STOCHASTIC CONTINUOUS-TIME CASH FLOWS

A COUPLED LINEAR-QUADRATIC MODEL

BY

JOHANNUS GERARDUS JOSEPHUS VAN DER BURG

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Synopsis

Substantiated by theoretical considerations and empirical evidence, this study corroborates the suitability of a stochastic differential equation specification with a linear drift function and a quadratic diffusion function to model continuous-time cash flows.

Ignoramus et ignorabimus

ACKNOWLEDGEMENTS

Soon after completing my Masters, a considerable time ago, I embarked on a PhD that I unfortunately had to abandon prematurely. The decision, not taken lightly, was forced upon me by a rapidly developing and very time-consuming career in (financial) management. However, a completed PhD was something that always lingered in my mind, and a few years into a second career in academia, this opportunity arose. I took it with both hands and invested every spare moment until reaching the finish line. In hindsight, I must admit that there are benefits in doing a PhD later in life. Blessed with the rich professional and personal life experience that I was able to bring this into this dissertation, I have likely avoided some challenges and pitfalls that I may have encountered earlier in life.

The process of writing this dissertation proved to be overall a satisfying and enjoyable experience. Yes, there were a few frustrations and desperate moments on my journey but, fortunately, never a true low. It felt like playing chess though: finding solutions to a problem usually required a series of well-considered moves. At each step, there was the anticipation to discover if the selected approach was going to work. More often than not, this was not the case. In chess-parlance: every time I made a move, my imaginary opponent came up with an unexpected, equally brilliant counter-move. That forced me to reformulate and rethink the problem, sometimes fundamentally, in order to develop a new solution strategy. However, never did I belief ‘the other player’ would deliver a checkmate: somewhere, there was always a solution waiting to be discovered, frequently in unanticipated, hidden places.

Despite being an activity of solitude, undertaking a PhD is never a solitary process. Here is the place to express my gratitude to all those people who have supported me to realise my academic dream. First, my sincere thanks and appreciation goes out to my supervisors, professors Mark Tippett and Tony van Zijl. They stimulated me intellectually to go the extra mile and showed me new horizons to improve the quality of my work. I truly enjoyed the weekly Skype-conversations with Mark: beside this dissertation, our discussion topics ranged from last week’s achievements of the All Blacks and Wallabies, monetary policy and free markets, to econometric models based on nonsense correlations. Tony had always a listening ear, taught me how to improve my academic writing style and make publications more accessible, and, importantly, gently guided me through the intricacies of a university's bureaucracy.

Patiently accepting my mental absence for over two years, my wife Angela deserves the ultimate gratitude, respect and appreciation. Without her loving support and deep involvement, this dissertation would never have been completed. Thank you, my dearest Angie.

A special word of thanks to my family: my Dad, almost ninety years, and Corine, Andries and Niels, who are all immensely proud. Sadly, my Mum passed away at the end of last year, too early to see this work completed, but she undoubtedly would have considered the PhD as a major achievement.

Last but not least, thanks go out to my employer Nelson Marlborough Institute of Technology for supporting this PhD-dissertation, not only financially but also by generously allocating research time. To my NMIT colleagues, some of whom are on a path to completing a PhD too: I wish them best of luck.

I hope that this study will make a contribution, however modestly, to a strongly growing, often challenging but always exciting discipline within the field of quantitative finance.

John Van der Burg, March 2018.

Not everything that counts can be counted – and not everything that is counted, truly counts.

Albert Einstein

STATEMENT OF ORIGINAL AUTHORSHIP

I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, or substantial proportions of materials which have been accepted for the award of any other degree or diploma at Victoria University of Wellington or any other educational institutions, except where due acknowledgement is made in this thesis. Any contribution made to the research by others, with whom I have worked at Victoria University of Wellington or elsewhere, is explicitly acknowledged in this thesis.

I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.

Signature:

Date:

ABSTRACT

The focal point of this dissertation is stochastic continuous-time cash flow models. These models, as underpinned by the results of this study, prove to be useful to describe the rich and diverse nature of trends and fluctuations in cash flow randomness. Firstly, this study considers an important preliminary question: can cash flows be fully described in continuous time? Theoretical and empirical evidence (e.g. testing for jumps) show that under some not too stringent regularities, operating cash flow processes can be well approximated by a diffusion equation, whilst investing processes -preferably- will first need to be rescaled by a system-size variable. Validated by this finding and supported by a multitude of theoretical considerations and statistical tests, the main conclusion of this dissertation is that an equation consisting of a linear drift function and a complete quadratic diffusion function (hereafter: "the linear-quadratic model") is a specification preferred to other specifications frequently found in the literature. These so-called benchmark processes are: the geometric and arithmetic Brownian motions, the mean-reverting Vasicek and Cox, Ingersoll and Ross processes, and the modified Square Root process. Those specifications can all be considered particular cases of the generic linear-quadratic model. The linear-quadratic model is classified as a *hybrid* model since it is shown to be constructed from the combination of geometric and arithmetic Brownian motions. The linear-quadratic specification is described by a fundamental model, rooted in well-studied and generally accepted business and financial assumptions, consisting of two *coupled*, recursive relationships between operating and investing cash flows. The fundamental model explains the positive feedback mechanism assumed to exist between the two types of cash flows. In a stochastic environment, it is demonstrated that the linear-quadratic model can be derived from the principles of the fundamental model. There is no (known) general closed-form solution to the hybrid linear-quadratic cash flow specification. Nevertheless, three particular and three approximated exact solutions are derived under not too stringent parameter restrictions and cash flow domain limitations. Weak solutions are described by (forward or backward) Fokker-Planck-Kolmogorov equations. This study shows that since the process is *converging* in time (that is, approximating a stable probability distribution), (uncoupled) investing cash flows can be described by a Pearson diffusion process approaching a stationary Person-IV probability density function, more appropriately a Student diffusion process. In contrast, (uncoupled)

operating cash flow processes are *diverging* in time, that is exploding with no stable probability density function, a dynamic analysis in a *bounded* cash flow domain is required. A suggested solution method normalises a general hypergeometric differential equation, after separation of variables, which is then transformed into a Sturm-Liouville specification, followed by a choice of three separate second transformations. These second transformations are the Jacobi, the Hermitian and the Schrödinger, each yielding a homonymous equation. Only the Jacobi transformation provides an exact solution, the other two transformations lead to approximated closed-form general solutions. It turns out that a space-time density function of operating cash flow processes can be construed as the multiplication of two (independent) time-variant probability distributions: a stationary family of distributions akin to Pearson's case 2, and the evolution of a standard normal distribution. The fundamental model and the linear-quadratic specification are empirically validated by three different statistical tests. The first test provides evidence that the fundamental model is statistically significant. Parameter values support the conclusion that operating and investing processes are converging to overall long-term stable values, albeit with significant stochastic variation of individual firms around averages. The second test pertains to direct estimation from approximated SDE solutions. Parameter values found, are not only plausible but agree with theoretical considerations and empirical observations elaborated in this study. The third test relates to an approximated density function and its associated approximated maximum likelihood estimator. The Ait-Sahalia- method, in this study adapted to derive the Fourier coefficients (of the Hermite expansion) from a (closed) system of moment ODEs, is considered a superior technique to derive an approximated density function associated with the linear-quadratic model. The maximum likelihood technique employed, proper for high-parametrised estimations, includes re-parametrisation (based on the extended invariance principle) and stepwise maximisation. Reported estimation results support the hypothesised superiority of the linear-quadratic cash flow model, either in complete (five-parameter form) or in a reduced-parameter form, in comparison to the examined five benchmark processes.

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1. A General Stochastic Continuous-time Cash Flow Model

Chapter 1 sets-out what a stochastic continuous-time cash flow model is and how it can be formulated in mathematical terms. Particular attention is given to the question if, and under what conditions, cash flow processes can be approximated by a continuous stochastic process as opposed to more general Lévy processes including jumps. Conclusions are supported by empirical evidence.

1.1. Introduction to cash flow models

A cash flow model is an abstract (usually mathematical) representation of real-world cash flows with the aim of describing and explaining how cash flows can be predicted, managed and controlled. Modelling cash flows is also useful because cash flows are often difficult to observe directly; commonly they are measured as the change in cash balances at subsequent times.

Cash flow models can be categorised according to various criteria; in this study (Figure 1-1) the dimensions of time (discrete-time versus continuous-time) and uncertainty (deterministic versus stochastic) are employed. Historically, and still in many current practical applications, a discrete deterministic model with largely subjective point estimates of cash flows, is the model of choice. Over time, other increasingly sophisticated cash flow models have emerged. These models aim at describing cash flow processes more completely and more objectively. Two developments are important to mention. The first development is the application of ordinary differential equations (ODEs) to model deterministic continuous-time cash flows. The second is the introduction of probabilistic models in which cash flows are represented as random variables governed by an appropriate probability distribution.

In the past decades these two developments have merged to advance much richer stochastic models capable of describing a wide variety of stochastic characteristics and properties. This synthesis is embedded in stochastic differential equations (SDEs) that are used to mathematically model stochastic continuous-time cash flow processes.

Figure 1-1 Categories of cash flow models

The field of stochastic continuous-time cash flow models is still at a relatively early stage of development, which is apparent from the small number of publications addressing these models relative to the other categories of cash flow models to be found in the literature. The consequent opportunity to contribute to advancement of the field and practice is the main motivator for focusing this study exclusively on stochastic continuous-time cash flow models. As mentioned before, (stochastically) predicting future cash flows is an important reason to utilise cash flow models. Following from those forecasts, is the want to better manage and control cash flows. Here one can think of, *inter alia*, the optimisation of cash balances held by firms, dividend pay-out decisions and the selection of the best funding options available to the firm. Whilst this study is primarily concentrated on models to (stochastically) predict cash flows, managing and controlling cash flows is of no less importance. An example of how cash balances can be optimised in a stochastic environment by using a particular version of the model developed in this study, can be found in J. van der Burg et al. (2018).

When modelling cash flows, it is good practice to take an accounting model as a starting point, if only because businesses report cash flows accordingly. A similar approach is advocated in, for example, Kruschwitz and Loeffler (2006). The cash flow component model adopted in this study is illustrated in Figure 1-2 above.

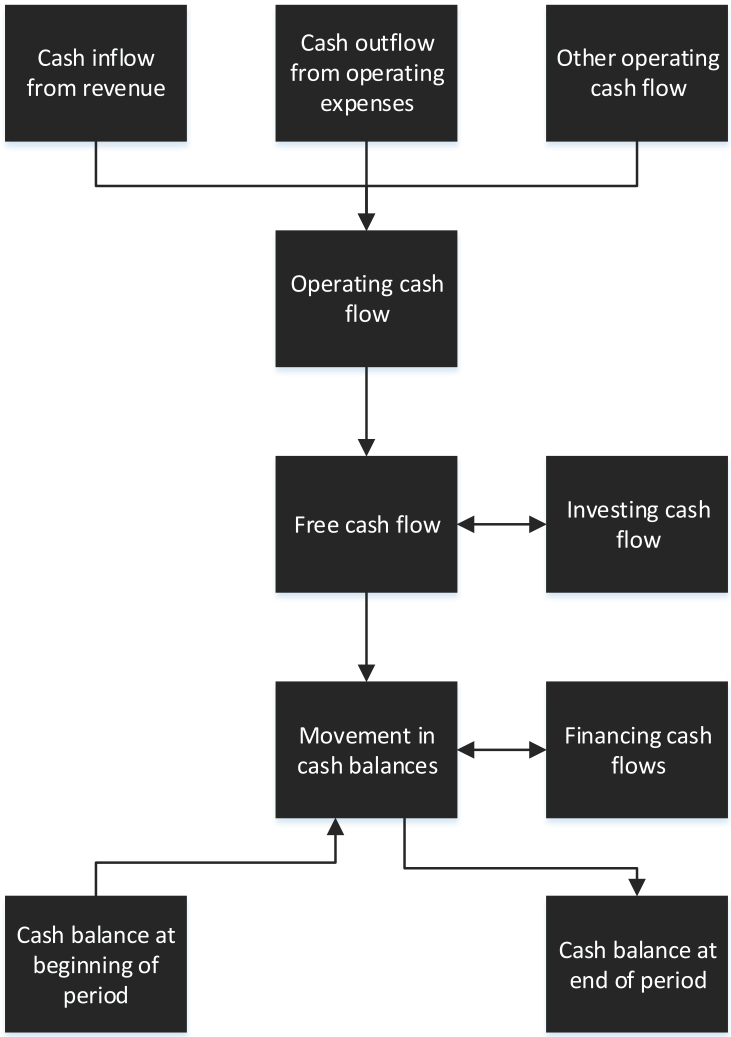

Figure 1-2 Cash flow component model used in this study

Each of the building blocks in Figure 1-2 can be seen as a separate cash flow process (or a cumulative cash flow process in the case of cash balances). In practice (discrete) cash flows are customarily predicted from a more general forecast model that explicitly describes the set of relationships between cash flow components and a myriad of underlying business and financial variables, see, for example, Fight (2005) or Tangsucheeva and Prabhu (2014) who include working capital management parameters in a stochastic cash flow model. Thus, cash flow randomness is directly related to the assessment of uncertainty in each of the underlying components. In a stochastic continuous-time environment, however, such an approach would often become mathematically and computationally burdensome. Hence, the cash flow models considered in this study describe cash flow processes in isolation (disregarding the relationships with the underlying explanatory business and financial variables) and consider randomness as the accumulation of uncertainty in each explanatory variable.

