these formulas are derived.

Q8. "Let f represent the rate at which the amount of water in Phoenix's water tank changed in 100's of gallons per hour in a 12 hour period from 6 am to 6 pm last Saturday (Assume that the tank was empty at 6 am (t=0)). Use the graph of f, given below, to answer the following.



- How much water was in the tank at noon?
- What is the meaning of $g(x) = \int_0^x f(t)dt$?
- What is the value of g(9) ?
- During what intervals of time was the water level decreasing?
- At what time was the tank the fullest?
- Using the graph of f given above, construct a rough sketch of the graph of g and explain how the graphs are related" (Carlson, et al., 2003, pp. 168-169).

This question is chosen from a Carlson, et al. (2003) study looking at the teaching and learning of the FTC. It explores how well students can use the FTC in a contextual problem. This question might help show whether students have a geometrical interpretation of the FTC. Students in both Cases have not previously solved contextual problems in relation to the FTC in their classes

(Section 6.1); therefore, it might be a challenge for a number of them. However, they might have seen similar questions to some parts of this question in other courses (i.e., physics courses), or in a differentiation topic.

Q8 might activate a range of cognitive processes in students' minds. As several questions were asked about the graph of rate of change, analysing might be activated for answering the questions. Students might also need to interpret the given graph for themselves to be able to answer the question; therefore, understanding might be involved in the question. Similarly to other questions, remembering and applying might also be necessary for answering some parts of the question.

Q9. Please can you write a problem based on the following graph whose solution would require using the Fundamental Theorem of Calculus?

In relation to the Christou et al., (2005) taxonomy of problem posing processes, Q9 is classified as translating quantitative information because students need to pose an appropriate problem based on the given graph. This question might be challenging for a number of students because they have not practised problem posing in this topic (Section 6.1). Similarly to Q5, creating might be activated for this question for the student to be able to create a mathematical problem based on the FTC. At least remembering should also be activated to recall what the FTC is.

5.2.2 Metacognitive knowledge questions asked in the interviews

Metacognition has three facets (i.e., metacognitive knowledge, skills, and experiences) (Section 3.1.4). Metacognitive knowledge questions are described in this section, and the way metacognitive skills and experiences are measured in the study are explained in the next section.

Knowing what can be considered as different types of RBT knowledge in integral calculus (Section 5.1), especially metacognitive knowledge, affected how the students' interview was designed. For exploring students' metacognitive knowledge, questions were designed (Table 5.1) based on RBT's structure of metacognitive knowledge. Questions 1 to 9 address learning, monitoring and problem-solving strategies that relate to strategic knowledge (Section 5.1.9).

Table 5.1

Questions that probe metacognitive knowledge in relation to integral calculus

Item	Area to be examined	Interview questions used
M1	Learning strategy	Did you attend the lectures/tutorials/classrooms in these topics? If so, could you please describe what you typically did when attending them? Why/Why not? Did you take notes in class? Why/Why not? Did you just listen to the instructor/tutor? Why/Why not? Did you talk to your classmate while the instructor/tutor teaches these topics? Why/Why not? Do you do any pre-reading before attending sessions in relation to these topics? Why/Why not? Do you look at your previous lecture notes, or Anton calculus textbook, etc before coming to the classes? Why/Why not?
M2	Learning strategy	Which source(s) of information do you use for learning about these topics? Why?
M3	Learning strategy	How do you help yourself to learn to calculate the area enclosed between curves? How about when you are learning about the Fundamental Theorem of Calculus? Please justify your answer.
M4	Learning strategy	What prior knowledge do you think you need to be able to solve problems related to finding the area enclosed between curves? How about the Fundamental Theorem of Calculus?
M5	Learning strategy	Do you use any practice or memory strategies for these topics, such as using an acronym for remembering formulas, procedures, and concepts in these topics (e.g., BEDMAS: for order of operations in algebra?
M6	Learning strategy	When you are studying integration do you think about the justification or rationale behind the formulas or do you just try to apply the formulas? Why?
M7	Learning strategy	Have you made a summary of the concepts, formulas, or procedures presented in these topics for yourself? Please explain your answer.
M8	Monitoring strategy	How do you check your answers when solving problems involving finding the area enclosed between curves? How about the Fundamental Theorem of Calculus?
M9	Problem-solving strategy	Do you have a plan for solving problems related to enclosed area between curves (See Appendix 1)? Why/why not? How about a plan for solving problems related to the Fundamental Theorem of Calculus? If not, can you create one now?
M10	Knowledge about cognitive task	Do you solve all questions in relation to finding enclosed area in the same way or do you use different strategies for solving different questions in this topic? How about the solving questions using the Fundamental Theorem of Calculus?
M11	Strength and weaknesses in the topic	What difficulties do you have in learning how to calculate the area enclosed between curves? What are your strengths and weaknesses in this topic? How about the answers to these two questions in relation to the Fundamental Theorem of Calculus?
M12	Goal orientation and attitudes	Why are you taking this course?
M13	Goal orientation and attitudes	Do you like calculus, especially integration?
M14	General question	Have you heard about metacognitive knowledge? If so, what is metacognitive knowledge in terms of learning how to calculate the area enclosed between curves? What is it in terms of learning about the Fundamental Theorem of Calculus?

Note. Questions are ordered based on the structure of RBT's metacognitive knowledge, not the order used in the students' interview (Appendix 1).

M1 explores to what extent students use the teaching in the classes for learning the topics and how they spend their time while they are in the classes. M2 focuses on the resources students use for learning the topics. Are they relying on only one or two resources, or do they use a variety of resources? For instance, use of textbooks as one of the learning resources was reported as being very low by engineering students in a calculus course (Randahl, 2012). Randahl (2012) reported students preferred lecture notes over textbooks. However, the number of studies which have explored the resources used by students at Year 13 and first year university is limited. M3 is a complementary question for M1 and M2 to provide an opportunity for students to give more information about how they learn these topics.

M4 explores to what degree students are aware of the importance of mathematical prior knowledge in learning new mathematical topics (Hailikari, Nevgi, & Komulainen, 2008) as students' difficulties with the FTC are reported to be related to students' prior knowledge about concepts of function, limits, and rate of change (Section 3.2.2).

M5 was designed to explore whether students use any memorising strategies for learning the topics. Memory strategies refers to "techniques specifically tailored to help the learner store new information in memory and retrieve it later" (Oxford & Crookall, 1989, p. 404). Memory strategies have been used by University students for language learning (Mochizuki, 1999); however, there is a lack of research regarding to what degree this type of strategy is being used by students for learning mathematics at Year 13 and first year university. Memory strategies can be helpful if part of assessment in classes focuses on remembering certain information. Considering the fact that formula sheets were given to students in both Cases, and the nature of mathematical questions, it would be interesting to find out to what extent students use this type of strategy for learning mathematics.

M6 was designed for exploring whether students are aware of the importance of proofs and rationales behind the formulas. Proofs in mathematics are an intersection between students and experts (Wilkerson-Jerde & Wilensky, 2011) and are a main part of professional mathematical practice (Hanna & de Villiers, 2008). Proofs have also been identified as a mechanism for people of connecting different parts of the mathematical knowledge (Cuoco, Goldenberg, & Mark, 1996).

M7 explores to what extent students use summarising strategies for learning the topics. The literature in educational research identifies summarising as a tool that helps students to learn (King, 1992; Susar & Akkaya, 2009; Thiede & Anderson, 2003). It helps students to comprehend knowledge and transfer knowledge to long-term memory more easily as it motivates them to do reading for understanding and identify important ideas and express them using their own words (Susar & Akkaya, 2009). Summarising can help students to make connections between the new concepts and their prior knowledge (King, 1992). However, there is a lack of research about how useful summarising strategies are for learning mathematics and how much they are being used by students. Therefore, this question was designed to shed light on this aspect of metacognitive strategies.

M8 explores which monitoring strategies students are aware of that can be used for checking answers in questions related to the integral-area and the FTC. However, their use of these strategies during problem solving, metacognitive skills, was explored using a think aloud protocol (Section 5.2.3). For instance, whether the antiderivative is calculated correctly can be checked by finding the derivative of it and check whether the integrand is obtained or not (Barton & Laird, 2002).

M9 explores the complexity of students' plans for solving the integral-area and the FTC problem. Making a solution plan is part of metacognition in the domain of mathematics (Schoenfeld, 1992). Asking students to make a plan also identifies whether verifying answers is part of their plan or not.

M10 relates to knowledge about cognitive tasks, including appropriate contextual and conditional knowledge (Section 5.1.10). This question explores to what extent students are aware that different strategies might be necessary for solving different types of problems.

M11 to 13 address the third subtype, self-knowledge (Section 5.1.11). M11 was asked so as to explore whether students are aware of their difficulties in learning integral calculus. If they are not aware, it is unlikely they will make an effort to understand the materials they have difficulty with (Section 5.1.11). M12 and 13 were used to identify students' goal orientation toward learning integral calculus. Students' goal orientation has been found to influence students' mathematical achievement (Keys et al., 2012). Students who have a mastery approach toward learning are more engaged in learning and use deeper cognitive strategies. However, students who have performance-avoidance approach are less motivated towards learning (Keys et al., 2012), and the negative correlation between this approach and achievement has been reported (Harackiewicz, Barron, & Elliot, 1998).

M14 is a general question to explore whether students are familiar with the term metacognitive knowledge in general, and in particular, in integral calculus.

5.2.3 Measuring metacognitive experiences and skills

For measuring metacognitive skills and experiences during mathematical problem solving a think-aloud protocol was used. A think-aloud protocol is an effective method for having insight into students' metacognition. In this method, students are asked to verbalise their thoughts during solving/working on tasks (Jacobse & Harskamp, 2012). An advantage of using think-aloud protocols in comparison to metacognitive questionnaires, is that students' metacognitive experiences and skills are gathered directly when it is executed in students' minds; therefore, it is less vulnerable to students' memory distortions (Jacobse & Harskamp, 2012). Several activities are suggested for analysing metacognitive experiences and skills (Veenman, Kerseboom, Imthorn, 2000; Jacobse & Harskamp, 2012). The study used five items from Veenman et al. (2000) and Jacobse & Harskamp (2012) (Table 5.2). Two are related to metacognition experience (ME1 and ME2), and the remaining three items are related to metacognitive skills (Section 3.1.4).

Table 5.2

Items explored in relation to metacogntivie experiences and skills

	Themes of analysis in relation to metacognitive experiences and skills
ME1	Having an accurate pre-judgment of whether student is able to solve the problem
MS1	Making a drawing related to the problem
MS2	Making a calculation plan and systematically doing it
MS3	Checking calculations and answer
ME2	Having an accurate post-judgment of how student solved the problem

In relation to measuring ME1 and ME2, before solving the integral questions students were asked to answer a question: How well do you think you can solve this problem? They could choose one of the following judgments: I am sure I will solve this problem; I am not sure whether I will solve this problem correctly or incorrectly; or I am sure I cannot solve this problem (Appendix 1). After choosing one of these, they were encouraged to provide reasons for their choice. A similar question was asked after they solved the problem; rate your confidence for having found the correct answer. Again, students had three choices and were asked to provide reasons for their choice. These two questions formed the instrument for measuring students' metacognitive experiences and were adapted from the VisA instrument (Jacobse & Harskamp, 2012).

MS1 was chosen because making a drawing in relation to the given problem was found to be an important factor in mathematical problem solving (See Jacobse & Harskamp, 2012). MS2 was selected because it shows to what extent students use their metacognitive knowledge and regulate their problem solving strategies while solving problems. Finally, MS3 was chosen to explore whether students monitor their problem solving during solving mathematical problems.

5.3 Chapter summary

In this chapter, the RBT knowledge dimension has been contextualised for integral calculus and the detailed structure of students' interviews is described and justified. The next Chapter describes the context of the study.

Chapter Six: Context of the Study

This chapter has two sections for describing the context of the study. The first section describes how integral calculus was taught - in particular, how the integral-area relationship and the FTC were taught in the University and the College, to set the scene for the student results described in Chapter Seven to Nine. The second section presents analysis of the opinions of lecturers and teachers towards students' difficulties in these two topics.

6.1 The calculus courses in the University and the College

This section shows to what degree the teaching and assessment of the integral-area relationship and the FTC differed across the two Cases. First, the structure of the calculus course at the University, Case 1, is presented (Section 6.1.1), followed by the structure of the calculus course at the College, Case 2 (Section 6.1.2). In the discussion section (Section 6.1.3), the major differences and similarities between these two Cases are highlighted, and possible reasons for such differences/similarities are discussed.

6.1.1 The calculus course in Case 1

The calculus course, which includes the topics of limits, derivatives, partial derivatives, integrals, and differential equations, consisted of three one-hour lectures and four one-hour tutorials per week. Two tutorials were run by the lecturer and two by post-graduate students. Attending lectures and tutorials was not mandatory, but students were asked to sign up for tutorials and were encouraged to attend (Figure 6.1). Assignment and tutorial questions were set every week relating to the content that was covered by the lecturer.

In the lectures, the lecturer did the mathematics (e.g., solved the mathematical examples and proved theorems while students observed him). However, on some occasions, (on average approximately three times per session[3]), the lecturer asked students how an example could be solved (e.g., "How would you suggest that we do [$\frac{d}{dz}[\int_6^{z^2} \sin s \, ds]$? Any suggestions?"(L11[4])). On those occasions, on average, approximately half of the time[5], one or two students responded to

[3] This information is obtained from the observational tool (Appendix 2).
[4] Lecturer 1 from University 1, see Table 4.3
[5] This information is obtained from the observational tool (Appendix 2).

the lecturer's questions, and during the other half no students responded. After presenting a concept, the lecturer usually asked students whether they had any questions (e.g., "any questions about the proof" (L11)). For each topic, the teaching structure was as follows. First, the lecturer provided the motivation behind teaching the topic (e.g., why the Riemann sums are useful for finding area under curves), then, necessary definitions and theorems were provided. The theorems were proved by the lecturer, followed by several examples for illustration.

In the tutorials, tutorial questions were discussed to help students have a better understanding of the topics. Depending on the tutor's teaching style either the students or the tutor, or students and tutor together, solved the questions in the class. If the tutor preferred group work, groups of three or four students were formed and tried to solve the questions together. If they had a problem, they asked the tutor to help them. Tutors who preferred a tutor-centred style solved all questions on the board with the help of students, and students who had problems with any these asked questions about them. The third style was that students had a chance to work on each question for a few minutes, individually or as a group, then a volunteer came to the board to solve the question. If no one volunteered, the tutor solved the question on the board, and during the process encouraged students to suggest the next steps for solving it.

In relation to assignments, several assignment questions which were similar to tutorial questions were given to students at the start of each week. Students were expected to solve them by Friday of that week. All assignment questions were marked by post-graduate students and results contributed towards 10 percent of students' final grades. In addition, the lecturer encouraged marker to provide feedback on student-working, however, providing feedback was not mandatory. The marked assignments were given back to students to help them understand their difficulties. The lecturer also provided a model answer for assignment questions on the website of the course after the due date of solving the assignment questions. Students could also attend several Helpdesks[6] that were run by postgraduate students and faculty members. Attending Helpdesks was not compulsory, but students were encouraged to attend.

[6] During Helpdesk time, students could go to a staff member/post graduate's office and ask questions about the content of the course.

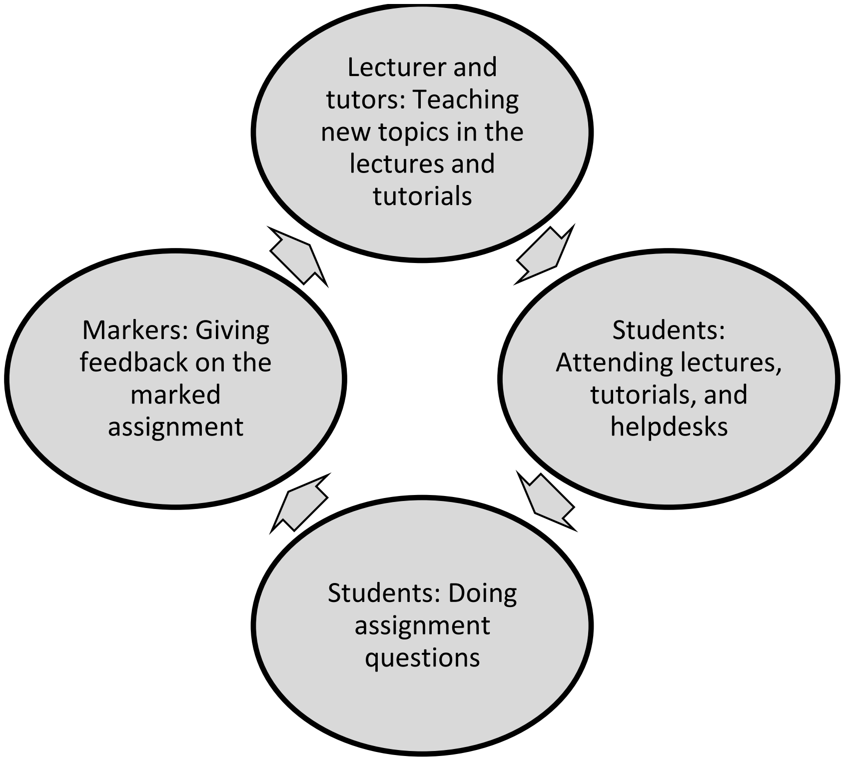


Figure 6.1 Structure of the University teaching

The course had a textbook (i.e., Anton et al., 2012) but it was not mandatory for students to have. A summary of topics covered in the course based on the textbook was also provided as a PDF file in the webpage of the course.

Teaching of the integral-area relationship and the Fundamental Theorem of Calculus in the University calculus course

In this section, first, the overall structure of the relevant sections of the course to the study is presented, then the detailed structure of topics related to the study is described.

Overall structure

The integral-area relationship and the FTC were taught in the second and third weeks of the course, after reviewing the following topics: limits, derivatives, and functions (Table 6.1). These three topics were revised by providing relevant theorems (e.g., Rolle's theorem) and by working through examples.

Table 6.1

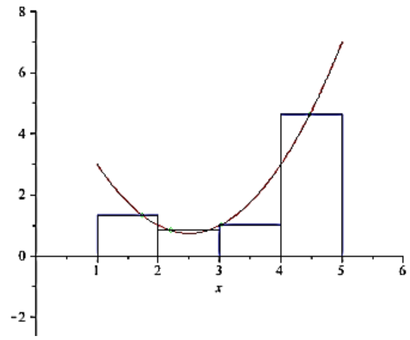
Overall structure of the first half of the University course

Week	Session	Topic
1st week	14.07.2014	Reviewing limits
		Closed/ Open intervals
	16.07.2014	Implicit differentiation
		Reviewing differentiation, trigonometric functions, and inverse functions
	18.07.2014	Reviewing trigonometric functions and inverse functions continued
		Rolle's Theorem
2nd week	21.07.2014	Rolle's Theorem (continued)
		Computing area under a curve using Riemann integral
	23.07.2014	Calculating Riemann sums
		Properties of integrals
	25.07.2014	The mean value theorem
		The Fundamental Theorem of Calculus
3rd week	28.07.2014	The Fundamental Theorem of Calculus (continued)
		Integration by substitution
	30.07.2014	Calculating area between curves
	01.08.2014	Volumes by slicing: Disks and washers methods
4th week	04.08.2014	Volumes by slicing: Disks and washers methods (continued)
	06.08.2014	Volume by Cylindrical shells
		Length of planar curve
	08.08.2014	Area of a surface of revolution
5th week	11.08.2014	Hyperbolic and Inverse hyperbolic trigonometric functions and their integration
	13.08.2014	Integration by parts
	15.08.2014	Integrating trigonometric functions
6th week	18.08.2014	Integrating trigonometric functions continued
		Trigonometric substitution
	20.08.2014	Trigonometric substitution continued

The lecturer started the integral-area relationship topic by introducing Riemann sums and Riemann integral. Then, the FTC and integration by substitution were introduced. Calculation of volumes, length of planar curves, and area of revolution were taught before techniques for finding antiderivatives of different types of integrand. Students' interviews were conducted after the sixth week of the course and before the start of the second half of the course. Partial fractions techniques and the improper integral were taught in the second half of the course.

Detailed structure

In the University course, how areas under a curve and areas enclosed between curves can be calculated using integral calculus was justified using Riemann sums and Riemann integral. The lecturer started the topic using the following example:



…A consideration for the integration part [of the course] is to calculate the area. So what do we want to do? Assuming that we have a function f, and let us for simplicity assume the value of f is non-negative…and we want to calculate the area under this graph from 1 to 5… How could we do that? Again, we have situations that we have not seen before and we try to do calculate the area. So we try to pull out as much as we can from what we know... we know how to calculate the area of simple figures like triangle, square... In this case, what could we do? Well, we could divide the region that we want to calculate the area of into small pieces of the form that we know, triangle, square, rectangle, etc, and we [could then] add up the area. That process should work for any region... (L11).

The lecturer continued by presenting the ideas (proof) of Riemann sums and Riemann integral. Before teaching the FTC and how it can be used for finding areas, he explained how the area under a curve can be approximated using Riemann integral. Then, integrable functions based on Riemann sums were defined and a counter example and an example were presented for illustration (Example 1 and 2, Table 6.2). Before teaching the FTC, selected properties of integrals were introduced to students including:

$\int_a^b [\alpha f(x) + \beta g(x)]\, dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$ and $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ assuming α and β are constant and f and g are integrable functions.

Table 6.2

Examples solved during the teaching of the University course related to the topics of the study

Session	Examples
23.07.2014	1) Example of non-integrable function $f(x) = \begin{cases} 0, & if\ x\ rational \\ 1, & if\ x\ irrational \end{cases}$
	2) $\int_1^4 (3x+1)dx$ using the Riemann integral.
25.07.2014	3) $\int_1^4 (3x+1)dx$ using the FTC.
	4) $\frac{d}{dx}\left[\int_1^x \sin t\, dt\right]$ 5) $\frac{d}{dy}\left[\int_y^7 \sin v\, dv\right]$
	6) $\frac{d}{dz}\left[\int_6^{z^2} \sin s\ \ ds\right]$ 7) $\frac{d}{dt}\left[\int_{-t}^{t^2} \sin x\, dx\right]$
28.07.2014	8) $\int (3x+1)\, dx$
	9) $\int (3t+1)\, dt$
	10) $\int_0^x (3t+1)\, dt$
30.07.2014	11) Calculate the area enclosed between $y = x + 1$ and $= -\sin x$ between $x = \frac{\pi}{2}$ and $x = \pi$.
	12) Calculated area of the region enclosed by $y = x^3$, $y = x$, $x = -\frac{1}{2}$ and $x = \frac{1}{2}$.
	13) Calculated area of the region enclosed by $y = x^3$, $y = x$.
	14) Calculate the area enclosed by $= \sqrt{x}$, $y = -\sqrt{x}$, and $y = x - 2$.

The lecturer taught the first part of the FTC and then proved it using the Mean-Value theorem. It seems for illustration, and to show how useful the theorem is, the lecturer calculated Example 2 from the previous session using the FTC. Teaching the FTC was continued by introducing the second part of the FTC. However, when the lecturer presented the second part, the geometric interpretation was not provided. "…The derivative of the big function F is equal to small function f for every x in the interval I. So, that is the conclusion of the fundamental theorem of calculus, part two" (L11). For proving this part, the Integral Mean Value Theorem was also presented and proved. Then several examples were solved by the lecturer to show how the second part of the FTC could be used (Examples 4 to 7, Table 6.2). The lecturer also provided three examples (Examples 8 to 10, Table 6.2) to show the role of dummy variables in integrals. However, the lecturer, while solving those examples, did not highlight the geometric interpretation of the FTC, only focusing on how those examples could be solved.

The lecturer continued teaching the integral-area relationship after introducing the integration by substitution method. He showed how the area enclosed between two curves, $f(x)$ and $g(x)$, could be calculated using Riemann sums and Riemann integral. It seems several examples (Example 11 to 14, Table 6.2) were solved for illustration and to point out the importance of determining which function is the upper/lower function (for integration with respect to the x-axis) or which function is on the left/right (for integration with respect to the y-axis). The lecturer after finding the area with integration with respect to the x-axis for Example 14, explained how the area could be calculated using integration with respect to the y-axis, and also solved Example 14 using this method.

Assessment of the integral-area relationship and the Fundamental Theorem of Calculus in the University Calculus course

In this section, tutorial, assignment, and midterm questions used for assessing student integral calculus performance at the University course are presented to provide more information about the context of the study. The questions are described in three categories including Riemann sums and Riemann integral, the integral-area relationship, and the FTC.

The Riemann sums and Riemann integral

Three tutorial questions and three assignment questions of the second week and one question in the mid-term exam were related to Riemann sums. In the midterm exam, one question was also related to Riemann sums (Table 6.3).

Table 6.3

Assessment questions related to Riemann sums and Riemann integral

Question type	Question
Tutorial	Apply the definition of the integral via [the] Riemann sums to compute the following using [the] left endpoints: $\int_1^3 x^2 dx$
	Use Riemann sums to show that if f and g are integrable on $[a,b]$ and $f(x) \geq g(x)$ for $x \in [a,b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
	Evaluate the following limit by interpreting it as a Riemann sum and then computing the corresponding integral using the Fundamental Theorem of Calculus: $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{4n}\ sec^2(\frac{\pi k}{4n})$$
Assignment	Apply the definition of the integral via Riemann sums to compute the following using [the] right endpoints: $\int_{-2}^{1}(x+2)^3 dx$
	Use Riemann sums to show that if f is integrable on $[a,b]$ and α is a constant then $\int_a^b [\alpha f(x)]dx = \alpha \int_a^b f(x)dx$
	Evaluate the following limit by interpreting it as a Riemann sum and then computing the corresponding integral using the Fundamental Theorem of Calculus $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}$$
Mid term exam	Express the following as the limit of a sequence of Riemann sums, and then find this limit by calculating the appropriate integral. $$\lim_{n\to\infty}\sum_{k=1}^{n^2}\frac{k^{\frac{3}{2}}}{n^5}$$ Note that the number of the terms in the summation is n^2.

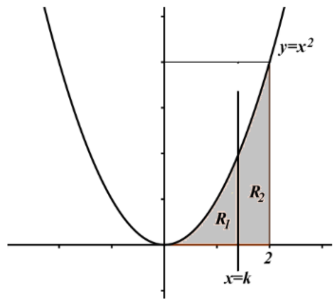
These questions show that Riemann sums and Riemann integral were one of the foci of the assessment of integral calculus in Case 1.

The integral-area relationship

In the University course, four tutorial and two assignment questions were related to the integral-area relationship, particularly focussing on the area enclosed between curves (Table 6.4). However, there was no question in the midterm exam in relation to this topic.

Table 6.4

Assessment questions related to the integral-area relationship at the University course

Question type	Question
Tutorial	1) Consider the region R enclosed by the curves $y = cos\, x$, $y = sin\, x$, $x = 0$, and $x = \frac{\pi}{4}$. Sketch the described region and find its area. 2) Consider the region R enclosed by the curves $y = x^2$, $x = y^2$. Sketch the described region and find its area. 3) Consider the region R enclosed by the curves $y = x^3 - 3x$ and $y = 2x^2$. Sketch the described region and find its area. 4) Suppose that the shadow area of the Figure below was split into two regions by the line $x = k$. What should be the value of k such that both regions have the same area? 
Assignment	5) Consider the region R enclosed by the curves y = e^{-2x}, $y = 0, x = 0$, and $x = 1$. Sketch the described region and find its area. 6) Consider the region R enclosed by the curves $x = y^2$ and $x = y + 2$. Sketch the described region and find its area.
Mid term exam	No question in the midterm exam

The questions covered different types of regions such as regions under the x-axis, and regions bounded by the y-axis. In addition, to find the correct answer students also needed to be able to split the region, because the upper function changes across the enclosed area (e.g., Tutorial Question 3).

The Fundamental Theorem of Calculus

Questions where the FTC was the focus of the assessment are presented here (Table 6.5). Questions related to finding the definite integral are not covered completely in this section as it is not the focus of the study. A few examples (six out of 13 definite integrals) are presented here to illustrate the level of complexity of integral techniques needed for solving them.

Table 6.5

Assessment questions related to the FTC in the University course

Question type	Question
Tutorial	1) Use the Fundamental Theorem of Calculus, Part 1 (hence using antiderivatives) to evaluate the following integrals: $(a)\ \int_1^4 \left(x^2 - 6x + 8\sqrt{x}\right)dx \quad (b)\ \int_0^{\frac{1}{\sqrt{2}}} \frac{dx}{\sqrt{1-x^2}} \quad (c)\ \int_2^3 (e^t + 1)dx$ 2) Use the Fundamental Theorem of Calculus, Part 2 to find the following: $(a)\ \frac{d}{dx}\left[\int_e^x \tan(t^3)dt\right] \quad (b)\ \frac{d}{dx}\left[\int_4^{x^4} e^{\sqrt{u}}du\right] \quad (c)\ \frac{d}{dx}\left[\int_{3x-1}^{3x+5} t^7\right]$
Assignment	3) Use the Fundamental Theorem of Calculus, Part 1 (hence using antiderivatives) to evaluate the following integrals: $(a)\ \int_{-1}^1 \left(3^x + \sqrt[3]{x^2}\right)dx \quad (b)\ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2}{sin^2\theta}d\theta \quad (c)\ \int_1^4 \left(\frac{3+x^3}{x} + \sqrt{x}\right)dx$ 4) Use the Fundamental Theorem of Calculus, Part 2 to find the following: $(a)\ \frac{d}{dx}\left[\int_2^x \frac{dt}{t+\ln t}\right] \quad (b)\ \frac{d}{du}\left[\int_2^{e^u} \frac{1}{1+\|x\|}dx\right]$ $(c)\ \frac{d}{dx}\left[\int_x^1 \sin(t^2)dt\right] \quad (d)\ \frac{d}{dx}\left[\int_{\sin x}^{\cos x} \sqrt{s^4+9}ds\right]$
Mid term exam	5) Find the following integral: $(a)\ \int_0^1 \left(2x^2 + x + \sqrt{x}\right)dx \quad (b)\ \int_0^{\sqrt{\frac{\pi}{2}}} xsec^2\left(x^2 - \frac{\pi}{4}\right)dx$ 6) Find the following: $(a)\ \frac{d}{dx}\left[\int_3^x \sin(e^t + 2^t)dt\right] \quad (b)\ \frac{d}{dy}\left[\int_{y^2}^1 e^{x^2}dx\right]$

Table 6.5 shows that all aspects of the FTC were not covered in the assessment. For instance, answering these FTC questions did not require geometric understanding of the FTC. Students could answer these questions with instrumental understanding (Section 3.1.2) of, and procedural knowledge (Section 3.1.3) about the FTC.

6.1.2 The calculus course in Case 2

The calculus course at this College consisted of five one-hour sessions every six school days. Normally, the teacher, before teaching a new topic, encouraged students to ask questions arising from previous topics, especially the topic of the last session (Figure 6.2). Students were encouraged to ask questions about homework that they found hard, and the teacher discussed those questions in the class and solved them with the help of the students. Then, the teacher typically taught a new concept for around 20 minutes and solved examples related to the topic with the help of the students. Then, the teacher selected a range of questions from the textbook for students, who were encouraged to solve those questions in the remaining session time and for homework. On those time, some students worked alone and some worked with their peers (in a group of 2 or 3) for solving the questions. During the time students worked on the textbook questions, the teacher circled around the classroom and talked to students individually, helping them with their difficulties in the topic/exercise questions.

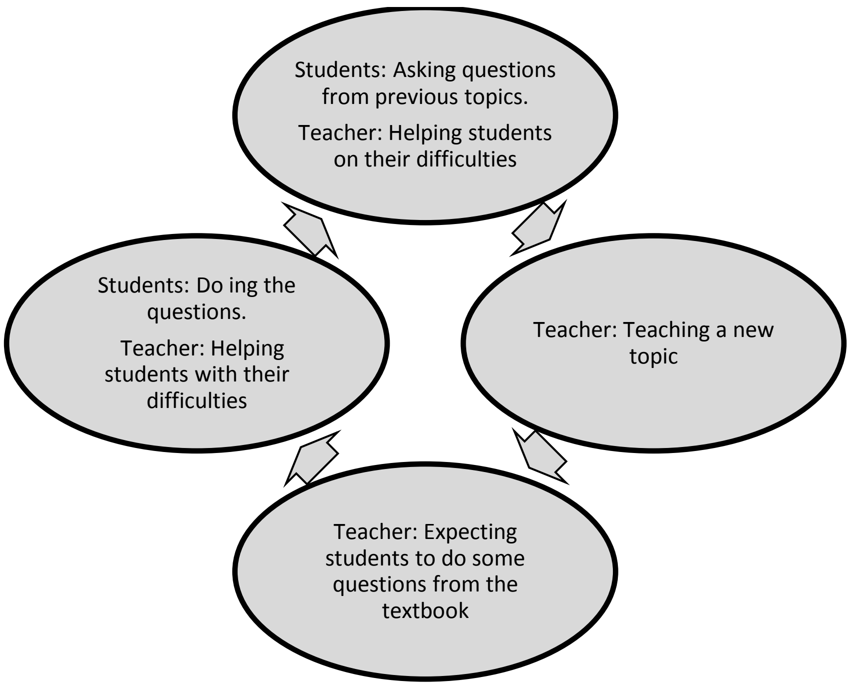


Figure 6.2 Structure of the College teaching

The homework was not marked by the teacher[7] or anyone else and therefore is not discussed here. Only homework questions that the students asked about and that had been discussed in the classroom are presented in this section.

Teaching of the integral-area relationship and the Fundamental Theorem of Calculus in the College calculus course

In this section, similar to Case 1, the overall and detailed structure is presented.

[7] Only the final answers, not the step by step solution, are available at the back of the textbook.

Overall structure

The integral-area relationships and the FTC were taught in the College course after teaching how to find antiderivatives of different types of integrands (Table 6.6).