The general stochastic model specification can be expressed as: development of cash flow in time = deterministic trend + “noise” around this trend, where “noise” stands for integral randomness at an aggregate level. The risk of this approach is that the choice of the specification of the cash flow model is treated as somewhat arbitrary, without economic theory suggesting a concrete functional form for the process. As a consequence, model misspecification may yield misleading conclusions on the dynamics of the process and with the likelihood of large errors in practical applications (B. Chen and Song (2013, p. 1)). To partially avoid this critique, the focus of this study is on theoretically and empirically analysing the interdependencies between *operating cash flow* processes and *investing cash flow* processes, two important components of the cash flow model in Figure 1-2. In addition, being the resultant of these two processes, in several sections of this study *free cash flow* processes are examined because of their relevance to, for example, business valuation.

This study pursues to show the usefulness of stochastic continuous-time cash flow models. The trade-off between discrete-time and continuous-time models is discussed in detail in Section 1.3. Until then it suffices to cite Kruschwitz and Loeffler (2006) whose book deals with discrete-time cash flow models: *“(Continuous-time models) are much more popular than discrete models. But the mathematical tools required in the continuous time models are far more demanding than those which can be used in discrete time models. …… In this book we always apply the framework of a discrete time model. This is purely and solely for practical reasons*”.

Lastly, the relationship between cash flows and cash balances deserves further attention. Obviously, cash flow is a flow variable whilst a cash balance is a stock variable. In discrete time the relationship between the two is clear: the change in the cash balance is seen as cumulative cash in and out flows over a period of time. In continuous-time, however, the concept of cash flow is slightly more problematic, as will become clear in Section 1.3.

1.2. Real-world cash flows

As a preliminary to examining the concept of continuous-time cash flows, Figures 1-3 to 1-6, below, show the behaviours of cash flow processes in discrete time for a sample[1] of 16 small and large North-American firms that consistently reported cash flow data over the period 1987-2016. The sample was drawn from the dataset model for this study: 5,202 listed North-American firms with cash flow data reported during at least 20 consecutive quarters (for a detailed description of the data set used see Appendix S1).

Figures 1-3 to 1-6 (below) suggest the following conclusions:

- Operating cash flows of most firms show a strong seasonal (inter-quarter) pattern; for investing cash flows there is no corresponding seasonal pattern. The reported (dis)investment flows are significantly more irregular than the operating cash flows. However, measured over a longer period, a positive correlation between operating cash flows and (cumulative) investments seems plausible;
- Whilst the vast majority of the sample firms experienced modest growth during the reported 29 years, a few exceptions (Johnson & Johnson, Lilly (Eli) & Co) achieved high operating cash flow growth rates. One firm (ITT Inc) was significantly restructured including a massive disinvestment programme that resulted in a strong reduction of operating cash flow;
- For most (but not all) firms, the variability of operating cash flow tends to increase with time (and with cash flow size); for investing cash flows this conclusion is also mostly true but the pattern is less regular.

[1] The firms included in the sample were selected on the basis of having 118 consecutive reporting quarters to show cash flow behaviour over the longest period possible. Accordingly, the sample is biased towards firms with longevity.

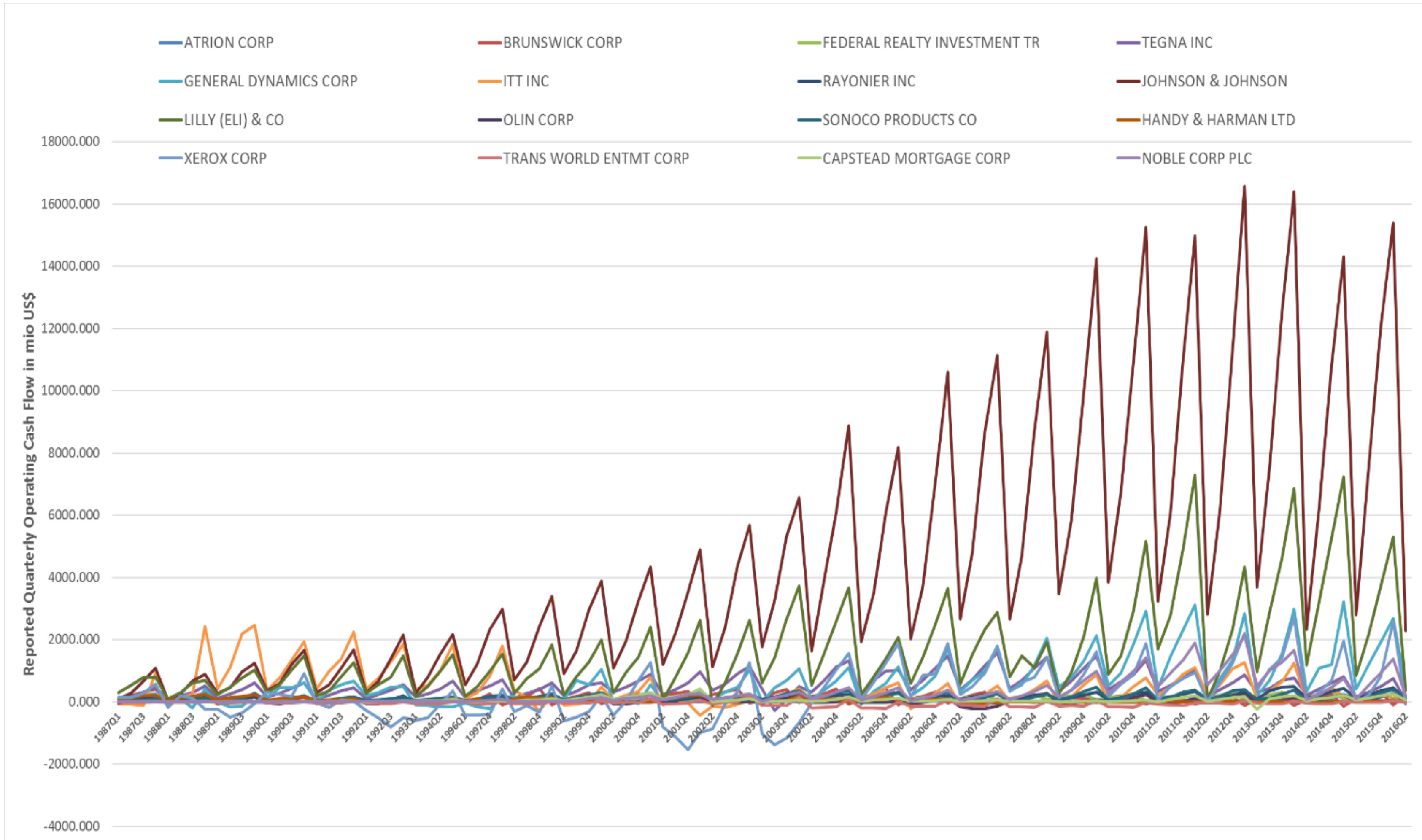

Figure 1-3 Quarterly operating cash flows of 16 North American firms, over the years 1987 – 2016

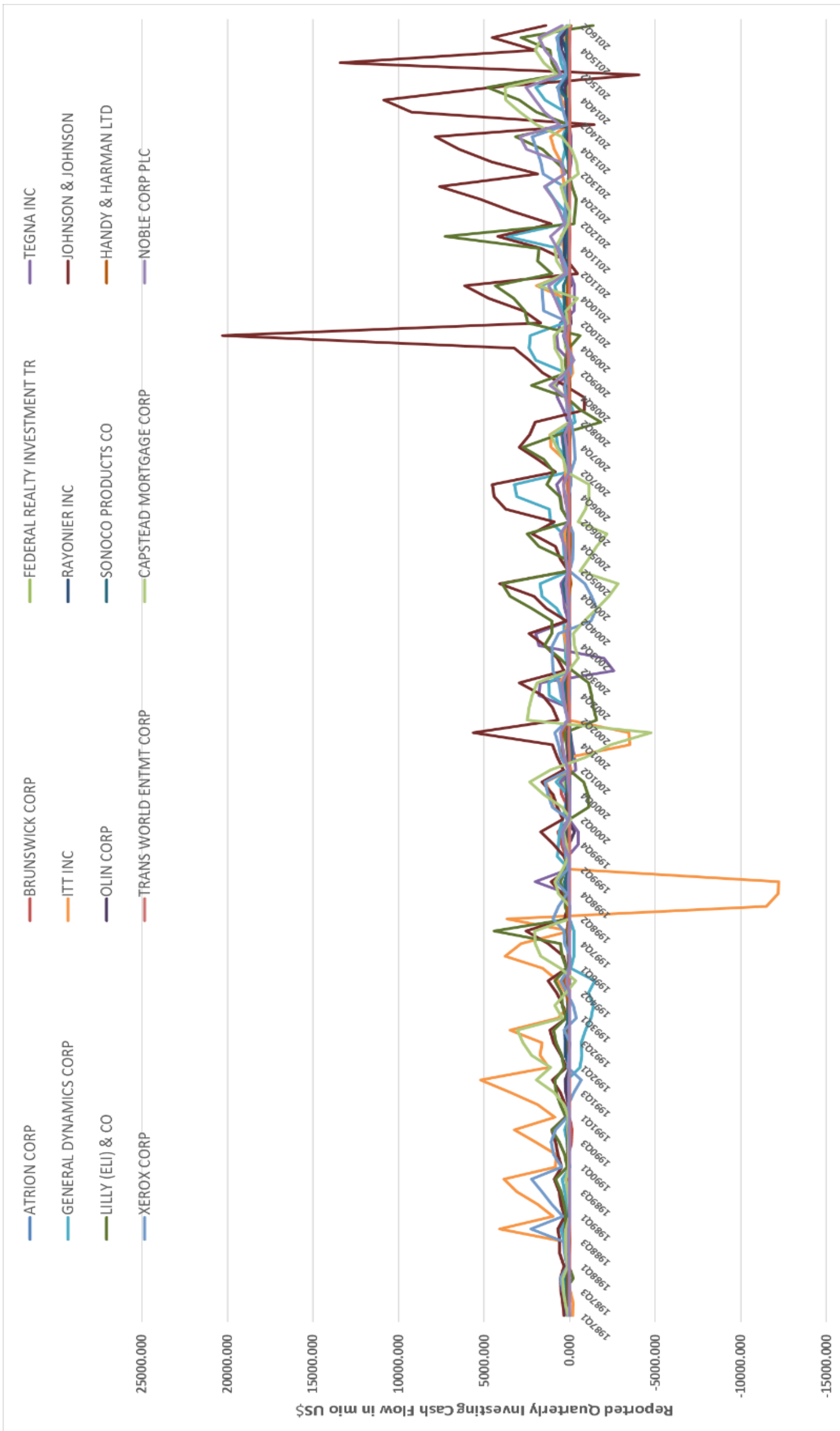

Figure 1-4 Quarterly investing cash flows of 16 North American firms, over the years 1987 – 2016