Table 6.6

Overall structure of the course at the College

Date	Topics
30.07.2014	Introducing antidifferentiation (integration)
	Integration of polynomials
31.07.2014	Integration of exponential and trigonometric functions (session 1)
	Integration by substitution (session 1)
01.08.2014	Integration by substitution (session 2)
	Particular and general solution of an integral
5.08.2014	Integration of trigonometric functions (session 2)
07.08.2014	Integration by substitution (session 3)
	Integration of trigonometric functions (session 3)
08.08.2014	Integration of trigonometric functions (session 4)
	Integration by substitution (session 4)
11.08.2014	Integration by substitution (session 5)
13.08.2014	Definite and indefinite integral
15.08.2014	Definite integral (session 2)
18.08.2014	Calculating areas under a curve (session 1)
19.08.2014	Calculating areas under a curve (session 2)
21.08.2014	Calculating areas enclosed between curves (session 1)
22.08.2014	Calculating areas enclosed between curves (session 2)
25.08.2014	Numerical integration: The trapezium rule
26.08.2014	Numerical integration: The Simpson's rule
27.08.2014	Comparing the trapezium and Simpson's rule
	Introducing Riemann integral
02.09.2014	Using Calculus for solving Kinematics problems
03.09.2014	Differential equations

Not all aspects of the FTC, in particular the second part of the FTC, were covered in the teaching. Those that were covered were taught in the name of the definite integral (Table 6.6).

Detailed structure

The FTC was not covered completely in the College course. The teacher introduced the FTC as follows:

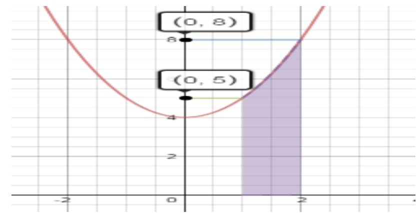
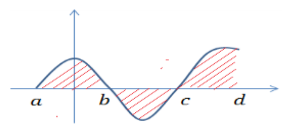
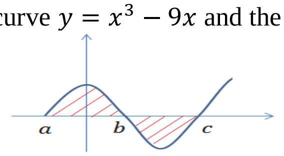
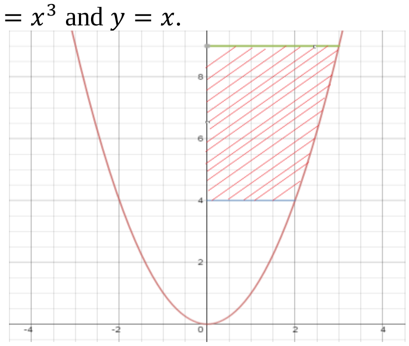
$\int f(x)dx$ and $\int_a^b f(x)dx$ are two different things but related. The first is the indefinite integral and gives us a function. This [the second one] is called a definite integral and it is a fixed value, a number. If $F(x)$ is the integral of $f(x)$, then, (T5)

$\int_a^b f(x)dx = F(x) + C]_a^b = (F(b) + C) - (F(a) + C) = F(b) - F(a)$ (Boardwork, T5).

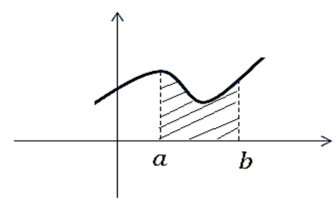
The teacher illustrated the definite integral using several questions (Table 6.7).

Table 6.7

Examples solved during the teaching of the College course related to the topics of the study

Date	Examples solved during the teaching
13.08.2014	1) $\int_1^3 (x^2 + 5x)dx$ 2) $\int_1^2 x\sqrt{x^2+1}dx$
15.08.2014	3) $\int_{-2}^{-1} \frac{x}{1-x}dx$ 4) $\int_0^1 \frac{6x+4}{3x-2}dx$
18.08.2014	5) Find the shaded area. [The equation of the graph was given to students]  6) Find the area bounded by (enclosed by) the curve$y = x^2 - 3x$ and the x-axis.
19.08.2014	7) Calculate $\int_0^{\pi} \cos x \, dx$ 8) The area bounded by the curve $y = \cos x$, the x −axis, and between the coordinates $x = 0$ and $x = \pi$.  9) Find the shaded area. 10) Find the area bounded by the curve $y = x^3 - 9x$ and the x-axis.
21.08.2014	 11) Find the shaded area. 12) Calculate the area enclosed by the curve $y = \cos 2x$, both axes, and the line $x = \frac{\pi}{2}$. 13) Find the area bounded by $y = x^3$ and $y = x$.
22.08.2014	 14) Find the shaded area.

The area under curves was presented as an application of integration. The teacher did not provide Riemann integral for students. He said, I am going to take the expedient route in introducing this to begin with. I am going to give you the application, it is like doing the differentiation, saying without proving…I am going to give you the results, what integration is used for and then go back...and even then I won't be able to spend enough time on showing you why to bring the power down, reduce the power by one, I just want to show you where it comes from (T5).

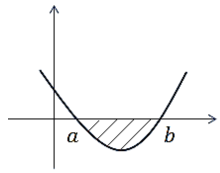
Areas: $\int_a^b f(x)dx \sim$ 

(Board work, T5)

Is that a big surprise to anyone? [The class was silent]. This thing [$f(x)$] that you evaluate is the shaded area. What that says, is the area trapped between the curve lies above the x-axis between the ordinates a and b, is what comes out of that. That's what it is. There are a few other things we need to [care about], if the curve is underneath the x-axis, we need to be careful about that. At that moment just let's say that is a continuous function (T5).

It seems the teacher provided several examples to show how areas could be calculated using integration to facilitate students' learning (Table 6.7). Example 5 was designed in a way that drew students' attention to the fact that the shaded area is bounded by lines on the x-axis, not the y-axis. Example 6 was posed in a way that emphasised the fact that drawing is important for finding the area using integration. He said, "Have you got a visual on this? Would that be helpful?". Then, he sketched the curve and found the area. It appeared to the researcher that the teacher also posed Example 6 in a way that taught how areas under the x-axis should be considered and treated. Then, the teacher sketched the graph of Cos x between 0 to π, presumably to point out the fact that an area under a curve between two points is not always equal to the corresponding integral of that curve between those points. He told students that the area is not zero here, but the integral is. The teacher highlighted the need for splitting the integral for finding the area using integration.

The teacher then continued with his previous topic, how to solve area questions. "Areas below the x-axis are negatively signed, i.e., if we have this [the figure below] is it equal to area? It is equal to -(area)" (T5).



(Board work, T5)

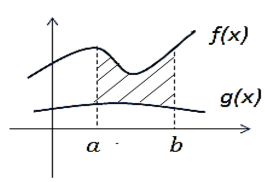
The teacher provided one of the properties of integral calculus which is "$\int_a^b = -\int_b^a$ " and derived it using the following rationale:

$$\int_a^b f(x)dx = F(x) + C]_a^b = (F(b) + C) - (F(a) + C) = F(b) - F(a)$$

$$\int_b^a f(x)dx = F(x) + C]_b^a = (F(a) + C) - (F(b) + C) = F(a) - F(b) = -[F(b) - F(a)] = -\int_a^b f(x)dx$$ (Board work, T5)

The teacher continued his teaching by providing Examples 7 and 8 that helped students understand the difference between finding areas and calculating integrals. The teacher did not draw the graph at the start. A few students mentioned, "these are the same questions", and pointed out the integral is zero as well as the enclosed area. Then, one student mentioned that the integral needed to be split for finding the enclosed area. Then the teacher placed emphasis on the fact that the graph should be drawn so as to explain what the student said. Next, the teacher calculated the area. He showed the students four ways to calculate the enclosed area and encouraged them to try these to see which way they preferred. The ways are $\int_0^{\frac{\pi}{2}} \cos x \ dx + \left|\int_{\frac{\pi}{2}}^{\pi} \cos x \ dx\right|$, $\int_0^{\frac{\pi}{2}} \cos x \ dx - \int_{\frac{\pi}{2}}^{\pi} \cos x \ dx$, $\int_0^{\frac{\pi}{2}} \cos x \ dx + \int_{\pi}^{\frac{\pi}{2}} \cos x \ dx$, and $2\int_0^{\frac{\pi}{2}} \cos x \ dx$. Afterward, two other examples (Examples 9 and 10) were presented to students presumably to help them to have a better understanding of how the areas under the x-axis should be treated.

The teacher showed how the enclosed area between two curves can be calculated as follows. "If $f(x)$ and $g(x)$ are continuous functions over $[a, b]$ and if $f(x) > g(x) \ \ \forall x \in [a, b]$ area enclosed by each graph= $\int_a^b f(x) - g(x)dx$"(T5).

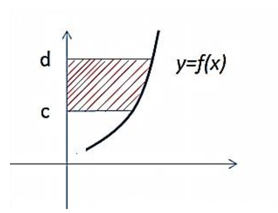


(Board work, T5)

He illustrated this by providing Example 13 (Table 6.7). As he solved the example, the teacher pointed out that if the curves are below the axis, the procedure could still be used. He said, "It does not matter if you are underneath the axis or what, if you got the ascendant function minusing the lower function, it always works out properly for you". The teacher also justified why that formula works by considering the area underneath each curve separately and subtracting them.

The teacher then showed how the area with the y-axis as a boundary could be calculated by providing Example 14 (Table 6.7). First, two methods were presented for calculating the shaded area in the example using integration with respect to the x-axis. The first method, suggested by a student, was subtracting the area under the curve $y = 4$ between zero to two, and the area under the curve $y = x^2$ between zero and three from the area under the curve $y = 9$ between zero and three. The second method was to bring the curve down four units and subtract the area under the curve $y = x^2 - 4$ between two to three from the area under the curve $y = 5$ between zero and three. For calculation of the area using integration with respect to y-axis, the teacher provided the following formula without any justification.

$\int_c^d x\, dy \quad \sim$



(Boardwork, T5)

The teacher, after finishing the numerical integration topics, introduced the idea of Riemann sums and Riemann integral for a decreasing function above the x-axis. However, he did not solve any example using Riemann integral and did not mention the name of the method.

Assessment of the integral-area relationship and the Fundamental Theorem of Calculus in the College calculus course

In this section, integral questions (Table 6.8) that were used in the internal assessment[8] of the regular and the scholarship classes are provided to describe assessment in the context of the study.

[8] The examination administered at the end of the topic within the College for preparing students for the external assessment.

Table 6.8

Assessment questions related to integral calculus at the College

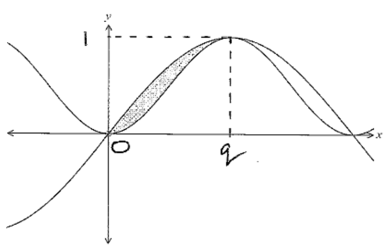
<table>
<tr><th>Type</th><th>Questions</th></tr>
<tr><td>Scholarship</td><td>1) i) Find $\frac{d}{dx}\,(x\cos(x))$ and use this result to find $\int x\sin(x)\,dx$.
ii) Hence find the value of $\int_0^{n\pi} x sin(x) dx$ for positive integer value of n.</td></tr>
<tr><td>Internal</td><td>2) Find the following integrals. You do not need to simplify your answers. Do not forget the constant of integration.
a) $\int \frac{4x^4+3}{x} dx$ b) $\int \left(\sqrt[5]{x} - \frac{2}{x^2}\right) dx$
c) $\int 4\sec(2x)\tan(2x)dx$ d) $\int (e^{3x} - cosx)dx$
3) The following table shows some values of the function $y = \frac{1}{\sqrt{x^2+6}}$
<table>
<tr><td>x</td><td>0</td><td>2</td><td>4</td><td>6</td><td>8</td><td>10</td><td>12</td></tr>
<tr><td>y</td><td>0.4082</td><td>0.3162</td><td>0.2132</td><td>0.1543</td><td>0.1195</td><td>0.0971</td><td>0.0816</td></tr>
</table>
Use the Trapezium Rule to estimate the value of the integral $\int_0^{12} \frac{1}{\sqrt{x^2+6}} dx$
4) Find the following integral.
a) $\int \frac{5x^2}{4x^3+3} dx$ b) $\int \frac{4x}{\sqrt{2x-1}} dx$
5) Calculate the area of the region enclosed by the functions $x = y^2$ and $y = x - 2$.
[The picture of area was given in the exam.]
6) The shaded region in the given graph is bounded by the two functions
$y = sin\ kx$
$y = sin^2\ kx$ where k is a natural number.
Find the area of the shaded region in term of k.
Given the results of any integration needed to solve this problem.

</td></tr>
</table>

Table 6.8 shows that the integral-area relationship and finding of antiderivatives using integral techniques were the focus of the assessment. The FTC was not the focus of the assessment in this College. There was no assessment question related to the Riemann integral. However, one question was related to the one of the numerical integration methods (i.e., the Trapezium rule).

6.1.3 Discussion

With reference to the literature relating to the teaching of integral calculus (Section 3.2.1 and Section 3.2.2): in the both Cases described, diagrams and graphs were used to help students have a better understanding of the relationship between definite integral and area. In both Cases, the examples solved in classes in relation to finding areas, the given curves were sketched before or during the solution process. However, in Case 1, when solving questions that related to finding the derivative of integrals using the second part of the FTC (Example 4 to 7, Table 6.2), graphs were not used for illustrating why $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$. This did not happen in Case 2, as the second part of the FTC was not taught to the students.

In relation to the suggestion in the literature that the focus of integral calculus teaching should be on determining the enclosed area as a limit of a sum rather than on integration techniques for solving different types of integrals: in Case 1, Riemann sums and Riemann integral were taught and used in the assessment; whereas in Case 2, Riemann sums were introduced at the end of the teaching of integral calculus without being illustrated, and did not appear in the assessment. In terms of the FTC, in both Cases, the accumulation area function was not the focus of assessment and teaching.

The reason for not focusing on Riemann sums and the accumulation function in Case 2 is likely to be related to the *New Zealand Curriculum* (Section 1.2.2). As stated in Section 1.2.2 above, Ministry of Education (2007a) and the NCEA level 3 mathematics achievement standards (New Zealand Qualifications Authority, 2013) focus on problem solving that requires using antidifferentiation techniques. The FTC and Riemann sums are not highlighted in the curriculum, and therefore are not being taught by teachers and did not appear in the assessments.

In relation to the lens of teaching as transmission versus constructive teaching (Section 3.1.1): in Case 1, the lecturer was the centre of the lectures; that is, an instance of teaching as transmission. There were only a few interactions between the lecturer and students, which may have been related to the structure of the lecture theatre or the large number of students in the lectures. However, in the tutorials more interactions were possible between tutors and students. Students had this chance to work on mathematical questions both individually and as a group to construct their knowledge. In all three styles mentioned (Section 6.1.1) consistent with constructivist teaching tutors tried to encourage students to come up with a solution, rather than providing the answers to the tutorial questions to the students.

In comparison to Case 1, in Case 2, there were more interactions, (one to one between the teacher and students, between students during group work, and talking about the work on the whiteboard between the teacher and students. However, during the teaching of the core concepts, the teacher acted as a knowledge dispenser (Section 6.2.1, Detailed structure). In addition, the teacher presented the core concepts without justifying them, whereas in Case 1, the lecturer followed the traditional way of teaching mathematics, providing Definition, Theorem, Proof (DTP) (Weber, 2004), and examples. It seems there was a higher probability that instrumental learning (Section 3.1.2) happened to students in Case 2 because proofs of theorems were not in the curriculum and were not taught to them, Riemann sums and Riemann integrals were not the focus of the teaching and the assessment, and because of how the teacher taught the core concepts. However, in Case 1, proofs were taught to students, and Riemann sums and Riemann integral were the focus of the teaching and the assessment. Therefore, students interested in having a relational understanding (Section 3.1.2) of the topic could consider those approaches, whereas such opportunity was not provided in Case 2.

6.2 Perspectives of lecturers and teachers towards students' difficulties in learning integral calculus

This section describes lecturers' and teacher's perspectives towards students' difficulties in the integral-area relationships, and the FTC in particular, for first year calculus courses. Their perspective towards students' difficulties is important because those perspectives affect their teaching, therefore, knowing them will provide more information about the study context. In addition, their views are useful for discussion in the following chapters, in order to explore to what extent lectures and teachers are aware of students' difficulties in these topics. Responses of a number of lecturers/teachers were coded in relation to more than one theme/subtheme.

6.2.1 Students' difficulties with the integral-area relationships

Three main themes were found in terms of students' difficulties in understanding the integral-area relationships based on the perspectives of lecturers and teachers. These were: students' learning and life style; setting up integrals; and students' prior knowledge (Table 6.9).

The themes are explained in the order of those mentioned most to least frequently by lecturers and teachers.

In terms of students' approaches to learning, four (L15, 23; T1, 5) believed students do not solve enough questions in this topic:

> I think the main difficulty is not practising enough. I actually think when something like that [integral-area relationship], when we do it in class, is actually not a difficult concept for them. What is difficult is that one concept and we have lots of other concepts and to get good at all of them you need to practise all of them, so the problem is that they do not practice (T1).

Table 6.9

Lecturers' and teachers' perspectives toward students' difficulties in understanding the integral-area relationships

Main themes	Sub themes	Lecturers	Teachers	Total
Student learning approach and lifestyle	Not practising enough questions	2 (L15,23)	2 (T1,5)	4
	Relying on procedural knowledge and having performance approach toward learning the topic	2 (L11,22)	2 (T4,5)	4
	Being dependent on graphic calculators	1 (L22)	1 (T4)	2
	Students' lifestyle	1 (L15)	0	1
Setting up the integral	Difficulty with the concept of upper (bigger) function	2 (L12,14)	1(T4)	3
	Setting up the integral when the enclosed area needs to be split	1 (L13)	2 (T2,3)	3
	Setting up the integral when the integral boundaries are not given	1 (L14)	1 (T2)	2
	Difficulty with choosing the method (i.e., with respect to x or y-axis), and not familiar with integration with respect to y-axis	1 (L21)	0	1
Students' Prior knowledge	Lack of relevant prior knowledge	4 (L12,15,23,24)	1 (T2)	5
	Lack of skills of sketching functions	3 (L14,15,24)	1 (T4)	4

The second sub-theme relates to the fact that a number of students rely on procedural knowledge about integral-area relationships and have a performance approach toward learning, mentioned by four lecturers/teachers (L11, 22; T4, 5). For instance, L22 believed it is hard for students to "grasp the conceptual understanding" of the subject and said, "they [students] just rely on the procedural knowledge and how they pass the course, [is] they feel like they understand it". L11 mentioned "my feeling is that they are not doing properly, they just look at whatever formula they can remember and then apply [it] without trying to understand that". T4 mentioned "These boys [students] are very results driven".

The third sub-theme is about students' dependency on graphic calculators pointed out by two lecturer/teachers (L22; T4):

> a lot of students coming from high school who are just attached to graphic calculators and that's kind of detrimental in some cases that having graphic calculators, especially the last five years, has intensified. We feel that they [students] just, they do not have, need to have a picture in their head of elementary functions (L22).

T4 highlighted, " [the] graphic calculator is a bad tool. It is bad, Farzad. [The] calculator has become an addiction. I am very old fashioned. No computer devices in my classrooms, no computers. I do not like it".

The final sub-theme highlighted by one lecturer (L15) is related to students' lifestyle in that students do not pay enough attention to their study:

> Too much drinking, I am thinking of lifestyle, student lifestyle. I guess this is a general thing, the main difficulty for doing a study... A guy came to the states, and talked about teaching first year calculus. He said the main difficulty with teaching first year calculus is students are in states [where] they are recovering, they are sick because they are drinking too much the night before, or they were sleepy, because they did not get enough sleep the night before, because they are drinking and he said the lifestyle, we always fighting the lifestyle. They are not applying themselves. They are applying themselves to socialisation, not to working.

The second theme relates to setting up the integrals, mentioned by seven lecturers/teachers (L12, 13, 14, 21; T2, 3, 4). Three of them (L12, 14; T4) believed a number of students have difficulties in identifying which function is upper (bigger) when finding the enclosed area between curves. For instance, L14 believed students have difficulty in "getting the functions the right way around, realising which one is the larger, and which one is the smaller". T4 mentioned, "It is interesting that some students have difficulty when you took one function that is bigger than another, what that actually means”.

Three (L13; T2, 3) believed a number of students have difficulty when they need to split the area. For instance, T2 said, “the harder conceptual question would be where I have to break the area and why I need to do that”. One lecturer and one teacher (L14; T2) believed finding the appropriate bounds for an integral is hard for a number of students. Finally, L21 mentioned that a number of students have difficulty in choosing the most efficient method of finding the enclosed area (i.e., integration with respect to the x or y-axis) and are not familiar with the integration with respect to the y-axis:

> What is the most efficient way of doing it? The little dy or dx at the end cause a lot of problem [for students]. All they know is that, I suppose to integrating with respect to the x. They do not know that you are setting up a coordinate system which they can integrate with it... I would say that the kids are so used to using the dx at the end of integral that as soon as they see dy they start double thinking. It is a new thing for them.

The third theme is students’ lack of prior knowledge, mentioned by five lecturers (L12, 14, 15, 23, 24) and two teachers (T2, 4). The prior knowledge necessary for solving integral-area problems from their perspectives are knowledge about functions (L15, 23, 24), graphing functions (L14, 15, 24, T4), algebraic manipulation (L12, T2), and the x-y coordinates (L23). For instance, L23 highlighted:

> You have students with different backgrounds. So you have people who still struggling with the definition of the function, some of them struggling with the x-axis and the y-axis, some of them are fine but they cannot do the derivative.
>
> L15 mentioned:
>
> The difficulty would be graphing functions and knowing what the graph is that is going with the functions. So the difficulty would be the fundamental one, we need to know about the functions, what it means to have $f(x)$, what the graphs tell you about the function…

6.2.2 Students' difficulties with the FTC

From the perspectives of lecturers and teachers, two main themes were found in terms of students’ difficulties in understanding the FTC (Table 6.10). The themes are described in order, based on the number of times they are mentioned.

Table 6.10

Lecturers’ and teachers’ perspectives toward students’ difficulties in understanding the FTC

Main theme	Sub themes	Lecturers	Teachers	Total
Learning approach	Relying on procedural knowledge and having performance approach toward learning the topic	2 (L11, 24)	2 (T1,2)	4
Fundamental Theorem of Calculus	Difficulty with the conceptual knowledge of the FTC	2 (L15, 22)	1 (T2)	3
	Difficulty with the notational aspects of the FTC	2 (L14, 15)	0	2
	Difficulty with the second part of the FTC when the chain rule needs to be used	2 (L11,12)	0	2
	Difficulty with the second part of the FTC	1 (L21)	0	1
	Not knowing the difference between definite integral and antidifferentiation	1 (L14)	0	1
	Difficulty with understanding the accumulation area function	1 (L14)	0	1

In relation to learning approach, two lecturers (L11, 24) and two teachers (T1, 2) believed the difficulty students have with the FTC is related to the fact that students focus on understanding the procedural knowledge underpinning this topic and have a performance orientation toward learning the topic:

> I do not think even they [students] have tried [to understand the FTC]. They are more interested in what is in the exam? What do questions look like? What is the procedure I need to know to get to the end? (T1).

In relation to the FTC and its first sub-theme, two lecturers (L15, 22) and one teacher (T2) mentioned that students have difficulty with understanding the concepts involved in the FTC. For instance, L15 said, “the difficulty would be I think understanding of the fundamental concepts of derivative and integral” and T2 mentioned, “conceptual connection because they [students] do not really understand where it has come from, they just use the results”.

The second sub-theme related to the FTC is students’ difficulties in understanding the notations involved in the FTC, mentioned by two lecturers (L14, 15):

[the difficulty for the students is] understanding what an integral with the variable bounds mean as a function and understanding that there is a distinction between making the bound variable and the variable of integration and why they have to be different things (L14).

The third sub-theme relates to the second part of the FTC; in particular, questions that asked students to find the derivative of integrals using the FTC (e.g., Q2b in Table 6.5). Two lecturers (L11, 12) believed a number of students have difficulty with solving those problems when they need to use the chain rule. L12 mentioned students have difficulty with the “chain rule part, when the top bound is a function of x”.

One lecturer believed that considering the two parts of the FTC, students have difficulty with the second part:

I would say [in] the second part that they have problems…when you start looking at giving the derivative of the particular function and giving the integral and asking the kids to realise that the actual function being differentiated is the one that is the integrand, that requires a few more steps and I think from my experience if the kids have a problem that's the area they have a problem (L21).

L14 mentioned another two difficulties that students have with the FTC. He pointed out that for a number of students the difference between definite integral and antidifferentiation is not clear, “you have to make the point quite strongly that integration is not antidifferentiation”. The second point he mentioned was that students have difficulty with understanding the accumulation area function, “…understanding what an integral with the variable bounds means as a function”.

While answering the interview question, two lecturers (L12, L22), from different Universities, mentioned that conceptual knowledge of the FTC is not focussed on in the assessment in the first year of University:

I think the majority of them [students] in the first year they do not grasp the conceptual understanding and we [lecturers] kind of do not insist on that. We do not really check that. I think they [students are] just terrified. The FTC I think it is worse for students. We relying on procedural understanding (L22).

And L12 highlighted, “we [lecturers] tend to test the applications, not the idea, we do not test the proof and when we go to the assignments and the test, we usually test on calculating”.

One teacher (T4) mentioned students have no difficulty with the FTC which could be because he does not cover all aspects of the FTC in his teaching. T4 told students the FTC is “just a magic” that enables finding area using anti-differentiation.

6.2.3 Discussion

Some of the students’ difficulties with integral calculus mentioned in the literature (Section 3.2) were also mentioned by the lecturers and teachers. For instance, in terms of integral area relationships, difficulties with finding the area when the integrand is below the x-axis, the graph of the integrand is not given in the questions, and reliance on procedural knowledge, were mentioned both in the literature and by lecturers and teachers. In terms of the FTC, difficulty with notations was highlighted by both lecturers and the literature.

Presenting the FTC as a “magic” (T4) or not covering all aspects of it in teaching may cause several problems for students. The first problem might be that students would not be helped to understand the structure of mathematics and think there is no rationale or justification about how formulas are derived. Second, having such an approach may affect students’ attitudes toward mathematics, because if students think mathematics is magic and there is no rationale behind the formulas and theorems, they may tend towards instrumental learning, focusing on memorising and applying procedures. Previous studies revealed that students, both at secondary and tertiary level, (e.g., Grundmeier, Hansen, & Sousa, 2006; Vinner, 1976), mathematics teachers (e.g., Leikin, & Zazkis, 2010), and prospective mathematics teachers (e.g., Levenson, 2012) had confusion about the structure of mathematics (e.g., the difference between definition and theorem, or the distinction between proof and motivation). One of the reasons for this may be related to the fact that the research methodologies of mathematics are not in the focus of curricula, therefore, calculus instructors do not spend, or only spend a short time, informing students about the structure of mathematics (Vinner, 1976).

6.3 Chapter summary

This chapter has described how the integral-area relationship and the FTC were taught in the Cases. In addition, it has shown the perspectives of lecturers and teachers towards students’ difficulties in these two topics. These results have been described to set the scene for the results and analysis provided in the following Chapters.

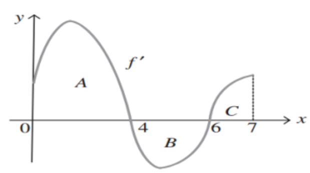
Chapter Seven: Students' Factual, Conceptual, and Procedural Knowledge about Integral-area Relationships and the FTC

In this chapter, students' learning of the integral-area relationships and the FTC is explored in relation to the RBT types of factual, conceptual, and procedural knowledge. The chapter presents results drawn from student interviews (Section 5.2.2). These results provide answers for the second and fourth research questions (Section 1.4). First, the qualitative analysis is presented, and at the end of the chapter the quantitative analysis is described. The integral-related questions used for the interview (Appendix 1) were

Q1. Please calculate the area enclosed between the curve $x = y^2$ and $y = x - 2$ in two ways. Which way is better to use? Why?

Q2. What do you understand by A = $\int_a^b [f(x) - g(x)]dx$ and B = $\int_c^d [w(y) - v(y)]dy$? Can you justify how these formulas are derived? Can you justify when each one is used?

Q3. The graph of f'(x), the derivative of $f(x)$, is sketched below. The area of the regions, $A, B,$ and C are 20, 8, and, 5 square units, respectively. Given that $f(0) = -5$, find the value of $f(6)$. (Mahir, 2009, p. 203).



Q4. Are these examples solved correctly? Please justify your answer.

Ex.1: Find, if possible, the area between the curve $y = x^2 - 4x$ and the x-axis from $x = 0$ to $x = 5$.

$\int_0^5 (x^2 - 4x)dx = [\frac{x^3}{3} - \frac{4x^2}{2}]_{x=0}^{x=5} = \left[\frac{5^3}{3} - \frac{4(5)^2}{2}\right] - \left[\frac{(0)^3}{3} - \frac{4(0)^2}{2}\right] = \frac{-25}{3}$.

Ex.2: Find, if possible, the area enclosed between the curve $y = \frac{1}{x^2}$ and the x-axis from $x = -1$ to $x = 1$.

$$\int_{-1}^{1} \frac{1}{x^2} dx = \int_{-1}^{1} x^{-2} dx = [\frac{(x)^{-1}}{(-1)} = \frac{-1}{x}]_{x=-1}^{x=1} = \frac{-1}{1} - \frac{(-1)}{(-1)} = -2.$$

Q5. Please can you pose a problem about the area enclosed between a curve and a line with any two arbitrary bounds that will give an answer of 1 (i.e., the enclosed area will be equal to one)?

Q6. Find the derivative of the following functions.

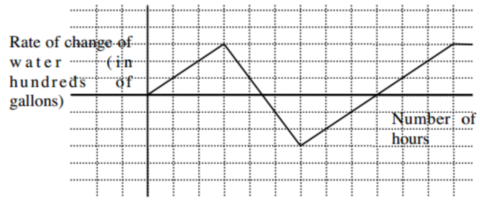
$O(x) = \int_1^x \frac{1-t}{t^2-2t-9} dt$ G$(x) = \int_0^{x^2} r^2\sqrt{1 + r^3}\, dr$ $D(x) = \int_{2x-5}^{4x+4} t^3 dt$

Q7. What do you understand by $F(x) = \int f(x)dx$? What do you understand by $F(x) = \int_a^x f(t)dt$?

What do you understand by $\int_a^b f(x)dx = F(b) - F(a)$? When do you use this formula? Can you justify how it is derived? What do you understand by $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$? When do you use this formula? Can you justify how it is derived?

Q8. Let f represent the rate at which the amount of water in Phoenix's water tank changed in 100's of gallons per hour in a 12 hour period from 6 am to 6 pm last Saturday (Assume that the tank was empty at 6 am (t=0)). Use the graph of f, given below, to answer the following.

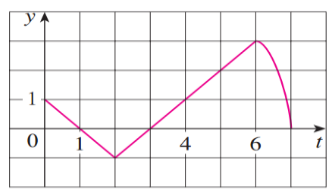
- How much water was in the tank at noon?
- What is the meaning of $g(x) = \int_0^x f(t)dt$?
- What is the value of g(9) ?



- During what intervals of time was the water level decreasing?
- At what time was the tank the fullest?

Using the graph of f given above, construct a rough sketch of the graph of g and explain how the graphs are related. (Carlson, et al., 2003, p. 168-169).

Q9. Please can you write a problem based on the following graph whose solution would require using the Fundamental Theorem of Calculus?



In the mathematical tasks/questions, several aspects of the factual, conceptual, and procedural knowledge are addressed at the same time. Defining notions such as procept as an amalgam of process, symbol, and concept support this idea (Section 3.1.7). In this chapter, the responses of students to the nine questions were provided in terms of 20 themes (Table 7.1), rather than in terms of factual, conceptual, and procedural knowledge because each question exposes different types of knowledge. These themes were inspired by the interim knowledge dimension of integral calculus (Section 5.1).

Table 7.1

Themes of analysis of students' integral calculus performance

Main type of knowledge investigated	Theme
Factual knowledge	Recognising and using symbol of integral, integrand, lower and upper bound, and dx in $\int_a^b f(x)dx$ correctly
Factual knowledge	Definite integral can be used for finding area under curve(s)
Conceptual knowledge	The net signed area under $f(x)$, enclosed by the x-axis, $x = a$, and $x = b$ is equal to $\int_a^b f(x)dx$
Conceptual knowledge	The net signed area under $f(y)$, enclosed by y −axis, $y = c$, and $y = d$ is equal to $\int_c^d f(y)dy$
Conceptual knowledge	Integrand should be continuous on the interval that is being integrated
Conceptual knowledge	Understanding the relationship of enclosed areas between curves, $f(x)$ and $g(x)$, and the definite integral ($\int_a^b [f(x) - g(x)]dx$)
Conceptual knowledge	Understanding the relationship between the definite integral and the limit of Riemann sums: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_i)\Delta x$
Conceptual knowledge	Being able to use the integral-area relationships for a graph of derivative function
Factual/conceptual knowledge	Understanding the meaning of $F(x) = \int f(x)dx$
Factual/conceptual knowledge	Understanding the meaning of $F(x) = \int_a^x f(t)dt$
Factual/conceptual knowledge	Understanding the meaning of $\int_a^b f(x)dx = F(b) - F(a)$
Factual/conceptual knowledge	Understanding the meaning of $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$
Procedural knowledge	Be able to do necessary algebraic manipulations for simplifying integrands, finding intersection points, etc.
Procedural knowledge	Find the antiderivative of $f(x)$ using integral techniques $\int f(x)dx = F(x) + c$
Procedural knowledge	$\int_a^b f(x)dx = F(b) - F(a)$, where $F'(x) = f(x)$
Procedural knowledge	Knowing when it is efficient to use integration with respect to the x or y-axis for solving integral-area problems
Procedural knowledge	Being able to use $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$ for finding the derivative of integrals
Conceptual/procedural knowledge	Being able to use FTC in a contextual problem
Conceptual/procedural knowledge	Being able to pose a question for the integral-area relationship
Conceptual/procedural knowledge	Being able to pose a question in relation to the FTC

The results in terms of factual knowledge indicate that students had difficulty with notational aspects of the FTC. There is evidence that conceptual knowledge was less developed than procedural knowledge for students both in terms of the integral-area relationships and the FTC. To compare the Cases at a macro level (Section 4.1), for each type of knowledge that was discussed in the following sections one point was given. Some sections discussed several aspects of knowledge about integral calculus, therefore, more than one point was allocated to them (e.g., Section 7.17). The total points each student scored was then calculated for comparing students' result in the Cases (Section 7.20).