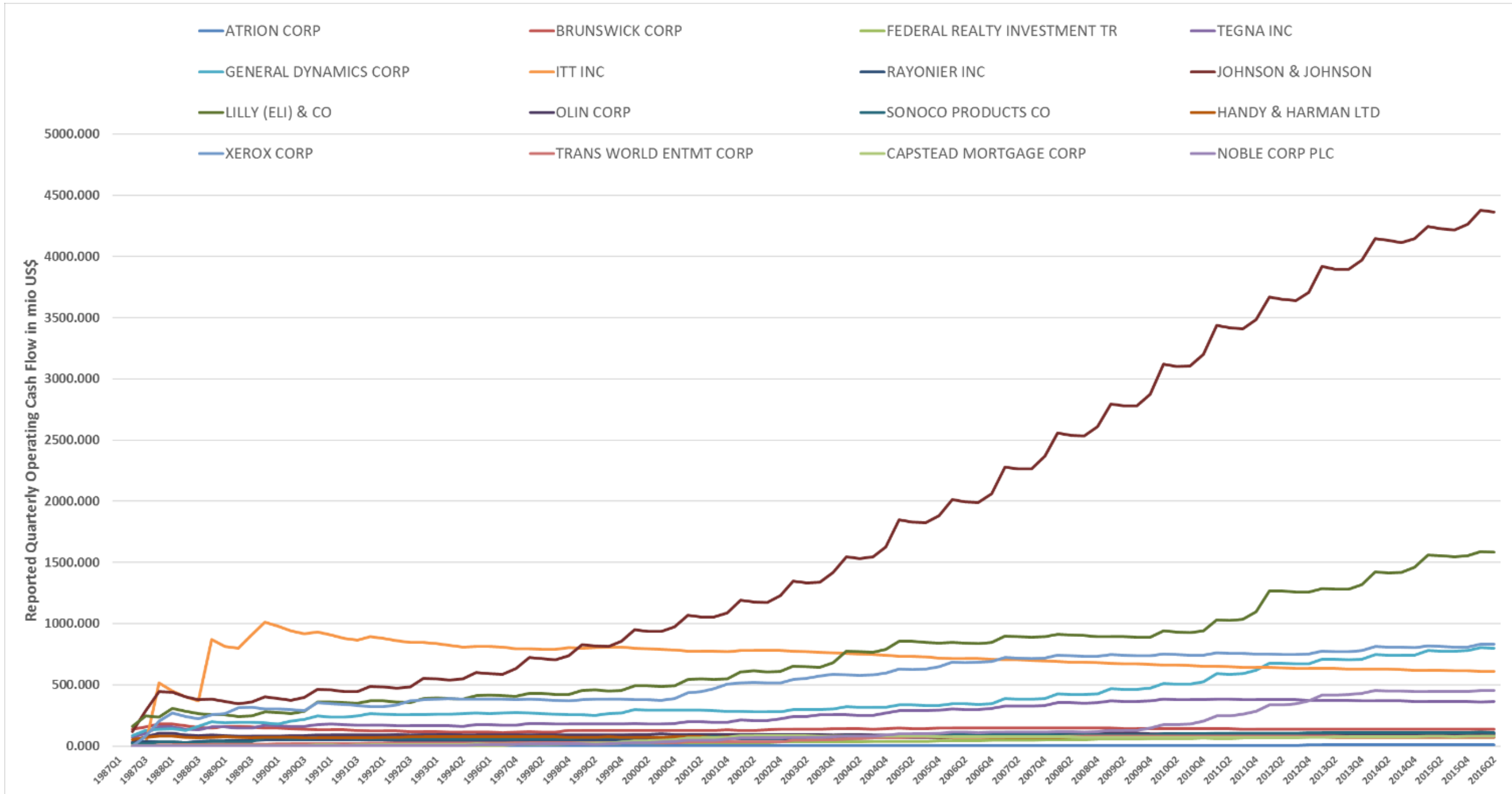

Figure 1-5 Variability (measured as cumulative standard deviation) of quarterly operating cash flows of 16 North American

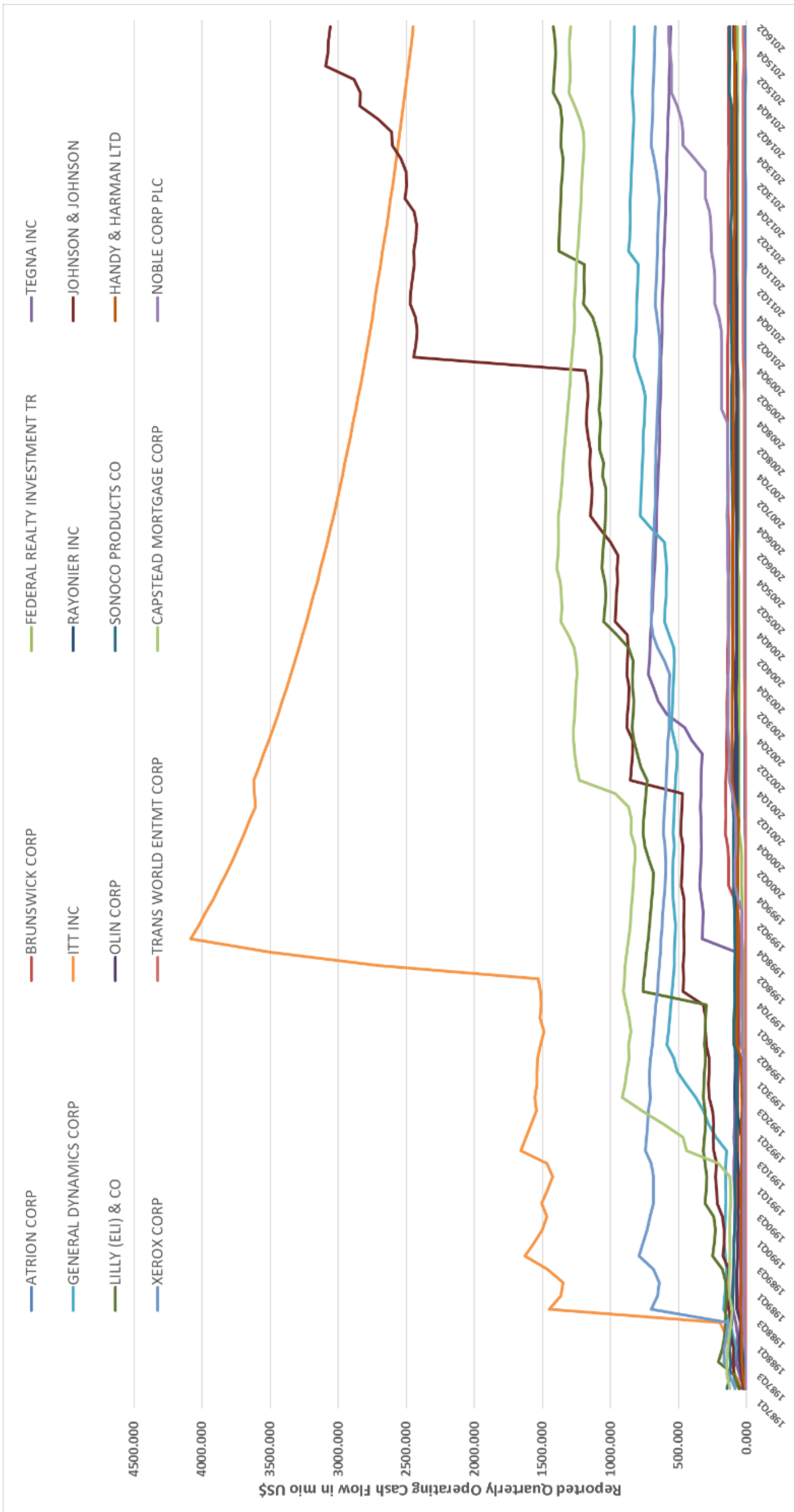

Figure 1-6 Variability (measured as cumulative standard deviation) of quarterly investing cash flows of 16 North American firms, over the years 1987 - 2016

From visually comparing Figure 1-3 to Figure 1-4 the impression is that investing cash flow follow a stochastic process that is different from that of operating cash flows. Sections 2.2. (drift function), 2.3. (diffusion function) and 2.4 (link between operating and investing cash flows) present a more rigorous analysis of some of the above observations.

1.3. Cash flow models in continuous time

The term 'continuous-time stochastic cash flows' is a *contradictio in terminis*: in reality cash flows are discrete random variables, both in amount and timing, regardless of the time-interval over which they are measured. Therefore, the question is valid why cash flows should be modelled in continuous time.

Some of the arguments made by Gandolfo (2012) can be adopted for the importance of studying cash flow models in continuous-time.

1. Discrete-time changes in cash flows and cash balances are the single-time outcomes of the interaction of a great number of other underlying stochastic variables and decisions taken by the firm's managers, all occurring at different times. Bergstrom (1990, p. 1) notes that for the financial and other aggregates controlled by large publicly listed firms "there will be thousands of small changes during random intervals of time on a single day, and the changes occur at any time during that day. A realistic aggregate model which, accurately, takes account of these microeconomic decision processes must, therefore, be formulated in continuous time". In financial markets, continuous *information flows* often provide a justification for using continuous-time models; a similar reasoning applies to cash flow processes since managers' decisions and other stochastic variables are influenced by continuously updated information flows. Hence it is appropriate to treat cash flow processes as if they were continuous.

2. The result of analysis of cash flows in discrete time can be dependent on the chosen time-interval. That is, if the model is not well-defined and consistent, the properties of stochastic variables can vary with respect to the time-length of the period being considered. Continuous-time models are not subject to these potential issues.

3. If adjustment speeds in the model are high, it may be difficult if not impossible to estimate these properly in discrete time since the set value practically coincides with the observed value over the period. Consider for example the relationship between operating cash flow and investing cash flow: most investments take a longer time before they come to fruition. However, there are also investments that almost instantaneously produce (additional) cash flows for which the adaptation time is unmeasurably small. With continuous-time cash flow models, it is always possible to obtain asymptotically unbiased estimates of the adjustment speed even for relatively long observation periods.

4. Analytically, differential equation systems in continuous time are usually more easily handled than corresponding difference systems in discrete time. Once parameter estimates are obtained, differential equations enable the forecast and simulation of sample paths regardless of the chosen time interval. Discrete models, on the contrary, cannot provide more information than included in the time-unit inherent to the data. Moreover, Cox and Miller (1977, p. 235) note that a "… useful procedure … is one of using a diffusion process to study a discrete process. This procedure is useful because mathematical methods associated with the continuum (e.g. differential equations, integration) very often lend themselves more easily to analytical treatment than those associated with discrete coordinate axes." Similarly, Karlin and Taylor (1981, p. 356) acknowledge that "A great advantage in the use of continuous stochastic differential equations versus discrete models in describing certain … economic processes is that explicit answers are frequently accessible in the continuous formulations. The dependence and sensitivity of the process on the parameters are therefore more easily accessible and interpretable. The process realisations (or expectation, variance and distributional quantities) for discrete time models rarely admit explicit representations and so their qualitative discussion is more formidable".

In continuous-time the change of the cash balance is also a continuous-time variable expressed as: $\mathrm{dCB_t} = \int_{\mathrm{t}}^{\mathrm{t+dt}} \mathrm{C_s ds}$ where $\mathrm{dCB_t}$ is the change in the cash balance at time t, $\mathrm{C_s}$ is a cash flow process (a particular solution to the corresponding SDE) with s defined on

some time interval [0, T]. The variable C_s should therefore be interpreted as cash flow *intensity*, in other words: the instantaneous change of the cash balance[2].
Importantly, cash flow intensity is a construct that cannot be measured directly, so that what actually is observed, is the integral of the cash flow process over the observation period(s). This has serious but not insurmountable implications for expressing continuous-time cash flow processes in a mathematical equation.

One of the methodological challenges that one needs to overcome is estimating the parameters of a continuous-time cash flow model with what are essentially discrete-time data (Lo (1988), Chambers (1999), Davidson and Tippett (2012, pp. 170-171)). Most firms record cash transactions on a daily basis, in larger firms on an hourly basis (and in some instances even on a minute time scale). Nonetheless, firms publically report their cash positions and flows over a quarterly if not annual time-interval. It suffices to say that if much more granular cash flow observations were available, say on a daily or even an hourly time-basis, then the stochastic behaviour of cash flow processes could be analysed far more precisely. It is the objective of this study to nevertheless demonstrate that by using continuous-time models and statistical inference from publicly available reported data, a useful contribution can be made to the knowledge of cash flow processes.

In his seminal contribution to the 'Handbook of Econometrics', edited by Griliches and Intriligator (1983), Chapter 20, Bergstrom convincingly laid the mathematical and statistical foundations for estimation of continuous-time stochastic models from discrete data that are reported on a quarterly or annual basis. These, and other methodological challenges of using continuous-time models, will be discussed in this dissertation, where appropriate.

[2] In some applications cash balances (excluding overdrafts) are used instead of cash flows. The advantage of cash balances is that by definition amounts must be positive and the data can be fitted to a wider range of models including those that admit non-negative values. In the view the author, this comes at the price of loss of information that is embedded in the composite parts of a cash balance, in particular operating and investing cash flows.

1.4. Deriving the general cash flow model

If cash flow processes are considered in continuous-time, the question arises whether cash flows can be modelled as a *continuous Markov* process (in which future data depend on the most recent realisation only). Do cash flow processes have a (short-term) memory in very small time? If so, seemingly the mathematical representation of cash flow processes will become considerably more complicated. Recall the observation in point 1. in Section 1-3, that cash flows are the single-time outcomes of the interaction of a great number of other underlying stochastic variables. Longer-term these interactions can be regarded as independent forces each acting separately on cash flows. In the very short term, however, a closer dependency and coordination is plausible and therefore the immediate history of the whole system is required to predict its probabilistic future. In Gardiner (1985, p. 45), the same issue is discussed which led the author to the conclusion that *"… there is really no such thing as a Markov process; rather, there may be systems whose memory time is so small that, on the time scale on which we carry out observations, it is fair to regard them as being well approximated by a Markov process*". Gardiner's conclusion that continuous Markov processes do exist mathematically and can reasonably approximate identified real-world processes (Gardiner provides examples from physics and chemistry), is in this study adopted for cash flows; for now this is considered a workable assumption but its validity will be more rigorously discussed in Section 6.2.

Markov cash flow processes are commonly modelled as a stochastic processes that permit cash flows c_t to increase as well as to decrease; the simplest of these processes is the well-known Birth and Death process, see for instance Karlin and Taylor (2012, p. 131). In the following, the derivation of a general cash flow model will be constructed in a few steps:

(i) first a generic n-state discrete Markov model at a firm-level will be presented;
(ii) then this model will be developed into a continuous-state firm-level model that includes (combinations of) pure diffusions and pure jumps;
(iii) after which the extended model will be elevated to a multi-firm equation; and
(iv) lastly, it will be shown how an Itô process, a specification essential to the development of the linear-quadratic cash flow model, can be derived from the prior steps.

In a discrete state-time setting an n-state-probability equation is given by

$$P(c_i, t_2) = p(c_i, t_2 \mid c_j, t_1)P(c_j, t_1) + p(c_i, t_2 \mid c_i, t_1)P(c_i, t_1) \tag{1.1}$$

where c_i is state i of an n-state discrete-time cash flow ($i \leq n$) at time t_1, and c_j is state j of an n-state discrete-time cash flow ($j \leq n$) at time t_2 with $t_2 > t_1$, $P(c_i, t_2)$ is the probability of the system having a (future) cash flow of c_i at t_2, $P(c_i, t_1)$ is the probability of the system having a (realised) cash flow of c_i at t_1, $P(c_j, t_1)$ is the probability of the system having a (realised) cash flow of c_j at t_1, $p(c_i, t_2 \mid c_j, t_1)$ is the transition probability that the cash flow changes from c_j to c_i in $\Delta t = t_2 - t_1$, $p(c_i, t_2 \mid c_i, t_1)$ is the transition probability that the cash flow remains c_n in Δt, which latter probability is equal to $[1 - p(c_j, t_2 \mid c_i, t_1)]$. Using the fact that the transition probability is the multiplication of a transition *rate* w and Δt: $p(c_j, t_2 \mid c_i, t_1) = w(c_i \rightarrow c_j)\Delta t$ and $p(c_i, t_2 \mid c_j, t_1) = w(c_j \rightarrow c_i)\Delta t$, it is well-known (Van Kampen (2011, p. 96)) that Equation (1.1) can be re-written as

$$\frac{\Delta P(c_i, \Delta t)}{\Delta t} = \sum_{j \neq i} w(c_j \rightarrow c_i)P(c_j, t_1) - w(c_i \rightarrow c_j)P(c_i, t_1) \tag{1.2}$$

where equation (1.2) describes the complete probability spectrum. Taking the limit $\Delta t \downarrow 0$, Equation (1.2) reduces to the following discrete state, continuous-time expression, also called the Master Equation (Gillespie (1992, pp. 381-383), Risken and Frank (2012, pp. 11-12))

$$\frac{\partial P(c_i, \Delta t)}{\partial t} = \sum_{j \neq i} w(c_j \rightarrow c_i)P(c_j, t_1) - w(c_i \rightarrow c_j)P(c_i, t_1) \tag{1.3}$$

Similarly, the equivalent Master Equation for $P(c_j, t_2)$ is

$$\frac{\partial P(c_j, \Delta t)}{\partial t} = \sum_{i \neq j} w(c_i \rightarrow c_j)P(c_i, t_1) - w(c_j \rightarrow c_i)P(c_j, t_1) \tag{1.4}$$

The analysis above considers only two time-steps $\{t_1, t_2\}$ to describe the dynamics of the process. The analysis to follow interposes a third time step t_3 and uses the Chapman-Kolmogorov equation to connect the conditional probabilities and describe the processes more accurately in a localised time setting (Appendix M1 provides a detailed derivation of the result). Notice that the cash flow variable c is now defined as a continuous variable and the cash flow subscripts $\{c_1, c_2, c_3\}$ denote specific values (out of a number of infinite states) that the cash flow variable can take at times $\{t_1, t_2, t_3\}$

$$\frac{\partial p(c_2,t_2 \mid c_1,t_1)}{\partial t} = \frac{-\partial\alpha(c,t)p(c_2,t_2 \mid c_1,t_1)}{\partial c} + \frac{1}{2}\frac{\partial^2\beta(c,t)p(c_2,t_2 \mid c_1,t_1)}{\partial c^2} + \int [\, J(c_2|c_3,t_2)\; p(c_3,t_2 \mid c_1,t_1) - J(c_3|c_2,t_2)\; p(c_2,t_2 \mid c_1,t_1)]\, dc \quad (1.5)$$

where:

c_1 is realised cash flows at t_1, c_2 and c_3 are future cash flows at t_2, where $t_2 > t_1$;

$p(c_2, t_1 \mid c_1, t_1)$ is the conditional transition probability between a realised cash flows c_2 at t_2 and a future cash flow c_3 at t_1 *after* a (discontinuous) jump between c_2 and c_3 at t_2;

$p(c_3, t_2 \mid c_1, t_1)$ is the conditional transition probability between a realised cash flow c_3 at t_2 and a future cash flow c_3 at t_1 *before* a (discontinuous) jump between c_2 and c_3 at t_2;

$\alpha(c, t)$ is a continuous, once-differentiable function of c and t;

$\beta(c, t)$ is a continuous, twice-differentiable function of c and t;

$J(c_2|c_3, t_2)$ is a jump function from c_3 to c_2;

$J(c_3|c_2, t_2)$ is a jump function from c_2 to c_3.

Notice that function J represents the probability distribution of the size of jumps and the expression $\int [\, J(c_2|c_3,t_2)\; p(c_2,t_2 \mid c_1,t_1) - J(c_3|c_2,t_2)\; p(c_2,t_2 \mid c_1,t_1)]\, dc$ is the Master Equation of the jump process. Also notice that the step size $c_2 - c_1$ is relatively small in comparison to step size $c_3 - c_2$ to distinguish a smooth diffusion from jump.

In Gardiner (1985, pp. 47-51) this equation is called the *differential* Chapman-Kolmogorov equation. It is powerful equation since it can model a wide variety of stochastic cash flow processes as a combination of diffusion and jump processes. The conditional probability $p(c_2, t_1 \mid c_1, t_1)$ function includes Gaussian and non-Gaussian density functions alike. The general equation has at least three separate special processes (for details see Appendix M1):

a. a deterministic process described by an ordinary differential equation $\frac{dz(c_1)}{dt} = \alpha(z(c_1), t)$ where $z(c_1)$ is a non-random function of cash flows.

b. a diffusion process represented by the generic Fokker-Planck equation $\frac{\partial p(c_2,t_2 \mid c_1,t_1)}{\partial t} = \frac{-\partial \alpha(c,t) p(c_2,t_2 \mid c_1,t_1)}{\partial c} + \frac{1}{2}\frac{\partial^2 \beta(c,t) p(c_2,t_2 \mid c_1,t_1)}{\partial c^2}$ which equation describes cash flows as a continuous process with solutions of $p(c_2, t_2 \mid c_1, t_1)$. Solutions to the generic Fokker-Planck equation are a Gaussian fluctuation $\beta(c,t)$ superimposed on a systematic drift $\alpha(c,t)$.

c. A pure jump process $\frac{\partial}{\partial t} p(c_2, t_2 \mid c_1, t_1) = \int [\, J(c_2|c_3, t_2)\; p(c_3, t_2 \mid c_1, t_1) - J(c_3|c_2, t_2)\; p(c_2, t_2 \mid c_1, t_1)]\, dc$ which, in a very small discrete time Δt, can be approximated by the following discrete equation: $p(c_2, t + \Delta t \mid c_1, t) = \delta(c_2 - c_1)\,[1 - \Delta t \int J(c_2|c_1, t_2)\, dc\;] - \Delta t\, J(c_3|c_1, t_2)$
with initial condition $p(c_2, t_2 \mid c_1, t_1) = \delta(c_2 - c_1)$ where $\delta(\,)$ is Dirac's delta function (Hoskins (2009)).

An analogy can be drawn between the physics of multiple particles and the analysis of cash flows by observing cash flows at microscopic, mesoscopic and macroscopic levels (see Figure 1-7); for a robust mathematical discussion of the topic, refer to Kotelenez (2007)). At a *microscopic* level, individual realisations of cash flows over time are the primary object of study. The emphasis is on a detailed specification of cash flow paths of individual firms, including jump processes, i.e. solutions to Equation (1.1) or its corresponding Stochastic Ordinary Differential equation ("SDE"). By implication the size of the system has to be small otherwise the system becomes unmanageably complex.

At a *mesoscopic* level the focus is expanded to include all possible paths that a cash flow process can follow. The analysis moves from a small-size system to a large-size system in which discrete movements of cash flows can be neglected and the detailed description of cash flow processes of individual firms can be approximated by a diffusion process (see b. above) representing an 'average' of all possible individual jump processes. The system can be described by a Stochastic Partial Differential Equation ("SPDE") of the Fokker-Planck type which class of equations are based on the principle of conservation of probability over time[3]. The solution of the Fokker-Planck equation is the evolution of a probability density function.

[3] At any moment in time the sum of the probabilities of all possible events should always equal 1.

When elevating the analysis to a *macroscopic* level, all firms in the ensemble under study are included. Now the emphasis shifts completely to studying the aggregate behaviour of the system ignoring cash flow variability of individual firms. A deterministic ordinary differential equation such as the one under a. above, forms an approximation of the average behaviour of the cash flow system.

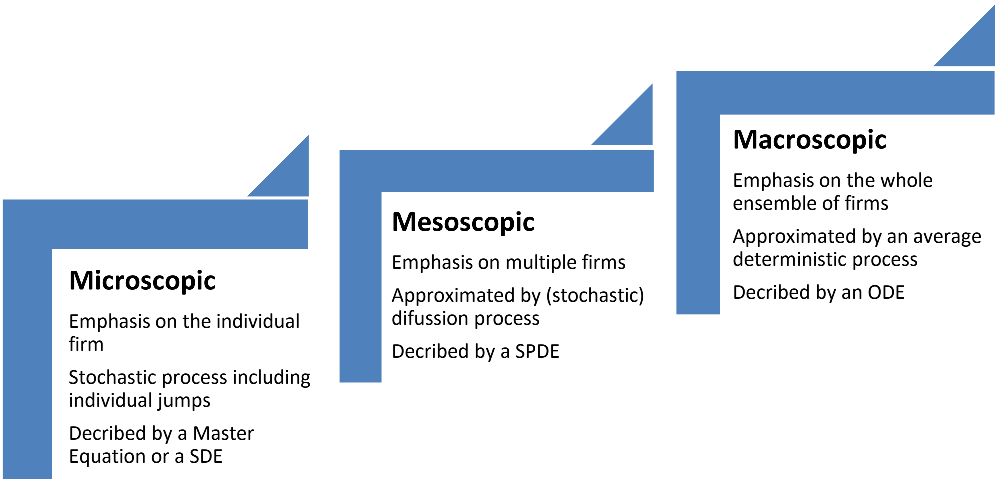

Figure 1-7 Levels of analysing cash flow processes

Unquestionably, the microscopic approach describes cash flow processes most accurately, followed by the mesoscopic and then by the macroscopic approach.

The distinction between observing cash flow processes from a single-firm or from a multi-firm perspective is important. Consider Equation (M1.7) in Appendix M1, which was used in the derivation of Equation (1.5) in this section above.

$$\frac{\partial \int f(c,t)p(c_3,t_2 \mid c_1,t_1)dc}{\partial t}$$
$$= \int \left[\frac{\partial f(c,t)\alpha(c,t)}{\partial c} + \frac{1}{2}\frac{\partial^2 f(c,t)\beta(c,t)}{\partial c^2}\right] dc \int \left[\frac{\partial f(c,t)\alpha(c,t)}{\partial c}\right.$$
$$\left. + \frac{1}{2}\frac{\partial^2 f(c,t)\beta(c,t)}{\partial c^2}\right] dc \; p(c_2,t_2 \mid c_1,t_1)$$
$$+ \iint f(c,t)\,[\,J(c_2|c_3,t_1)\;p(c_3,t_2\mid c_1,t_1)$$
$$- J(c_3|c_2,t_2)\;p(c_2,t_2\mid c_1,t_1)]\;dc_2\;dc \tag{1.6}$$

where $f(c_2)$ is replaced by $f(c,t)$. Recall that $f(c,t)$ is an arbitrary space-time function of cash flow which essentially describes the cash flow processes of multiple firms. If it is assumed that jump processes of individual firms approximately even-out and therefore can be ignored, then replacing probabilities $p(c_2, t_2 \,|c_1, t_1)$ and $p(c_3, t_2 \mid c_1, t_1)$ in Equation (1.6) by a more generally formulated conditional probability density function $p(c, t \,|c_0, t_0)$, and using the cash flow expectations operator $\mathbb{E}_c$, Equation (1.6) can be re-written as

$$\frac{\partial \mathbb{E}_c f(c,t)}{\partial t} = \mathbb{E}_c \left[\alpha(c,t) \frac{\partial f(c,t)}{\partial c} + \frac{1}{2} \beta(c,t) \frac{\partial^2 f(c,t)}{\partial c^2} \right] \tag{1.7}$$

Equation (1.7) is the *expected value* of Itô's formula applied to the arbitrary function $f(c(t), t)$ in which $dc(t)$ is specified as $dc(t) = \alpha(c,t) + \sqrt{\beta(c,t)}\xi(t)$ where $\xi(t)$ is an i.i.d. standard normal process. Using the fact that $\frac{\partial \mathbb{E}_c f(c,t)}{\partial t} = \frac{\mathbb{E}_c \, \partial f(c,t)}{\partial t}$ and abstracting out the expected value operator on both sides, Equation (1.7) becomes

$$dc(t) = \alpha(c,t)dt + \sqrt{\beta(c,t)}dW(t) \tag{1.8}$$

where $dW(t) = \xi(t)dt$.