7.1 Factual knowledge: ***Recognising*** and ***using*** symbols of integral, integrand, lower and upper bound in the definite integral

University (U) and Year 13 (Y) students in the sample did not ask the interviewer any questions about the symbols (integral, integrand, lower and upper bound, and dx in $\int_a^b f(x)dx$) while engaging with integral-area questions. Also, through transcript analysis of students' running commentary during their solution of the problems, no comment was noted that indicated that a student did not understand the questions. However, for the FTC questions, some students reported that they had difficulty with understanding the symbols used in the questions and in the FTC (e.g., Section 7.16). For these reasons, it is assumed that all students recognised symbols in $\int_a^b f(x)dx$.

From the study sample, only two students (U8; Y7) used correct notations and symbols while solving mathematical questions. Two errors were found in students' mathematical writing in relation to integral calculus. First, 15 students (Table 7.2) did not use brackets when integrating more than one term (e.g., instead of $\int_a^b [f(x) + g(x)]dx$, they wrote $\int_a^b f(x) + g(x)dx$). Second, three students (Table 7.2) did not write dx or dy, the symbol that shows integration should be done with respect to x or y, when working on the mathematical questions (e.g., Figure 7.1).

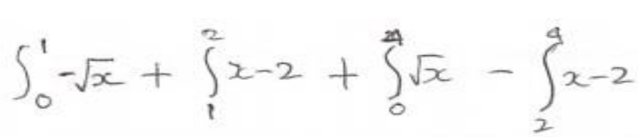


Figure 7.1 An example of missing dx in a solution

Table 7.2

Errors within students' writing in relation to the definite integral

Errors with writing definite integrals	Case 1	Case 2
Have not used brackets when integrating more than one term	8 (U12345679)	7 (Y1234568)
Have not used dx for showing integration should be done with respect to which variable	1 (U2)	2 (Y38)

One possible reason for the school students not using brackets for integrating more than one term is that the teacher did not use these brackets during teaching. However, in Case 1, this does not explain the lack of bracket use by the University students as the lecturer did use them during teaching. Another possibility for the exclusion of brackets and dx was that students used a 'shortcut' by leaving them out of in their working.

Recognising and using symbols contributes two points towards students' mathematical performance scores in integral questions; one for recognising the symbols and one for using them correctly for solving questions. All the students scored one point for recognising the symbols, however only two students, one from each Case, scored one point for using symbols correctly.

7.2 Factual knowledge: ***Remembering*** the definite integral can be used for finding the area under curve(s)

All students, when solving integral questions, remembered that when finding the area under a curve the definite integral should be used. However, there was one student in Case 2 (U2) who used $\int_a^b \pi f^2(x)dx$, the disk formula for finding volume, for finding areas under curves. This section contributes one point to students' mathematical performance ratings in integral questions so all students scored that point.

7.3 Conceptual and procedural knowledge: The net signed area under a curve is equal to the definite integral

In the first section, the way students used integration with respect to the x-axis for finding the enclosed area is explored; in the second section, students' learning in relation to how integration can be used with respect to the y-axis is explored.

7.3.1 ***Calculating*** area using integration with respect to the x-axis

In relation to Q1, only four students (U589; Y8) were able to find the enclosed area with respect to the x-axis correctly. Drawing the graph incorrectly can affect how students set up their integral. Therefore, to investigate whether students had the ability to find the area with respect to the x-axis, the integrals students set up were compared with their drawings. Four students (U13; Y67) set up integrals correctly according to their drawings. One student (Y2) did not use this method and the remaining eight students (U2467; Y1345) had difficulties with finding the enclosed area by using integration with respect to the x-axis.

U4 had difficulty with the fact that $x = y^2$ is not a function, and set up the integral incorrectly (i.e., $\int_0^4 x - 2 - \sqrt{x}\, dx + \int_1^0 x - 2 - \sqrt{x}\, dx$); even though he had sketched the graph correctly. U7 had not realised that if the upper/lower limit changes during the integration interval, the area function will change accordingly. He said, "I find a general formula that would give me the correct answer with whatever bound I put in. So having integrated it, then I can decide what bound I can use" (Figure 7.2).

$$\int x - 2 + \sqrt{x}\, dx + \int x - 2 - \sqrt{x}\, dx$$

$$= \left[\frac{x^2}{2} - 2x + \cancel{\frac{1}{2}x^{3/2}}\right] + \left[\frac{x^2}{2} - 2x - \cancel{\frac{1}{2}x^{3/2}}\right] + C$$

$$= x^2 - 4x + C$$

Figure 7.2 Part of U7's working on Q1

U2 used a wrong formula for finding the enclosed area (Section 7.2). The last University student who had difficulty with finding the area with respect to the x-axis, U6, could not find the second intersection point for the two curves, only found $x = 4$ as one of the intersection points. Also, she set up an incorrect integral for finding the area with respect to the x-axis (i.e., $\int^4 \sqrt{x} - x - 2\, dx$).

In terms of Case 2, Y1 considered only $\int_0^2 x - 2\, dx$ as the integral for finding the enclosed area. Y5 had difficulty with finding the area below the x-axis, mentioning "I am not too sure whether it [$x = y^2$] actually stops at $x = 0$ or would continue around like a parabola in which case there will be some area there that is missed out. I am not sure how to calculate that". Two (Y34) had difficulties in relation to finding intersection points (Section 7.8), preventing them from setting up the correct integrals. This section contributes one point to students' mathematical performance scores in integral questions. Five from Case 1 and three from Case 2 scored that point for finding the enclosed area correctly according to their graph, by using integration with respect to the x-axis.

7.3.2 ***<u>Calculating</u>*** area using integration with respect to the y-axis

In relation to Q1, 11 students (U23456789; Y278) attempted to find the area with respect to the y-axis. Of those, seven students (U45689; Y78) correctly found the enclosed area by using integration with respect to y-axis, while four of these (U569; Y2) used this method as their first choice for finding the enclosed area.

The four students who answered incorrectly (U267; Y2) made various errors in setting up the integrals with respect to the y-axis. Three students (U67; Y2) made a mistake about which function is the top function, considering the curve as the top function. Two students considered the bounds of integration with respect to the x-axis (Figure 7.3) rather than the y-axis (U2; Y2). In addition, U2 used an incorrect formula for finding the enclosed area (Section 7.2). U3 created an incorrect drawing of the graph of $x = y^2$ which led an incorrect lower bound for the integral, -2. The remaining six students (U1;Y13456) did not use this method for finding the enclosed area.

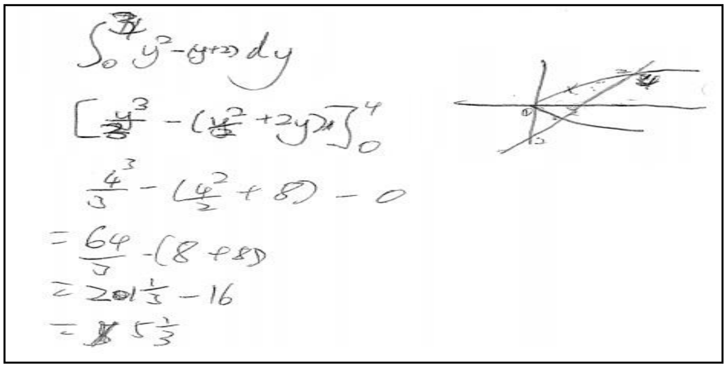


Figure 7.3 Part of Y2's working on Q1

Ten students (U23456789; Y78) who found the enclosed area with respect to both axes (whether correct or incorrect) highlighted that integration with respect to the y-axis is easier for finding the enclosed area in this question, because one integral is involved and they did not need to work with square root and negative area: "I could do it in one equation and also it is all above, all positive, [and] I don't have to deal with negative area" (U5). This section contributes one point to students' mathematical performance scores in integral questions and five students from Case 1 and two from Case 2 scored that point for finding the enclosed area using integration with respect to the y-axis.

7.4 Conceptual knowledge: ***<u>Understanding</u>*** Integrand should be continuous on the interval that is being integrated

Three students (U579) had the conceptual knowledge that the integrand should be continuous on the interval that is being integrated. This was shown by students' responses to Q4, Ex 2, in which they were asked to evaluate whether the area enclosed between the curve $= \frac{1}{x^2}$, the x-axis from $x = -1$ to $x = 1$ were solved correctly. U5 showed that the area is diverging by using the improper integral. U9 had the misunderstanding that if an integrand is not continuous at a given point, the calculation is not possible: "Why this work, I thought it will be something in the calculation that wouldn't work [sic]". U7 sketched the graph of $y = \frac{1}{x^2}$ and said, "we do not have the division by zero problem here", indicating he did not notice that the function is undefined at zero. This section contributes one point to students' mathematical performance in integral questions and three students from Case 1 scored that point for remembering the integrand should be continuous on the integration interval.

7.5 Conceptual knowledge: ***Understanding*** the relationship of enclosed areas between curves and the definite integral

All students in Case 1 except U2, the student who thought he was dealing with the volume topic, and five in Case 2 (Y45678), understood that $A = \int_a^b [f(x) - g(x)]dx$ in Q2 means the area enclosed between functions $f(x)$ and $g(x)$, $x = a$, and $x = b$ considering $f(x)$ as the upper function: " [A is] the area enclosed between the function $f(x)$ and $g(x)$ between a and b and $f(x)$ is the upper function" (Y6). In addition, seven students in Case 1 (U3456789) and five in Case 2 (Y45678) were able to correctly illustrate A on x-y plane using two arbitrary functions and bounds. U2, while answering Q2, still thought he was dealing with volume the topic and said "[A] is [the volume] rotating around the x-axis". U1 only considered the area above the x-axis and neglected the area enclosed between $f(x)$, $g(x)$, and $x = a$ (Figure 7.4). Y3 also failed to consider the area under the x-axis. Students in Case 2 besides this misconception, had other difficulties, including misconceptions about the upper/lower function and the bound of integration (Figure 7.5).

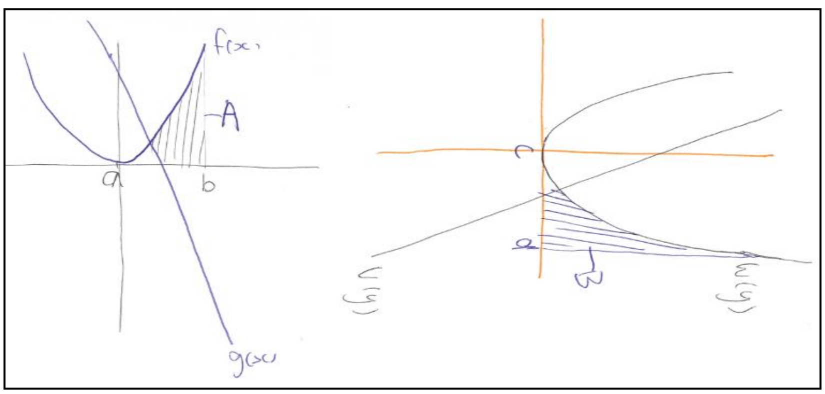


Figure 7.4 U1's incorrect illustration of $A = \int_a^b [f(x) - g(x)]dx$ and B=$\int_c^d [w(y) - v(y)]dy$

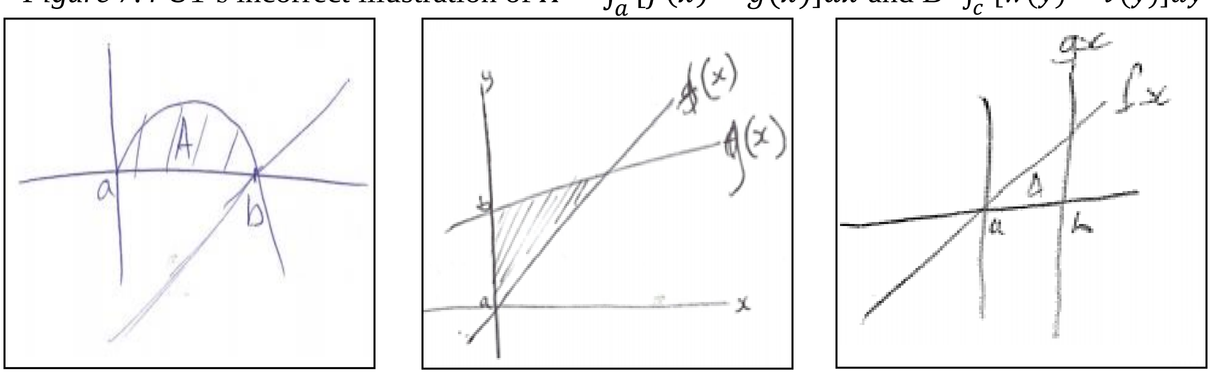


Figure 7.5 Misconceptions about illustrating $\int_a^b [f(x) - g(x)]dx$ in Case 2

In relation to B=$\int_c^d [w(y) - v(y)]dy$ in Q2, seven students in Case 1 (U3456789) and four in Case 2 (Y5678) understood that B is the area enclosed between functions, $w(y)$ and $v(y)$, $y = c$, and $y = d$ considering $w(y)$ is the upper function. Furthermore, seven students in Case 1 (U3456789) and three in Case 2 (Y578) were able to illustrate B correctly for two arbitrary functions and bounds. For the two remaining students in Case 1, the reasons for failure were the same as those reported for A. For students in Case 2, misconceptions about illustrating the enclosed area were related to the upper/lower functions, and neglect of the area under the y-axis (Figure 7.6). The reasons why students in Case 2 had more difficulties in understanding A and B in comparison to students in Case 1 might be related to them not being exposed to Riemann sums until the last few sessions before the end of the integral calculus topic (Table 6.6), and that no examples were solved for students using Riemann integral.

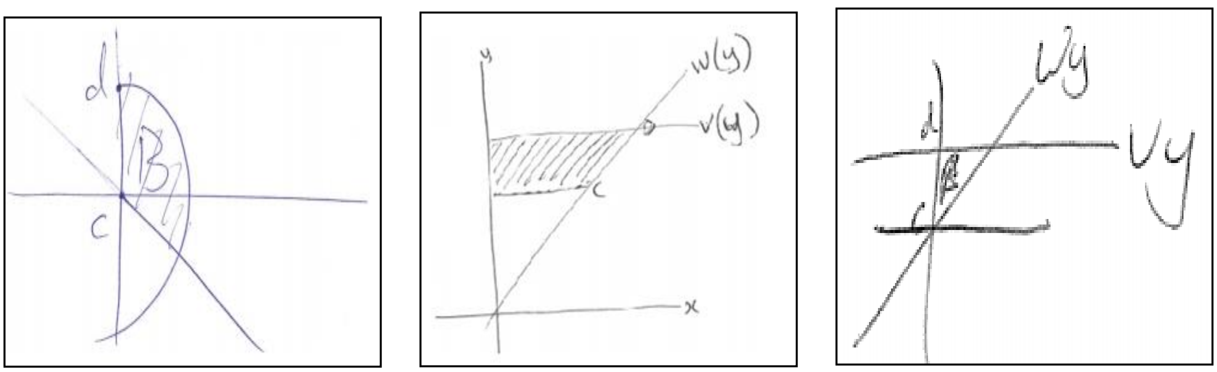


Figure 7.6 Misconceptions about illustrating B=$\int_c^d [w(y) - v(y)]dy$ in Case 2

This section contributes four points to students' mathematical performance scores in integral questions, comprising one point for understanding A, one for illustrating A, one for understanding B, and one for illustrating B. Students from Case 1, achieved the combined scores of 8, 7, 7, and 7 points respectively for these items; whereas students in Case 2 achieved combined scores of 5, 5, 4, and 3 points in these items.

7.6 Conceptual knowledge: ***Understanding*** the relationship between the definite integral and the limit of Riemann sums

In responding to Q2, seven students in Case 1 (U3456789), and one in Case 2 (Y7), showed they understood the relationship between the definite integral and the limits of Riemann sums. The other justification that four students in Case 2 (Y5678) provided for how A and B could be derived was they could be found by calculating the difference between the area under each curve:

> The way we derive these formulas is we find the area under the curve of one graph and the area under the curve of the other graph, and minus the two, and then, through distributive property of integrals we can condense them [sic] (Y7).

Students in both Cases had been taught how the definite integral is related to Riemann sums (Section 6.1). However, more materials are taught in Case 1 in relation to Riemann sums in comparison to Case 2. This section contributes one point to students' mathematical performance scores in integral questions, and seven students from Case 1 and one from Case 2 scored that point for understanding the relationship between the definite integral and the limits of Riemann sums.

7.7 Conceptual and procedural knowledge: ***Using*** the integral-area relationships for a graph of derivative function

Students' responses to Q3 were used to explore whether they were able to use the integral-area relationships for $f'(x)$. Seven students in Case 1 (U3456789) and four in Case 2 (Y5678) realised that the $\int_a^b f'(x)dx$ is equal to the area under the graph of $f'(x)$ between $x = a$ and $x = b$. However, only six students in Case 1 (U345689) realised this integral is equal to the signed net area underneath the graph of $f'(x)$. Furthermore, Y7 while solving Q3, made a mistake and considered $\int_a^b f'(x)dx = f'(b) - f'(a)$.

This section contributes one point to students' mathematical performance scores in integral questions, and only six students from Case 1 scored that point for using the integral-area relationships for finding the area under the graph of derivative function correctly.

7.8 Procedural knowledge: ***Doing*** algebraic manipulations necessary for simplifying integrands, finding intersection points, etc.

Across several questions a lack of proficiency in algebraic manipulations was found to be one of the students' major barriers to be a successful problem solver in these topics. In Q1, six students (U136; Y356) had difficulties in solving $x - 2 = \sqrt{x}$ for finding the intersection points. Five students (US13; Y145) made a mistake in finding y from $x = y^2$, assuming $y = \sqrt{x}$. Y6 made an error in squaring $x - \sqrt{x}$ assuming it is equal to $x^2 - x$. Y5 thought these two curves have no intersection point, and Y3 wanted to solve $x - 2 = y^2$ to find intersection points rather than $-2 = \sqrt{x}$.

Lack of proficiency in algebraic manipulation was also observed in solving Q4. In response to this question, four students in Case 2 (Y1456) used a calculator for checking whether the bounds were substituted correctly in the integrand, indicating students were not confident at doing calculations. Examples of calculations done with calculators include 25×2 and $\frac{5^3}{3}$. Moreover, Y1 made a mistake in simplifying $\frac{5^3}{3} - 2 \times 5^2$ and wrote that it is equal to $\frac{125}{3} - 20$. Then he asked to use a calculator to find the end results.

In Q6, Y3, to find the antiderivative of $G(x) = \int_0^{x^2} r^2\sqrt{1+r^3}\,dr$, simplified the integrand as follows $G(x) = \int_0^{x^2} r^2\sqrt{1+r^3}\,dr = \int_0^{x^2} \sqrt{r^4+1+r^3}\,dr$, neglecting the fact that r^4 should be multiplied by $1 + r^3$.

In Q8, two students (U1; Y6) were unable to find the equation of a line from the given graph (Section 7.17). This section contributes one point to students' mathematical performance ratings for integral questions, and six students from Case 1 and three from Case 2 scored that point for correctly doing the necessary algebraic manipulation when solving integral questions.

7.9 Procedural knowledge: ***Using*** integral techniques for finding antiderivatives

All of the students were able to find the antiderivative of different powers of $y = x^n$ when n is positive as evidenced by Q1 in which, after setting up the integrals (whether correct or incorrect), students were able to find the antiderivative correctly. However, when n was negative, difficulties in finding the antiderivative were found. Y5 had a misconception as to how to find the $\int_{-1}^{1} \frac{1}{x^2}\,dx$ and used the integration of natural logarithm for finding the antiderivative (Figure 7.7).

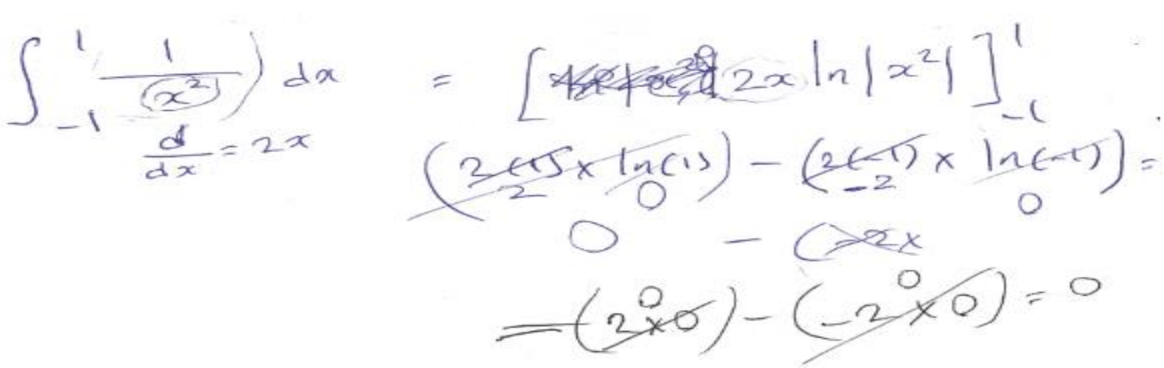


Figure 7.7 Misunderstanding of finding the antiderivative of negative power of x^n (Y5)

Also, U1 had difficulty in finding the antiderivative for this integral and used the U-substitution method by considering $u = x^2$. He found the antiderivative correctly; however, university students are recommended to master how to find the antiderivative of basic functions in order to be able to cope with more complex integrals.

While solving Q5, Y1 had difficulty with $x^2 + 0.5$ and asked whether it is a function or not, saying “I do not know how to integrate that”, thus showing a lack of understanding about the properties of integrals.

Among those students who did not use FTC, part two, for solving Q6, two students in Case 2 (Y23) had difficulties with finding the antiderivative of $O(x) = \int_1^x \frac{1-t}{t^2-2t-9}\, dt$ and $G(x) = \int_0^{x^2} r^2\sqrt{1+r^3}\, dr$. Y3 did not use the U-substitution integration technique for finding the antiderivative of $G(x)$ and after bringing r^2 under the square root incorrectly (Section 7.8) solved the problem as follows (Figure 7.8).

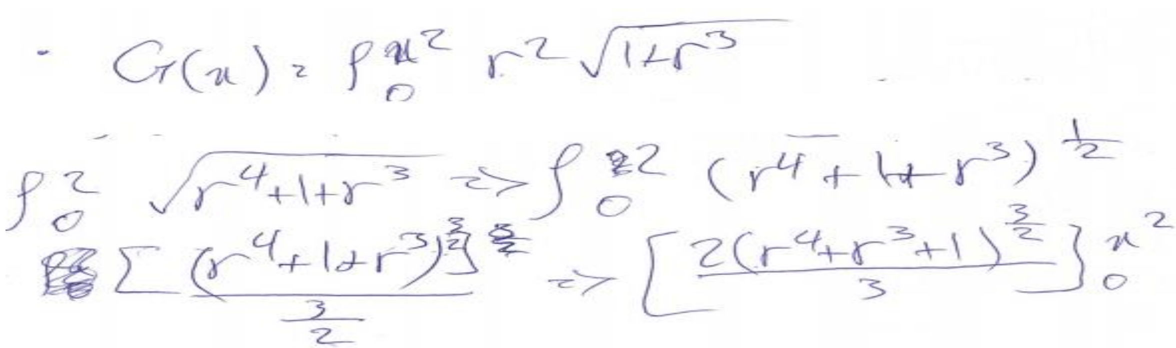


Figure 7.8 Example of lack of proficiency in using integral techniques

Y2 had a misconception about the definite integral properties and thought $\int_a^b f(x)g(x)dx = \int_a^b f(x)dx * \int_a^b g(x)dx$. Therefore, for finding the antiderivative of $O(x)$, he found the antiderivative $1 - t$ and $\frac{1}{t^2-2t-9}$ separately and multiplied them together. This section contributes one point to students’ mathematical performance scores in integral questions and all nine students from Case 1 and four from Case 2 scored that point for using the integral techniques correctly to find antiderivatives in the questions.

7.10 Procedural knowledge: ***Using*** the first part of the FTC to find the definite integral

All students in the sample, after finding the antiderivative, were able to use the FTC for finding definite integrals. This section contributes one performance point and all students scored the point for using the FTC to find the definite integral.

7.11 ***Understanding*** when it is efficient to use integration with respect to x and y for finding integral-area problems

Two themes were found in relation to how students chose which method should be used for solving the integral-area problems, including:

- choosing based on the graph of curves (U34589; Y1234); and
- necessary algebraic manipulations for solving the problem (U123456789; Y567).

Regarding the graph, five students (U3489; Y2) noted they chose the method that involved the lesser change in the lower/upper function. For example, U8 said "when you have to break it up less times like the previous example [Q1 that is efficient to use integration with respect to the y-axis]". U5 mentioned she would make her choice based on which alternative had the lesser negative signed area. Three students in Case 2 (Y134) said that their choice would be based on the graph being enclosed by the x or y axis, indicating they were thinking about the area under one curve rather than two curves, which is what is asked in the questions. This misconception was also found in illustrating A and B in Q2 (Section 7.5). In problems related to finding the enclosed area between curves and two bounds, the axes are not important. However, as most Y13 students did not understand the idea behind the integral-area relationships through Riemann sums (Section 7.6), such misconceptions had been created among students.

In relation to the second theme, students mentioned they made their choice of method based on the least algebraic manipulation being necessary for finding the lower and upper bounds. In other words, if the bounds were given in terms of $x = a$ and $x = b$, they would choose integration with respect to x, and if it was given in terms of $y = c$ and $y = d$, they would choose integration with respect to the y-axis. Another factor that affected their decision was how the integrand was given in a question. If the integrand was presented as a function of x, they would choose integration with respect to the x-axis, and if it was given in terms of y, they would choose integration with respect to the y-axis: "I would use it [the method] depending on what formula I have for the functions. I am integrating relative to the thing the function is of" (U7).

Two students in Case 2 (Y58) mentioned their first choice was always integration with respect to the x-axis. Y8 believed it is easier to conceptualise when you are integrating with respect to the x-axis. He mentioned:

> By default, I go x because it is easier to conceptualise, because you have positive to negative, but y, sort of is inverted in terms of positive to negative. But sometimes it is just easier to do in terms of y [sic]".

Y5 mentioned, "In all questions it is easier to use the integration with respect to x-axis because you don't need to write the function in terms of $x = f(y)$, at least in questions that we see in our school".

This section contributes one point to students' mathematical performance scores in integral questions. As all students gave a reason for choosing the most efficient method (i.e., their choice was based on the graphs, the necessary algebraic manipulations, and which method was easier to conceptualise) of integrating with respect to the x or y-axis, one point wss given to all students.

7.12 Factual and conceptual knowledge: ***Understanding*** the meaning of $\boldsymbol{F(x) = \int f(x)dx}$

Four students in Case 1 (U1356) and two in Case 2 (Y36), in response to the first part of Q7, read the mathematical statement rather than telling what they understood by it: "the function of $F(x)$ is equal to the integral of $f(x)$ [with] respect to dx" (Y6). After observing such responses, I tried to engage students so as to provide more responses by asking "Is $F(x)$ the antiderivative of $f(x)$?" Their answers showed that all students in Case 1 and four in Case 2 (Y1678) understood that the antiderivative of $f(x)$ is $F(x)$. Three students in Case 2 (Y234) were unsure about it, and Y2 said "I cannot see how they are related". Only U4 understood that f is the function that describes the rate of change of $F(x)$, saying "$f(x)$ is the formula for the gradient of $F(x)$". However, Y5 held a misconception and said "the F is the rate of change of f".

This section contributes two points to students' mathematical performance scores in integral questions, including one point for understanding $F(x)$ is the antiderivative of $f(x)$, and one point for understanding $f(x)$ as the function that describes the rate of change of $F(x)$. All students in Case 1 and four in Case 2 scored the first point, while only one student in Case 2 achieved the second point.

7.13 Factual and conceptual knowledge: ***Understanding*** the meaning of $\boldsymbol{F(x) = \int_a^x f(t)dt}$

Four students (U18;Y23), in response to the second part of Q7, read the equation as opposed to telling what they understood by it: "the integration of $f(t)$ between x and a is $F(x)$" (Y3). Five students (U9; Y1678) said "$F(x)$ is the antiderivative of f". Y1 was not sure what the a and x stand for. Three university students (U356) mentioned that the $F(x)$ shows the area between a, x, and the curve. Y7 had difficulty with $f(t)$ and said "I cannot remember what the $f(t)$ is". He could not understand the relationship between the area under the graph of $f(t)$ and $F(x)$. The responses of two students in Case 1 (U12) were influenced by procedural aspects of FTC that they were asked about in Q6, saying "It is FTC again because the bottom part doesn't matter over there" (U1), and "I understand by this you sub x to get the derivative of this" (U2). U7 and Y4 had no idea about the equation.

Students, after providing their initial responses, were asked to illustrate $F(x) = \int_a^x f(t)dt$. Seven students in Case 1 (U1345689) and two in Case 2 (Y56) illustrated it by considering the $F(x)$ as the area under the graph of $f(t)$ between a and x (Figure 7.9). Out of these students, three in Case 1 (U348) also showed that they understood $F(x)$ is the accumulated area under the curve. U3 said "a is constant and x is moving. $F(x)$ is depending on x but there is a specific lower bound". Two of them (U48) sketched $F(x)$ as an increasing function that shows their understanding of accumulated area function (Figure 4.10). Y3 made a mistake in bounds and considered x as the lower bound in his drawing. Two university (U27) and four Year 13 (Y1248) students were unable to provide any illustration for this part.

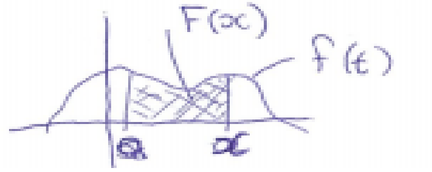


Figure 7.9 An example of students' illustration of $F(x) = \int_a^x f(t)dt$ (U6)

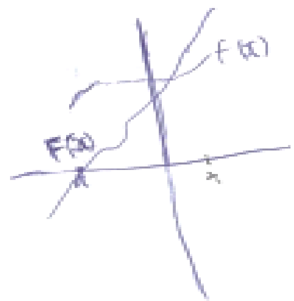


Figure 7.10 An example of considering $F(x)$ as an accumulated area function (U8)

Three points were conferred in this section for students' mathematical performance ratings in integral questions. One point was for having an understanding of the meaning of $F(x) = \int_a^x f(t)dt$, whether as an area under a curve or as accumulated area function; one for illustrating $\int_a^x f(t)dt$ as an area under the graph of $f(t)$; and one for illustrating $F(x)$ as an accumulated area function. Three students from Case 1 scored the first and third point, and seven students from Case 1, and two from Case 2 scored the second point.

7.14 Factual and conceptual knowledge: ***Understanding*** the first part of the FTC

In response to Q7, part three, eight students in Case 1 (U23456789) and four in Case 2 (Y1358) understood that $\int_a^b f(x)dx = F(b) - F(a)$ could be used to find the area under the graph $y = f(x)$ between $x = a$ and $x = b$. Five students in Case 1 (U34568) and four in Case 2 (Y1578) were able to sketch the area under the graph of $f(x)$ between $x = a$ and $x = b$ correclty. However, U1 and Y6 held misconceptions about the statement and thought it meant find the area between two curves, $F(b)$ and $F(a)$ (Figure 7.11).

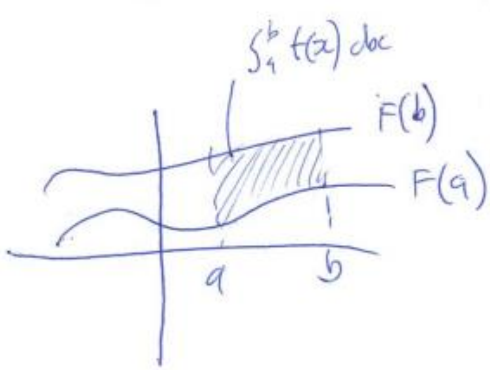


Figure 7.11 An example of students' misconceptions about $\int_a^b f(x)dx = F(b) - F(a)$ (Y6)

Only U8 was able to provide the formal proof of how the statement is derived. Three Y13S (Y457) initially could not recognise the statement, and when I asked which formula you should use to find the area under curves, they realised that they had seen and used this formula.

Three points were considered for this section regarding students' mathematical performance in integral questions. One point was for their understanding that the formula could be used to find the area under the graph of $f(x)$ between $x = a$ and $x = b$, one point for illustrating its use for an arbitrary function, and one point for proving the first part of the FTC. Eight students from Case 1 and four from Case 2 scored the first point, five from Case 1 and four from Case 2 scored the second point, and only one student from Case 1 scored the third point.

7.15 Factual and conceptual knowledge: ***Understanding*** the second part of the FTC

In response to Q7, part four, four students (U2; Y146) did not understand anything from reading the statement, and provided responses such as "I do not know" (Y1). Five (U18; Y235) initially read the statement rather than stating what they understand by it. Three students (U37; Y5) acknowledged that integration and differentiation are inverse processes. Procedural understanding about the statement was mentioned by U1: "I know effectively you just get the top part and just subs into $f(t)$ and the a part, the bottom one, is being ignored [sic]". Apart from solving questions that asked them to find the derivative of integrals (e.g., Q6), students did not know any applications of the statement, mentioning "I actually not sure when we use this formula. I do lots of questions on that, but I am not actually sure when it used for [sic]" (U6).

There was no student who could justify how this statement could be derived by using the formal proof. Four students (U689; Y8) provided the following justification.

> That is like what we have been doing in the last question basically the $f(a)$ doesn't mean anything because when you differentiate with respect to x the a is going down to zero. Basically, it goes to $F(x)$ and then the derivative of that is $f(x)$(Y8).

U5 justified the statement using $\frac{d}{dx}[\int_{u(x)}^{v(x)} f(t)dt] = f\big(v(x)\big)v'(x) - f\big(u(x)\big)u'(x)$, by considering $u(x) = a$ and $v(x) = x$, which indicated procedural understanding about the topic.

Two students in Case 1 (U37) held a misconception about the statement and thought the constant should be considered in it. A sample response was "I think it should have +c as the right side because even if it is the same thing, still [I] get lost when you differentiate and integrate"(U3). No student mentioned that the instantaneous rate of change of the accumulation function at x is equal to the value of the rate of change function at x.

Two points were considered for this section toward students' mathematical performance scores in integral questions, including one point for understanding the second part of the FTC, and one point for proving it. No students scored those points.