The step from Equation (1.7) to Equation (1.8) has important implications. The arbitrary function $f(c, t)$ in Equation (1.7) represents uncountable many (future) realisations of a cash flow process as opposed to a single (but random) realisation in Equation (1.8). The two equations are made compatible by using the expected value operator in Equation (1.9) which transforms a multi-firm equation into a single-firm equation[4]. The remainder of this section further investigates single-firm cash flow specifications.

For a complete description of the general cash flow model, consider the third term of Equation (1.6) which expresses a jump process

$$\int [\, J(c_2|c_3, t_2)\; p(c_3, t_2 \,|c_1, t_1) - J(c_3|c_2, t_2)\;\; p(c_2, t_2 \,|c_1, t_1)]\, dc \tag{1.9}$$

In time dt, $p(c_2, t_2 \,|c_1, t_1)dt$ is the probability that the cash flow remains in c_2 or jumps from c_3 to c_2 with a step size governed by the probability function $J(c_3|c_2, t_2)$. Similarly, $p(c_3, t_2 \,|c_1, t_1)dt$ is the probability that the cash flow remains in c_3 or jumps from c_2 to c_3 with probability $J(c_2|c_3, t_2)$. Equation (1.9) is often modelled as a compound Poisson process with the timing of the jump *and* the jump size governed by a Poisson process.

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[4] Also known as Langevin equation

Define $\nu_1(c_t,t)dN_{1,t} = \int[\,J(c_3|c_2,t_2)\;\; p(c_2,t_2\,|c_1,t_1)]\,dc$ and $\nu_2(c_t,t)dN_{2,t} = \int[\,J(c_2|c_3,t_2)\; p(c_3,t_2\,|c_1,t_1)]\,dc$ and call the aggregated jump process $\nu_2(c_t,t)dN_{2,t} - \nu_1(c_t,t)dN_{1,t} = \nu(c_t,t)dN_t$ (1.10)

Finally, combine Equation (1.8) with Equation (1.10) to express the general cash flow process as a single-firm stochastic differential equation

$$dc_t = \alpha(c_t,t)dt + \sqrt{\beta(c_t,t)}dW_t + \nu(c_t,t)dN_t \tag{1.11}$$

In accordance with the Doob-Meyer decomposition theorem, see Doob (1990), $\alpha(c_t,t)$ represents the deterministic (predictable) component and $\sqrt{\beta(c_t,t)}dW_t$ the continuous random component. The Lévy- Itô decomposition is an extension of Doob-Meyer and relates to all terms of Equation (1.11) by combining a Brownian motion in the first two terms with a jump process in the third term in order to expand the continuous process to the more general class of Lévy processes $\zeta(c_t,t)dZ_t$[5]. (Refer to D. Applebaum (2004) and Bertoin (1998) for a general introduction to Lévy processes).

In the preceding analysis, a general cash flow process (including jumps) was employed to (indirectly) derive an Itô cash flow specification (a pure continuous-time diffusion process without jumps). Irrespectively, a Birth and Death Markov process is already sufficient to directly define an Itô-process. In a two-state Birth and Death process, state j is replaced by state $i+1$. Often, a binomial grid in discrete time (subsequent combinations of up ticks and down ticks in small-time Δt) is advanced to a continuous-time version by taking the limit $\Delta t \downarrow 0$. The system of transition rates $\{w(c_i \to c_{i+1}), w(c_{i+1} \to c_i)\}$ is an important process characteristic and for a homogeneous Birth and Death Markov process it defines the complete dynamics of the cash flow process. For instance, Risken and Frank (2012, p. 76) explain how the system of transition rates are connected to the drift function $\alpha(c(t),t)$ and diffusion function $\beta(c(t),t)$ in small Δt if the process is approximated by a diffusion equation with a one-step change

$$\alpha(c(t),t) = \Delta c[w(c_i \to c_{i+1}) - w(c_{i+1} \to c_i)]\,,$$

[5] The class of Levy processes consists of all stochastic processes with stationary, independent increments. It extends the Gaussian conditional pdf typical for Wiener processes to other probability density functions.

$$\beta(c(t), t) = \frac{\Delta^2 c}{2}[w(c_i \rightarrow c_{i+1}) + w(c_{i+1} \rightarrow c_i)] \tag{1.12}$$

For cash flow processes, the following observations can be made about the system of one-step transition rates:

1. The generation rate $w(c_i \rightarrow c_{i+1})$ is expected to be (on average for all firms) larger than the recombination rate $w(c_{i+1} \rightarrow c_i)$ which on balance ensures growing cash flows in a deterministic environment. Observed over the whole lifetime of the firm, cash flow growth is necessary for the long-term survival of firms.
2. Both the generation rate and recombination rate should be large relative to Δt, which underpins the jumpiness that cash flow processes typically display. By definition, jumps are discontinuous but under certain conditions they can be approximated by a continuous cash flow process. See Sections 1-5. and 1-6.

1.5. Approximated continuous cash flow processes – theoretical foundation

Departing from the general single-firm cash flow Equation (1.11), the question can be asked under what conditions this equation can be reduced to the following general continuous-time Itô specification

$$dC_t = \alpha(C_t, t)dt + \sqrt{\beta(C_t, t)}dW_t \tag{1.13}$$

If justified, the modelling of cash flow processes can benefit from an arsenal of well-known and already developed mathematical methods and techniques, such as the Fokker-Planck equation, which greatly increases the chances of finding tractable, analytical solutions.

Indeed, at first glance the dynamics of cash flow processes will tend to indicate jump-like behaviour. This is most evident for investing cash flow processes where the cash flow size is relatively large and the occurrence of changes is infrequent compared to that of operating cash flows. On the other hand, operating cash flows can also exhibit large fluctuations from time to time, for instance when a large sales order is paid at once.

The starting point of the analysis is to consider three different approaches to approximate a general Lévy cash flow process by a more specific Itô cash flow process. These approaches are: (i) Time-step size reduction, (ii) Diffusion approximation, and (iii) Instantaneous variance increase.

Time-step size reduction

If the size of the time-step of the cash flow process is reduced, at least for some processes, jumps become relatively smaller and the process smoother. This is the case if the Lindeberg condition (Lindeberg (1922), W. Feller (1971, pp. 93-94)) is met, i.e. that a cash flow process is continuous if for all c_t its conditional probability $p(c_2, t_2 | c_1, t_1)$ between two cash flows $\{c_1, c_2\}$ goes faster to zero than $\Delta t = t_2 - t_1$ does, or mathematically expressed: $\lim_{\Delta t \to 0} \frac{\int p(c_2, t_1 + \Delta t | c_1, t_1) dc}{\Delta t} = 0$ where the integral is taken over $|c_2 - c_1| > \varepsilon$[6]. The Lindeberg condition implies that jump sizes are infinitesimally small compared to the total size of the system or, in different words that the transition rate of the processes varies slowly with the cash flow[7]. If the Lindeberg condition is not obeyed, then the process has true (discontinuous) jumps. In reality, cash flow processes will still exhibit finite size jumps which, provided these are sufficiently small, can be approximated by a diffusion process. For instance: a monthly cash flow report shows large jagged movements but if these amounts are broken down to daily cash deposits and disbursements, cash patterns often turn into much smoother flows. Nonetheless, large (relative to the system-size) one-off amounts will still show significant discontinuities and therefore cannot be approximated by a diffusion process.

Diffusion approximation

As will become apparent in Section 1.5., most well-known stochastic cash flow specifications have a set of linear functions $\{\alpha(C_t, t), \beta(C_t, t)\}$. Introducing nonlinear functions for α and β widens the scope of this study considerably, but at the price of methodological and mathematical complications, even if the cash flow process under examination meets all the conditions to qualify as a continuous-time process.

The issue, raised in Van Kampen (2011, p. 244), is the following. If the transition rates (step functions) $w(c_1 \to c_2)$ and $w(c_2 \to c_1)$ are nonlinear, which is likely for cash flow processes,

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6 In a slightly alternative expression: for all c_t, $\int p(c_2, t_1 + \Delta t | c_1, t_1) dc = \mathcal{O}(\Delta t)$ where the integral is taken over $|c_2 - c_1| > \varepsilon$

7 It can be shown that for example a Brownian motion obeys the Lindeberg condition.

then the corresponding Master Equation cannot be solved *exactly*, and one has to resort to approximation methods of which the Fokker-Planck equation[8] is the best known. This is explained in detail in Scott (2013, pp. 128-133). When stepping-up from a microscopic (Master Equation) to a mesoscopic-macroscopic level (Fokker-Planck equation), a single-firm cash flow transforms into a multi-firm average cash flow (and its associated probability density function): $C_t \rightarrow \overline{C_t} = \int_{-\infty}^{\infty} C_t p(C_t, t)\, dC_t$. The evolution of this ensemble average is $\frac{d\overline{C_t}}{dt} = \int_{-\infty}^{\infty} C_t \frac{\partial p(C_t,t)}{\partial t} dC_t$. From the general Fokker-Planck equation it follows that the expansion of the first moment is $\frac{d\overline{C_t}}{dt} = \int_{-\infty}^{\infty} \alpha(C_t, t)p(C_t, t)\, dC_t = \mathbb{E}_t\alpha(C_t, t)$. The two preceding expressions are equal if $\frac{d\overline{C_t}}{dt} = \alpha(\overline{C_t}, t)$ or $\alpha(\overline{C_t}, t) = \mathbb{E}_t\alpha(C_t, t)$ which is only true for specifications linear in C_t. A similar reasoning can be developed for $\beta(C_t, t)$: $C_t^2 \rightarrow \overline{C}_t^2 = \int_{-\infty}^{\infty} C_t^2 p(C_t, t)\, dC_t$ and $\frac{d\overline{C}_t^2}{dt} = \frac{\partial}{\partial t}\int c^2 p\, dc = -\int c^2 \frac{\partial p\alpha(\overline{C_t},t)}{\partial c} dc + \int c^2 \frac{1}{2}\frac{\partial^2 p(\overline{C_t},t)}{\partial c^2} dc = \mathbb{E}_t\big(\beta(c, t)\big) + 2\mathbb{E}_t[c.\alpha(c, t)] = \beta(\overline{C_t}, t) + 2\alpha(\overline{C_t}, t) + 2\frac{d\alpha(\overline{C_t},t)}{dc}\mathbb{E}_t(c^2)$ which last equality is only exact if $\{\alpha(C_t, t), \beta(C_t, t)\}$ are linear functions. Therefore, if a nonlinear set of functions $\{\alpha(C_t, t), \beta(C_t, t)\}$ is transformed into the corresponding Fokker-Planck equation, it is implicitly assumed that $\alpha(C_t, t)$ is a smooth function with relatively small fluctuations around its average, i.e. $\alpha(\overline{C_t}, t) \approx \mathbb{E}_t\alpha(C_t, t)$[9] which also pertains to $\beta(C_t, t)$. If one of $\{\alpha(C_t, t), \beta(C_t, t)\}$ is nonlinear, or both functions are, a diffusion process (as described by a Fokker-Planck equation), then $\alpha(C_t, t)$ can only be an *approximation* of the true process.

Various methods have been developed to derive a proper diffusion approximation in cases of nonlinearity. Van Kampen (2011, Chapter X) discusses the Linear Noise Approximation (LNA) which rests on the assumption that the stochastic variable, in this study cash flow, can be separated into a deterministic part and a random part. Both parts relate to the system size N, and that the fluctuations of the random part scale about proportionally to the square root of the system size: $C_i = c_i N + \sqrt{N}\xi_i$ where C_i is a cash flow, c_i is a cash flow relative to the system size ('cash flow concentration') and ξ_i is a random variable. The full probability

[8] Here, it should be noted that the Fokker-Planck equation is nothing but a Kramers Mayol expansion, truncated after two terms, of the underlying Master Equation (based on the Chapman-Kolmogorov functional equation)

[9] Taking the average of a Taylor expansion of α around the average of C_t: $\alpha(C_t, t) = \alpha(\overline{C}_t, t) + \alpha'(\overline{C}_t, t)(C_t - \overline{C}_t) + \frac{1}{2}\alpha''(\overline{C}_t, t)(C_t - \overline{C}_t)^2 + \cdots$. $\mathbb{E}_t\alpha(C_t, t) = \alpha(\overline{C}_t, t) + \frac{1}{2}\alpha''(\overline{C}_t, t)(C_t - \overline{C}_t)^2 + \cdots$ and assuming that $C_t \approx \overline{C}_t$ it follows that $\mathbb{E}_t\alpha(C_t, t) \approx \alpha(\overline{C}_t, t)$.

density function of the process is replaced by the probability distribution function of the fluctuations on the macroscopic trajectory. Furthermore, Fuchs (2013, chapter 4) investigates five different methods[10], including LNA, that can be used to approximate a diffusion process. All these methods result in a de-scaled diffusion equation

$$dC_t = \alpha(C_t, t)dt + \sqrt{\frac{\beta(C_t, t)}{N}}dW(t) \tag{1.14}$$

where $C_t = Nc_t$, C_t is the original cash flow process, c_t is the de-scaled process and N (independent of C_t) is some proxy of the system size. The system size parameter N bridges the scale on which jump-like conditions do matter and the other scale on which macroscopic properties of the process are measured and jumps can be neglected. For cash flow processes, N could be represented by the firm's overall revenue or the value of total assets. Implicit in the derivation is the assumption that the transition probability of the original system scales proportionally to the probability rate of the de-scaled system: $p_N(c_2, t_2 | c_1, t_1) = N\, p(c_2, t_2 | c_1, t_1)$.

Another approximation technique, called linear approximation, is described in Socha (2007); linear approximation techniques extend to linearization of higher-order or lower-order replacement systems, and linearization of probability and spectral density functions.

Instantaneous variance increase

This approach views the question from a different angle: if it is assumed that cash flow processes are continuous and governed by a general Itô process $dC_t = \alpha(C_t, t)dt + \sqrt{\beta(C_t, t)}dW_t$, which specifications will best mimic cash flow behaviour? Alternatively worded: which drift and diffusion functions admit adequate variability around a (deterministic) trend to simulate the jump-like behaviour that is typical for cash flow processes?

[10] The other methods are: convergence of the Master Equation, convergence of the infinitesimal generator, the Langevin approach and the Kramers-Moyal expansion.

Since continuity implies that the term $\nu(C_t, t)dN_t$ is dropped from Equation (1.11), the Fokker-Planck equation can be used to express the evolution of moments, in this case the second moment[11].

$$\frac{\partial}{\partial t}\int c^2 p\,dc = \int \left[\frac{-c^2\,\partial\alpha(c,t)p}{\partial c} + \frac{c^2}{2}\frac{\partial^2\beta(c,t)p}{\partial c^2}\right]dc \tag{1.15}$$

where p is the conditional probability density function $p(c, t)$ in a continuous setting.

After integrating in parts the RHS of Equation (1.15), the evolution of the second moment is:

$$\frac{\partial}{\partial t}\int c^2 p\,dc = \mathbb{E}_t\big(\beta(c,t)\big) + 2\mathbb{E}_t[c\,\alpha(c,t)] \tag{1.16}$$

If the LHS of Equation (1.15) is re-written in terms of the variance, $\mathbb{v}_t(c) = \mathbb{E}_t(c - \mathbb{E}_t(c))^2 = \mathbb{E}_t(c^2) - \mathbb{E}_t^2(c)$, then Equation (1.16) becomes an equation for the evolution of the fluctuation of the cash flow process

$$\frac{d\mathbb{v}_t(c)}{dt} = \mathbb{E}_t\big(\beta(c,t)\big) + 2\mathbb{E}_t[(c - \mathbb{E}_t(c))\,\alpha(c,t)] \tag{1.17}$$

In the following step, the functions $\beta(c, t)$ and $\alpha(c, t)$ are approximated by a Taylor expansion to the third term around the average (deterministic trend) $\mathbb{E}_t(c)$ to (partially) circumvent the issue identified above for nonlinear $\{\alpha(c, t), \beta(c, t)\}$. This technique is described in Cox and Miller (1977, pp. 236-237) and is known as a Bartlett expansion (Bartlett (1955, p. 83)).

$$\beta(c,t) = \beta(\bar{c}) + (c-\bar{c})\beta'(\bar{c}) + \frac{1}{2}(c-\bar{c})^2\beta''(\bar{c}) + \mathcal{O}[(c-\bar{c})^3] \tag{1.18}$$

$$\alpha(c,t) = \alpha(\bar{c}) + (c-\bar{c})\alpha'(\bar{c}) + \frac{1}{2}(c-\bar{c})^2\alpha''(\bar{c}) + \mathcal{O}[(c-\bar{c})^2] \tag{1.19a}$$

and directly multiply Equation (1.19a) by $(c - \bar{c})$ to get to Equation (1.19b)

$$(c-\bar{c})\alpha(c,t) = (c-\bar{c})\alpha(\bar{c}) + (c-\bar{c})^2\alpha'(\bar{c}) + \mathcal{O}[(c-\bar{c})^3] \tag{1.19b}$$

where $\bar{c} = \mathbb{E}_t(c)$.

[11] For notational convenience C_t is written as c.

Applying the expected value operator $\mathbb{E}_t$ to Equations (1.18) and (1.19b) and then substituting the results into Equation (1.17) gives the following equation

$$\frac{d\mathbb{v}_t(c)}{dt} = \left(\beta(\bar{c}) + \frac{1}{2}\beta''(\bar{c})(\bar{c})\mathbb{E}_t(c-\bar{c})^2\right) + 2\alpha'(\bar{c})\mathbb{E}_t[(c-\bar{c})^2] = \left[\frac{1}{2}\beta''(\bar{c}) + 2\alpha'(\bar{c})\right]\mathbb{v}_t(c) + \beta(\bar{c}) \qquad (1.20)$$

Equation (1.20) can be interpreted in a mesoscopic-macroscopic context: $\bar{c}$ is the 'average' cash flow of the population of firms and only variable in t, $\alpha'(\bar{c})$ is an approximation of the first derivative of the 'average' drift function and $\beta(\bar{c})$ and $\beta''(\bar{c})$ are approximations that relate to the 'average' diffusion function.

The solution to Equation (1.20) can be expressed in the following generic form

$$\mathbb{v}_t(c) = \mathrm{K}\, e^{\int_0^t(\frac{1}{2}\beta''(\xi)+2\alpha'(\xi))d\xi} + e^{\int_0^t(\frac{1}{2}\beta''(\xi)+2\alpha'(\xi))d\xi}\int_0^t\left[\beta(\Xi)e^{-\int_0^t(\frac{1}{2}\beta''(\xi)+2\alpha'(\xi))d\xi}\right]d\Xi \qquad (1.21)$$

where K is an integration constant.

From Equation (1.21) it follows that the first RHS term, $\mathrm{K}\, e^{\int_0^t(\frac{1}{2}\beta''(\xi)+2\alpha'(\xi))d\xi}$, is the *dominant* (i.e. exponential) determinant of the cash flow variance if $\frac{1}{2}\beta''(\bar{c}) + 2\alpha'(\bar{c}) > 0$ or $\beta''(\bar{c}) > -4\alpha'(\bar{c})$. In other words, the variance of cash flow processes, where the diffusion term $\beta(\bar{c})$ is (at least) quadratic in $\bar{c}$ or the drift term is (at least) linear in $\bar{c}$, or preferably the combination of both, is significantly higher than the variance of lower order cash flow processes that are dominated only by the second RHS term of Equation (1.21).

A related, but not less important, question is under what conditions the variance of cash flow processes evolves to a steady-state. From Equation (1.20) it can be shown that a steady-state equilibrium holds if

$$\mathbb{v}_s(c) = \frac{-\beta(\bar{c})}{\frac{1}{2}\beta''(\bar{c}) + 2\alpha'(\bar{c})} \qquad (1.22)$$

where $\frac{1}{2}\beta''(\bar{c}) + 2\alpha'(\bar{c}) < 0$ since $\beta(\bar{c}) \geq 0$ and $\mathbb{v}_t(c) \geq 0$. The equation is singular at $\frac{1}{2}\beta''(\bar{c}) + 2\alpha'(\bar{c}) = 0$. Therefore, no steady state is defined if $\frac{1}{2}\beta''(\bar{c}) + 2\alpha'(\bar{c}) \geq 0$.

Combining the conclusions from Equations (1.21) and (1.22) it follows that:

1. If $\frac{1}{2}\beta''(\bar{c}\,) + 2\alpha'(\bar{c}\,) > 0$, the variance process is dominated by an exploding (diverging) exponential term;
2. If $\frac{1}{2}\beta''(\bar{c}\,) + 2\alpha'(\bar{c}\,) = 0$, the variance process is dominated by an exploding (diverging) non-exponential term with variances that are significantly below variances dominated by processes with an exponential term;
3. If $\frac{1}{2}\beta''(\bar{c}\,) + 2\alpha'(\bar{c}\,) < 0$, the variance process is dominated by a mean-reverting exponential term (converging to a constant K). The domination of the exponential term diminishes strongly over time.

Discussion

If cash flow processes can be adequately approximated by a diffusion process the resulting mathematical tractability gives enhanced prospect of finding a closed-form solution. The key question is under what conditions a better approximation can be found. Firstly, reducing the time step-size is accommodating but only if the underlying process is sufficiently continuous regardless of step-size. Secondly, re-scaling cash-flows in proportion to the system-size is another technique that can be applied (with or without process linearization). Finally, assuming that the processes can be sufficiently approximated by a diffusion process, it was shown that specifications with a quadratic diffusion term and linear drift term are superior in mimicking jump-like continuous cash flow processes over specifications with lower-order diffusion and drift terms.

1.6. Approximated continuous cash flow processes – empirical evidence

This section presents analysis of the quarterly changes to operating and investing cash flows of the 5,202 listed North American firms comprising the dataset for this study. The analysis includes Q-Q plots and a test to detect jumps. Notice that not all 5,202 firms consistently report operating cash flows and investing cash flows, or that in some instances the number of observations is too small to perform a specific statistical test.

Normal Q-Q plots

A normal Q-Q plot provides a first visual impression as to whether a process is approximately continuous, i.e. its increments are about normally distributed.

Figure 1-8 (below) clearly shows that changes in operating cash flows (left-hand plot) are strongly non-Gaussian with significant left-side and right-side fat tails. For changes in the *de-scaled* operating cash flows (right-hand plot), the tails are less pronounced but still notably fat.

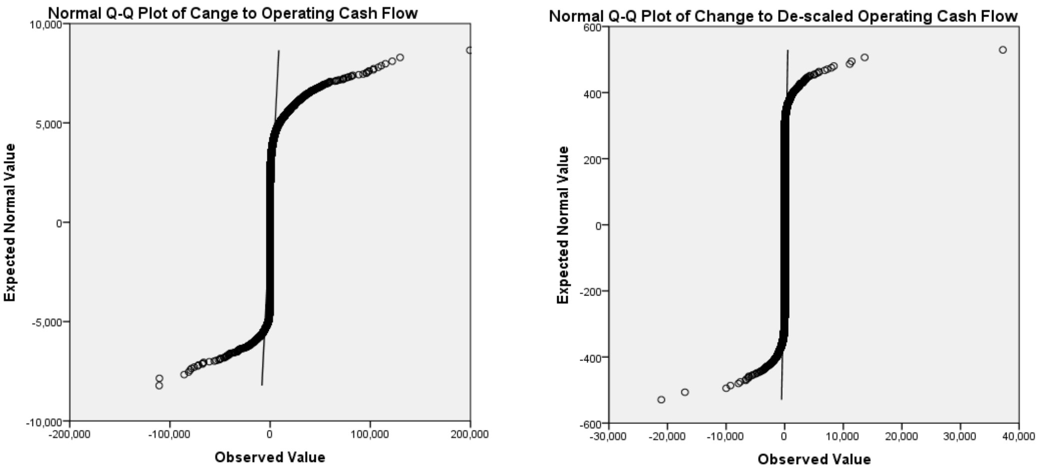


Figure 1-8 Normal Q-Q plots of Changes in Operating Cash Flow

A similar conclusion can be drawn for investing cash flows (see Figure 1-9).