7.16 Procedural knowledge: ***Using*** the second part of the FTC for finding the derivative of integrals

In response to Q6, five students in Case 1 (U14589) had the procedural knowledge for finding the derivative of different integrals using the second part of the FTC (Table 7.3). U5 instead of using the chain rule, used $\frac{d}{dx}[\int_{u(x)}^{v(x)} f(t)dt] = f\big(v(x)\big)v'(x) - f\big(u(x)\big)u'(x)$ correctly for solving the problem, indicating instrumental learning (Section 3.1.2) about the FTC. Four students (U67; Y25) acknowledged integration and differentiation are inverse processes while they worked at solving the question. The rest of the students held misconceptions about the statement or didn't know it.

Table 7.3

Number of correct responses to each part of Q6

Q6	Case 1	Case 2
$\frac{d}{dx}\left[\int_1^x \frac{1-t}{t^2-2t-9}dt\right]$	8 (U12345689)	1 (Y8)
$\frac{d}{dx}\left[\int_1^{x^2} r^2\sqrt{1+r^3}dr\right]$	5 (U14589)	0
$\frac{d}{dx}\left[\int_{2x-5}^{4x+4} t^3dt\right]$	5 (U14589)	1 (Y7)

Students who did not know the second part of the FTC had different approaches to solve the question. Five students in Case 2 (Y23578) first tried to integrate the integral, then, differentiate the answer. Y6 made a mistake and instead of integrating, differentiated the integrand, then substituted the bounds in it. Two students in Case 2 (Y14) did not try to solve the question. Y1 said “I am confused by the integral sign being [*dt*, *dr*] and the bound is a variable. If it was without that then I think I might be able to do it”. Y4 said, “I am sure I cannot because we have not been taught it”. Having different variables for integration and differentiation caused difficulty for another student. Y5 wrote $o'(x) = \frac{1-t}{t^2-2t-9}$. Then, after realising one side is based on x and one side based on t, used the substitution method by considering $1 - t = x$ and changing the right hand side based on x, which indicated instrumental learning about the substitution method.

Four students in Case 1 (U2367) and three in Case 2 (Y257) held misconceptions about the FTC, substituted the band in the integrand, and did not use the chain rule (Figure 7.12).

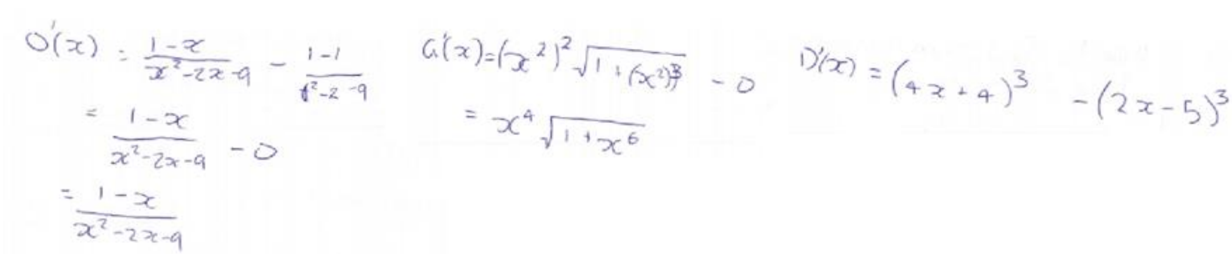


Figure 7.12 Example of students’ misconception about FTC (U6)

U3’s misconception (Figure 7.13) was related to not knowing the geometrical interpretation of the FTC (i.e., the instantaneous rate of change of the accumulation area function at x is equal to the value of the rate of change function at x):

> I am thinking about how suddenly you have the derivative of a function, but you are moving the bounds... my intuition is what I would do, but I think it is wrong is stating at lower bound the derivative at lower bound is this and at the upper bound is this. Because I am not sure what meaning it would be behind taking two different points of the graph and say here is the average derivative, it's just like does not have any meaning to me [sic].

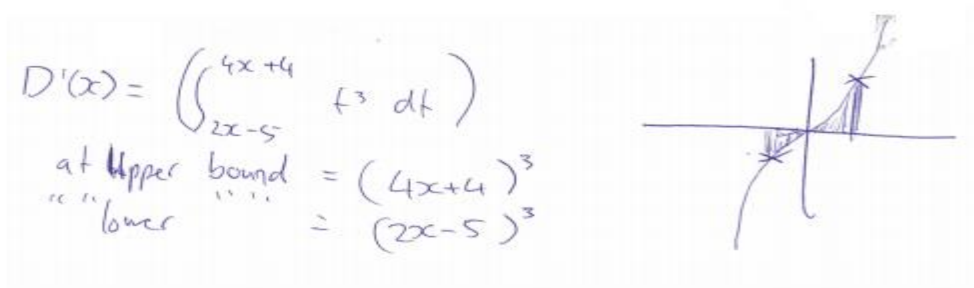


Figure 7.13 An example of students' misconceptions about the FTC (U3)

And U7s thought that showed misunderstanding of the FTC was

> If you integrate the derivative of something you end up with the something. If I integrate some function and then I take the derivative of the results I get back to the original function. But because this is the definite integral I feel like I want to plug x and 1 somewhere into it, but I don't have two slots to put into.

This section contributes one point to students' mathematical performance scores in integral questions, and five students from Case 1 scored that point for correctly using the second part of the FTC to find the derivative of integrals in all three items in Q6.

7.17 Conceptual and procedural knowledge: ***Using*** the FTC in a contextual problem

Students' abilities to use the FTC in a contextual problem were explored in Q8 (Table 7.4).

Table 7.4

Frequency of correct responses to each part of Q8

	Case 1 (total=9)	Case 2 (total=8)	Did not answer: Case 1	Did not answer: Case 2
8.1 How much water was in the tank at noon	6 (U345679)	2 (Y78)	1 (U1)	1 (Y6)
8.2 The meaning of $g(x) = \int_0^x f(t)dt$	8 (U13456789)	5 (Y24568)	0	2 (Y37)
8.3 The value of $g(9)$	7 (U3456789)	2 (Y28)	1 (U1)	2 (Y67)
8.4 Interval of time the water level decreasing	8 (U13456789)	3 (Y478)	0	0
8.5 Time the tank was the fullest	7 (U1345689)	2 (Y78)	0	0
8.6 Sketch the graph of $g(x)$	7 (U3456789)	2 (Y48)	2 (U12)	3 (Y367)

Students' responses are described in the following sections.

7.17.1 Item 8.1

Eight students (U345679; Y78) found that 450 gallons of water were in the tank at noon. Of these students, Y8 solved the question by finding the equation of $f(t)$ for each interval, then integrating it between 0 to 9. U7 only found the equation from 0 to 3, then, integrated on this interval, and for the remaining interval he calculated the area under the graph of $f(t)$ using the area of geometry shapes. The remainder calculated the area underneath the graph of $f(t)$ by using the area of geometry shapes for the whole interval. U1 and Y6 could not find the answer because they were unable to find the equation for $f(t)$, indicating lack of prior knowledge. Therefore, they were not able to find the amount of water by integrating $f(t)$ between 6 am to noon. In addition, these students did not use the fact that the net signed area under the graph of $f(t)$ shows the amount of water in the tank. Three students (U2; Y12) did not consider $f(t)$ as the graph of derivative function, and said the amount of water in the tank is 300 gallons (U2) and -300 gallons (Y12). U8 found the equation of $f(t)$, but, made a mistake in finding the slope of the equation of $f(t)$. Y3 said, "It says at 6 am it was zero and at noon is below zero, so still nothing", indicating a lack of understanding how to interpret the graph of $f(t)$. Y4 did not consider the fact that the graph of $f(t)$ changed in the interval, and used $\int_0^6 t\, dt$ for finding the amount of water.

7.17.2 Item 8.2

Thirteen students (U13456789; Y24568) noted that $g(x)$ is the amount of water in the tank after x hours. Two students in Case 2 (Y37) did not know the meaning of $g(x)$ and had difficulties with having both t and x in the equation. U2 and Y1 thought $g(x)$ is the rate of change of water in the tank. U7, despite his correct answer, tried to find a general formula for $g(x)$ by solving $\int_0^x t\, dt$. He did not notice that $f(t)$ has different equations between 6 am to 6 pm and found $\frac{x^2}{2}$ as the general formula for it.

7.17.3 Item 8.3

Thirteen students (U23456789; Y12358) found $g(9) = 0$. However, four of them (U2; Y135) had incorrect justifications, considering $g(9) = 0$ because the graph of $f(9) = 0$. Therefore, only nine students (U3456789; Y28) had a correct understanding of the meaning of $g(9)$. Y4 considered $g(9) = 900$ because he thought the rate of change is constant at 100 gallons per hour; therefore, after 9 hours $g(9)$ is equal to 900. Three students (U1; Y67) could not respond to this part. U1 and Y6 could not answer because they were unable to find the equation of $f(t)$, and said they could not find the value of $g(9)$. Y7 could not respond because of difficulty with the meaning of $g(x) = \int_0^x f(t)\, dt$ (Section 7.17.2).

7.17.4 Item 8.4

Eleven students (U13456789; Y478) made the correct interpretation of the graph of the rate of change in terms of the interval that the water level was decreasing. The remainder of the students (U2; Y12356) had difficulties in interpreting the graph of the derivative. Five (U2; Y2356) considered 9 am to 12 pm, and Y1 considered 9 am to 10 30 am and 3 pm to 6 pm as the intervals at which the water level was decreasing.

7.17.5 Item 8.5

Nine students (U1345689; Y78) mentioned correctly that the tank was the fullest at 10:30 am. Of the remaining students, six (U27; Y1246) considered 9 am and 6 pm as the times that the tank was the fullest, and two (Y3; Y5) said 9 am is the time that the tank was the fullest indicating a wrong interpretation of the graph of the rate of change.

7.17.6 Item 8.6

Nine students (U3456789; Y48) drew the graph of $g(x)$ correctly. Five students (U12; Y367) were unable to provide a drawing for it. The remaining three students (Y125) made wrong drawings for $g(x)$. Y5 made a mistake, saying “the integral of linear graphs is a straight line”, considering the differentiation of linear graphs instead of integration. However, he only made this mistake in this part and in other questions he knew the difference between differentiation and integration (Figure 7.14).

Figure 7.14 Y5’s incorrect drawing of $g(x)$

Y1 and Y2’s graphs were incorrect because of their failure to answer the previous parts of Q8 (Figure 7.15) correctly.

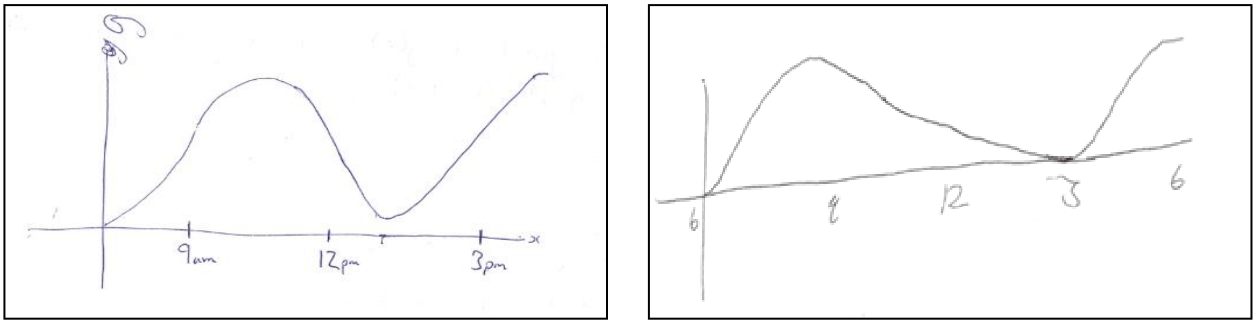


Figure 7.15 Y1 and Y2's incorrect drawings of $g(x)$ (Left graph: Y1, right graph Y2)

This section contributes six points to students' mathematical performance scores in integral questions, one point for each item. Students in Case 1 scored more points from these items in comparison to students in Case 2 (Table 7.3).

7.18 Conceptual and procedural knowledge: Posing a question for concerning integral-area relationship

In response to Q5, eight students (U3456789; Y7) posed a correct question about the area enclosed between a curve and a line using the given information. Of the remaining, seven students (U12; Y23458) could not pose a problem and two students (Y16) posed an incorrect problem (Table 7.5).

Table 7.5

Question posed by students for the integral-area relationship

Find the area enclosed between	Curve	Line	Bounds		N
			Lower	Upper	
Suitable problems	$y = x^2$	$y = 0$	$a = 0$	$b = \sqrt[3]{3}$	5 (U3456; Y7)
			$a = 1$	$b = \sqrt[3]{4}$	1 (U9)
	$y = 3x^2$	$y = 0$	$a = 0$	$a = 1$	1 (U7)
	$y = \sin x$	$y = 0$	$a = 0$	$b = \frac{\pi}{2}$	1 (U8)
Unsuitable problems	$y = x^2 + 0.5$	$y = 2x$	$a = 0$	$a = 1$	1 (Y1)
	$y = x^2$	$y = 1.31$	--	--	1 (Y6)

Posing a problem was not an easy task for students. For example, Y4 said, "I am struggling with only the area is given and you have too many things". Students who were not able to pose a problem, had not considered the line as the x-axis to simplify the task. For instance, Y8 chose $y = x^2$ as the curve and $y = x$ as the line for his first try, then tried $y = \sqrt{x}$ as the curve. After he was unable to find an area equal to one, he changed the functions to a more general form by considering the curve $y = a\sqrt{x}$ and the line as $y = bx$.

This section contributes one point to students' mathematical performance scores in integral questions and seven students from Case 1, and one from Case 2, scored that point for posing a correct question based on the given information in Q5.

7.19 Conceptual and procedural knowledge: Posing a question in relation to the FTC

In response to Q9, all the students except two (U17) posed a problem for the given graph. U1 did not pose a problem because he believed he had not "understand the FTC". U7 said "copy and paste question eight here", indicating a lack of knowledge about the FTC. He also commented, "I'd need to know my stuff a lot better to set problems". In addition, three students (U2; Y56) posed a problem that is not related to the FTC (Table 7.6).

Table 7.6

Student-posed problems not related to the FTC

Student	Students' problem
U2	Find the equation from 0 to 2 and from 2 to 6.
Y5	Please can you draw $f'(x)$ of the following graph?
Y6	$f(x)$ [the function corresponded to the given graph] is the rate of change of $g(x)$. Between what values is the rate of change of $f(x)$ changing?

Of those 12 students who posed a problem related to the FTC (Table 7.7), nine students (U345689; Y278) were able to solve their own problem correctly.

Table 7.7

Students' problems related to the FTC

Student	Students' problem
U3	What is the integral of the above graph on the range 2 to 6? To rephrase, assume the graph above is $f(x)$. What is $\int_2^6 f\,(x)\,dx$?
U4	The graph gives the rate of changes temperature in terms of degree of Celsius. Find temp at t=6. Describe the graph of T(t) between 2 and 6. Equation of T(t) between 6 to 7 (assuming the curve between 6 and 7 is a parabola.).
U5	The following graph describes a person's velocity from a tree where east is counted as positive and west is counted as negative. The unit of time is second and the unit of velocity is m/s. Please find how many meters did he walk in the first 2 seconds.
U6	f(t) rate water going into the tank. How much water goes into the tank between t=4 and t=6?
U8	The graph represents the rate at which a tank is filled in with $m^3$of water per minute. Assuming tank has 1 $m^3$of water at $t = 0$. Find how much water is in the tank after 5 mins.
U9	Let y be the velocity of a runner. His displacement at t=0 was 0 m. Find his position at t=6 hours.
Y1	Sketch the graph of $g(x)$ considering $g(x) = \int_0^7 f(x)dt$ using the graph of $f(x)$.
Y2	Graph f represents the rate of water change within a water tank. At what point is the tank empty? How much water is in the tank at 6? What is the meaning of g (x) = $\int_0^x f(t)dt$?
Y3	The tank has 1000 litre of water at the time of 0. When the amount of water is decreasing? When the amount of water is increasing? [sic] [Considering the conditions of question eight in terms of the graph.]
Y4	The graph above represents the change in velocity of a cart on a kids roller coaster over a 7 minute period. Sketch a graph showing the velocity of the cart and find the point where the cart reaches its maximum velocity? [sic]
Y7	Between which intervals does f(x) have the greatest area under the graph?
Y8	The graph shows the rate of change of water in a harbour. At what point is the water level in the harbour at its greatest point?

Y3 provided an incorrect answer and two students (Y14) were unable to find an answer to their own problems. Y3 made an incorrect interpretation of the graph of the rate of change and said the amount of water was decreasing between $(0,2)$ and $[6,7)$, and increased between $[2,6)$. Y1's problem is correct, but it seems this student had used an incorrect notation for what he meant by $g(x)$. Y1 considered $g(x) = \int_0^7 f(x)dt$ which is equal to $g(x) = f(x)\int_0^7 1.dt = 7f(x)$. It means that for sketching $g(x)$ he needed to multiply the values of $f(x)$ by seven. However, what he had sketched as an answer to his problem (Figure 7.16), shows that he sketched $g(x) = \int_0^x f(t)dt$ for $x \in [0,7]$ (he made a mistake for the interval of 6 to 7).

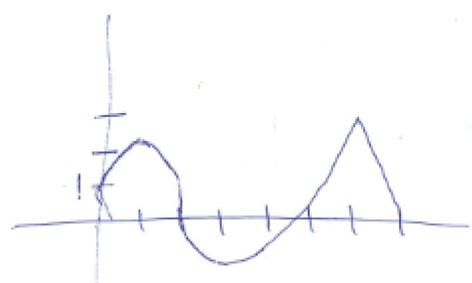


Figure 7.16 Y1's drawing of $g(x)$ for the answer to his posed problem

Y4 could not solve his problem because he did not use the integral-area relationship to find the velocity; instead he found the equation for the graph for $[0,2]$ and $[2,6]$, then integrated them and considered C for each equation. Then, he would have liked to find the C for each line, but could not. Among students' posed problems based on the FTC, two students (U3; Y7) posed a problem that is more related to the integral-area relationship than the FTC, especially the second parts of the FTC.

This section contributes two points to students' mathematical performance scores in integral questions, including one point for posing a question related to the FTC, and one for solving their own problem. Six students from Case 1 and six from Case 2 scored the first point, and six from Case 1 and three from Case 2 scored the second point.

7.20 Overall mathematical performance of students in integral questions

The points students obtained from the sections were added together. The maximum number of points that a student could score was 36. The minimum number of points students obtained in

Case 1 was 8 (U2), and in Case 2 was 6 (Y3) (Table 7.8). The highest points achieved in Case 1 was 30 (U58), and in Case 2 was 23 (Y8).

Table 7.8

Descriptive statistics for students' mathematical performance

	N	Min	Max	Mean	Standard division
Case 1	9	8	30	23.4	8.0
Case 2	8	6	23	12.8	6.3
Total	17	6	30	18.5	8.9

The total points each student achieved are shown in the following Table (Table 7.9). The distribution of students' mathematical performances was not normal in Case 1 according to the Shapiro-Wilk test of normality (P-value=0.026); therefore[9], the equivalent non-parametric test, Mann-Whitney test, was used to compare students' mathematical performance in the Cases.

Table 7.9

Students' total points in integral questions

	U1	U2	U3	U4	U5	U6	U7	U8	U9	Y1	Y2	Y3	Y4	Y5	Y6	Y7	Y8
Total scores	13	8	27	29	30	26	20	30	28	8	9	6	11	12	12	22	23

Students in Case 1 had significantly better performance in comparison to students in Case 2 according to the Mann-Whitney test (P-value=0.01) (Table 7.10).

[9] For using parametric tests, the distribution of sample in both groups should be normal.

Table 7.10

Results of the Mann-Whitney test for students' mathematical performance

	N	Mean rank	Sum of ranks
Case 1	9	11.8	106.5
Case 2	8	5.8	46.5

Eight students (47.05%), including two from Case 1 (U12) and six in Case 2 (Y123456), obtained less than half of the possible points that student could have got from the integral questions (i.e., 18 points).

7.21 Discussion

In both Cases, students had more difficulties with the FTC than the definite integral and the integral-area relationships. Students' problem posing ability (Q5 and Q9) and their ability to evaluate whether a question has been solved correctly (Q4) have been not studied for integral calculus in previous studies. In the following, the study findings are compared to previous studies in relation to factual, conceptual, and procedural knowledge.

Regarding factual knowledge, students recognised the symbols involved in the definite integral and understand its basic terminologies. However, they had difficulties with symbols and terminologies involve in the FTC, especially the second part of it. This finding is in line with previous research showing that the role of t in $\int_a^x f(t)dt$ is confusing for students (Thompson & Silverman, 2008).

In terms of conceptual and procedural knowledge, students' procedural knowledge was better developed in comparison to their conceptual knowledge in both definite integral and the FTC. This is also consistent with previous studies showing students are able to do routine procedures to find area using integral techniques, however their knowledge about why such a procedure is used are limited (Artigue, 1991; Grundmeier, Hansen, & Sousa, 2006; Mahir, 2009; Orton, 1983; Rasslan, & Tall, 2002; Thomas & Hong, 1996).

Lack of algebraic manipulation skills and prior knowledge were a barrier for several students to solving the questions correctly, due to their not being able to find the intersection points, sketch the graph correctly, or find the equation of lines and curves. These findings, also highlighted by previous studies, indicate that several of students would benefit from improving their knowledge of functions, and/or algebraic manipulation and/or graph sketching prior to starting integral calculus (Kiat, 2005).

In terms of conceptual knowledge, the fact that the integrand should be continuous was ignored by most students during problem solving. In Case 2, several students were unsuccessful at illustrating $\int_c^d [w(y) - v(y)]dy$ correctly, which indicates they had not developed conceptual knowledge about the definite integral. Several students considered area or net area underneath the graph of $f'(x)$ is equal to the $\int f'(x)dx$, rather than its signed net area. More evidence supporting this claim is that only one student in Case 2 was able to explain the relationship between Riemann sums and the definite integral. Lack of conceptual knowledge about the definite integral is also reported in literature (e.g., Mahir, 2009; Thomas & Hong, 1996). Previous research has also highlighted that students had difficulty in understanding the definite integral as the limit of a sum (Orton, 1983; Grundmeier, Hansen, & Sousa, 2006).

In terms of the FTC, in line with previous research (Thompson, 1994; Thompson & Silverman, 2008), the notion of accumulated area function was not developed in the students' minds, and their understanding of the first and second part of the FTC was limited in both Cases. For instance, several students did not understand the fact that f is the function that described the rate of change of the accumulated area function, $F(x)$. A number of students in Case 1 had difficulty with using the FTC to find the derivative of integrals when bounds of the integral were functions of x.

7.22 Chapter summary

This chapter has described the research findings in terms of students' factual, conceptual, and procedural knowledge of RBT. It provides responses to the second research questions (i.e., using RBT as a lens, what are students' difficulties in solving integral questions in Year 13 and first year university?) and the fourth (i.e., what differences exist between student learning of integral calculus in Year 13 and first year university?) of the study (Section 1.4). The next Chapter explores students' metacognitive knowledge in relation to integral calculus.

Chapter Eight: Metacognitive Knowledge in Relation to the Integral-area and the FTC Problems

In response to the question *Have you heard about metacognitive knowledge?*, all the students except U9 said they had not. U9 said, "it is knowledge about understanding how you know things, how you learn things". However, he stated he did not know what metacognitive knowledge (MK) is in the context of the integral-area and the FTC problems. Despite the fact that students thought they did not know what the term MK means and what it is in the context of integral calculus, they showed in their responses to interview questions related to MK that they did in fact have a variety of knowledge about it in the context of integral calculus. In this chapter, students' MK is explored in relation to questions whose design was based on the structure of RBT's metacognitive knowledge, including questions about strategic knowledge ((learning strategies (Section 8.1), monitoring strategies (Section 8.2), problem solving strategies (Section 8.3)), knowledge about different cognitive tasks (Section 8.4), and self-knowledge (Section 8.5).

8.1 Strategic knowledge: Learning strategies

In this section, students' learning, monitoring, and problem-solving strategies are explored in relation to the integral-area and the FTC problems.

8.1.1 Attending lectures, tutorials, and school classes

In this section, students in Case 1 and 2 are compared separately, as students in Case 1 attended lecturers and tutorials while students in Case 2 attended school classes. This section shows most students in both Cases were aware that attending classes is important and useful.

Attending lectures and tutorials

Of the nine students in Case 1, five attended both lectures and tutorials in these topics (U23789). Two only attended lectures (U46), and one only attended tutorials (U5). U1 did not attend either lectures or tutorials (Table 8.1)[10].

[10] If the actual attendance data were available, it would be valuable to cross check it with the interview data.

Table 8.1

Attending lectures, tutorials, and school classes

Sources	Case 1	Case 2	Total
Attending lectures & tutorials	5 (U23789)	NA	5
Attending lectures only	2 (U46)	NA	2
Attending tutorials only	1 (U5)	NA	1
Not attending lectures & tutorials	1 (U1)	NA	1
Attending school classes	NA	8 (Y12345678)	8

Students who attended both lectures and tutorials believed attendance was useful for learning the topics (U789), getting support from lecturers, tutors, and peers (U8), knowing how to do examples, problems, and assignment questions (U2), and exploring the structure of the course to find out whether it could be learnt using self-study (U3). The two students who only attended lectures had different reasons for doing so. U6 believed learning through attending lectures is easier than doing self-study. U4 attended lectures to write notes towards learning the topic later. These two had different reasons for not attending the tutorials, including that only the easy questions were solved by the tutor (U4); and, the time of the tutorial was too close to the time that assignments needed to be handed in (U6).

U5, who only attended tutorials, believed attending lectures without pre-reading the materials is a waste of time. She stated that she was also too busy with other courses, so that she could not do pre-reading for this course; therefore she only attended tutorials that she had time to prepare for. U1, who did not attend either lectures or tutorials, was behind with his other courses and had therefore decided he had not the time to attend this course's lectures and tutorials.

Attending school classes

Students in Case 2 had varied reasons for attending classes in these topics. Despite the fact that attending the classes is compulsory, which was mentioned by two students (Y24), three reasons were found as to why attending the classes are important.

- Four students said attending the classes is useful for learning the topic (Y1456). Three students (Y258) believed it is easier to learn in the classes rather than doing their own reading. Y6 believed that, to have a better understanding and do well in exams, you need to attend classes; otherwise by self-reading you could just pass the course.
- Getting support from their teacher and peers was another reason for attending the classrooms, mentioned by three students (Y128).
- Y7 and Y8 said they were more motivated to study when they come to classes, and Y7 also added that he attended the classes so as not to lose his skills in calculus and to reinforce his understanding. An example of the students' responses is:

> I find it easier to learn if I am getting taught face to face and you can ask questions and you can see the person doing it. But when you read the textbook you cannot get the full idea of it. If I was at home, I would not have such motivation to do work, but if I am in class it helps me to do more work. Also, if I am stuck or my friend stuck we can help each other. If I make a mistake, I can compare my work with other people (Y8).

In comparison, Y3 believed attending the classes is not useful because he could do self-study: "I can learn these topics at home as well. Because the teacher told us the same thing that is in the textbook".

Discussion

The importance of attending classes is highlighted in the literature and most students in the sample were aware of it. Therefore the results show the presence of this aspect of metacognitive knowledge for a majority of students. Several studies highlight that class absence is negatively correlated with overall course grades (e.g., Brown, Graham, Money, & Rakoczy, 1999; Clump, Bauer, & Whiteleather, 2003). The study findings are consistent with previous research, in that U1, who did not attend the lectures and tutorials, had scored the second lowest points in integral questions in Case 1 (13 out of 36). In addition, Y3, who believed that attending the classes is not useful, scored the lowest performance score in integral questions of the sample (i.e., 6 out of 36).

8.1.2 Taking notes in lectures, tutorials, and classes

Most students in the sample took notes while attending classes and were aware of the importance of taking notes. In detail, thirteen students (U1234567; Y124568) took notes when attending lectures, tutorials, and classes. The remaining four students (U89; Y37) did not take notes. The reasons for taking notes included that they were:

- useful for studying later, including reviewing and revising the materials (U4; Y23568);
- useful for learning the steps of problem solving and understanding the topic (U2; Y16);
- helpful for remembering the topic (U6; Y4);
- good for "engaging brain" (U25);
- useful to "supplement textbook and online notes" (U37);
- easier to follow than online notes (U46); and
- good for keeping students focused (U4).

The four students who did not take notes had several reasons for this. Y3 believed it is a waste of time and he could use the textbook to understand the topic. Y7 said he did not take notes this year because he had done this course last year. U8 did not take notes because he only used the online notes and said "…because we have online notes and I find it easier if I have got full attention to the material. But if it is an example, I try to follow along and compare them". U9 said because calculus is computational and the material is not new, he did not need to write notes for this course; however, he noted, he took notes for other courses like discrete mathematics.

Discussion

Two main reasons are highlighted that support the value of taking notes during classes, including encoding and external storage (Anderson & Armbruster, 1986). Encoding is important because it helps students learn and remember topics. Taking notes is considered as an external storage because it preserves information for later use (e.g., for studying before examinations) (Anderson & Armbruster, 1986). These two reasons were mentioned by students, indicating that students in both Cases were aware of the usefulness of taking notes. Therefore, the results indicate the presence of this aspect of metacognitive knowledge for a majority of students.

8.1.3 Engaging with lecturers, teachers, and tutors

Five students (U23578) only engaged with instructors when they attended lectures, tutorials, and school classes. The rest, except Y3, both engaged with the instructor and talked with their classmates (Table 8.2). The reason Y3 did not engage with his teacher was that he preferred self-study. For other students, talk was about the course topic, on task, or off task. On task it was about helping each other when one of them was confused. Off task talk happened when the materials were boring (U9), not new to students and students were "comfortable" with it (U49), or students were "completely lost" (U46).

Table 8.2

Engagement during attendance at classes

	Case 1	Case 2	Total
Only engaging with instructors	5 (U23578)	0	5
Engaging with the instructor and talking to their classmates	4 (U1469)	7 (Y1245678)	11
Not engaging with instructors	0	1 (Y3)	1

Students' statements about their reasons for engaging with the teaching are provided in Table 8.3. Four themes were found in relation to students' engagement with the teaching, including learning, efficiency, behaviour, and affect. Most of the students in the sample engaged with teaching because they wanted to learn the topics. The most frequent sub-themes within their learning were to get the full idea and not miss anything from the teaching.

Table 8.3

Reasons for engaging with the teaching

Themes	Sub-themes/ examples	Case 1	Case 2	Total
Learning	"To get the full idea and don't miss anything"	3 (U789)	1 (Y8)	4
	"knowing how to do the questions"	1 (U2)	2 (Y16)	3
	To be able to take notes effectively	1 (U4)	1 (Y4)	2
	Easier to learn from the teacher rather than reading textbooks	0	1 (Y2)	1
Efficiency	"Don't want to waste time"	3 (U365)	0	3
Behaviour	It is a habit to only listen to teachers in class	0	1 (Y5)	1
Affect	The teacher was "very interesting"	0	1 (Y7)	1

Discussion

Most of the students were aware of the importance of engaging with the instructor during teaching. Y3, who did not value engaging with the teaching, achieved the lowest performance in the integral questions.

8.1.4 Pre-reading before attending lectures, tutorials, and classrooms

Most of the students in the sample did not use pre-reading before attending the classes. Four students (U358; Y2) did pre-reading about the materials that would be taught in classrooms before attending them. Students had different reasons for doing so. U8 did this to be introduced to the idea and be able to understand it a lot better the second time in lectures. U5 believed if she did not do the pre-reading, she would not understand the topic in the lecture. U3 preferred self-learning, therefore, "read around the topic" before attending sessions. He added, "I go to lectures to pick up odd things, information, stuff, that might not come up naturally". Y2 said, "It is easy to understand when you have taken a look ahead".

Those who did not do any pre-reading had different reasons for not doing so. Three students (U6; Y14) believed it was a good idea to do this, but they had not done it before: "I did not think about it. Thinking about it, it would be a good idea, probably" (Y1). Y1 also said he did have time to do so. Two students (U2; Y6) believed they did not have time to do it. Two University students (U47) believed that they were not organised enough to do pre-reading before attending lectures. Four students (U9; Y758) believed the teaching of the instructor was enough for learning the topic, indicating reliance on the instructor: "the teacher does everything" (Y5) and U9 said, "I think it is not necessary. I expect the lecturers to stand by themselves". Y3 did not do any pre-reading because he preferred self-reading; therefore, he only attended classes because it was compulsory. Y8 said doing pre-reading may cause misunderstanding about the topic:

> We cover all of the workings in class. We do not require to do pre-reading. I think it would be quite difficult because a lot of things that we are learning is complicated, so if we go and try to understand it ourselves, we might get the wrong idea, or we might not able to understand it. But if the teacher teaching it to us we might get a better idea and starting off from the right places rather than us starting off from a wrong place and have to do re-work to understand it.

Discussion

Of the four students who had done pre-reading, three of them (U358) scored more than half of the possible points in the integral questions. U5 and U8 scored the highest points in the sample for the integral questions. Comparing these four students with the other students in the sample in terms of mean score in integral questions, there was no significant difference between students who had done pre-reading (24.0) and those who had not done pre-reading (16.76) according to the Mann-Whitney test (P-value=0.1) which may have been due to the small sample size.

The literature highlights that there is a direct relationship between pre-reading and academic achievement (e.g., Hwang & HSU, 2011; Spies & Wilkin, 2004). Pre-reading re-inforces students' prior knowledge and relates new topics to students' prior knowledge (Hwang & HSU, 2011). Pre-reading also relates to the idea of a flipped classroom approach to the teaching, "students prepare for class by engaging with resources that have been pre-prepared by their teachers" (Muir & Geiger, 2015, pp. 149-150), an approach that has gained more attention in the past few years (See Muir & Geiger, 2015 for more information). As most of the students in the sample had not done pre-reading, and were not aware of its importance, the results indicate that students may benefit from this learning strategy, which therefore could be suggested to them by lecturers and teachers.

8.1.5 Reviewing previous materials before attending the next session

This section is different from the previous section as it relates to studying topics that have already been taught in the classes, not new topics. Twelve students (U13457; Y1245678) had regularly studied previous materials before attending the next sessions. These students had different reasons for doing that including that they found it:

- reinforced learning (U1345; Y678);
- helpful for doing homework/assignments (U7; Y1246); and
- helpful for remembering the materials (Y25).

Two sample responses are: "it allows me to filter to unique lecture note. I can filter information that is not trivial, but, can be easily adapted [sic]" (U3); and, "help me understand it and makes it easier to understand new content which is based on it" (U4).