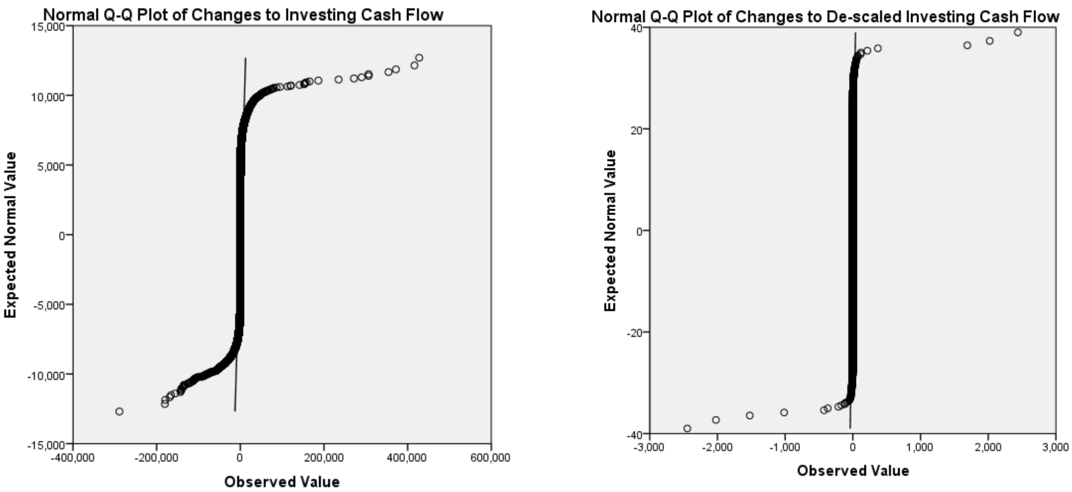


Figure 1-9 Normal Q-Q plots of Change in Investing Cash Flow

Note that the above conclusions are only valid for quarter-to-quarter observations: if a finer time grid had been used then the cash flow changes may well have been more normally distributed.

Testing for jumps

In the literature, few statistical tests are found that can be used to determine whether a stochastic process is (sufficiently) continuous, that is, excludes sizeable jumps. Most tests are parametric and require an ex-ante specification of the continuous process; an early example is Ait-Sahalia (1996)'s 'double-transformation' specification test which was applied to diffusion processes in Aït-Sahalia (2002a, 2002b). This author also examined nonparametric continuity tests in Aït-Sahalia et al. (2009) and Ait-Sahalia and Jacod (2009). Another (fairly convoluted) diffusion process test is described in B. Chen and Song (2013), which has its theoretical foundations in Barndorff-Nielsen and Shephard (2004) and practical applications are discussed in Barndorff-Nielsen and Shephard (2006).

The Barndorff-Nielsen Shephard test splits the overall bi-power variation of time series into two components: (i) a component that is contributable to a continuous process and (ii) a component that captures the specific variation of jumps. The test is explained in more detail in Appendix S2. The two test statistics (linear and ratio) must be interpreted with caution when calculated from low-frequency realised discrete-time time series since the statistics asymptotically converge slowly to a normal distribution and therefore the authors recommend using the test with high-frequency data only. In addition, restricting the test to quarterly data excludes its ability to find jumps within a quarter[12]. Notwithstanding an average number of no more than 75 data points per firm, it is presumed that the Barndorff-Nielsen Shephard test provides at least a good indication of the occurrence of significant jumps in cash flow processes.

Seemingly, from Table 1-1, operating cash flows can be reasonably well approximated by a continuous process: 30.2% of examined firms test positively for jumps at a 5% significance level which drops to 19.2% at a 1% significance level. Merely 7.8% of firms show indications

[12] If these inter-quarterly jumps balance each other out, then they will not be detected by measuring only (cumulative) quarterly jumps. In contrast, the test could detect what appears to be a quarterly jump but in reality is a series of very small jumps observed over smaller time intervals.

of significant jumps, here defined as greater than 4 times the standard deviation. For investing cash flows, the opposite conclusion is arrived at. Jumps are present as is evident from the fact that 67.2% at a 5% significance level and 58.7% at a 1% significance level of all firms exhibited jump conditions in time series, and 42.5% of firms provide indications of significant jumps, again defined as greater than 4 times the standard deviation.

Table 1-1 Barndorff-Nielsen Shephard test results for cash flows

CASH FLOW SERIES WITH JUMPS	N		FIT *		FIT **		FIT 4 σ		MEAN	STD
OPERATING CASH FLOW	5138	100%	1550	30.2%	985	19.2%	402	7.8%	218.99	1811.49
INVESTING CASH FLOW	5189	100%	3487	67.2%	3046	58.7%	2207	42.5%	195.80	3907.56

These results give rise to the following question: if jumps do matter, can this be mitigated by re-scaling cash flows as suggested in the prior section? Table 1-2 gives the answer. Re-scaling operating cash flow by revenue and investing cash flow by total assets has a slightly negative (but barely significant) impact on the jump characteristics of the cash flow processes as such; however, process *variability* as measured by standard deviation is, as expected, greatly reduced.

Table 1-2 Barndorff-Nielsen-Shephard test results for re-scaled cash flows

CASH FLOW SERIES WITH JUMPS	N		FIT *		FIT **		FIT 4 σ		MEAN	STD
OPERATING CASH FLOW	5138	100%	1750	34.1%	1205	23.5%	621	12.1%	2.87	110.30
INVESTING CASH FLOW	5189	100%	3592	69.2%	3112	60.0%	2240	43.2%	0.02	6.12

In more recent papers the full power of the Barndorff-Nielsen Shephard test to detect jumps is questioned (refer for example to Andersen et al. (2012) and Buckle et al. (2016))[13]. Therefore, the tests are rerun by amending the original Barndorff-Nielsen Shephard test statistic to include a bi-power variation term based on the *median* of five consecutive quarterly changes in cash flows (the 'modified Barnhoff-Nielsen Shephard test', see Appendix S2 for a further explanation).

Table 1-3 Modified Barndorff-Nielsen Shephard test results for cash flows

CASH FLOW SERIES WITH JUMPS	N		FIT *		FIT **		FIT 4 σ		MEAN	STD
OPERATING CASH FLOW	5141	100%	4466	86.9%	4153	80.8%	2760	53.7%	218.99	1811.49
INVESTING CASH FLOW	5188	100%	4991	96.2%	4871	93.9%	4339	83.6%	195.80	3907.56

[13] Which leads to the conclusion that currently there is no generally accepted test for jump detection.

Indeed, Table 1-3 shows that the modified Barndorff-Nielsen Shephard test is much more demanding in detecting jumps[14]. For both cash flows, as reported in Table 1-3, the modified test identifies a significantly higher number of firms with jumps: 86.9% and 96.2% (at a 5% significance level). Notice that for investing cash flow almost all jump statistics qualify as significant (greater than 4 standard deviations) while operating cash flow jumps appear to be more consistent with above.

Table 1-4 Modified Barndorff-Nielsen Shephard test results for re-scaled cash flows

CASH FLOW SERIES WITH JUMPS	N		FIT *		FIT **		FIT 4 σ		MEAN	STD
OPERATING CASH FLOW	5141	100%	4142	80.6%	3908	76.0%	2980	58.0%	2.87	110.30
INVESTING CASH FLOW	5191	100%	4997	96.3%	4924	94.9%	4521	87.1%	0.02	6.12

Table 1-4 shows that there is also a material difference in outcomes between the original and the modified Barndorff-Nielsen Shephard test when applied to re-scaled operating cash flow, albeit not as pronounced as for re-scaled investing cash flows.

Discussion

Within restrictions inherent to the test methodology, in particular the ambiguity in the power of the two test variants, it can be concluded that operating cash flow displays jumps of a relatively modest size. Significant jumps, however, are a prominent feature of investing cash flow of most firms.

The impact of jumps on estimating parameters can be markedly reduced by re-scaling cash flows (in proportion to total revenue or total asset value as proxies of a system-size variable) resulting in strongly deflated jumps which is observable from the difference in volatility. Re-scaling does not mitigate jump patterns as such, only their size, as is obvious from the reported jump statistics. Therefore, the methods to re-scale cash flows described in Section 1.5. are of importance to approximate cash flow processes by a continuous stochastic process.

[14] The point can be made that the modified Barndorff-Nielsen Shephard test may be overly demanding in a low-frequency data environment.

1.7. Conclusions from Chapter 1

After motivating the importance of continuous-time stochastic models to describe cash flow processes, a generic cash flow model that includes a diffusion and a jump component is introduced. The generic model is able to describe a broad range of stochastic cash flow behaviour. It is firmly anchored in well-known principles of a continuous-time Markov processes.

An important question that follows, is if -and under which conditions- the generic cash flow model can be expressed as a pure diffusion model, thus considerably enhancing the mathematical tractability and availability of solution techniques. Empirical testing supports the conclusion that under some not too stringent regularities, operating cash flow processes can be well approximated by a continuous process whilst investing processes will first need to be rescaled by a system-size variable, for instance when estimating model parameters, after which the incremental variance is small enough to permit such approximation. If cash flow processes are approximated by a diffusion processes, then it was shown that a specification consisting of a linear drift function and a quadratic diffusion function is an adequate general specification to mimic important stochastic properties of cash flow processes.

2. Characteristics of Stochastic Continuous-time Cash Flows

After examining cash flow specifications found in the literature, Chapter 2 discusses in detail which specifications are appropriate to describe the drift function and the diffusion function of cash flow processes. Ample consideration is given to a theoretical foundation supported by a multitude of empirical evidence. The possible (bi-causal) relationship between operating cash flow and investing cash flow is also investigated.

2.1. Cash flow specifications found in the literature

This section, discusses the cash flow specifications that are commonly found in the literature. The majority of cash flow models are continuous without addition of a jump term. In the literature, a limited number of studies consider cash flows models that include a (non-Wiener) jump process. These mixed continuous-jump specifications will be noted if they are important to cash flow models. Of the well-known cash flow models, the most important stochastic characteristics will be identified.

There are surprisingly few finance papers on stochastic continuous-time cash flow models as such. Most publications are limited to *applications* of cash flow models. Hardly any author(s) provide(s) compelling reasons for choosing a particular specification above others; often words are used like “cash flows are governed by the usual GBM”. This is remarkable because the choice of a cash flow specification has a fundamental impact on the results from analyses.

The accounting literature is another source of publications around the theme of cash flow forecasts, particularly those that refer to the quality and sophistication of analysts’ cash flow forecasts. Givoly et al. (2009) examine properties of analysts’ cash flow forecasts in relation to earnings forecasts. They find that analysts’ cash flow forecasts are less accurate than analysts’ earnings forecasts and appear to be a naïve extension of earnings forecasts. Call et al. (2013) largely contradict that view. After detailed research Call et al. (2013) conclude that when analysts prepare cash flow forecasts they attempt to adjust earnings for a number of financial variables, such as the movement in working capital and accruals, and do not merely add back depreciation. Their findings suggest that investors view analysts’ forecasts as sufficiently sophisticated.

One of the early statistical cash flow models is an ARIMA time-series model to predict quarterly cash flow from operations and is described in Brown and Rozeff (1979). A succinct overview of more recent cash flow prediction models can be found in Lorek (2014) who classifies those models into the following classes: (i) complex, cross-sectional estimated disaggregated-accrual models, (ii) parsimonious ARIMA models, (iii) disaggregated-accrual regression models, and (iv) parsimonious ARIMA models with both adjacent and seasonal characteristics. However, none of these models use stochastic continuous-time cash flow specifications. Ashton et al. (2004) is one of the few papers in the accounting literature that apply stochastic models to forecasting dividends (and hence indirectly to cash flow forecasts). Nevertheless, the remainder of this review will focus on applications of stochastic continuous-time cash flow models in the finance literature given the limited number of similar publications found in the accounting literature.

In the finance literature five specifications frequently feature as cash flow models in applications such as cash management models, capital project analysis, and business valuation:

1. Geometric Brownian Motion (GBM); $dC_t = \mu C_t dt + \sigma C_t \, dW_t$ (2.1)
2. Arithmetic Brownian Motion (ABM); $dC_t = \mu dt + \sigma dW_t$ (2.2)
3. Mean-reverting Vasicek process (Vasicek process); $dC_t = \alpha(m - C_t)dt + \sigma dW_t$ (2.3)
4. Mean-reverting Cox, Ingersoll and Ross process (square root or CIR process); $dC_t = \alpha(m - C_t)dt + \sigma\sqrt{C_t}dW_t$ (2.4)
5. Modified Square Root process (MSR process); $dC_t = \mu C_t dt + \sqrt{k_1^2 + k_2^2 C_t^2}dW_t$ (2.5)

The section below draws on selected core papers and/or books to provide a description of each of the above five specifications.

Cash flow models based on GBM

Selected core literature:

Dixit, R. K., & Pindyck, R. S. (2012). *Investment under Uncertainty*: Princeton University Press.

This book is considered one of the standard works on stochastic investment processes. In Chapter 3 a generalised Brownian motion - Itô process is defined with the GBM as an important special case. The authors show how the present value of a so called ‘profit flow’ that follows GBM, can be calculated. Interestingly, they criticise the GBM for excluding negative project values. The application of mean-reverting processes and jump processes in investment analysis is also covered.

Kanniainen, J. (2008). Can properly discounted projects follow geometric Brownian motion? *Mathematical Methods of Operations Research, 70*(3), 435-450. doi:10.1007/s00186-008-0275-0

Geometric Brownian motion is routinely used as a dynamic model of underlying project value in real option analysis. The author concludes that the conditions of a Geometric Brownian motion can only rarely be met, and therefore real option analysis should be based on models of cash flow factors rather than a direct model of project value. The paper specifies necessary and sufficient conditions for project volatility and drift to be time-varying, and shows how fixed costs can cause project volatility to be mean-reverting.