Students who did not study previous materials regularly, had different reasons for not doing so. U6 believed it is a good idea, but she had not done that before. U2 said he only looked at them if it helped him with assignments. Two students (U89) said they looked at these materials if they were confused or the topic was hard: “If I was confused, I would, but if I have got it, I am not looking at it again” (U9). U8 also looked at them if he had a problem with assignments.

Discussion

Reviewing previous materials before attending classes is one of the study techniques suggested in the literature (e.g., Gurung & McCann, 2011; Gurung, 2004). Most of students in the sample had done that and were aware of its effectiveness. Therefore, the results indicate the presence of this aspect of metacognitive knowledge for a majority of students.

8.1.6 Learning resources

In response to M2 (Table 5.1), students said they had used different resources for learning integral calculus (Table 8.4). The resources can be categorised into offline, online, and human resources. The most frequent resource category used was offline resources in both Cases.

The textbook was used by all the students except U2, who only used published lecture notes and the internet. For students in Case 1, in addition to the textbook, published lecture notes written by the lecturer, and the internet, were the main resources used to learn the topic. For students in Case 2, textbook, personal notes, past exam papers, and the teacher were the main resources for learning the topic. Relying on the instructor is more dominant in Case 2 in comparison to Case 1. No students in Case 1 mentioned the lecturer or the tutors as a learning resource for the topic, whereas four students in Case 2 said “the teacher” in response to question two: “[I] use the teacher because he understands the topic and can help me face to face” (Y8).

Table 8.4

Learning resources mentioned by students in response to M2

Theme	Resources	Case 1	Case 2	Total
Offline resource	Textbook	8 (U13456789)	8 (Y12345678)	16
	Published lecture notes	8 (U12356789)	N/A	8
	Personal notes	1 (U4)	6 (Y124568)	7
	Past exam papers	1 (U2)	4 (Y2467)	5
	Extra workbooks/test books	2 (U57)	3 (Y136)	5
	Tutorial and assignment solutions	2 (U12)	N/A	2
	Teacher's handout	N/A	1 (Y6)	1
Online resources	The internet	8 (U12345679)	2 (Y58)	10
Human resources	Teacher/lecturer/tutor	0	4 (Y2478)	4
	Classmates/friends/girlfriends	2 (U47)	2 (Y48)	4
	Family: Sister	0	1 (Y4)	1
	Help Desk	1 (U6)	N/A	1
	Extra tutor outside of the class	0	1 (Y1)	1

Students in Case 2 (Table 8.5) did not use the internet frequently as a resource for learning the topic. Only Y5 used the YouTube channels for learning the topic and Y8 used them to find mathematics formulas. The use of the internet by other students in Case 2 was for downloading past exam papers from NZQA or the College website. Y4, in response to my follow-up question, “Do you use the internet for learning the topic?” said: “No, [I] use the web to get questions, not information”, indicating a difference between students in Case 1 and Case 2 in terms of using resources available on the internet. Downloading past exam papers from NZQA or College websites is not coded as a usage of the internet, because the students went to a specific website that they were aware of. They had not used a wide range of the integral questions and notes available on the internet.

Table 8.5

Ways students used the internet for learning the topic

Types of using the internet	Case 1	Case 2	Total
YouTube channels	5 (U13457)	1 (Y5)	6
Wolfram alpha	6 (U123467)	0	6
Wikipedia	4 (U1379)	0	4
Googling the topic and using files that appeared	3 (U456)	0	3
Finding mathematics formula online	0	1 (Y8)	1
Online integral calculator	1 (U2)	0	1

In relation to internet resources, the Wolfram alpha website was mainly used for checking answers. In terms of YouTube channels, U5 critiqued a channel that is widely used on the internet for learning mathematics. She said, “It just provides basic ideas, not providing a deep understanding…the person who makes the videos are not prepared the topic, the length of the videos are longer than other YouTube files on the same topic”. U5 also showed that she was aware of learning theories, indicating MK in terms of cognitive function (Section 3.1.4), and said, “from a journal paper that I have read a long time ago, people cannot concentrate on some intense topic for more than five minutes. That's why some standard YouTube files are less than five minutes”.

In response to question two, the importance of using different resources for learning a topic was highlighted by four students (U1357). These students were aware that using different resources help students to explore a topic from different perspectives:

> A lot of time people will state their teaching in one way, and repeatedly they stated in one way. If you go to a lot of sources, you see it is stated in different ways and in might be one of them is more understandable to you or expressing in such a way that you can see what can be done with it (U3).

Discussion

Students in both Cases used several resources for learning integral calculus. There were two main differences between the resources used by students in the respective Cases: first, students in Case 2 relied more on the instructor as a learning resource compared to students in Case 1. This might be because the Case 1 environment, especially lecture size and number of students in the lectures, might prevent students talking to lecturers. However, the tutorial size and number of students attending tutorials in Case 1 were similar to College classes. Second, students in Case 1 used internet resources more frequently in comparison to students in Case 2. The potentials of the internet as a learning resource should be highlighted to students in College more frequently, as College students in the sample did not use it frequently for learning these topics, and also the teacher did not promote these resources in his teaching. One of the possible reason for this finding might be related to not having access to internet at home. If websites such as https://www.desmos.com/ for drawing curves or http://www.emathhelp.net/calculators/calculus-2/integral-calculator/[11] for finding antiderivatives were used in the classes, students might be more likely to use these tools for checking their answers.

8.1.7 Memorising strategies

In response to M5 (Table 5.1), thirteen students (U2345689; Y234678) said they did not have any memory strategies for learning these topics. The others had varying memory strategies (Table 8.6). Y1 said he used flip cards that he made to learn the steps. Y5 used the fact that he knew the rate of change of a parabola is a linear graph for doing rate-of change-questions related to other types of functions. He said, "the rate of change of a parabola is a linear graph. It can help me in other examples as well, even for changing cubic functions to parabola, I always remember this first".

[11] The latter is a very useful website because it provides the complete solution free of charge, while some of the most popular websites do not provide the complete solution free.

Table 8.6

Memory strategies used by students for learning the topics

Memory strategies for integral-area problems	Case 1	Case 2	Total
LIATE rule for substitution in integration by parts	2 (U17)	0	2
Using flip card to remember steps	0	1 (Y1)	1
Use one known information for remembering similar information	0	1 (Y5)	1
Use a Table for remembering the values of Sin x and Cos x	1 (U1)	0	1
None	7 (U2345689)	6 (Y234678)	13

The LIATE (Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential) rule for choosing u in integration by parts $\int udv = uv - \int vdu$ was mentioned by two students in Case 1 (U17). Despite this, U1 said he used the following table (Table 8.7) to remember the value of basic trigonometric functions. He said that, for $\sin x$, the values start by $\frac{\sqrt{0}}{2}$, and for each major angle the value under the square root increase by one. For $\cos x$, the process is inverse, and for $\tan x$, and $cotan\ x$, he said he used the fact that $tanx = \frac{\sin x}{\cos x}$, and $cotan = \frac{\cos x}{\sin x}$.

Table 8.7

A table used by U1 for memorising values of basic trigonometric functions

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$Sin\ x$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
$Cos\ x$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$

Those students who did not use any memory strategy for learning these topics had different reasons for not doing so (Table 8.8). Four students in Case 1 (U3489) said they tried to understand the idea behind the concepts and formulas, then they derived it when necessary rather than memorising the formulas. In addition, U3 was against using memory strategies for learning mathematics:

> I think rote learning is the worst idea ever, formula memorisation, etc. [Learning] without understanding is useless because I am a big believer of you should be able to reproduce all of the formulas you use from the scratch; that is my goal in any papers to be able to recreate everything as you need. Understanding can reproduce everything.

Table 8.8

Reasons given by students for not using memory strategies

Reasons for not using memory strategies	Case 1	Case 2	Total
Use understanding the idea behind the concepts for reconstructing materials	4 (U3489)	0	4
Do not know any	0	3 (Y268)	3
Formula sheet	0	2 (Y34)	2
Do questions/practice	1 (US4)	1 (Y6)	2
Do not need any	0	1 (Y6)	1

Practising questions was another reason for not using memory strategies mentioned by two students (U4; Y6). Formula sheets were given in exams for both students in Case 1 and 2. However, only Y3 and Y4 said that because they had the formula sheet in exams, they did not need to memorise the formulas. Y6 said he did not need any memory strategy: “I haven't really come across anything that I think I need it. The questions are about having method you can use and work through things and you just practicing by doing questions”.

Discussion

Most of students in the sample did not use any memory strategy for learning integral calculus. Three (U1; Y15) out of four students who used memory strategies for learning the topic scored less than half the possible points for integral calculus questions. The fourth student, U7, got the third lowest points of the students in Case 1. Comparing students who used memory strategies with the other students in the sample in terms of mean score in integral questions, there was no significant difference between students who used memory strategies (13.2) and those who had not used those strategies (20.1) according to the Mann-Whitney test (P-value=0.3) which may have been due to the small sample size. Therefore, the obtained results suggest memorising strategies are not associated with mathematical performance in this instance. No research has been found in this regard to compare this study findings with. In addition, four students in Case 1 (U3489) emphasised the importance of understanding the ideas behind the concepts and formulas, relational understanding (Skemp, 1976) (Section 3.1.2), indicating they had a better understanding of how mathematics should be learnt. This fact was not highlighted by students in Case 2. These four students achieved a higher mean score (28.5) compared to other students (15.4). The difference was significant according to the Mann-Whitney test (P-value=0.01).

8.1.8 Summarising strategies

Summarising strategies are different from memorising strategies, as the latter focus on learning one chunk of information, while the former focus on connecting several chunks of information. In response to M7 (Table 5.1), eleven students (U12358; Y235678) said that they did not use any summarising strategies for learning these topics. The rest said they had done some sort of summarising or intended to do so later (Table 8.9). Three students in Case 1 (U469) said they would make a summary of the materials close to their exams: "I write down everything in pages, then I make it smaller and smaller to a page. It is a process of writing and copying help me. It is how I study previously. Writing helps" (U9). U6 supported the idea that writing helped and added another reason why making summaries is useful: "writing it down helps you remember it. Also, for last minute study before going to the test, you can look at it [your summary]." Y1 used flip cards to remember the steps of problem solving (Section 8.1.7), which can also be considered a summarising strategy: "it kind of make it easier, rather than having a whole lot of information. You can break it down to particular things you need to remember. Make it easier to remember the procedures". Y4 said he added a summary to his class-notes to know what he needed to focus on. U1 and Y8 said having a summary is useful, but they had not made one so far.

Table 8.9

Summarising strategies given by students for learning these topics

Summarising strategies for learning these topics	Case 1	Case 2	Total
Make a summary close to exams	3 (U469)	0	3
Add a summary to class-notes	0	1 (Y4)	1
Make a summary of the course materials in a page	1 (U9)	0	1
Make flip cards of steps	0	1 (Y1)	1
Summarise mathematical statements and make a list of possible ways to deal with similar statements	1 (U7)	0	1
None	5 (U12358)	6 (Y235678)	11

Students who had not done any sort of summarising or had not intended to, had different reasons for this (Table 8.10). Four students (U8; Y278) said because they had the formula sheet they did not feel it necessary to make a summary:

> No, I just use the formula sheet. I found that you need to practise how to use each formula and then you can just see the formula there. You get faster at doing them. I think it is more effective than trying to memorise formulas (Y2).

Table 8.10

Reasons for not using summarising strategies for these topics

Themes	Sub-themes/ example	Case 1	Case 2	Total
Use other resources	Formula sheet	1 (U8)	3 (Y278)	4
Use other learning strategies	By practicing question, summarising is not necessary	0	3 (Y236)	3
	Read lecturer/class notes rather than summarising	1 (U8)	2 (Y68)	3
	Not for these topics because it is not complicated	1 (U5)	0	1
	“Never felt the need”	1 (U4)	0	1
Negative attitude toward summarising strategies	Dislike rote learning	1 (U3)	0	1
	“Mathematics is not a memorising stuff”	0	1 (Y5)	1

U5 said she had done that before, but because these topics are not complicated she did not feel it necessary to make a summary of them:

> I do that for the trig, inverse trig, hyperbolic trig, and inverse hyperbolic trig. But for the FTC I did not do that because it is not that complicated…when you have lots of materials that look similar, you need to read it and compare.

U3 was against using summarising strategies because he felt it is related to rote learning: “No, I just make sure I am comfortable going through everything without it…I dislike rote learning…with understanding the relationships are obvious. The reason I like math is there is no real rote learning”. Y5 also made a similar claim and said, “Mathematics is not a memorising stuff [sic]”.

Discussion

The literature suggests summarising as a tool that helps students to comprehend knowledge, transfers knowledge to long-term memory more easily, and makes connections between the new concepts and students’ prior knowledge (King, 1992; Susar & Akkaya, 2009) (Section 5.2.2). However, it was not used by most of the students in the sample. Students with good integral calculus performance can be seen in both the groups of students who had used summarising strategies and those who had not; therefore, the study’s findings cannot show any association between using these strategies and mathematical performance. However, students who used summarising strategies received a higher mean score for integral questions (20.3) compared to those who had not used such strategies (17.4); but the difference was not significant according to the Mann-Whitney test (P-value=0.7).

It seems some of the students had negative attitudes toward summarising strategies, and that should be reconsidered. Summarising strategies can facilitate learning, and their usefulness should be highlighted to students. If websites such as https://bubbl.us/ could be used in classes by lecturers and teachers as an example to show how different mathematical concepts and theorems in a topic are related to each other, students might also realise its usefulness and use summarising strategies for other topics as well.

8.1.9 Thinking about justifications behind the formulas

In response to M6 (Table 5.1), eight students (U345689; Y57) said they had thought about the justification/rationale behind the formulas rather than just applying them. The remaining students said they had only applied the formulas. In this section, all the students except U9 did not differ between the integral-area relationships and the FTC when answering these questions, and said they had the same opinion regarding the FTC problems. However, U9 had thought about justifications for the integral-area relationships, but not for the FTC because he did not know the justification for that: "…not for the FTC, I have no ideas what it means geometrically. As far as I know there is no geometric interpretation [for the FTC]".

Students who just applied the formulas had different reasons for doing so (Table 8.11). The main reason for not thinking about justification was because it was not asked for in exams, therefore, they did not need it:

> I think it can be quite interesting, but I guess it probably not might be necessary at this level if it is not asking questions about it in the exam… I think it is an extra; I think for most people, including myself, want to do well in exams...rather spending time making sure I am better at things which gonna be in the exam (Y6).

Table 8.11

Reasons for not thinking about justifications

Themes	Sub-themes	Case 1	Case 2	Total
Negative attitude towards justifications	Do not need to know it/ not in the examinations or questions	0	4 (Y1368)	4
	Not have time to think about them	2 (U27)	0	2
	Sometimes justifications are more complicated than memorising them	1 (U1)	0	1
	Justifications confuse me	0	1 (Y1)	1
No access to justification	Have not seen a thorough justification for the topic	0	3 (Y248)	3
	Does not mention it in the textbook	0	1 (Y3)	1

The second reason was not seeing a thorough justification for the materials in the classrooms. Y8 thought the integral calculus topic did not have thorough justifications behind it: “Not in the integration probably because they do not have a thorough understanding. I do in differentiation”. Y4 said he would have liked to see the justification, but the teacher did not provide it because of time constraints:

> I do wonder about the justification behind it… It allows to have a deep understanding what you are actually doing with the formulas. I do not ask specifically, but, I noticed people asking about it, but, the teacher said we do not have time to explain it.

Two students in Case 1 (U27) said they did not have time to go over the justifications: “I wish I had time to think more about the rationale, but I am too busy applying trig identities in every conceivable combinations” (U7). Y3, who said the justifications are not stated in the textbook, said: “when I do [questions from] the textbook, I am using the formulas, and I do not care about anything behind or around the question. I just do the question and get the answer”.

Those students who had thoughts about the justifications behind the formulas had different reasons for doing so (Table 8.12). Three main themes were found, including that they are helpful for remembering, applying, or reproducing formulas, to have a better understanding about the topic, and better performance in exams and answering questions.

Table 8.12

Reasons of thinking about the justifications behind the formulas

Themes	Sub-themes	Case 1	Case 2	Total
Helpful for remembering, applying, or reproducing formula	Help to reproduce them when necessary	4 (U3458)	0	4
	Help to remember the formula	2 (U69)	0	2
	Don't need to remember the formula	1 (U4)	0	1
	It is easier to apply formula when you understand it	1 (U8)	0	1
Have a better understanding	To have a better understanding about the topic	2 (U89)	1 (Y7)	3
	Remembering the formula is not sufficient	1 (U5)	0	1
Have a better performance in exams, and answering questions	It is easier not to make a mistake	1 (U8)	0	1
	Helpful for checking workings	1 (U5)	0	1
	To answer some questions, knowing justification is necessary	0	1 (Y5)	1
	Maybe a question about the justification being asked in the scholarship exam	N/A	1 (Y7)	1

A sample response from the first theme is, "I do not have to remember the formula. If I understand the concept I can come up with the formulas for different questions" (U4). In terms of the second theme, U8 said:

> It just makes sense a lot better when I understand what is going on. That is what I like about the math. It is theorems and proofs, not just the tools. If I understand it, it is easier to apply it correctly and not make a mistake. If I do forget it and if I understood it, it is easier to reproduce it.

In relation to the last theme, U5 believed by knowing justification you could check your workings: "…otherwise, how can you double check it is legit?"

Discussion

Half the students in Case 2 thought they did not need to know about the justifications behind the formulas. This may have prevented them from having a relational understanding about mathematics and how mathematics should be learnt. The importance of knowing the rationale behind the theorems and formulas is addressed in the literature (e.g., Cuoco, Goldenberg, & Mark,

1996) and most students in Case 1 were aware of its importance and usefulness. Therefore, it seems that its importance should be highlighted to students in college.

A comparison of students' integral calculus performance between those who had thought about the justification behind the formulas and those who had not, shows the first group achieved a higher mean (20.1) in comparison to the second group (13.2), and this difference is significant according to the Mann-Whitney test (P-value=less than 0.01).

8.1.10 Prior knowledge necessary for learning the topics

In response to M4 (Table 5.1), students said different prior knowledge is necessary for learning these topics (Table 8.13). In terms of algebra, Y6 said it is necessary for "manipulating functions". Regarding differentiation, U6 believed differentiation should be taught before integral calculus: "if you jump straight into integration people think what is this?" Y8 believed differentiation as prior knowledge help him to understand the topic and could be used for checking the antiderivative: "differentiation can help you understand the integration and also if you differentiate [the antiderivative] you can see what the answer kind of be". Limits and summation were mentioned for understanding Riemann sums and Riemann integral. U9 said, "Physics can also be considered as a prior knowledge because several questions in this topic involve velocity, displacement, and acceleration".

Y4 mentioned justification about the formulas and applications of the topic in daily life as a prior knowledge for learning the topic: "I always like to know where these things come from. We are just dumps with equations and told to use them. [Also, I want to know] the reason why we learn this for future [sic]". Trigonometry, logarithm, natural logarithm, and exponential functions are the functions about which students believed knowledge was necessary for learning these topics.

Table 8.13

Prior knowledge necessary for learning the topics

Themes	Sub-themes	University students	Y13Students	Total
Mathematical topics	Algebra	5 (U13459)	5 (Y24568)	10
	Differentiation	6 (U123689)	3 (Y138)	9
	Skills of drawing function	4 (U1239)	3 (Y178)	7
	Basic numerical skills/ Arithmetic	4 (U1346)	3 (Y678)	7
	Functions	4 (U4679)	0	4
	Trigonometry	1 (U7)	3 (Y256)	4
	Euclidean geometry	2 (U79)	1 (Y3)	3
	Logarithm and natural logarithm function	1 (U7)	2 (Y26)	3
	Exponential functions	0	2 (Y26)	2
	Limits	2 (U38)	0	2
	Summation	2 (U38)	0	2
	Familiarity with Leibniz and Newton notations	1 (U3)	0	1
	Solving linear equation	1 (U5)	0	1
	The idea of area	1 (U4)	0	1
Other skills and knowledge necessary	Justification about the formulas	0	1 (Y4)	1
	Applications of the topic in daily life	0	1 (Y4)	1
	Being able to use a calculator effectively	1 (U1)	0	1
	Physics	1 (U9)	0	1

Discussion

Students in both Cases knew that various prior knowledge is necessary for being successful mathematical problem-solvers in integral calculus. However, several students had difficulties with this prior knowledge; in particular algebra, functions, and sketching functions when solving integral questions (Chapter Seven). Therefore, it seems that mastering the prior knowledge necessary for integral calculus should be highlighted to students. Similar findings have been reported in literature. Kiat (2005) reported many students have technical errors when solving integral problems and suggested remedial lessons and revision worksheets related to prior knowledge to prepare students for the integral calculus topic. In addition, diagnostic tests in relation to topics necessary for integral calculus could be used for helping students understand which part of their prior knowledge needs further development.

8.2 Strategic knowledge: Monitoring strategies

In response to M8 (Table 5.1), students in Cases 1 and 2 mentioned different strategies for checking their answers in integral-area (Table 8.14) and the FTC (Table 8.15) problems. However, three students in Case 1 (U137) highlighted they did not have time to do that in their exams.

Table 8.14

Monitoring strategies for integral-area problems

Themes	Sub-themes	Case 1	Case 2	Total
Monitoring strategies related to integral-area relationship	Approximating the area using geometric shapes	6 (U134689)	2 (Y78)	8
	Check the area is positive	2 (U39)	3 (Y156)	5
	Check the antiderivative by differentiating it	2 (U27)	3 (Y458)	5
General monitoring strategies	Double-check /redo working	6 (U134568)	4 (Y2457)	10
	Use the Wolfram alpha website to check answers	5 (U34567)	0	5
	Check answers with classmates	2 (U46)	2 (Y67)	4
	Use the answers at the end of textbook	0	4 (Y2346)	4
	Use assignment solutions	2 (U23)	N/A	2
	Use calculator to check answers/graph of curves	0	2 (Y36)	2
	Use the Maple software for checking answers	1 (U8)	0	1
	Long answer-simple equation: You probably do it wrong	1 (U6)	0	1
	Express the final answer as an English statement to see whether it makes sense	1(US7)	0	1
	Think about the answer before doing the problem and compare the answer with your thoughts	1(US6)	0	1

In terms of integral-area problems, the most frequent strategy for checking answers was going over the calculation (U134568; Y2457). The second was approximating using the geometric shapes to find out whether the answer makes sense (U134689; Y78). Using the fact that the area should be positive was another strategy mentioned by five students (U39; Y156) to check the final answer is correct. These students were aware that if the final answer is negative, then some parts of their workings were not correct. Because to find the area enclosed by the curves using integration, first the antiderivative of the curves should be found, five students (U27; Y458) said they check whether they found the correct antiderivative by differentiating it. In a non-exam situation, five students in Case 1 (U34567) used the Wolfram alpha[12] website and U8 used Maple software for checking answers. However, students in Case 2 did not use these resources, and only two (Y36) said that they had used a calculator to check their answers. Four students (U46; Y67) compared their answers with their classmates' answers. Another strategy mentioned by four students in Case 2 (Y2346) was using the answers given at the end of their textbook for checking answers. Using the assignment solution, mentioned by two (U23), was another strategy used for checking the answer in a non-exam situation. A strategy only mentioned by one of the students in Case 1 (U6), was, if you are dealing with a simple equation and your answer is long, you had probably done the question incorrectly. U7 wrote the final answer as an English statement to find out whether it makes sense for him. Finally U4, before solving the problem, thought about the possible answer, and then after finding the answer, he compared his answer with his initial thoughts.

[12] Wolfram alpha has the capability to solve indefinite and definite integrals and if a person subscribes to the website, he/she can access step-by-step solution methods.

Table 8.15

Monitoring strategies for the FTC problems

Themes	Sub-themes	Case 1	Case 2	Total
Monitoring strategies related to the FTC	Check the antiderivative using differentiation	4 (U45677)	2 (Y58)	6
General monitoring strategies	Double check /redo working	4 (U3458)	1 (Y8)	5
	Check whether the final answer makes sense	1 (U6)	2 (Y28)	3
	Use the Wolfram alpha website	1 (U7)	0	1
	Check with classmates	1 (U4)	0	1
	Express the final answer as an English statement to see whether it makes sense	1 (U7)	0	1
None	N/A	3 (U129)	5 (Y13467)	8

The most frequent strategy for checking answers for the FTC problems was checking the antiderivatives using differentiation (Table 8.15). This strategy is useful for questions that relate to the definite integral, the first part of the FTC. There was not any particular strategy mentioned by students for checking questions related to the second part of the FTC. U3 highlighted, "I feel there is little I can do [for checking the FTC problems]. Ensure understanding if you can. It is more abstract. Harder to check". Double-checking/redoing working, checking whether the final answer makes sense, using the Wolfram alpha site, checking with classmates, and expressing the final answer as an English statement are strategies mentioned by students that are not specifically related to FTC problems.

In terms of checking answers for the FTC problems, eight students (U129; Y13467) said they were not sure how to check answers in relation to FTC problems, suggesting they may not have had a good understanding of this theorem.

Discussion

It seems online resources and technology for checking answers should be suggested to students in College. Five students in Case 1 had used Wolfram alpha and one used Maple software for checking answers on this topic; however, students in Case 2 had not used online resources for checking answers, and only two students in Case 2 used calculators for checking their answers.

There are several websites that can be used for checking answers. Some of them also provide step-by-step solution free of charge (e.g., http://www.integral-calculator.com/). If teachers in Colleges use those websites for checking answers to questions that have been solved on the board, College students might also use them for checking their answers. The step-by-step solutions that are provided by some websites are useful for checking answers to textbook questions as well, because the textbook that has been used for teaching in the College (i.e., Delta mathematics (Barton & Laird, 2002)) only provides the final answer to the questions at the back of the book.

Approximating area using geometric shapes, differentiating antiderivatives, and the fact that area should be positive are monitoring strategies that should be highlighted to students for checking answers. More than half the students in the sample did not mention those strategies for checking answers. If lecturers/teachers use those strategies for checking their answers in classes, students might be encouraged to use them.

Approximately half the students (8 out of 17) had no idea how they could check their answers related to the FTC problems. In addition, there was no particular strategy mentioned by students for checking questions related to the second part of the FTC. Strategies that can be used for checking answers in questions related to the second part of the FTC are not fixed, unlike questions related to the first part of the FTC or integral-area relationship. However, using geometric interpretation of the second part of the FTC might be useful for checking whether the answers make sense.

8.3 Strategic knowledge: Problem-solving strategies

In response to M9 (Table 5.1), students were asked to pose two plans, including a plan for how they would solve enclosed area between curves problems, and a plan for the FTC problems. All the students were able to provide a plan for the area between curves problems. Six plans (U23; Y1247) were general, and the remaining 11 plans were detailed (U1456789; Y3568). Those plans that had three, or less than three, steps were considered to be “general plans”, and those with more than three steps were considered to be “detailed plans”. A sample of a general plan is: “1. Look for the bounds. 2. Integrate the function. 3. Use appropriate formula to find the area” (Y4). An example of a detailed plan is “1. Visualise/sketch. 2. Find the top and bottom function. 3. Calculate $f(x) - g(x)$. 4. Integrate $f(x) - g(x)$ 5. Put the bounds in [the anti-derivative]” (Y6).

Two items were considered for exploration of students' plans, including checking the integrand being continuous on the interval of integration, and checking/evaluating the processes and answers. Only U1 mentioned the process of checking the integrand is continuous as part of his plan. The first step in his plan is "check that both curves are continuous on the closed interval - where you are trying to find the area". In terms of checking the process and answer, three students in Case 1 (U489) said checking was part of their plans. U8, at the end of his plan wrote, "make sure the answer makes sense". In U9's plan, the final step was checking the answer is positive. U4 had more checking processes in his plan. After he set up the integral for the enclosed area between curves, he checked that he had not made a mistake in writing $f(x)$ and $g(x)$ because he said I "usually write them wrong". Then, after finding the area by solving the integral, he asked himself "whether it [the answer] is reasonable?" He also said that before solving the problem he tried to guess the answer to compare it with his final answer. If the final answer makes sense for him, he took "a quick look at the working", if not, he took "a quite thorough look at the working".

In terms of having a plan for the FTC problems, thirteen students (U134569; Y123467) had no plan for solving those problems. For example, U9 said, "no [I do not have any strategy, just] cry", and U3 said, "if obvious, then solve, if not try something with no confidence". It is worth mentioning the FTC problem is a broad term in integral calculus and several questions can be considered as an FTC problem, including the integral-area problems. During the interviews, if a student asked about the statement of the FTC, I only told them it was the relationship between differentiation and integration, in order to direct students' focus onto the second part of the FTC. For the remaining four students, one plan was related to the first part (Y5), two related to the second part of the FTC (U8; Y8), and one was a general plan (U7). Y5's plan for the FTC was: "1. [consider the] power of the variable. 2. Raise or reduce depends on the question. 3. Change the shape of the graph. 4. Add/derive the power. 5. Check the answer maybe". Y8's plan for the FTC was: "Try to get my head around stationary points and points of inflection". His plan was suitable for FTC questions similar to question eight of this study. U8's plan was related to FTC questions where students are asked to differentiate an integral, similar to question six of this study, "If necessary, split up, use the chain rule to find resulting function". The general plan mentioned by U7 was "1. Express the problem as a function. 2. Do all the maths stuff".

All students would have liked to see a plan for how the integral-area problem could be solved while the materials were being taught to them. Students said this after Figure 2 of the interview question (Appendix 1) had been shown to them and I had talked to them about the steps. This happened after they had posed their plans. For instance, U5 said, “It is super,” and U4 said, “It could be quite helpful to see something like this”. Some of the students provided more comments about the plan. For example, Y5 said, “It can show you step by step what you should do to get the correct answer”. Two students in Case 1 (U25) and all students in Case 2 said they would like to see a plan after some questions were solved in the class, and U1 said he would like to see it at the start of the topic. Others did not provide any comment about when they wanted to see the plan. For instance, Y8 said, “…probably after you have been taught because if I did not know what I was doing and I got this, probably overwhelming. Probably have a few practical demonstration of it”. U1, who would have liked to see a plan at the start said

> Right at the start, to be honest, and when you go along at the end of each lecture you can come back and say this part is checked off and move on. Because we have a lecture on calculating the intersection points, one session finding the limits between two parts, you could refer back to the lecture and at the end you have got your list and you can track through it. So, you feel equipped.

Discussion

Since for most of the students, checking whether the integrand was continuous or not, and monitoring problem-solving were not part of students’ plans for solving integral problems, those should be addressed with students in both Cases. The first check can be done in classes by asking students to find the area under curves that has discontinuity points such as $\int_{-3}^{3} \frac{1}{x} dx$ or $\int_{-\pi}^{\pi} \tan x \, dx$ .The second check can be addressed by using monitoring strategies (e.g., approximating the area using geometric shapes) while solving questions on the board for students in classes.

The fact that more than half the students (13 out of 17, including five university students) had no plan for solving the FTC problems is another piece of evidence that students have difficulty with this topic. In addition, using problem-posing tasks in teaching is useful for student learning (Lavy & Shriki, 2007) as it enhances students’ problem-solving skills; reduces their dependency on their lecturers/teachers and textbook, and "fosters more diverse and flexible thinking" (p. 130). Therefore, it should be used more frequently in teaching and assessment.

8.4 Knowledge about different cognitive tasks

In response to M10 (Table 5.1), five students (U129; Y13) said they solved all questions relating to the enclosed area in the same way. Twelve students (U345678; Y245678) said they used different strategies for solving this type of question based on:

- the number of curves (Y7);
- the shape of enclosed area (U4; Y26);
- representation of functions (Y5);
- the given functions (U3; Y8); and
- the difficulty of questions (U3567; Y4).

Regarding the FTC questions, eight students (U2589; Y1357) said they used the same strategy for solving these questions. Four students (U16; Y26) said they had no strategy for solving the FTC questions, indicating a lack of knowledge about this type of question. Five students (U347; Y48) said they used different strategies for solving this type of question. For instance, Y4 changed the order of steps of his strategy when he solved different types of problems. U7 said he rearranged the formula in different ways according to the given questions. The other three students did not mention what their different strategies were.

Discussion

Acknowledgment of having different strategies for solving different questions in a topic can be an indication of the presence of metacognitive knowledge (Anderson et al., 2001) (Section 2.2.1). This question was asked to explore whether students were aware of this aspect of metacognitive knowledge. For the integral-area relationship, most of the students highlighted that they had different strategies for the integral-area problems; however, for the FTC problems only five students said they had different strategies for this type of problem. This may be related to the fact that their knowledge about the FTC is limited, as indicated in Chapter Seven.

8.5 Self-knowledge

Students' self-knowledge was explored in terms of three items; that is, students' knowledge/perception about their difficulties in learning the integral-area relationships and the FTC, their orientation toward taking the calculus course, and their attitude toward calculus especially the integral calculus topic.

8.5.1 Students' knowledge about their difficulties in learning the integral-area relationships and the FTC

Students' responses to M11 (Table 5.1) were considered to explore their knowledge about their difficulties in learning the integral-area relationships and the FTC.

The integral-area relationships

In terms of the integral–area problems, most of the students believed they knew how to solve these problems. Therefore, their main difficulty was finding the antiderivative of complex functions (U3689; Y28, Table 8.16), rather than setting up integrals to find the enclosed area (U7).

Table 8.16

Students' opinions about their difficulties with integral-area problems.

Themes	Sub-themes	Case 1	Case 2	Total
Integral-area topic	Finding the antiderivative of some complex functions	4 (U3689)	2 (Y28)	6
	Not confident in using $\int_a^b [f(x) - g(x)]dx$	0	2 (Y68)	2
	Finding area using integration with respect to the y-axis	0	1 (Y4)	1
	Setting up the integral based on the given information	1 (U7)	0	1
	Several formulas for integral-area problems	0	1 (Y5)	1
Prior knowledge	Sketching the integrand	1 (U1)	1 (Y1)	2
	Finding the upper/lower limits	1 (U6)	1 (Y7)	2
	Manipulating the graphs in terms of y or x	1 (U7)	1 (Y7)	2
	Finding the intersection points	1 (U8)	0	1
	One of the curves is not a function	1 (U4)	0	1
	Non-real-world problem	0	1 (Y4)	1
Nothing	N/A	2 (U25)	1 (Y3)	3

U9, in response to question five said, "no, because it seems very intuitive, unless the integral itself is hard to integrate". Three students (U25; Y3) said they had no difficulty in solving integral-area problems. Two students (Y68) were not confident of using the $\int_a^b [f(x) - g(x)]dx$ to find the enclosed area between two curves, and said they found the enclosed area between each curve and the x-axis separately, and then subtract the areas: "I think I need more practice to be more confident in using $\int_a^b [f(x) - g(x)]dx$ instead of using $\int_a^b f(x)\,dx$ minus $\int_a^b g(x)\,dx$ when solving questions in relation to the area between curves" (Y6). Y4 believed he had difficulty with finding the area using integration with respect to the y-axis. U7 thought he had difficulty with setting up the integral for finding the area: "I find it hard to translate the question into a mathematical statement". Y5 had difficulty with several formulas that exist for finding the area, indicating he did not understand the intuition behind the integral-area problems: "we have many different kinds of areas, between [the] y-axis, x-axis, between two curves, and they all have a specific formula. There are so many formulas just for the area and it is a little bit confusing".

The rest of the difficulties mentioned were not related to the integral calculus topic specifically, rather they related to students' prior knowledge. U1 and Y1 said they had difficulty with sketching the curves of some functions: "I think my graphing needs to be stronger" (U1). Two students (U6; Y7) had difficulty with finding the bounds of integral, and two students (U7; Y7) had difficulty manipulating the graphs in terms of x or y. U8 believed he had difficulty with finding the intersection points, and U4 thought he had difficulty with finding the enclosed area when one of the curves is not a function such as a circle. Finally, Y4 made a general comment, saying he had difficulty with non-real-world problems: "with the applied questions, I find it easier to visualise which way to do [the integration]".

The FTC problems

Five students (U58; Y347) believed they had no difficulty in relation to solving the FTC problems. Three students (U1; Y16) believed they needed to learn more about the FTC (Table 8.17). Similarly, three students (U49; Y5) believed they had not understood the FTC. Two students (Y28) said it took more time to solve the FTC problems: "it takes me a while to do each question. I do not have any workout method. It takes me a lot longer I guess" (Y2). U6 made her comments based on the first part of the FTC, and believed finding the antiderivative was her difficulty in solving the FTC problems. U2 believed he had difficulty with understanding t and x in $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$ and said he was not confident with the second part of the FTC because of "the t and x thing [sic]". U3 said he did not know the intuition behind the second part of the FTC. However, he had seen a proof about the FTC, but highlighted that sometimes the intuition is different from the formal proof:

> Formal proofs are not intuitive all the time. I can have a formal proof, and knowing that there is a formal proof and even being able to reproduce a formal proof. The formal proof isn't always the logic behind what is happening. It is just a demonstration that is there is an entailment. The proof is not always the intuitive aspect of why it works...there is no guarantee the proof that I am looking at is the one that most help me, like it is just a proof, it is not the proof and it is not the intuitive proof always.

Table 8.17

Students' opinions about their difficulties with the FTC problems.

Themes	Sub-themes	Case 1	Case 2	Total
Difficulty with the conceptual knowledge	Not know enough about the FTC	1 (U1)	2 (Y16)	3
	Vague/not understanding the FTC	2 (U49)	1 (Y5)	3
	Understanding t and x in $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$	1 (U2)	0	1
	The intuition behind the second part of the FTC	1 (U3)	0	1
Difficulty with the procedural knowledge/problem solving	It takes a long time to solve them	0	2 (Y28)	2
	Finding the antiderivative	1 (U6)	0	1
Nothing	N/A	2 (U58)	3 (Y347)	5

Discussion

In terms of the integral-area relationship, seven students said they had difficulties with prior knowledge related to integral calculus, and those difficulties were also found in their performance in integral questions (Chapter Seven). As stated in Section 8.1.10, helping students to master prior knowledge seems to be necessary to make them successful problem-solvers in integral calculus. For the FTC, most students believed their difficulties were related to the theorem rather than the prior knowledge related to it. For the three students from Case 2 who said they had no difficulty with the FTC, their responses seemed to be related to the first part of the FTC, as they could not respond to questions related to the second part of the FTC correctly (Chapter Seven). The two University students who had no problem with the FTC answered most of the FTC questions correctly; however, they did not have geometric understanding of the second part of the FTC (Chapter Seven).

8.5.2 Students' orientation toward taking the calculus course

Students' responses to M12 (Table 5.1) were considered for exploration of students' orientation toward taking the calculus course. Three themes were found, including requirement for further study, liking for calculus, and calculus is a useful subject (Table 8.18).

Table 8.18

Students' orientation toward taking the calculus course

Themes	Sub-themes	Case 1	Case 2	Total
Requirement	Prerequisite for taking further courses in a major	7 (U1236789)	N/A	7
	Requirement for doing a major in a University	N/A	6 (Y123456)	6
	Get one of the University prizes	1 (U5)	N/A	1
	Needed for knowing general mathematics knowledge	1 (U3)	0	1
Like	Like/enjoy calculus	1 (U4)	2 (Y78)	3
Useful	Through calculus learn logical thinking	0	1 (Y4)	1

The main reason for taking the calculus course in Year 13 was the fact that calculus is one of the requirements for doing some majors in Universities that students want to do. For University students, the main reason was that the calculus course was one of the pre-requisites for taking other courses (Table 8.18).

For three students (U4; Y78) the main reason was not the fact that calculus is one of the requirements, rather they liked calculus:

> I enjoy it. It is one of my favourite subjects because I really like mathematics and I like the problems and working them out… I need to have calculus for that [doing the major in a University] but I was doing Year 13 calculus before I made that decision (Y8).

Y4 highlighted the fact that through learning calculus you would learn logical thinking: “they teach you logical processes and logical thinking”.

Discussion

Only four students in the sample said they took the calculus course because they found it enjoyable or useful. The remainder took the course because it was a requirement. This might lead to their having a performance approach toward the course, and could prevent their developing conceptual knowledge if this type of knowledge is not being assessed or taught. If the students only want to pass the course with even a good grade, and the assessment does not focus on conceptual knowledge, students might only focus on learning the procedures for solving questions and will not put enough effort into learning the conceptual knowledge underpinning the topic. Such orientation was also found in Section 8.1.9, where students mentioned to what extent they thought about the justification behind the formulas.

8.5.3 Students' attitudes toward calculus, especially integral calculus

Students' responses to M13 (Table 5.1) were considered in order to explore students' attitudes toward calculus, especially integral calculus. Students had differing attitudes towards calculus and integral calculus (Table 8.19).

Table 8.19

Students' attitudes toward calculus, especially integration

Row	Students' attitude toward calculus/ integral calculus	Case1	Case 2	Total
1	Like calculus when I understand/can do it	5 (U12469)	2 (Y13)	7
2	Like integration because of its application	2 (U58)	1 (Y5)	3
3	Like calculus because it is fun/enjoyable	1 (U4)	2 (Y78)	3
4	Like calculus and integration topic	2 (U47)	1 (Y4)	3
5	Like calculus, but not integration topic	0	2 (Y23)	2
6	Neutral feelings about calculus/integration	1 (U3)	1 (Y6)	2

Seven students (U12469; Y13) liked calculus and integration if they understood it: "I like it when I can do it. I feel satisfied if I get it right. I just get frustrating, if I constantly get it wrong" (Y1). Three students (U58; Y5) liked integration because of its application in different disciplines, sciences, and daily life: "[For] a lot of things in science, physic, daily life you need to involve this" (U5). Three students (U4; Y78) liked calculus because they believed it is fun: "I find it fun" (Y7). Two students (U3; Y6) had neutral feelings about calculus/integration: "It is not pretty the most fun subject, but it is not terrible" (Y6). U3 said, "calculus is a more of a tool kit for physicists and engineers than pure mathematics disciplines. A lot of it has a very applied focus where I do not have applied focus at all". U9 did not like integral calculus because some questions in this topic do not have clear procedures for solving them:

> I do not like integration because it is hard and tricky and requires you to have quite an intuitive sense of how you can solve it. Should I substitute, or I should do integration by parts? It is not very fun.

Discussion

In terms of comparing the attitudes of students in Case 1 and 2, approximately the same number of students liked calculus or integration (i.e., row 2, 3, and 4 in Table 8.19) (5 in Case 1 and 4 in Case 2) in these Cases. However, more students in Case 2 (1 in Case 1 and 3 in Case 3) had a neutral feeling about integration (i.e., row 5 and 6 in Table 8.19), and more students in Case 1 liked calculus when they understood it (five in Case 1 and two in Case 2).

There was no significant difference between students' mean scores in integral questions of those who liked calculus or integration (22.1), compared to students who felt neutral about integration (13.5) according to the Mann-Whitney test (P-value=0.1) because of the small sample size. The remaining students who were not classified in these two groups obtained a mean score (16.6) between them. The findings are in line with previous studies (e.g., Samuelsson & Granstrom, 2007) that attitudes toward mathematics and mathematical performance are related.

8.6 Chapter summary

In this chapter, students' metacognitive knowledge was explored in relation to questions that were designed based on the structure of the metacognitive knowledge dimension of RBT. The results described in Chapter Eight provide some answers to the third research question of the study (i.e., what metacognitive *knowledge*, *experiences*, and *skills* do students hold about integral calculus in Year 13 and first year university?). The results indicate the questions that were designed based on the structure of RBT's metacognitive knowledge are useful for exploring student learning and show some of the aspects of students' metacognitive knowledge that need further development (e.g., the importance of the rationale behind the formulas, and the usefulness of summarising strategies). The next chapter describes students' metacognitive experiences and skills in solving integral questions.

Chapter Nine: Metacognitive Experiences and Skills Relating to Solving the Integral-area and the FTC Problems

In this chapter, students' metacognitive experiences and skills are explored in relation to solving integral-area and FTC problems. Metacognitive experience, what a person is aware of and feels when coming across a task (Efklides, 2008), has several aspects (Efklides, 2001, 2006, 2008; Schneider, & Lockl, 2002) (Section 3.1.4). The aspects explored in this chapter relate to feelings of knowing, familiarity, difficulty, confidence, judgment of learning, and estimating the correctness of solution (Efklides, 2006, 2008). Metacognitive skills, activities that help individuals to control and regulate their cognitive activities (Schraw, 1998), also has several aspects such as task planning, and monitoring (Section 3.1.4). Metacognitive experiences and skills are related to the metacognitive knowledge row of RBT, such as the applying metacognitive knowledge cell (Section 3.1.4). It is important to know how students feel and use their judgment when facing different types of mathematical problems, to help them not make wrong decisions in problem-solving, nor to oversimplify or over-complicate the question.

The results discussed in this chapter indicate that the students in this study used differing metacognitive experiences and skills when solving different types of integral-area and FTC problems. Students' metacognitive experiences and skills are explored regarding integral Questions that are problem-solving in nature, i.e., all the integral questions except Q2 and Q7 (Section 5.2.1). Students' metacognitive experiences and skills are explored for all of these seven Questions because the types of questions are different (e.g., typical, atypical, and problem posing) (Section 5.2.1). For measuring students' metacognitive experiences and skills a think aloud protocol was used (Section 5.2.3). For each question, firstly, students' pre-judgment of their ability to solve the question is explored (Sections 9.1.1 to 9.7.1); then, the three items related to metacognitive skills (i.e., making a drawing related to the problem; making a calculation plan and systematically doing it; and checking calculations and answer) are described (Section 9.1.2 to 9.7.4). The fifth section relating to each question is students' post-judgment of how accurately they had solved the question (Section 9.1.5 to 9.7.5). The last section relating to each question is a discussion of the results obtained for the question (Section 9.1.6 to 9.7.6). After presenting the results for the seven questions, a general discussion is provided (Section 9.8).

9.1 Students' metacognitive experiences and skills concerning a typical question of integral-area relationship

Students' metacognitive experiences (Section 9.1.1 & 9.1.5) and skills (Section 9.1.2 to 9.1.4) regarding Q1 are explored in this section.

9.1.1 Having an accurate pre-judgment of whether they can solve the problem

Students had different levels of metacognitive experiences in dealing with Q1. Eight students (U127; Y12567) made their judgment based on their familiarity with how to find an area using integral calculus. An example of these responses is "we have recently learnt this [topic] in class and I am practising these questions at the moment" (Y1). Four students (U568; Y8) based their judgment on their ability to integrate the form of the integrand. They provided reasons such as "equations [are] not particularly difficult to integrate" (U6). Three students (U3; Y34) based their decision on their familiarity with the shape of the graph, providing such reasons as "I can imagine it graphically" (U3). U9 highlighted the importance of the shape of the enclosed area for making his judgment about his ability to solve the problem:

> What I will do before I would have known if I am sure or not I will draw the graph. Then I decide whether I am sure I can solve it or not. That is hard for me to look at those two functions and say, oh yes, it is easy I can find the area between functions.

Drawing the enclosed area is an important part of solving integral-area problems, as it helps students decide whether integration should be done with respect to which axes, and shows whether the curves have any discontinuity point. In addition, drawing a graph is an element of metacognitive skills (Jacobse & Harskamp, 2012), and U9 said he does this before solving the problem.

The judgement of U4 was affected by the fact that $x = y^2$ is not a function, and therefore he was unsure whether or not he could solve this problem. A comparison between the results of students in Cases 1 and 2 (Table 9.1) shows that students in Case 1 had higher metacognitive experience compared to students in Case 2 in relation to predicting their ability to solve Q1.

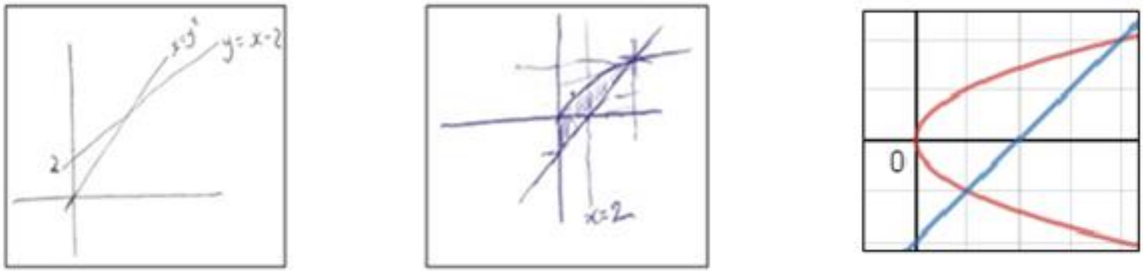
Table 9.1

Students' prediction of ability to solve Q1

		Find area with respect to x-axis			Find area with respect to y-axis		
		Correct	Incorrect	Didn't use the method	Correct	Incorrect	Didn't use the method
I am sure I will solve this question (N=12)	Case 1 (N=6)	3	3	0	4	2	0
	Case 2 (N=6)	1	5	0	1	1	4
I am not sure whether I will solve this question correctly or incorrectly. (N=5)	Case 1 (N=3)	0	3	0	1	2	0
	Case 2 (N=2)	0	2	0	1	0	1
I am sure I cannot solve this question. (N=0)	Case 1 (N=0)	0	0	0	0	0	0
	Case 2 (N=0)	0	0	0	0	0	0

9.1.2 Making a drawing related to the problem

For Q1, all students drew the curves while solving this question. Eleven (U2456789; Y2378) made a correct drawing of the curves. For the six remaining students there were two major reasons for their failure to draw the curves correctly. Firstly, neglecting the part of x=y^2 which is under the x-axis (U13; Y456) (Figure 9.1). Secondly, it seems students had tried to remember the graph, and have not checked their drawing by considering each curve as a function/relation and substituting some values in the domain of the function/relation for finding the relationship between x and y. The reason for making such a claim is that during the think alouds none mentioned they would substitute some values in the function/relation to sketch the graph. If students checked their drawing by substituting some values in the function/relation, they were a higher chance that they would identified their errors.



An example of trying to remember the curves (Y1) An example of neglecting the negative part of x=y^2(U3) Correct drawing of the curves Sketched using desmos.com

Figure 9.1 Examples of students' mistakes in drawing the curves and a correct drawing of it

The students' ability to find the area enclosed between the curves correctly were closely related to their drawing of the curves, especially when the upper and lower functions do not change in the enclosed area. Seven students (U45689; Y78) out of those who sketched the curves correctly and tried to find the area with respect to the y-axis were successful. Of those integrated with respect to the x-axis, four of the students (U589; Y8) who correctly drew the graph, were successful (Table 9.2). The lower success level was due to changes in the lower function at $x = 1$ from $y = -\sqrt{x}$ to $y = x - 2$. The final piece of evidence that supports the importance of curve sketching when using integral to find area, is that all the students who did not draw the curves correctly were unsuccessful with the item (Table 9.2).

Table 9.2

Relationship between correct drawing and finding area respect to the axes

		Find area respect to x-axis			Find area respect to y-axis		
		Correct	Incorrect	Didn't use the method	Correct	Incorrect	Didn't use the method
Correct sketch (N=11)	Case 1 (N=7)	3	4	0	5	2	0
	Case 2 (N=4)	1	2	1	2	1	1
Incorrect sketch (N=6)	Case 1 (N=2)	0	2	0	0	1	1
	Case 2 (N=4)	0	4	0	0	0	4

9.1.3 Making a calculation plan and systematically performing it

According to students' think alouds, all students in the sample had a plan for solving Q1, and started by either drawing the curves or finding the intersection points. Another evidence supporting this claim is all students in their responses to M9 (Section 8.3), described a plan for how they would solve enclosed area between curves problems.

9.1.4 Checking calculations and answers

For Q1, more than half of the students in the sample did not check their solutions. Four students in Case 1 (U1589) and three in Case 2 (Y246) did some sort of checking. They had only checked their calculations and not their drawing of the curves. Of these, four (U1; Y246) made errors in their working; and three (U1; Y46) could not find their errors. Y2 was able to amend his drawing for x=y^2 (Figure 9.2) on his fourth try. The three remaining University students (U589) who solved the question correctly, had checked their answer to find out whether they found the same answer using both axes (two out of three), and one checked his calculation to ensure he not made any mistake in finding the intersection points.

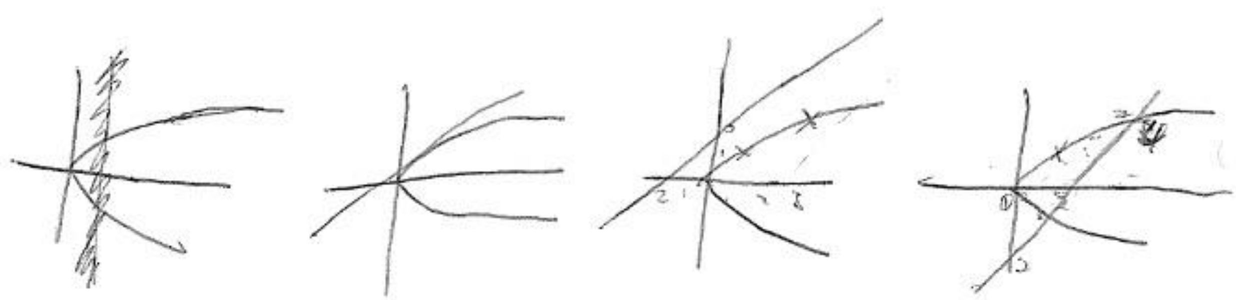


Figure 9.2 Y2's attempts in drawing $x = y^2$

9.1.5 Having an accurate post-judgment of how effectively the problem was solved

Students' judgments of how well they answered the problem were varied for Q1 (Table 9.3). Four students (U589; Y8) were sure they had solved the question correctly because they could find the same answer using both methods (and their answers were correct). Similarly, two university students (U27) were sure they had solved the question incorrectly as they found different answers using the two methods (and both their answers were incorrect). Two students in Case 2 (Y15) were sure they had solved it incorrectly because their answers were negative.

Table 9.3

Students' post-judgment of their ability for solving Q1

		Find area with respect to x-axis			Find area with respect to y-axis		
		Correct	Incorrect	Didn't use the method	Correct	Incorrect	Didn't use the method
I am sure I solved this question correctly (N=7)	Case 1 (N=4)	3	1	0	3	1	0
	Case 2 (N=3)	1	1	1	1	1	1
I am not sure whether I solved this question correctly or incorrectly (N=5)	Case 1 (N=3)	0	3	0	2	1	0
	Case 2 (N=2)	0	2	0	1	0	1
I am sure I solved this question incorrectly (N=5)	Case 1 (N=2)	0	2	0	0	2	0
	Case 2 (N=3)	0	3	0	0	0	3

Two students (U3; Y6) made an inaccurate post-judgment, saying they were sure they had found the correct answer because "it makes sense graphically" (U3), and "looking at graph it seems right visually" (Y6). However, their drawings were incorrect. Use of drawing the curves to check their answers and intersection points was used by other students. Y4 was unsure whether or not he had solved the question correctly because the intersection points he found did not match the curves.

Three students (U46; Y7) were unsure if they had solved the question correctly for the following different reasons; U6 could not distinguish which function is the top function: "not sure which one [in] $\int f(x) - g(x)\, dx$ is $f(x)$" (U6). U4 had difficulty with $x = y^2$ (U4) and said: "$x = y^2$ is not a function so that confused me"; Y7 was unsure because "[I] forget to account for the other part [the part which is under x-axis] of $x = y^2$" (Y7).

U1 was not confident with his problem solving, saying, "usually with math question you are pretty sure when you have got it right. I was pretty hazy when I go through. I was over confident when I started". Finally, Y2 was sure he had solved the question correctly but could not explain why. However, he had not found the correct answer.

9.1.6 Discussion

The fact that several students made their pre-judgment of being able to solve this question – based on knowing the techniques for finding the antiderivative of integrands, or knowing how to find the enclosed area in general – suggests it is important to highlight the shape of the enclosed area to students to enable their pre-judgment of whether they are able to solve integral-area problems. The reason is, the shape of the enclosed area affects the method that can be used to find the enclosed area, and a number of students had made errors in setting up the integral in terms of which curve is the top function. However, all students, during their problem-solving, had made a drawing to help solve the problem, which could be an indication of the presence of that metacognitive skill. Checking the calculations and answers were not done by more than half of the students; therefore this aspect of metacognitive skills could be regularly modelled for students. In relation to their post-judgment of having found the correct answer, the ten students who were unsure they had found the correct answer, or were sure they had solved the problem incorrectly, were not found to be revisiting and checking their solution, showing a lack of persistence in mathematical problem solving.

9.2 Students' metacognitive experiences and skills about an atypical question of integral-area relationship

Students' metacognitive experiences (Section 9.2.1 & 9.2.5) and skills (Section 9.2.2 to 9.2.4) about Q3 are explored in this section. Students' experiences and skills in this question might be different from that for previous questions, as this question might be unfamiliar to the student since the function is not given explicitly in the question.

9.2.1 Having an accurate pre-judgment of whether they can solve the problem

Four students (U348; Y4) were sure they could solve question three. However, only two of them were correct (U34). Three (U348) of them had that feeling because they thought they understood why the information is given. For instance, U8 said, "we know the area of important parts of the $f'(x)$ and area is related to the anti-derivative". Y4 had this feeling because he thought he had seen similar questions; however when he started solving the problem he said he had not understood correctly what the question asked. He said, "I have seen question like this before… I think I misinterpret what is asking [sic]".

Eleven students (U12567; Y123568) were not sure whether they could solve this question correctly. Of those, only U5 solved the question correctly. U5 was unsure because she had not "encountered any question like this". Other students had different reasons for that feeling. Two students (U7; Y3) were unsure because they had not seen a question like this before. Two students (Y68) were unsure because they needed to think more about it to know whether they were able to solve it. For instance, Y6 said,

> I am not sure because I need time to think about it mentally in my head to understand the question better... I think I might be able to do it in time...I know it seems there are some familiar parts in it but all together in one question [I am not sure whether I am able to solve it].

Three students (U1; Y12) recognised the question, but were unsure how to solve it. For instance, U1 said, "I have seen questions like this before, but cannot remember how to do it". U6 was unsure because she thought maybe guessing is involved for solving the problem: "not sure, because some guessing may be required". U5 had some misunderstanding about the given information, and thought the given graph is the graph of $f(x)$. He said, "$f(0) = -5$ is confusing me". Y1 was unsure because he did not know "where to start". Y5's feeling was specifically related to how the problem needed to be solved. He said, "I am not sure because I have to change the area to $f'(x)$. but I am not sure I can calculate it correctly [sic]".

Two students (U9; Y7) were sure they were not able to solve this question; however, U9 did solve the question correctly. U9 had that feeling because he had not seen a similar question before. He said, "I never seen this before". Y7's feeling was related to how the problem should be solved. He said "I do not know how to find out the $f(x)$ because I do not recognise the graph type of $f'(x)$" .

9.2.2 Making a drawing related to the problem

For this question, students did not make any drawing related to this problem.

9.2.3 Making a calculation plan and systematically performing it

Students had three different plans for solving Q3 (Table 9.4). Eight students had planned to use the integral-area relationships for the graph of derivative function (Section 7.7) and three students planned to use the given information to find the equation of $f(x)$. Three students, when they failed to find the equation, used the integral-area relationship to solve the question. The remaining three had no plan for solving Q3.

Table 9.4

Students' plans for solving Q3

Plans for solving Q3	University students	Year 13 Students	Total
Plan A: Use the integral-area relationships for the graph of derivative function	6 (U345689)	2 (Y58)	8
Plan B: Use the given information for finding the equation of $f(x)$	1 (U2)	2 (Y23)	3
First tried to use plan A, then, plan B	1 (U7)	2 (Y67)	3
No plan	1 (U1)	2 (Y14)	3

9.2.4 Checking calculations and answers

During their solving of Q3, only U5 did some sort of checking. After she found the correct answer, she revisited her working to make sure did not make any mistake.

9.2.5 Having an accurate post-judgment of how effectively the problem was solved

Five students (U58; Y568) thought they had solved this question correctly. However, from those, only U5 made an accurate post-judgment about solving this question correctly. She was sure because she used all of the given information in the question. U5 said, "I have utilised all the information I have in the question". Y5 could not provide a reason why he thought he had solved the question correctly. The remaining three students did not provide any reliable justification for their judgment. For instance, Y6 said, "It works logically and reasonably I am confident with the working", and Y8 believed he had solved the question correctly because he found the answer more easily than he expected. He said, "I think I solved it correctly… because it was a lot simple one to do once I was start looking at it in more depth [sic]".

Four students (U349; Y7) were unsure whether they had solved the question correctly. From these students, three of them (U349) solved the question correctly. The reason for not being sure was different for each of them. U4 was unsure because he did not use a part of the information given in the problem, the area of region C. U9 was unsure because he could not justify the method he had used. He said, “I am not sure because I do not feel I can justify the method I have used”. U3 was not confident because he had not solved a non-calculation question in this topic for a long time, indicating that most of the questions he dealt with were procedural or calculative questions:

> It is a question that I haven't come across for a long time…It is just being so long since I have to do a non-calculation based integration and or differentiation [question] that it's pretty much taken me by surprise. Always expect calculation heavy question. Not prepared.

Y7 was unsure because he was not confident whether he could integrate both sides of $f'(6) + 12 = f'(0)$. In his solution, he integrated both sides and wrote $f(6) + 12x + c = f(0)$, which is not correct. The remaining eight students (U1267; Y1234) could not come up with an answer to this question, therefore, they have been considered automatically as if they were sure they had solved the question incorrectly.

To sum up, five factors were found in relation to students’ feelings about whether they had solved Q3 correctly, including

- how much of the given information was being used;
- how well the method could be justified;
- how confident the student was with his/her working;
- how familiar the student was with the question; and
- how easily the answer was found.

9.2.6 Discussion

For this question the number of students who had checked their answers was fewer than for Q1, only one having checked, again suggesting it could be useful for lecturers/teachers to more strongly encourage students to use this metacognitive skill. The fact that six students did not use the integral-area relationship to solve this question suggests lecturers and teachers should also use the graph of derivative for integral-area questions they solve in their classes to help students to understand that $\int_a^b f'(x)dx = f(b) - f(a)$ is another representation of the first part of the FTC and equivalent to $\int_a^b f(x)dx = F(b) - F(a)$ where $F' = f$. This practice might provide a better chance for students to realise that the relationship between the graph of the function and the area under the curve also exists for the derivative function. The number of students who were sure they were able to solve this question was less than for Q1 (12 for Q1 and 4 for Q3), indicating that students are not confident about answering questions that focus on conceptual knowledge (as stated in Section 5.2.1, this question evaluates students' conceptual knowledge). In addition, students' reasons for their post-judgment about solving this conceptual question correctly show that they are not confident in their problem solving concerning this question. This suggests that more conceptual questions should be included in class practices.

9.3 Students' metacognitive experiences and skills about an evaluation task of integral-area relationship

Students' metacognitive experiences (Section 9.3.1 & 9.3.5) and skills (Section 9.3.2 to 9.3.4) regarding Q4 are explored in this section. Students' experiences and skills for this question might be different from those for previous questions because the question is an evaluation task.

9.3.1 Having an accurate pre-judgment of whether they can solve the problem

All the students thought they were able to do this task. However, only U5 identified that the first example in Q4 was solved incorrectly. In relation to the second example in Q4, only two students (U59) identified it as solved incorrectly. Students made that judgment because they thought the examples were familiar to them (U124; Y356), saying the integrals are "simple" and "straightforward to integrate" (U28; Y7), and they knew how to solve these questions (U9; Y248). U5 made that judgment because she thought only one curve is involved in each example. She said, "sure, in both questions, they are only one equation involved so it is not as tricky as the ones with more equations". Y1 made that judgment because he thought he did not need to graph the curves. He said, "Also, I do not need to work out the formula from the graph".

Some of the students' responses related to the fact that the task was an evaluating task. For instance, U7 said, "…easier to find someone wrong than to prove me right". U6 said, "I think I can find any wrong steps".

9.3.2 Making a drawing related to the problem

For this question, two students (U57) made a drawing for this question. U5 made a drawing for both parts of the question and solved the question correctly. U7 made a drawing only for the second part, but could not identify that the function is not defined at zero (Section 7.4).

9.3.3 Making a calculation plan and systematically performing it

10 students (U12345; Y13458) had chosen to solve the examples and compare their final answer in order to check the examples. The remaining seven students (U6789; Y267) went through the steps and checked the steps rather than solving the examples.

9.3.4 Checking calculations and answers

Five students (U3; Y1456) had done some sort of checking for this question. Year 13 students used their calculators for checking the calculation involved in the problem (Section 7.8). U3 checked the calculation after he had solved the question, without using a calculator, to explore whether he had substituted the bounds in the anti-derivative correctly (because the answers were negative).

9.3.5 Having an accurate post-judgment of how effectively the problem was solved

15 students were sure they had solved the task correctly. However, only U5 made the correct judgment for the first example, and only two students (U59) made the correct judgment for the second example. The remaining two students (U12) were unsure whether they had solved the task correctly. They were unsure because they had got a negative area. In addition, U1 was unsure whether the antiderivative of $\frac{1}{x^2}$ is $\frac{-1}{x}$ or not, indicating a lack of procedural knowledge for finding antiderivatives.

The 15 students were sure for several reasons. 13 students (U34679; Y12345678) had that judgment because they had got the same answer (for one or both examples). U8 had that feeling because was “confident” with his workings. Three students (U5; Y15) were sure because they had found a different answer from what is written in the task. The two Year 13 students had found a wrong answer due to wrong calculation (Y1) and a misconception about the integral-area relationship (Y5).

9.3.6 Discussion

The fact that this task is an evaluation task does not negatively affect the confidence of students as all students were confident they could solve this question. Students’ pre-judgments, similarly to Q1, show that most of them made their judgment based on familiarity with the integrand and knowing how to find the antiderivatives, and not by focusing on the shape of the enclosed area. In addition, most of the students did not make any drawing of the given curves in the question, with the result that they were unable to find the mistake in the question. This suggests that is may be useful for lecturers and teachers to more strongly highlight the importance of drawing curves in the integral-area problems and encourage them to focus on the enclosed area's shape to solve these types of questions. In relation to checking; since most of students did not check the fact that integrand should be continuous, this checking strategy also needs to be suggested to students. Most of the students' post-judgments were incorrect, related to the fact that they had not used the necessary checking strategies (e.g., that the integrand be continuous, the area be positive, and approximating the area using geometric shape) to understand their mistakes in finding the solution.

9.4 Students' metacognitive experiences and skills regarding a problem posing task about integral-area relationship

Students' metacognitive experiences (Section 9.4.1 & 9.4.5) and skills (Section 9.4.2 to 9.4.4) regarding Q5 are explored in this section. This question is a problem posing task, therefore, students' experiences and skills in answering this question might be different from those used for previous ones.

9.4.1 Having an accurate pre-judgment of whether they can solve the problem

Nine students (U3456789; Y57) were sure they were able to pose a problem based on the given information. From these, all but Y5 posed a correct problem. These students had these feelings because they thought they could find an example (U689; Y57), or thought they (U357) could use simple functions for posing a problem. For example, U7 said, "sure, I am going to use simple stuff", and Y7 said, "I can think of an example". Apart from those reasons, U4 was sure he could pose a problem because he "understand the theory behind" the task.

Seven students (U12; Y24568) were unsure whether they were able to pose a problem based on the given information. All of them could not pose a problem or posed an incorrect problem based on the given information (Section 7.18). Students were unsure for several reasons including:

- unsure they could find an example that fitted the given information (U1; Y26);
- unsure they could "do it backward" (U2; Y8);
- unsure because they had not posed a problem before (Y3); and

- unsure because they “may make a mistake” (Y4).

It was only one student (Y1) who was sure he could not pose a problem based on the given information because he had not posed a problem before.

9.4.2 Making a drawing related to the problem

11 students (U34579; Y235678) made a drawing for Q5. In their drawing they had tried to sketch a curve and a line to have a better understanding of what they should consider a curve and a line. Of those, six were successful in posing a correct problem (Table 9.5). However, making a drawing was not as closely related to being successful in this task in comparison to Q1.

Table 9.5

Relationship between making a drawing and being successful in Q5

	Posed correct problem	Posed incorrect problem	Did not pose a problem
Made a drawing	6 (U34579; Y7)	1 (Y6)	4 (Y2358)
Did not make a drawing	2 (U68)	1 (Y1)	3 (U12; Y4)

9.4.3 Making a calculation plan and systematically performing it

14 students (U3456789; Y1345678) for posing a problem had tried to choose simple curves (e.g., $y = x^2$) and lines (e.g., $y = 0$) (Section 7.18). Four students (U12; Y24) had no specific plan for solving Q5, i.e., they did not fix the bounds or functions and just tried some random functions. For instance, Y4 drew $y = x$ and $y = x^2$ on his calculator to see whether he could find an enclosed area of one, and after he could not, gave up. Two students (Y1; U7) tried to fix the bounds and then choose a curve and a line in a way that the enclosed area be equal to one. For instance, Y1 fixed the bounds, considering the lower bound equal to zero and the upper bound equal to one. Five students (Y23568) tried to fix a curve and a line, then chose both bounds in a way that fitted the task. Five students (U34569; Y7) tried to fix a curve, a line, and the lower bound, and chose the upper bound so that the enclosed area would be equal to one. For instance, Y7 considered the enclosed area between $y = x^2$, the x-axis, $x = 0$ and $x = b$, and solved $\int_0^b x^2 dx = 1$ to find the b. U2 had a misconception about which formula should be used for the area, considered $f^2(x) - g^2(x) = 1$, and then did not continue working on this question.

9.4.4 Checking calculations and answers

Six students (U45689; Y6) had done some sort of checking for Q5. These students, after posing their problems, solved it to see whether they had got 1 as the answer. However, Y6 who made a mistake in his problem posing, could not identify where he had made the mistake.

9.4.5 Having an accurate post-judgment of how effectively the problem was solved

Eight students (U345689; Y67) were sure they had posed the problem correctly. Of those, all were correct except Y6. Five of them (U45689) had that feeling because they had solved their posed problems and had got one as the answer. Y6 had that feeling because he believed if someone solves that problem, the answer is one. Y7 was sure because he thought he had set up the integral correctly. U3 thought his solving "was thorough and multiple wrong solutions excluded".

One student who was unsure posed a correct problem (U7), but his reason was not related to the content and indicated that he was not confident with his working. He said, "still I can see a smile on the researcher's face". Eight students (U12; Y123458) were sure they had solved this question incorrectly because they could not pose a problem.

9.4.6 Discussion

The problem-posing task is not included in the assessment and the examples solved regarding teaching of integral calculus in the Cases (Section 6.1). However, students in Case 1 were more confident they were able to pose a problem in comparison to students in Case 2. This may be due to the fact that they had more experience of working with integral calculus questions, as they had passed Year 12 and Year 13 calculus courses. Another possible reason might be related to the more conceptual teaching in Case 1 in comparison to Case 2.

In relation to their metacognitive skills, more than half of the students did not check their calculation for this question, similarly to previous ones. However, several students showed that they had metacognitive skills by trying to sketch graphs to be able to pose a problem that met the given conditions.

9.5 Students' metacognitive experiences and skills relating to a typical question of the second part of the FTC

Students' metacognitive experiences (Section 9.5.1 & 9.5.5) and skills (Section 9.5.2 to 9.5.4) concerning Q6 are explored in this section. This question is about the FTC; therefore, students' experiences and skills in this question might be different from those found in the previous five questions.

9.5.1 Having an accurate pre-judgment of whether they can solve the problem

Six students (U58; Y2358) were sure they were able to do this task. Of those, three of them (U58; Y8) were correct. Others misinterpreted what had been asked in the question, thinking they only needed to differentiate the expressions or do the definite integral. For example, Y3 said, "sure, because I did this in the class". However, these types of questions about the FTC were not taught in Year 13, consequently, they had not done it in their class.

Eight students (U1234679; Y6) were unsure whether they were able to do the task. Of those, three students (U149) solved the question correctly. The remaining four University students solved some parts correctly (Section 7.16). Y6 did not solve any part correctly. He was unsure because the task was unfamiliar to him. Three students (U239) were unsure because they had had difficulties in learning the FTC. For instance, U2 said, "I am struggling with the second part of the fundamental theorem of calculus", or U9 said, "not sure, these are [the] FTC questions". U7 was unsure because he could not remember the process for solving such questions. U1 was unsure because he had not done "much practice" of these types of questions. U6 was unsure, but could not provide a reason. She just said, "not sure, I think I can but not sure. We did this in one of the assignment". Only U4's reason was specifically related to the details in the question. He said, "not too sure, because [the] bounds that are functions are a bit confusing for me".

Three students (Y147) were sure they were not able to solve this task. The task was "very unfamiliar" for Y7. Y4 had that feeling because the task had not been taught in their classroom, and Y1 did not provide any reason for the feeling.

9.5.2 Making a drawing related to the problem

For Q6, two students (U37) made a drawing for the question. However, their drawings did not help them to understand how the FTC could be used to solve the problem [see Section 7.16 for

U3]. For U7, the drawing was not specifically related to the information given for the problem (Figure 9.3).

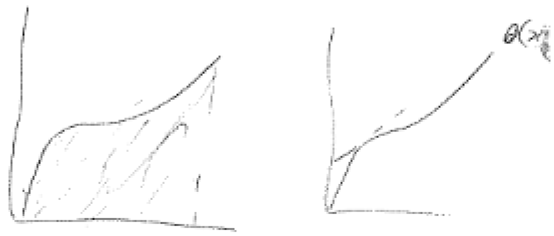


Figure 9.3 U7's drawing for Q6

9.5.3 Making a calculation plan and systematically performing it

Students' plans for Q6 are covered in section 7.16 that described how the second part of the FTC was used for finding the derivative of integrals.

9.5.4 Checking calculations and answers

In relation to Q6, only U5 had done some sort of checking. After she found the correct answer, she double-checked her workings to make sure she had not made a mistake.

9.5.5 Having an accurate post-judgment of how effectively the problem was solved

Six students (U1568; Y28) were sure they had solved the task correctly. Of these, four students (U158; Y7) solved the question correctly, U6 partially solved the question correctly, and Y2 had an incorrect judgment. Three students (U168) were sure because they had "recognised the question and the path needs to get through" (U1). U5 was sure because she had double- checked her working. Y2 was sure but did not provide any reason for that judgment. Y7 had that feeling because he thought "there was not much room for errors" and the answer "came out very nicely".

Six students (U2349; Y67) were unsure whether they had solved the task correctly. Of those, two students (U49) had solved the question correctly, three students (U23; Y7) solved some parts correctly, and Y6 had not solved any parts correctly (section 7.16). Students had different reasons for that feeling. U2 had the feeling because his answer was obtained "too easy". U4 was unsure because the bounds in the questions were functions rather than numbers. U9 was unsure because he was "confused about the FTC". U3 thought he "did not do what was asked" in the question and he "missed the rule" for solving the task. Y6 was unsure because the task was unfamiliar to him. Y7 was unsure because he thought he had not solved the first two parts correctly and for the third one, he only "figured out how to actually solve them".

Four students (U7; Y145) were sure they had solved the question incorrectly. All of them had that feeling because they did not finish solving that task or did not try to solve it (section 4.17). The remaining one student did not answer this part of Q6.

9.5.6 Discussion

Students' pre- and post-judgments for this question showed that students in both Cases were not confident with solving questions related to the second part of the FTC. Students in Case 2 had not seen the second part of the FTC, which could be why they were not confident at solving this type of question. However, students in Case 1 had seen this theorem completely in the teaching (Section 6.1), but only two students were confident of being able to solve this typical FTC question. Similar questions to this question were solved in the lectures and tutorials and also were in the assignment questions. This suggests that the teaching of the FTC needs to be reconsidered so that students can be more confident when dealing with this type of question. That might be done by emphasising the development of conceptual knowledge about the FTC.

Similarly to previous questions, the number of students who used checking strategies was limited, as only one student used this type of strategy. Most of the students did not make any drawing to help solve this question, as making drawings is not as essential as the integral-area question for being able to find the correct answer.

9.6 Students' metacognitive experiences and skills about a contextual problem of the FTC

Students' metacognitive experiences (Section 9.6.1 & 9.6.5) and skills (Sections 9.6.2 to 9.6.4) concerning Q8 are explored in this section. This question is a contextual question about the FTC, therefore, students' experiences and skills in this question might be different from those used for Q6.

9.6.1 Having an accurate pre-judgment of whether they can solve the problem

Five students (U1358; Y4) were sure they were able to solve the task. Of those, two of them (U35) solved the task correctly, and three students (U18; Y4) partially solved the task. Students had different reasons for that feeling. Y4's reason was not related to the information given in the task, and he was sure just because he thought he was "better at applied questions". U5 and U8 said, "the f' represents the velocity of the water at the tank" and they knew how to solve this type of question. U1 and U3 believed it is not a hard task to perform. U1 said, "sure, it doesn't look complex integration problem. It is just given a real world application", and U3 said, "sure, simple graph about a simple rate of change".

Ten students (U4679; Y123568) were unsure whether they were able to solve the task. Of those, four students (U469; Y8) solved the task correctly. The remaining six students partially solved the task correctly. These students had that feeling because of different reasons. The task was unfamiliar for two students (U7; Y3). U9 was unsure because the task "look like a physic question". U6 was unsure but did not provide any reason. She said, "not sure, if I know how, but I think I can". Five students (U4; Y1268) were unsure because they were not confident about being able to solve all parts of the task. For example, Y6 said he was unable to sketch $g(x)$, or Y1 said, "I probably need to find the equation of the graph which I am not good at doing that". Y5 was unsure because x in the $\int_0^x f(t)dt$ "confused" him.

Two students (U2; Y7) were sure they were unable to solve the task. Of those, Y7 partially solved the task. They had this feeling because the task was unfamiliar to them. U2 said, "I have not seen a question like this", and Y7 said, "…not familiar with $\int_0^x f(t)dt$", indicating a lack of knowledge about the FTC.

9.6.2 Making a drawing related to the problem

Apart from part six, which students were asked to sketch a graph for $g(x)$ (Section 7.17), they did not sketch any other drawing for this question.

9.6.3 Making a calculation plan and systematically performing it

Similar to Q6, students' plans for solving Q8 are covered in Chapter Seven (section 7.17) that described how the FTC was used in a contextual problem.

9.6.4 Checking calculations and answers

Three students (U35; Y8) had done some sort of checking for Q8. Two of them double-checked their working. U3, by doing that amended his answer to part 5 of this question. Y8 when he found a negative answer for the first part, checked his workings and amended his answer (Section 7.17).

9.6.5 Having an accurate post-judgment of how effectively the problem was solved

Seven students (U3568; Y467) were sure they had solved the task correctly. Of those, three students (U356) solved the task completely, and the remaining four students (U8; Y467) partially solved the task. These students made that judgment for several reasons. Two students (U3; Y7) thought it was not a hard task. For instance, Y7 said for the parts he answered, “it only involved geometry and simple understanding of the rate of change”. U3, apart the fact that he thought the task is not a hard one, had that feeling because all of his “answers agree with each other”. U8’s feeling was related to the answer to one part of the task. He said, “I could work out the area from looking at the graph”. U6’s reason was a general one. She said, “looks like the right answer. It makes sense”. Three students (Y64; U5) did not provide a reason for their feelings. For instance, Y6 said, “I do not know. These are the things that I remember”.

Seven students (U249; Y1235) were unsure whether they had solved the task correctly. Y3 could not provide a reason for his feeling. Two students (Y25) were “confused” by the questions that caused that feeling. For instance, Y5 said, “I am just so confused about the question that I am not sure I am right or wrong”. Y1 had that feeling because he was not confident with his answer and his drawing. U4 had that feeling because the task was a bit different from previous questions he had solved. He said, “done similar stuff before, but not quite like this”. Two students (U29) were unsure because they were not confident about their answers to all parts. For instance, U2 said, “I am not sure, some bits I think I did, some bits not”.

Two students (U17) who were sure they had solved the task incorrectly had partially solved the task. U7 had that feeling because he was not sure about the value of $g(9)$. U1 was sure he had solved the task incorrectly because he could not draw the function of $g(x)$.

9.6.6 Discussion

In relation to pre- and post-judgment, as for Q6, this analysis shows several students were not confident about the FTC problem, indicating changes to the teaching of the FTC might be beneficial for students, as also highlighted in previous chapters. As for previous questions, the number of students who used checking strategies was limited, i.e., 3 out of 17. For this question, students did not make any drawing apart from part six, probably because they had the graph of f' in the question and for part six they had to draw the graph of $\int_0^x f'(t)dt = f(x) - f(0) = f(x)$. These two graphs are enough for answering all parts of the question.

9.7 Students' metacognitive experiences and skills regarding a problem posing question of the FTC

Students' metacognitive experiences (Section 9.7.1 & 9.7.5) and skills (Section 9.7.2 to 9.7.4) in relation to Q9 are explored in this section. This question is the second problem-posing task, and as it is about the FTC, student's experiences and skills shown in response to this question might be different from Q5.

9.7.1 Having an accurate pre-judgment of whether they can solve the problem

Three students (U5; Y48) were sure they were able to pose a problem based on the FTC for the given graph. Of those, all posed a question based on the FTC. However Y4 could not solve his problem. These students had different reasons for having that feeling. U5 had that feeling because she was sure she could use the graph to pose a problem related to the velocity. Y7 had that feeling because it was an open question. He said, "It is quite open so I think I am able to pose a problem". Y4's feeling was not related to the given information in the task. He said, "I have done questions so perhaps I can write one".

Nine students (U23468; Y1356) were unsure whether they were able to pose a problem based on the FTC for the given graph. Of those three students (U2; Y56) posed a question that is not related to the FTC. The remaining six students posed a problem related to the FTC. Y1, however was not able to solve his problem and Y3 solved his problem incorrectly. Students had different reasons for having such a feeling. Four students (U234; Y6) were unsure because they had not understood the FTC thoroughly. For instance, U2 said, "not sure, because, I am not quite familiar with the FTC part 2" and U4 said, "not too sure because I am not too confident with the FTC". One student was unsure because he believed he was not good at posing questions: "I am not sure, because I am not good at posing questions, but might be able to use the previous question" (Y1). Y5 was unsure because the graph had three components. U6 made a general comment and said, "not sure, it looks complicated". U8 was unsure because the task was open: "not sure. It is a bit trickier because, like the last pose a problem question, infinite many possible answers [exist]".

Five students (U179; Y27) were initially sure they were unable to pose a problem based on the FTC for the given graph. Of those, two students (U17) could not pose a problem, and the remaining three students were able to pose a problem and also solved their own problem correctly. All of them had that feeling because they thought they did not understand the FTC: "I am sure, I cannot, because I do not have a geometric understanding of the FTC" (U9).

9.7.2 Making a drawing related to the problem

For Q9, there was no student who made a drawing to help them to pose a problem. In relation to answering the problem they had posed, students did not make any drawing to help them solve their problems. The two students (Y13) who posed their problems in such a way that drawing a graph is part of the answer were not considered for this section.

9.7.3 Making a calculation plan and systematically performing it

Seven students (U45689; Y28) posed their problems by considering the given graph as the graph of the rate of change, and solved their problems by using the areas under the curve. Two students (Y34) posed their problems using the given graph as the graph of the rate of change but did not use the area under the curve for solving their problems; and were unsuccessful with their problem as a consequence. Two students (U3; Y7) did not consider the given graph as the graph of the rate of change; instead, they posed their problems based on the first part of the FTC. Two students (Y56) posed problems about the rate of change, and U2 posed a problem about finding an equation of lines using the given graph. Two students (U17) did not pose a problem and Y1 used the given graph as the graph of the rate of change, but had a misunderstanding about the FTC (Section 7.19).

9.7.4 Checking calculations and answers

Only U2 did some form of checking for this task. When he was asked to rate his confidence that he had found the correct answer, he re-visited his working and made some changes. After the changes, however, his answer to the problem that he had posed was still incorrect (Section 7.19).

9.7.5 Having an accurate post-judgment of how effectively the problem was solved

Six students (U456; Y358) were sure they had posed a corrected problem based on the FTC. OF those, five students (U456; Y38) posed a problem related to the FTC and four (U456; Y8) solved it correctly. Students had different reasons for that judgment. Two students were sure because they believed their problems related to the FTC. U5 said, “I utilise this theorem and posed a basic problem utilising the velocity and the distance”. Y8 said, “my question requires an understanding of the relationship between the rate of change and actual amount there”. U6 was sure because the question was similar to question eight. U4 was sure because he liked the applied questions and the problem and the answer make sense for him. Y3 was sure because he believed he should have been able to solve the problem he had posed. Y5 did not provide a reason for his feeling.

Five students (U238; Y47) were unsure whether they had posed a correct problem based on the FTC. Of those, all posed a problem related to the FTC and solved their problems, except U2, who posed a problem which is not related to the FTC. U8 was unsure because he was not sure he “used the FTC”. U3 had difficulty in understanding the FTC: “I am not used to it. My two weaker points: the FTC and writing problems”. Y2 was unsure because he was not sure “…the questions covers all aspects of the FTC”, and whether he had solved it correctly. Y7 was unsure because his question does not “need [an] understanding of the FTC to solve but it is part of it”. U2 did not provide any reason for his feeling.

Five students (U179; Y16) were sure they had not posed a correct problem related to the FTC. Of those, two of them (U9; Y1) had posed a problem related to the FTC, and U9 was able to solve his problem correctly. Two students (U17) had that feeling because they could not pose a problem. Two students (U9; Y6) were sure they did not pose an FTC problem because they thought they did not understand what the FTC is. U9 said, “I have absolutely no idea [whether that question is related to the FTC or not]”, and Y6 said, “I am not confident with the concept of the FTC and whether I have used the FTC”. Y1 had that feeling because he could not solve his problem correctly: “I did not know how to sketch it and I just sketch what I remember”.

9.7.6 Discussion

Similar conclusions apply to this problem posing question as applied to Q6 and Q8. Students’ pre- and post-judgments show that several students were not confident with the FTC problems, and several University students were among them. Most of the University students who were unsure of being able to solve this question, or sure they could not solve it said they had this feeling because they had not understood the FTC. As mentioned earlier, this suggests changes to the teaching of the FTC, could benefit students.

9.8 General discussion

Considering the number of students who were sure they could solve the integral-area and the FTC problems, it can be concluded that students at both levels were more confident of solving the integral-area problems in comparison to the FTC problems (Table 9.6 and Table 9.7), regardless of whether they had solved the problems correctly. Section 9.5 to Section 9.7 shows that several University students were not confident of solving the FTC problem despite the fact that this theorem is taught completely in the course, suggesting changes to the teaching of the integral calculus are warranted. This suggestion is supported by the literature (e.g., Thompson, 1994; Thompson & Silverman, 2008) and well supported by the results in Chapters Six to Nine. No research was found around students' metacognitive experiences using the VisA instrument (Jacobse & Harskamp, 2012) at secondary or tertiary level that could be compared with the results of this study. However, the results obtained show that this instrument is helpful for understanding what students feel when dealing with different types of integral calculus questions. In relation to post-judgment of having found the correct answer, most of the students who were unsure they had found it, or were sure they had solved the problem incorrectly, did not go back to check their solution. This result could be an indication of lack of persistence in mathematical problem solving in both Cases, considering the fact that there was no time constraint for solving the integral questions in the interviews. However, other reasons may also have contributed to the lack of checking. For example, it is possible that students had time constraints outside of the researcher's knowledge or that because they were responding to questions during an interview, rather than during a high-stakes assessment, they felt less need to check their work. If the checking process was routine for students, however, it could be expected that they would automatically do so.

University students, in comparison to College students, had a more accurate pre-judgment of their ability to solve the integral questions. In addition, they had a better post-judgment as to whether they had solved the questions correctly. However, several students in both Cases made an incorrect pre- and post-judgment of their ability to solve the problems. The accuracy of metacognitive experience is a very important factor as it has an effect on decisions which students make in learning situations regarding effort allocation, time investment, or strategy use (Efklides, 2006). The researcher believes if students think that, because they have successfully solved some integral-area relationship questions or they have seen similar questions in the class, they can solve all integral-area questions, such thinking is a barrier to improving their knowledge of this topic, and increases the possibility of making an incorrect pre-judgment. For instance, some students are able to solve integral-area questions where the upper function does not change in the enclosed area, but, when exposed to questions like Q1 that the upper function changed in the enclosed area (if you integrate with respect to the x-axis), they might just treat this question like questions where the upper function does not change. Therefore, they found the enclosed area incorrectly while they thought they could solve the question (incorrect pre-judgment). Encouraging using monitoring strategies, such as approximating the enclosed area using geometric shapes, might help students to identify their mistakes and errors, as would the systematic use of checking strategies when lecturer or teachers solve exemplar problems.

Literature in Mathematics education (e.g., Garofalo, & Lester, 1985; Silver, 1982; Verschaffel, 1999) also highlighted the importance of metacognition and its relationship to problem solving ability (Section 3.1.4). Several general activities are suggested to develop students' metacognitive awareness that might be also helpful for solving questions about integral calculus such as

- exposing students to general problem solving and thinking skills;
- encouraging students to think aloud the strategies they have used to solve questions;
- giving questions to students which require planning before solving and evaluating after solving; and
- encouraging students to use different methods for solving the same question (Ministry of Education, 2006). At least this study have addressed the second and fourth suggestions by asking Q1 (asking student to find area using two methods) and using think aloud protocol for exploring student learning. Therefore, these type of activities could be used during teaching integral calculus at Year 13 and first year university courses.

Table 9.6

Students' responses to metacognitive experiences questions related to the integral-area problems

	Q1		Q3		Q4		Q5	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
I am sure I will solve this question	6 (U135689)	6 (Y123678)	3 (U348)	1 (Y4)	9 (U123456789)	8 (Y12345678)	7 (U3456789)	2 (Y57)
I am not sure whether I will solve this question correctly or incorrectly	3 (U247)	2 (Y45)	5 (U12567)	6 (Y123568)	0	0	2 (U12)	5 (Y23468)
I am sure I cannot solve this question	0	0	1 (U9)	1 (Y7)	0	0	0	1 (Y1)
I am sure I solved this question correctly	4 (U3589)	3 (Y268)	2 (U58)	3 (Y568)	7(U3456789)	8 (Y12345678)	6 (U345689)	2 (Y67)
I am not sure whether I solved this question correctly or incorrectly	3 (U146)	1 (Y7)	3 (U349)	1 (Y7)	2 (U12)	0	1 (U7)	0
I am sure I solved this question incorrectly	2 (U27)	4 (Y1345)	4 (U1267)	4 (Y1234)	0	0	2 (U12)	6 (Y123458)

Table 9.7

Students' responses to metacognitive experiences questions related to the FTC

	Q6		Q8		Q9	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
I am sure I will solve this question	2 (U58)	4 (Y3258)	4 (U1358)	1 (U4)	1 (U5)	2 (Y48)
I am not sure whether I will solve this question correctly or incorrectly	7 (U1234679)	1 (Y6)	4 (U4679)	6 (Y123568)	5 (U23468)	4 (Y1356)
I am sure I cannot solve this question	0	3 (Y147)	1 (U2)	1 (Y7)	3 (U179)	2 (Y27)
I am sure I solved this question correctly	4 (U1568)	2 (Y28)	4 (U3568)	3 (Y147)	4 (U4567)	3 (Y358)
I am not sure whether I solved this question correctly or incorrectly	4 (U2349)	2 (Y67)	3 (U249)	4 (Y1235)	3 (U238)	2 (Y27)
I am sure I solved this question incorrectly	1 (U7)	3 (Y145)	2 (U17)	0	2 (U19)	2 (Y16)

In relation to checking calculations and answers (one of the aspects of metacognitive skills), in all questions more than half the students did not check calculations and answers (Table 9.8). Therefore, the importance of this aspect of metacognitive skills could be highlighted to students, and ways that students can use those strategies could be suggested to them (e.g., approximating area using geometric shapes, differentiating antiderivatives, and the fact that area should be positive). Encouraging students to challenge themselves with questions like "how do you know you are correct?" after solving questions can also help students to think about different strategies they can use for checking answers.

Table 9.8

Number of students who have done checking and calculations in integral questions

Q1	Q3	Q4	Q5	Q6	Q8	Q9
7 (U1589; Y246)	1 (U5)	5 (U3;Y1456)	6 (U45689;Y6)	1 (U5)	3 (U35; Y8)	1 (U2)

Make a drawing for the problem is known both as a metacognitive skill (Jacobse & Harskamp, 2012) and a thinking skill (Ministry of Education, 2006); and its importance is highlighted in the literature (Jacobse & Harskamp, 2012; Ministry of Education, 2006). For the first integral questions all students made a drawing towards solving the problem. However, for the evaluation task, Q4, only two students made a drawing to help them solve the question. It seems the importance of making a drawing to aid finding the solution of mathematical questions also should be highlighted to students, because in atypical questions some of the students in the sample did not make a drawing for solving the problem. For instance, in Q4 if students made a drawing for the integrands, there was a higher chance that they could understand in the first example that the area needs to be split and the integrand in the second example is not continuous.

9.9 Chapter summary

This chapter has explored students' metacognitive experiences and skills during solving integral questions for answering the third and fourth research questions of the study (i.e., what metacognitive *knowledge*, *experiences,* and *skills* do students hold about integral calculus in Year 13 and first year university? And what differences exist between student learning of integral calculus in Year 13 and first year university?). This chapter shows students' metacognitive experiences and skills could be further developed at both levels. The next and final chapter describes the overall discussion and conclusions of this study.

Chapter Ten: Discussion and Conclusion

In this chapter, an overview of the study is presented (Section 10.1) followed by a description of the limitations of the study (Section 10.2). Section 10.3 explains the main study findings and their implications. It also includes the answers to the research questions of the study. Directions for further research (Section 10.4) and concluding words (Section 10.5) form the other sections of this chapter.

10.1 Overview

In this study, a multiple case study approach was used to explore students' learning of the integral-area relationships and the FTC. The study sample comprised nine first year university and eight Year 13 students who each participated in an individual semi-structured interview. Ten undergraduate mathematics lecturers and five Year 13 mathematics teachers were also interviewed in relation to the teaching and learning of integral calculus. The teaching of integral calculus in a first year university course and a Year 13 college class were also video recorded and observed, to obtain a better understanding of the teaching and learning of integral calculus.

The study used RBT (Anderson et al., 2001) and Efklides's metacognition framework (Efklides, 2008) to explore students' learning of integral calculus. The combination of these two frameworks can be used to explore student learning in any mathematical topic. Conceptual and procedural knowledge (e.g., Mahir (2009)), factual, conceptual, and procedural knowledge (e.g., introducing procept by Gray and Tall (1994)), and metacognitive knowledge, experiences, and skills (e.g., Jacobse, & Harskamp, 2012), were the focus of research into how students learn mathematics and solve mathematical problems. However, RBT (Anderson et al., 2001) and Efklides's metacognition framework (Efklides, 2008) were not previously used together in a study for exploring student learning. Using these two frameworks provides a better understanding of how students learn mathematics and solve mathematical problems, as illustrated in Chapters Seven to Nine. Using them together can help researchers investigate student learning in relation to six items, i.e., factual, conceptual, procedural, metacognitive knowledge, experiences, and skills.

10.2 Main study findings and its implications

The work in this thesis adds to the literature about mathematics education in several ways. In relation to the first research question of the study, *what examples of factual, conceptual, procedural, and metacognitive knowledge in integral calculus based on RBT can be found in Year*

13 and first year university?, the 11 subtypes of RBT's knowledge dimension for integral calculus are described in Chapter Five (Section 5.1). These 11 subtypes have not been explored for integral calculus before. Also the researcher could not find a study which described these 11 subtypes for mathematics in detail. The implication of this contribution is that it will create opportunities for the design of educational objectives, teaching activities, and assessments based on RBT cells for integral calculus. In addition, it could help lecturers, teachers, researchers, and curriculum designers involved in calculus teaching in Year 13 and first year university to have a better understanding of RBT in the context of integral calculus. Having metacognitive knowledge as part of RBT's knowledge dimension and describing it in the context of integral calculus helps lecturers and teachers to have a better understanding of what each subtype of metacognitive knowledge means in integral calculus (Section 5.1.9 to 5.1.11).

The discussion and implications of the remaining research questions are described together, as they are related to each other. These research questions are:

2. Using RBT as a lens, what are students' difficulties in solving integral questions in Year 13 and first year university?

3. What metacognitive *knowledge*, *experiences*, and *skills* do students hold about integral calculus in Year 13 and first year university?

4. What differences exist between student learning of integral calculus in Year 13 and first year university?

5. What are the perceptions of lecturers and teachers towards students' difficulties in integral calculus?

Making a profile of students' metacognitive knowledge, experiences, and skills in relation to the topics of the study has not previously been explored in the literature. These findings show which aspects of students' metacognitive knowledge, experiences, and skills may need further development (Chapter Eight and Nine). In addition, the study makes contribution to literature in mathematics education in New Zealand by making a profile of students' conceptual and procedural knowledge in the FTC, compared to most previous research in New Zealand which focuses on the integral-area relationships, definite integral, and Riemann sums (e.g., Thomas & Hong, 1996).

In relation to the transition from secondary to tertiary education, the findings show that the teaching (Section 6.1) and learning (Chapter Seven and Eight) of mathematics in Year 13 was more procedural than conceptual in comparison to first year university; therefore, students who start studying at university should learn how to deal with more conceptual teaching. Year 13 students should develop their metacognitive knowledge in a way that gives them an understanding of how to learn mathematics effectively, and to understand the importance of the rationale and ideas behind the formulas and theorems which promote understanding (Cuoco, Goldenberg, & Mark, 1996; Hanna, 1995). Students need to come to see mathematics as more than just a collection of formulas and theorems, and that students need to learn more than the procedures for using them. If they want to be successful in mathematics, students understand how different ideas in mathematics are related to each other, and not forget to explore why formulas and theorems exist and how they are related to other concepts and topics in mathematics. To achieve this, teachers could also adapt practices that support students development of conceptual and metacognitive knowledge to prepare students for studying at universities.

The study findings show that students had difficulty with conceptual knowledge about the FTC (Chapter Seven). Results in relation to metacognitive experiences also reveal that students knew they had not understood the FTC (Chapter Nine). Students in both Cases were more confident about solving the integral-area problems in comparison to the FTC problems (Chapter Nine), regardless of whether they had solved the problems correctly. This is more important in Case 1, as these students were taught both parts of the FTC. There are several possible reasons for such performance. Firstly, the geometric interpretation of the FTC was not mentioned explicitly to students (Section 6.1.1). Secondly, the FTC questions used for assessment in both Cases related to procedural knowledge about the FTC (Section 6.1.1). No question was asked about the geometric interpretation of the FTC in their assessments; also students did not ask about it in the classes. Therefore students did not focus on understanding the meaning of the FTC. Understanding of the FTC for many students was limited to using it for finding the definite integral from the corresponding antiderivative, or being able to find the derivative of an integral using FTC, regardless of their understanding of the relationship between the accumulated area function and rate of change of accumulated area function. The obtained results are in line with the literature (e.g., Orton, 1983; Thompson & Silverman, 2008) which shows that students could apply the FTC for finding the definite integral, however, they did not have conceptual knowledge about it.

The fact that students' focus was on procedural knowledge was also highlighted by lecturers and teachers (Section 6.2) and the literature on mathematics education (e.g., Thomas & Hong, 1996). Avoiding presenting the FTC as magic (Section 6.2.3), and more focus on conceptual knowledge both in teaching and assessment might change the students' focus. This study's findings (Chapters Six and Eight) show that a number of students had a performance approach toward learning integral calculus; therefore, if conceptual knowledge was not involved in the assessment, they had not focused on this type of knowledge.

These findings suggest the advisability of designing and using teaching activities and assessment that focus on conceptual knowledge, particularly in relation to geometric interpretation of the FTC in calculus courses. This has been also been supported by the literature (Thompson & Silverman, 2008). Providing conceptual questions, both in examples that are solved in classrooms and in assignments and assessments, might influence students' focus because of their wish to obtain good scores in tests and assignments. If lecturers and teachers in their assessment, and the examples they solve in classes for students, use conceptual questions like *Why do you use definite integral for finding the area enclosed between curves?,* or *why* $\frac{d}{dx}\left(\int_a^x f(t)dt\right)$ *is equal to* $f(x)$*?,* or use questions like the integral questions used in this study, or similar conceptual questions asked in the literature about exploring student learning in integral calculus (e.g., Thomas & Hong, 1996); this might change students' approaches toward learning integral calculus.

The findings in Chapter Seven (Section 7.8) showed several students made technical errors while solving integral problems. This finding suggests remedial lessons and revision worksheets related to necessary prior knowledge (e.g., algebra, skills of drawing functions, and functions) could be used to prepare students for the integral calculus topic. This also has been suggested by literature regarding other contexts (e.g., Singapore: Kiat, 2005). Another suggestion could be to use a diagnostic test, before or at the start of the course, about the prior knowledge needed for this topic to help students identify their difficulties.

Considering the teaching of integral calculus in Case 2 that was more procedural and the perspective of teachers and lecturers (Chapter Six), it seems the importance of conceptual knowledge, particularly, the rationale behind the formulas, should be highlighted to instructors, particularly teachers. As several University students stated (Section 8.1.9), knowing the rationale helps students to remember, apply, or reproduce formulas when needed. It helps them to have a better understanding of the topic, and better performance in exams and questions. The study provides further evidence that pre-service and in-service teaching programmes benefit from including focus on conceptual knowledge and its usefulness for students' learning.

Students in Case 2 had more difficulties with conceptual knowledge about the definite integral and the integral area relationship in comparison to Case 1 (Chapter Seven). One possible reason for such performance might be related to the teaching of the Riemann sums (Section 6.1). In Case 1, Riemann sums were the focus of teaching definite integrals, and examples were solved in this regard. In addition, during the teaching of other topics such as finding volume by slicing and cylindrical shells, the proofs for volume formulas were taught to students. The ideas used in those proofs are related to Riemann sums, therefore, help students to have a better understanding of the Riemann sums and Riemann integral. However, other possible reason for such performance could be university students have more experiences with this topic as they exposed to it for a longer time.

In Case 2, the teacher did not introduce Riemann sums until end of the teaching of integral calculus and no example was solved in the classroom. This teacher started with teaching definite integrals, showing his procedural approach toward teaching, i.e., "I am going to take the expedient route...I am going to give you the application…saying without proving…" (Section 6.1.2). Similar to presenting the FTC as magic (Section 6.2.2), such approaches toward teaching mathematics are related to instrumental learning (Section 3.1.2), which can have negative consequences for students' learning (e.g., they influence their attitude toward mathematics and their understanding of the structure of mathematics (Section 6.2.3)). One of the possible reasons why this teacher did not focus on Riemann sums is related to the *New Zealand Curriculum* (Ministry of Education, 2007a) and the NCEA level 3 mathematics achievement standards (New Zealand Qualifications Authority, 2013). Being able to use the numerical method of integration is prescribed in these documents; however the Riemann sums are not highlighted (Section 1.2.2). In addition, the textbook used in the College (Delta mathematics (Barton & Laird, 2002) does not focus on the Riemann integral and only provides it in the appendix. Among the numerical integration methods, the trapezium and Simpson's rule are provided in the main body of the textbook and also are in the focus of teaching in the College. However, it seems the proofs behind these methods are more complicated in comparison to the Riemann sums as their formulas have more elements. Another possible reason is the numerical method was presented at the end of integral calculus topic in the textbook and was also taught by the teacher at the end of the integral calculus topic, whereas, in Case 1, students first learnt about Riemann sums, then were exposed to the FTC and integral techniques. Numerical methods are more conceptual and need more time to understand. Students in Case 1 had more time to understand them while being introduced to different topics in integral calculus, while this opportunity was not available to students in Case 2.

Study findings (Section 8.5.3) show that five University and two College students like calculus when they understand it. Teaching the Riemann sums to students can help them to have a better understanding of the definite integral and integral-area relationship (Sealey, 2006, 2014) and the FTC (Thompson & Silverman, 2008). Therefore, teaching about Riemann sums might be one of the useful ways of changing students' attitudes toward calculus as students may develop a better understanding of the structure of mathematics.

In relation to one aspect of metacognitive skills (that is, checking calculations and answers), in all integral questions, more than half the students did not check their calculations and answers in the interviews (Table 9.8). In addition, for most students, checking whether the integrand is continuous or not, and monitoring problem solving were not part of students' plans for solving integral problems (Section 8.3). Moreover, the number of monitoring strategies students knew for checking their answers were limited for several students (Section 8.2). Therefore, this aspect of metacognitive skills should be included as a key element of teaching, and ways in which students can use those strategies should be suggested to them (e.g., approximating area using geometric shapes, differentiating antiderivatives, and that the fact that area should be positive). If lecturers and teachers used monitoring strategies more often during the solving of questions in their classes, and asked students to do so as part of their problem solving, this metacognitive skill might be used by students more often.

In relation to making a drawing related to a problem as part of metacognitive skills in solving mathematical problems; in the first integral question, which is a typical question about integral-area relationship, all students made a drawing to help solve the problem. However, for non-typical questions, like Q4 of the integral question (Section 5.2.1), only two students made a drawing to help them solve the question. It seems the importance of making a drawing to assist in solving mathematical questions should also be highlighted to students, because in unfamiliar questions some students in the sample did not make a drawing to help solve the problem. For instance in Q4, if students made a drawing for the integrands, there was a higher chance that they could understand that in the first example the area needs to be split, and the integrand in the second example is not continuous. Lecturers and teachers should be encouraged to make a drawing for each question they solve in classes for students, and also to encourage students to do so. Free online resources such as https://www.desmos.com/ can be used/suggested for this purpose.

The differences between students' performance in Case 1 and 2 in relation to definite integral suggests changes in the curriculum document to focus more on teaching Riemann sums. This would help to increase students' conceptual and procedural knowledge. By knowing the rationale behind the relationship between definite integral and the area under a curve through Riemann sums, students will develop their conceptual knowledge as to why a definite integral can be used for finding area. Teaching Riemann sums would also help students develop their procedural knowledge by knowing what to integrate and how to set up the bounds of integral (Sealey, 2006, 2014).

Students in Case 1 in comparison to students in Case 2 had a more accurate pre-judgment of their ability to solve the integral questions and also had a better post-judgment of whether they had solved the questions correctly (Chapter Nine). However, in both Cases several students made an incorrect pre- and post-judgment. Therefore, highlighting not oversimplifying the question and using monitoring strategies might help students to understand where they make mistakes.

In relation to metacognitive knowledge, the importance of knowing the rationale behind the theorems and formulas is addressed in the literature (Section 5.2.2), and most of the students in Case 1 were aware of its importance and usefulness (Section 8.1.7 & 8.1.9). Therefore, it seems that its importance should be highlighted to students in College. The usefulness of pre-reading (Section 8.1.4) and summarising strategies (Section 8.1.8) could be highlighted to students in both Cases as the number of students who used them was limited. If a lecturer or teacher, after teaching mathematical concepts and theorems within a topic, made a concept map of the mathematical concepts and theorems in the topic (using websites such as https://bubbl.us/) to show how the concepts and theorems in the topic are related to each other, this can help students to realise that making a summary of the ideas in the course is helpful for their learning. The teacher or lecturer could also encourage students to do so, and then students could compare their concepts map of the topics with their peers to further develop their understanding of the topic.

Overall, Table 10.1 summarises the similarities and differences found between the teaching and learning in the study's Cases. This could help researchers, lecturers, and teachers who are involved and interested in the secondary-tertiary transition to have a better understanding of this transition.

Table 10.1

Similarities and differences between the teaching and learning in the Cases of the study

Similarities	Differences
Assessment in both Cases was more procedural than conceptual.	The teaching of integral calculus was more procedural in Case 2 in comparison to Case 1.
Students' metacognitive knowledge, skills, and experiences could be further developed in both Cases.	For a majority of students, their understanding of the definite integral and the integral-area relationship was procedural in Case 2, while it was conceptual in Case 1.
The use of monitoring strategies should be modelled to students in both Cases.	Online resources were used more often in Case 1 in comparison to Case 2.
Students' understanding of the FTC was procedural in both Cases.	Metacognitive knowledge of students in Case 2 needed further development in comparison to students in Case 1 in relation to the importance of the rationale behind the formulas.
Several students in both Cases had difficulties with the algebraic manipulation necessary for solving integral questions and sketching graphs.	Lecturers were more aware of the importance of the rationales behind the formulas and theorems in comparison to the teachers.
Several students in both Cases had a performance approach towards learning integral calculus.	Students in Case 1 had more accurate pre- and post-judgments in comparison to students in Case 2.

10.3 Limitations of the study

For the first time, this study makes a profile of students' use of factual, conceptual, procedural, and metacognitive knowledge in integral calculus. In addition, it shows how RBT can be used to explore students' learning in a mathematical topic. However, there are a number of limitations to the study's findings.

In this study, student learning of integral calculus is explored, based on the responses of 17 students from one University and one College; therefore, the results obtained may not represent all students' problem-solving behaviours in integral calculus. The University and College were chosen from a region that was geographically accessible to the researcher. The school decile of the College was 10; therefore, the findings might not be applicable to students who attend other colleges with different decile rating. Studies with larger sample sizes in different contexts are necessary to explore the findings further.

Gender differences are not explored in the study because of the small sample size with most of the interviewees were male. If further studies, exploration of gender differences in terms of students' mathematical problem solving, metacognitive knowledge, experiences, and skills could also be fruitful. In addition, it would be valuable to explore whether there is a difference between students of different ethnic groups in terms of the themes of the study.

In this study, to explore facets of metacognition in relation to integral calculus, the Efklides's (Efklides, 2006, 2008) metacognition framework and subtypes of RBT's metacognitive knowledge (Anderson et al., 2001) were used. If other metacognition frameworks or other definitions that existed for metacognitive knowledge were to be considered, further valuable information may be obtained that could add to the study findings.

The presence of the researcher during the observation and recording of the teaching of integral calculus in these Cases might have had an impact on lecturers, tutors, teachers, and students behaviour in classes. The researcher attempted to establish friendly relationships with participants, however, his presence might still have had an effect on their behaviour and practices.

The main data from students were gathered using a fairly lengthy interview. Students before the interview were informed whenever they feel tired tell the interviewer to do the rest of the questions on another session (Appendix 1), however, they might not have done that when they felt tired. Although, for all students except one, the interview were done in two sessions, and for the remaining student it was conducted in three sessions. In addition, students might not feel comfortable answering integral questions while being observed by the researcher and think aloud.

Students' metacognitive skills were explored using a think aloud protocol (Section 5.2.3). While the interviewer encouraged students to think aloud consistently during problem solving, there were moments where students did not think aloud. It was possible that students did not verbalise all their thinking, therefore, some of the students' metacognitive skills might not have been captured in the study.

The interim RBT knowledge dimension for integral calculus used in this study was developed solely by the researcher, using several documents under the supervision of his PhD supervisors (Section 5.1). In addition, the qualitative analysis of this study reported in Chapters Six to Nine was done by the researcher only. Therefore the study findings are reliant on his interpretation and biases. Different issues related to reliability, such as data, theory, and method triangulation (Section 4.3.4) and validity (Section 4.3.5) in qualitative research, were considered to strengthen the validity of the findings. However, the findings were still influenced by his beliefs and experiences (Section 1.2.1) and should be seen in that light.

10.4 Directions for further research

This study used RBT for exploring students' learning of integral calculus, while RBT has other applications, such as exploring alignment and planning teaching (Chapter Two). Evaluating the alignment between curriculum documents, teaching activities, and assessment can be done using RBT. Developing a tool for observing teaching in classrooms and analysing teaching activities based on RBT can also provide further insights about teaching and learning. In addition, RBT can be used for designing questions that explore teachers' perspectives toward teaching different mathematical topics.

While several studies focus on students' learning of definite integral and integral-area problems (e.g., Grundmeier, Hansen, and Sousa, 2006; Kiat, 2005; Mahir, 2009), further studies in relation to students' learning of the FTC seem necessary, as the amount of research in relation to this topic is still limited.

The interim RBT knowledge dimension for integral calculus in this study could be developed and refined by a research team including mathematics lecturers, teachers, and researchers in undergraduate mathematics, to incorporate several perspectives and ideas.

Designing teaching activities and assessment questions that focus on conceptual knowledge about the FTC is also required, as lecturers and teachers are mainly using procedural questions related to this topic (Section 6.1) and the amount of research which provides such teaching activities and assessment is limited. In a broader perspective, designing teaching activities that address different cells of RBT for the FTC can be useful.

Further research is necessary to explore at what stage numerical integration should be taught to students: at the start of teaching integral calculus before exposing students to the FTC and antiderivative (Case 1); or at the end of the topic after teaching the definite integral, FTC, and integral techniques (Case 2). The study findings show that presenting the numerical integration method at the end of teaching integral calculus at Case 2 did not provide conceptual knowledge for students, while the Riemann sums that were provided at the start of teaching integral calculus in Case 1 did provide conceptual knowledge to students (Chapter Seven). However, the fact that students in Case 1 had been exposed to integral calculus in Years 12 and 13 should be taken into consideration here. Therefore, further research is necessary to see whether prioritising each of these topics for students in the same schooling Year has an effect on their conceptual knowledge or not.

10.5 Concluding words

The number of researchers in mathematics education who have used RBT as a framework is limited. However, the comparison that has been done between the major theories and frameworks that influence the teaching and learning of mathematics (Section 3.1) shows that RBT fits with these theories and addresses several aspects of them. The study findings show that RBT is a useful tool for exploring student learning, therefore its potential should be reconsidered by researchers in mathematics education when they wish to explore students' learning and problem solving. Several students in the sample had not developed conceptual knowledge in relation to the definite integral in Year 13, therefore, changes in the teaching of integral calculus at this level seems necessary to provide more focus on conceptual knowledge. At both levels, University and College, changes to the teaching of the FTC are necessary in order to provide opportunities for students to develop their conceptual knowledge about the FTC. The study shows that several aspects of student's metacognitive knowledge, skills, and experiences need further development; those aspects should be highlighted during the teaching of integral calculus, to help students to improve their understanding of this topic and become successful mathematical problem solvers. All of these changes could help improve the quality of the teaching and learning of mathematics at Universities and Colleges, and make the secondary- tertiary transition more enjoyable for students.

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Appendices

Appendix 1: Interview questions with University students

Appendix 2: Classroom observational tool

Appendix 3: A sample of information sheet

Appendix 4: A sample of consent form

Appendix 1: Interview questions with University students

Dear participant

Thank you so much for accepting to participate in the interviews regarding learning area under curves and the Fundamental Theorem of Calculus. Please read information below and after you feel ready tell the interviewer to start the interview.

- Please note that there are 9 integration questions which you need to solve in these interviews, five related to the area under curves and four related to the Fundamental Theorem of Calculus.
- Some of these questions may be not taught in X[13], and they are asked for research purpose.
- If you need extra paper for solving the questions ask the interviewer to provide some for you.
- Remember that how you answer these questions will not affect your X grade and your responses are completely confidential with the researcher and his supervisors as informed you previously in the information sheet.
- For each question, please explain your thinking aloud while you solve the questions.
- Please do not share the interviews questions with your classmates because some of them are going to take part in the same interviews and for the research purpose, they should not see the questions before the interviews.
- Whenever you feel tired, please tell the interviewer to do the rest of the questions on another session.
- For some of the questions you need to read the questions and rate your confidence for finding the correct answer (without calculating the answer). When you finished, you also need to rate your confidence for having found the correct answer.
- After you finish answering the integration questions, there are a couple of general questions about your experience of learning area under curves and the Fundamental Theorem of Calculus. There are 14 questions in this regard which approximately take around 15 to 20 minutes to discuss them with the interviewer. This is the place you can say your opinion about teaching, learning, and assessment of integration, particularly in these two calculus topics.

Many thanks again for participating in the interviews

Yours sincerely

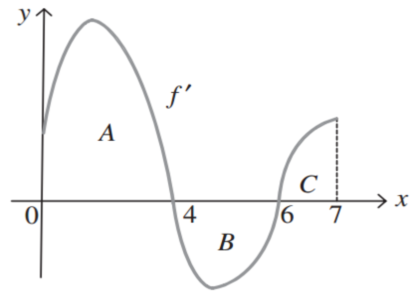
Farzad Radmehr

[13] The name of the course was mentioned here instead of X which is removed for not revealing the name of the University. All X in appendix 1 is the name of the course students were enrolled in it.

1. Please calculate the area enclosed between the curve $x = y^2$ and $y = x - 2$ in two ways. Which way is better to use? Why?	
How well do you think you can solve this question?	
I am sure I will solve this question.	
I am not sure whether I will solve this question correctly or incorrectly.	
I am sure I cannot solve this question.	
Please, explain why	
Solution:	
Rate your confidence for having found the correct answer.	
I am sure I solved this question correctly.	
I am not sure whether I solved this question correctly or incorrectly.	
I am sure I solved this question incorrectly.	
Please, explain why	

2. What do you understand by A $= \int_a^b [f(x) - g(x)]dx$ and B $= \int_c^d [w(y) - v(y)]dy$?

Can you justify how these formulas are derived? Can you justify when each one is used?

3. The graph of $f'(x)$, the derivative of $f(x)$, is sketched below. The area of the regions, $A, B,$ and C are 20, 8, and, 5 square units, respectively. Given that $f(0) = -5$, find the value of $f(6)$. 	
How well do you think you can solve this question*?*	
I am sure I will solve this question.	
I am not sure whether I will solve this question correctly or incorrectly.	
I am sure I cannot solve this question.	
Please, explain why	
Solution:	
Rate your confidence for having found the correct answer.	
I am sure I solved this question correctly.	
I am not sure whether I solved this question correctly or incorrectly.	
I am sure I solved this question incorrectly.	
Please, explain why	

4. Are these examples solved correctly? Please justify your answer. **Ex.1:** Find if possible, the area between the curve $y = x^2 - 4x$ and the x-axis from $x = 0$ to $x = 5$. $\int_0^5 (x^2 - 4x)dx = [\frac{x^3}{3} - \frac{4x^2}{2}]_{x=0}^{x=5} = \left[\frac{5^3}{3} - \frac{4(5)^2}{2}\right] - \left[\frac{(0)^3}{3} - \frac{4(0)^2}{2}\right] = \frac{-25}{3}$ **Ex.2:** Find if possible, the area enclosed between the curve $y = \frac{1}{x^2}$ and the x-axis from $x = -1$ to $x = 1$. $\int_{-1}^{1} \frac{1}{x^2} dx = \int_{-1}^{1} x^{-2} dx = [\frac{(x)^{-1}}{(-1)} = \frac{-1}{x}]_{x=-1}^{x=1} = \frac{-1}{1} - \frac{(-1)}{(-1)} = -2.$

How well do you think you can solve this question*?*	
I am sure I will solve this question.	
I am not sure whether I will solve this question correctly or incorrectly.	
I am sure I cannot solve this question.	
Please, explain why	

Solution:

Rate your confidence for having found the correct answer.	
I am sure I solved this question correctly.	
I am not sure whether I solved this question correctly or incorrectly.	
I am sure I solved this question incorrectly.	
Please, explain why	

5. Please can you pose a problem about the area enclosed between a curve and a line with any two arbitrary bounds that will give an answer of 1 (i.e., the enclosed area will be equal to one)?	
How well do you think you can solve this question*?*	
I am sure I will solve this question.	
I am not sure whether I will solve this question correctly or incorrectly.	
I am sure I cannot solve this question.	
Please, explain why	
Solution:	
Rate your confidence for having found the correct answer.	
I am sure I solved this question correctly.	
I am not sure whether I solved this question correctly or incorrectly.	
I am sure I solved this question incorrectly.	
Please, explain why	

6. Find the derivative of the following functions. • $O(x) = \int_1^x \frac{1-t}{t^2-2t-9} dt$ • $G(x) = \int_0^{x^2} r^2\sqrt{1+r^3}\, dr$ • $D(x) = \int_{2x-5}^{4x+4} t^3 dt$
How well do you think you can solve this question*?*

I am sure I will solve this question.	
I am not sure whether I will solve this question correctly or incorrectly.	
I am sure I cannot solve this question.	

Please, explain why
Solution:
Rate your confidence for having found the correct answer.
I am sure I solved this question correctly.
I am not sure whether I solved this question correctly or incorrectly.
I am sure I solved this question incorrectly.
Please, explain why

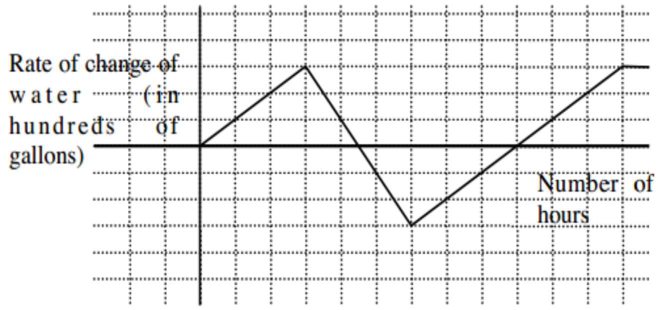
7. What do you understand by $F(x) = \int f(x)dx$?

   -What do you understand by $F(x) = \int_a^x f(t)dt$?

   -What do you understand by $\int_a^b f(x)dx = F(b) - F(a)$? When do you use this formula? Can you justify how it is derived?

   -What do you understand by $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$? When do you use this formula? Can you justify how it is derived?

8. Let f represent the rate at which the amount of water in Phoenix's water tank changed in 100's of gallons per hour in a 12 hour period from 6 am to 6 pm last Saturday (Assume that the tank was empty at 6 am (t=0)). Use the graph of f, given below, to answer the following.



- How much water was in the tank at noon?
- What is the meaning of $g(x) = \int_0^x f(t)dt$?
- What is the value of (9) ?
- During what intervals of time was the water level decreasing?
- At what time was the tank the fullest?
- Using the graph of f given above, construct a rough sketch of the graph of g and explain how the graphs are related.

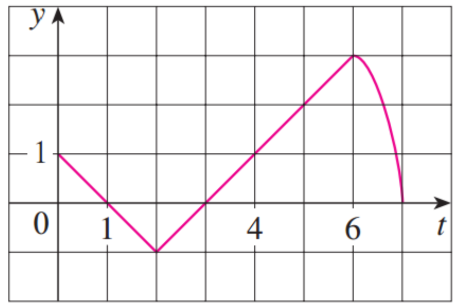
How well do you think you can solve this question*?*	
I am sure I will solve this question.	
I am not sure whether I will solve this question correctly or incorrectly.	
I am sure I cannot solve this question.	

Please, explain why

Solution:

Rate your confidence for having found the correct answer.	
I am sure I solved this question correctly.	
I am not sure whether I solved this question correctly or incorrectly.	
I am sure I solved this question incorrectly.	

Please, explain why

9. Please can you write a problem based on the following graph whose solution would require using the Fundamental Theorem of Calculus? 	
How well do you think you can solve this question*?*	
I am sure I will solve this question.	
I am not sure whether I will solve this question correctly or incorrectly.	
I am sure I cannot solve this question.	
Please, explain why	
Solution:	
Rate your confidence for having found the correct answer.	
I am sure I solved this question correctly.	
I am not sure whether I solved this question correctly or incorrectly.	
I am sure I solved this question incorrectly.	
Please, explain why	

End of first part of the interviews.

Part2: Please think about the calculus topics "**area between two curves, and the Fundamental Theorem of Calculus**" and answer the following questions.

1. Did you attend the lectures/tutorials in these topics? If so, could you please describe what you typically did when attending the lectures and tutorials of X in these topics? Why/Why not?		
Response : **Reason:**		
Did you take notes in class? Why/Why not?	Yes	No
Reason:		
Did you just listen to the instructor/tutor? Why/Why not?	Yes	No
Reason:		
Did you talk to your classmate while the instructor/tutor teaches these topics? Why/Why not?	Yes	No
Reason:		
Do you do any pre-reading before attending sessions in relation to these topics? Why/Why not?	Yes	No
Reason:		
Do you look at your previous lecture notes, or Anton calculus textbook, etc before coming to the classes? Why/Why not?	Yes	No
Reason:		

2. Which source(s) of information do you use for learning about these topics? Why?
Response :
Reason:

3. How do you help yourself to learn to calculate the area enclosed between curves? How about when you are learning about the Fundamental Theorem of Calculus? Please justify your answer.
Response about area:
Justification about area:
Response about FTC:
Justification about FTC:

4. Do you solve all questions in relation to finding enclosed area in the same way or do you use different strategies for solving different questions in this topic? How about the solving questions using the Fundamental Theorem of Calculus?
Response about area:
Justification about area:
Response about FTC:
Justification about FTC:
5. What difficulties do you have in learning how to calculate the area enclosed between curves? What are your strengths and weaknesses in this topic? How about the answers to these two questions in relation to the Fundamental Theorem of Calculus ?
Response about area:
Justification about area:
Response about FTC:
Justification about FTC:

6. How do you check your answers when solving problems involving finding the area enclosed between curves? How about the Fundamental Theorem of Calculus?
Response about area:
Justification about area:
Response about FTC:
Justification about FTC:
7. What prior knowledge do you think you need to be able to solve problems related to finding the area enclosed between curves? How about the Fundamental Theorem of Calculus?
Response about area:
Justification about area:
Response about FTC:
Justification about FTC:

8. Do you use any practice or memory strategies for these topics, such as using an acronym for remembering formulas, procedures, and concepts in these topics (e.g., BEDMAS: for order of operations in algebra?
Response about area:
Justification about area:
Response about FTC:
Justification about FTC:
9. Have you made a summary of the concepts, formulas, or procedures presented in these topics for yourself (e.g., Figure 1)? Please explain your answer.
Response about area:
Justification about area:
Response about FTC:
Justification about FTC:

10. When you are studying integration do you think about the justification or rational behind the formulas or do you just try to apply the formulas? Why?
Response about area:
Justification about area:
Response about FTC:
Justification about FTC:
11. Do you have a plan for solving problems related to enclosed area between curves (e.g., Figure 2)? Why/why not? How about a plan for solving problems related to the Fundamental Theorem of Calculus? If not, can you create one now?
Response about area:
Justification about area:
Response about FTC:
Justification about FTC:

Plan for solving enclosed area between curves:
Plan for solving FTC problems:

12. Have you heard about metacognitive knowledge? If so, what is metacognitive knowledge in terms of learning how to calculate the area enclosed between curves? What is it in terms of learning about the Fundamental Theorem of Calculus?
Response about area:
Justification about area:
Response about FTC:
Justification about FTC:
13. Why are you taking X?
Response :
14. Do you like calculus, especially integration?
Response:

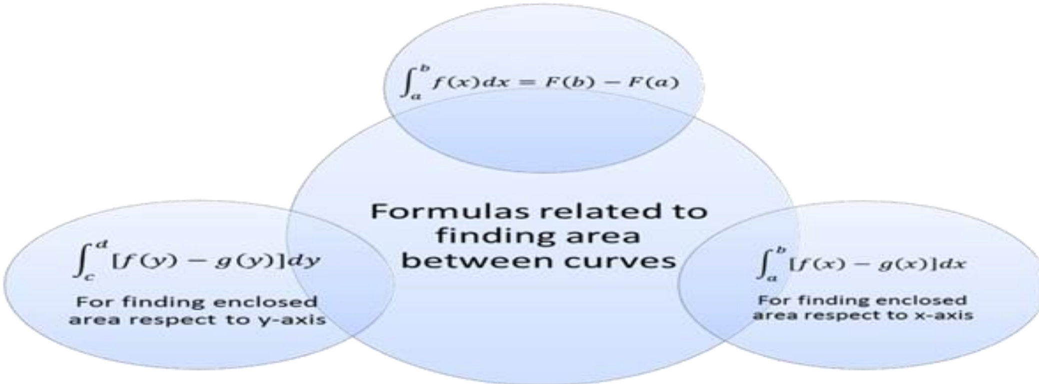

$\int_a^b f(x)dx = F(b) - F(a)$
Formulas related to finding area between curves
$\int_c^d [f(y) - g(y)]dy$
For finding enclosed area respect to y-axis
$\int_a^b [f(x) - g(x)]dx$
For finding enclosed area respect to x-axis

Figure 1

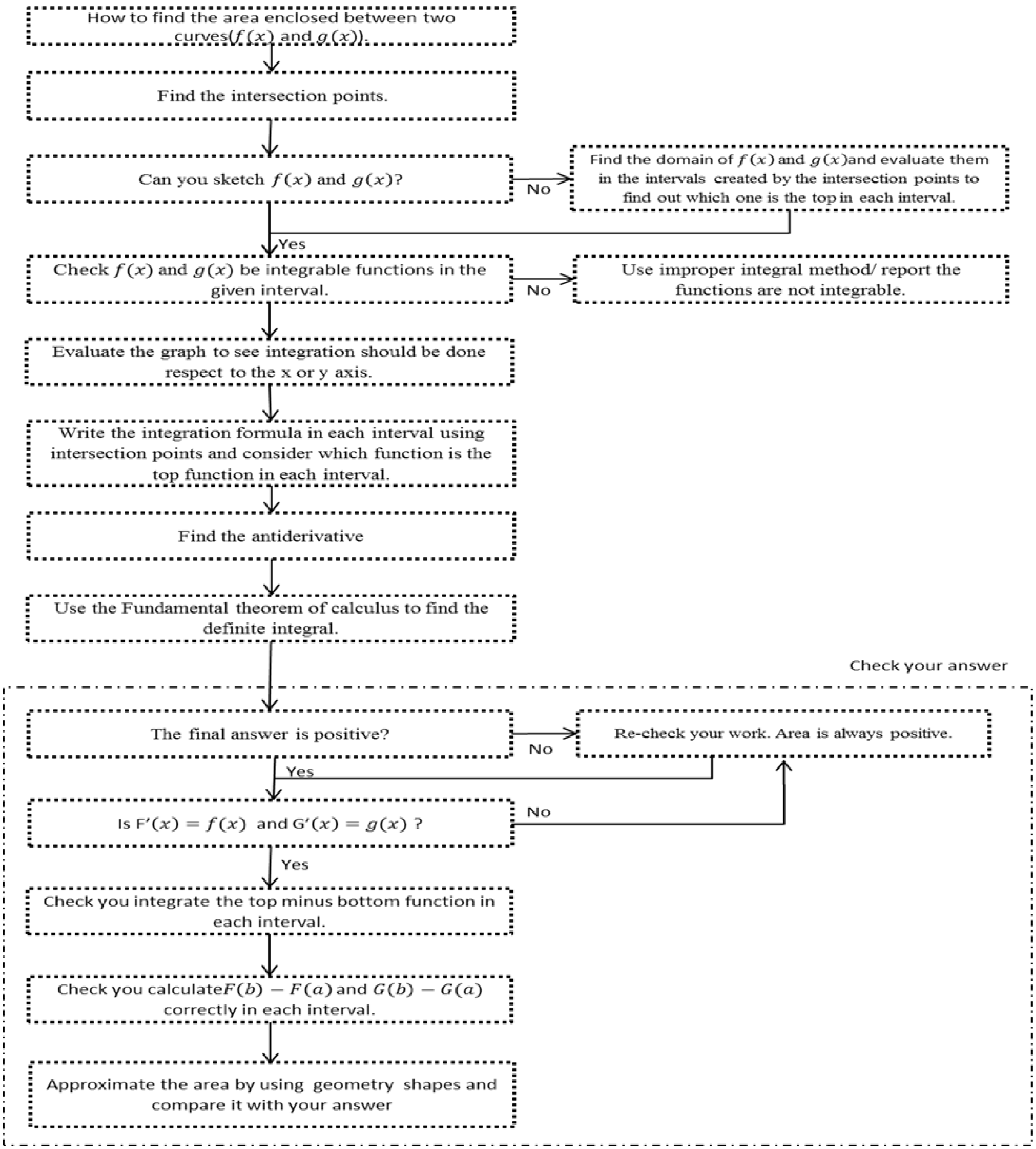

How to find the area enclosed between two curves($f(x)$ and $g(x)$).
Find the intersection points.
Can you sketch $f(x)$ and $g(x)$?
No
Find the domain of $f(x)$ and $g(x)$and evaluate them in the intervals created by the intersection points to find out which one is the top in each interval.
Yes
Check $f(x)$ and $g(x)$ be integrable functions in the given interval.
No
Use improper integral method/ report the functions are not integrable.
Evaluate the graph to see integration should be done respect to the x or y axis.
Write the integration formula in each interval using intersection points and consider which function is the top function in each interval.
Find the antiderivative
Use the Fundamental theorem of calculus to find the definite integral.
Check your answer
The final answer is positive?
No
Re-check your work. Area is always positive.
Yes
Is $F'(x) = f(x)$ and $G'(x) = g(x)$?
No
Yes
Check you integrate the top minus bottom function in each interval.
Check you calculate$F(b) - F(a)$and $G(b) - G(a)$ correctly in each interval.
Approximate the area by using geometry shapes and compare it with your answer

Figure 2

Appendix 2: Classroom observational tool

Course:______________ instructor:__________ Date:_________________ Time:______________

Exploring Students' Learning of Integral Calculus using Revised Bloom's Taxonomy

Learning objective

Time	Reflective note	Time	Descriptive note	RBT table for learning objective						
					R	U	Ap	Ana	E	C
				F						
				C						
				P						
				M						

Teaching activity

Time	Reflective note	Time	Descriptive note	RBT table for teaching activity						
					R	U	Ap	Ana	E	C
				F						
				C						
				P						
				M						

Instructor & students interaction

Time	Reflective note	Time	Descriptive note	RBT table for assessment						
					R	U	Ap	Ana	E	C
				F						
				C						
				P						
				M						

Appendix 3: A sample of information form

TE WHARE WĀNANGA O TE ŪPOKO O TE IKA A MĀUI
VICTORIA
UNIVERSITY OF WELLINGTON

Information sheet (Interview: Teachers/Lecturers/Tutors)

Researcher: *Farzad Radmehr,* School of *Education Policy and Implementation,* Victoria University of Wellington.

Title of project: *Applications of Revised Bloom's Taxonomy in Mathematics Education: The Teaching, Learning, and Assessment of Integration.*

I am a PhD student in mathematics education at Victoria University of Wellington. As part of this degree I am undertaking a research project leading to a thesis. The project I am undertaking is about using Revised Bloom's Taxonomy for improving teaching, learning, and assessment of integration. This research project has received approval from the Victoria University of Wellington Human Ethics Committee (Approval #20851).

As part of my research, I am inviting Year 13 mathematics teachers, undergraduate mathematics lecturers, and tutors who have at least 3 years experience of teaching integration to participate in this study. I would like to seek your opinions about different types of knowledge in integration and how we can enhance teaching, learning, and assessment of integration by focusing on higher level thinking. You are invited to participate in two interviews, each lasting around one hour.

Data will be used on an anonymous basis. It will not be possible for you or your institution to be identified. All material collected will be kept confidential. No other person besides me, my supervisors (*Dr Robin Averill and Dr Michael Drake*), and a transcriber who fill a confidentiality form will have access to the data. Once completed a copy of the thesis will be deposited in the University library. It is intended that one or more articles will be drawn from the thesis study. As is usual in research, all data will be destroyed five years after the end of the project. During the project, if you would like to withdraw from the project, you can do so without needing to give any reasons by sending an email to me or my supervisors up to the end of data collection (December 1st, 2014).

If you have any further questions or would like to receive further information about the project, please contact me at (0223895906 or farzad.radmehr@vuw.ac.nz) or my supervisors (*Dr Robin Averill and Dr Michael Drake*), at the Faculty of Education of Victoria University of Wellington (*Dr Robin Averill*: 04 463 9714 or robin.averill@vuw.ac.nz ; *Dr Michael Drake*: 04 463 9668 or michael.drake@vuw.ac.nz).

Please keep this letter for your information after completing and returning the consent page to me.

Sincerely,

Farzad Radmehr

Appendix 4: A sample of consent form

TE WHARE WĀNANGA O TE ŪPOKO O TE IKA A MĀUI
VICTORIA
UNIVERSITY OF WELLINGTON

Consent to participate in research (Interview: Teachers / Lecturers/ Experts)

Title of project: *Applications of Revised Bloom's Taxonomy in Mathematics Education: The Teaching, Learning, and Assessment of Integration.*

I have been given and have understood an explanation of the research project. I have had an opportunity to ask questions and have them answered to my satisfaction. I understand that I may withdraw myself (or any information I have provided) from this project without having to give reasons up to end of the data collection (December 1st, 2014). In addition, I understand that any information I provide will be kept confidential to the researcher and the supervisors. I understand any published results will not use my name, and that no opinions will be attributed to me in any way that will identify me or my institution.

- I consent to being involved in this research by participating in interviews regarding teaching, learning, and assessment of integration. **Agree** O **Disagree** O

If you are interested in receiving the summary of the results, please provide your email address: …………………………………………………………………………………

- **Signed:**
- **Name:**
- **Date:**