Other authors:

GBM is widely applied in Real Option Analysis; see, for example, the ground-breaking works of Brennan and Schwartz (1985) and McDonald and Siegel (1986) and also more recent papers such as Bolton et al. (2014). Cash flow processes directly described as GBM are found in the Marketed Asset Disclaimer method which was first introduced by Copeland (2001). They argue that the present value of the project’s cash flows without options is the best estimator for the market value of the project ‘‘were it a traded asset’’. To tackle the issue of negative project values, Câmara (2001) suggests that operating cash flows can be better described by a displaced log-normal distribution allowing for log-normal distributions that extend to negative cash flow values. In his dissertation about valuation of cash flows Armerin (2004) analyses the dynamics of cash flow processes specified as a GBM and as mean-reverting in their corresponding value processes. Su (2006) examines the application of a simple GBM and a GBM including a compound Poisson in the context of real option analysis. The underlying NPV process is modelled with output prices following a general Levy process. Jaimungal and Lawryshyn (2015) assume that there exists a non-tradable underlying process called the stochastic driver that determines the cash-flows and can be mapped to the distributions of cash flows specified by managers. The stochastic drivers they consider, are the GBM, the Uhlenbeck and Ornstein (1930) process and the William Feller (1951) process.

To include rare events and related jump processes, Merton (1976) develops a mixed GBM and Poisson process model, and Guimarães Dias (1999) a mixed mean-reverting and Poisson process model. A more recent paper Grenadier and Malenko (2010) proposes a mixed GBM and Poisson jump process which includes, in the context of Bayesian learning options, temporary and permanent shocks in the cash flow model.

Cash flow models based on ABM

Selected core literature:

Alexander, D. R., Mo, M., & Stent, A. F. (2012). Arithmetic Brownian motion and real options. *European Journal of Operational Research, 219*(1), 114-122. doi:10.1016/j.ejor.2011.12.023

This paper is one of the few applications of ABM to real option analysis. To model the project valuation process, the authors advocate ABM since it permits the occurrence of negative project values. They argue that variation in project values does not need to scale with the size of the project, a position they defend by referring as an example to values of capacity constrained physical assets, and hence a constant-volatility specification like ABM is more appropriate to describe project valuation processes.

Other authors:

If the valuation process is governed by an ABM process, then it can be shown that the underlying cash flow process needs to be ABM as well, see J. G. Van der Burg (2015).

Yang (2011) considers the pricing and timing of real options under partial information. The paper considers an ABM cash flow process with a (non-observable) variable drift parameter governed by a mean-reverting process.

Cash flow models based on mean-reverting processes (Vasicek and CIR)

Selected core literature:

Bhattacharya, S. (1978). PROJECT VALUATION WITH MEAN-REVERTING CASH FLOW STREAMS. *The Journal of Finance, 33*(5), 1317-1331. doi:10.1111/j.1540-6261.1978.tb03422.x

This paper appears to have been the first that applies a stochastic continuous-time cash flow model to capital budgeting decisions. The author proposes a mean-reverting stochastic cash flow process, in contrast to an extrapolative random walk cash flow process (as for example in Myers and Turnbull (1977)), on the grounds that 'in a competitive economy we should expect some long-run tendency for project cash flows to revert to levels that make firms indifferent across new investments in the particular type of investment opportunity that a given project represents rather than "wandering" forever'. The specification used in the paper is akin to the CIR model which does not permit negative cash flows. Not surprisingly, Bhattacharya comes to the conclusion that the results of the model are not as robust as hoped for as numerical simulations based on reasonable parameter values show an inaccuracy level of 8 – 10% in gross value. The author concedes that the theoretical rigour of the analysis suffered in the quest for simplicity of the model.

Other authors:

In a paper about multi-period firm valuation, T.-k. Chen and Liao (2004) combine a mean-reverting Wiener process with Poisson diffusion jumps to model the firm's abnormal changes to cash flow.

Cash flow models based on the modified square root process

Selected core literature:

Biekpe, N., Klumpes, P., & Tippett, M. (2001). Analytic solutions for the value of the option to (dis)invest. *R&D Management, 31*(2), 149-161. doi:10.1111/1467-9310.00205.

Klumpes, P., & Tippett, M. (2004). A Modified 'Square Root' Process for Determining the Value of the Option to (Dis)invest. Journal of Business Finance & Accounting, 31(9-10), 1449-1481.

In their 2001 paper the authors examine the value of investments with two underlying cash flow processes: a CIR process and an Uhlenbeck and Ornstein (Vasicek) mean-reverting random walk. Of the two processes, only the latter accommodates negative cash flows. One of the more interesting technical aspects of the paper is that it shows how a power expansion can be used to derive analytic expressions for the value of the firm's investment opportunities that otherwise would have been difficult to achieve. The paper clearly demonstrates the implied impact of different cash flow specifications on project valuation, leading often to non-trivial valuation outcomes.

The analysis is extended in Klumpes and Tippett (2004), by developing a novel specification in which the expected instantaneous change in cash flow (per time unit) scales linearly with cash flow size, and that the variance (per time unit) of the instantaneous change in cash flow scales quadratically with cash flow size. A benefit of this new specification is that negative cash flows are admissible. These assumptions are considered to be much more realistic than the ones underpinning other cash flow models. The authors call their specification the 'modified square root process'. They obtain a closed-form solution for a modified square root process with particular parameter values. Similar to Biekpe et al. (2001), this paper shows how a power series expansion can result in an approximated closed-form solution of the investment valuation function. The paper explains in a few examples the significant impact that the cash flow specification can have on the optimal investment criteria that should be applied by firms.

The Klumpes and Tippett (2004) paper is relevant to this study since it is (to the author's knowledge) the first paper that develops a cash flow specification which is based on stochastic characteristics of real-world cash flow processes.

In Chapter 9, Appendix O1, an overview is given of the stochastic characteristics of the five common cash flow processes described in this section.

2.2. Specification of the drift function

From Section 2-1, common cash flow specifications have one of the following three drift functions:

(a) $\alpha(C_t, t) = \mu$; a linear trend specification in t, also called *additive* growth process (ABM);

(b) $\alpha(C_t, t) = \mu C_t$; an exponential trend specification in t, also called *multiplicative* growth process with (i) a diverging (exploding) evolution (GBM, MSR), or, (ii) an evolution converging to a long-term value (Vasicek, CIR).

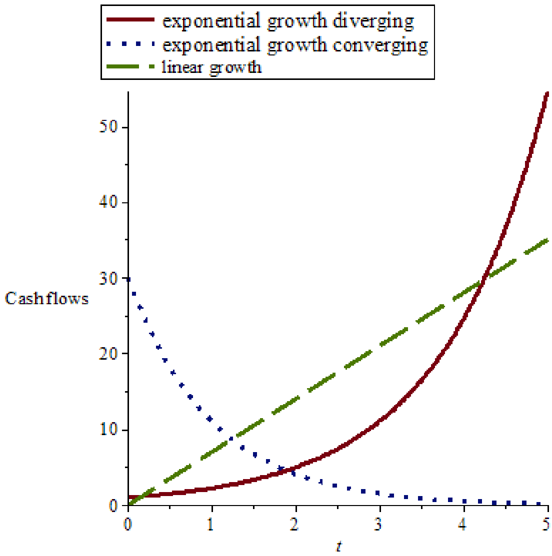


Figure 2-1 Common drift functions

This section attempts to answer two questions with regard to drift functions:

1. Are the above three specifications supported by general business research?
2. Can the aforementioned drift functions be empirically validated; i.e. do they resemble the ones found in real-world cash flow processes?

Business growth processes

If it is assumed that there exists a correlation[15] between the firm's size and its cash flow generating capacity, then the cash flow process can be linked to the business growth process.

Exponential business growth follows from the Law of Proportionate Effect, first described by Gibrat (1931), which states that the growth rate of a firm's size is proportional to the current size of the firm. Saichev et al. (2009) show that the drift function of the GBM specification is consistent with the Law of Proportionate Effect in continuous-time which implies that the instantaneous relative growth rate of the cash flow $\frac{dC_t}{C_t dt}$ in Equations (2.1), (2.3), (2.4) and (2.5) above is scale invariant. Interestingly, the authors show that the log-normal unconditional probability density function of the GBM, in long time asymptotically approximates a power function. Heavy-tailed distributions, for instance the ones related to

[15] The most likely relationship is positive time-lagged correlation provided that the firm's management is able to convert growth opportunities into additional cash flow.

power functions, provide a far better description than does the Gaussian distribution[16] for the occurrence of many multiplicative processes, such as the incidence rate of extremely successful and fast growing firms (see, for example, Sornette (2003), Rachev (2003), Malevergne and Sornette (2006) and Sornette (2007)).

Since the Law of Proportionate Effect is a compelling explanation of the firm's growth process, it has been thoroughly tested and challenged by many authors; for a detailed overview of publications including empirical research, see Santarelli (2006). Marathe and Ryan (2005) analyse how often stochastic processes (as varied as electric utility data, passenger data, cell-phone revenue data and Internet host data) follow a GBM (with constant drift and volatility). The authors conclude that the GBM occurs less frequently in practice than the literature assumes on a priori grounds. Pammolli et al. (2007) find evidence that the firm's growth rate follows a Laplace (symmetrical exponential) distribution with long tails that are more significant than predicted by Zipf's law. In other words: the vast majority of firms achieve growth rates close to zero whilst only few firms (but a larger number than predicted by a normal distribution) experience spectacular growth or decline rates. The paper shows that a Laplace distribution can be explained from modelling proportional growth in both scale (number of the same products sold) and scope (the number of different products sold) as independent stochastic processes[17]. Coad (2009) confirms that business growth rates, after adjusting for the control variables size and age, are randomly distributed with pronounced heavy tails. Growth rates not only vary substantially across firms but also the growth rate of individual firms displays variability over time with little evidence of persistent levels of growth despite continuing significant inter-firm differences in productivity, innovative capacity, and profitability. Aoyama et al. (2010) analyse growth rate data from different countries and come to the following conclusions: Gibrat's theory holds only for larger firms above a minimum size threshold, not for typical small and medium size enterprises for which the variance in the distribution for the growth-rate has a scaling relation with respect to company size.

[16] Often Gaussian distributions are explained by the Central Limit Theorem (CLT); however, this refers to the additive variant of the CLT and not the multiplicative (log transformed) CLT expression corresponding to a lognormal distribution.

[17] The normal-Laplace (NL) distribution results from convolving independent normally distributed and Laplace distributed components. It is the distribution of the stopped state of a Brownian motion with a normally distributed starting value if the stopping hazard rate is constant. See for example Balakrishnan et al. (2007), Chapter 4.

Empirical tests of common drift functions

The two common drift function specifications, outlined at the beginning of this section, were tested on this study's dataset of 5,202 listed North-American firms with cash flow data reported during at least 20 consecutive quarters (for a detailed description of the data set used see Appendix S1 and for the results of the analysis see Appendix S3).

A pure exponential growth process can mathematically only be defined on a positive range of cash flows C_t: $[0, \infty]$. Unsurprisingly, this constraint poses a serious limitation on the applicability of the multiplicative model as a single specification. Of the sample firms, no more than 511 (9.8%) have consistently positive operating cash flows during the observed period and even fewer firms, 410 (7.9%), have positive investing cash flows (no quarterly divestments on balance).

From Table 2-1 the conclusion can be drawn that for the subset of firms with positive cash flows, the exponential growth process is a good fit (measured by the F-statistic) for about 3 out of 4 firms.

Table 2-1 Exponential Growth Process - Goodness of Fit and Growth Rates

		SIGNIFICANCE LEVEL								ANNUALLY COMPOUNDED GROWTH RATE		
	N	Fit *		Fit **		Fit ***		Fit ****		Average	Max	Min
OPERATING CASH FLOW	511	409	80.0%	375	73.4%	326	63.8%	0	0.0%	9.1%	63.7%	-15.9%
INVESTING CASH FLOW	410	292	83.1%	252	74.6%	207	64.9%	0	0.0%	-7.7%	42.5%	-50.7%

The average annually compounded growth rate of operating cash flows is 9.1% in a 95% confidence interval between [8.4%, 9.8%] while the comparable numbers for investing cash flows are: -7.7% [-8.7%, -6.7%]. Of significant importance is that if cash flows are modelled as an exponential growth process, operating cash flows are *diverging* and investing cash flows *converging*, see Figure 2-1 above. A theoretical explanation of this finding will be given in Chapter 3.

A linear cash growth process is a good fit (as measured by the F-statistic) for the operating cash flow of about half of all firms, and for investing cash flow of about one third of all firms, as can be observed in Table 2-2. Recall that the ABM with an unconditional Gaussian probability distribution describes a linear growth process.

Table 2-2 Linear Growth Process - Goodness of Fit and Growth Rates

		SIGNIFICANCE LEVEL								ANNUALLY COMPOUNDED GROWTH RATE		
	N	Fit *		Fit **		Fit ***		Fit ****		Average [1]	Max [2]	Min[2]
OPERATING CASH FLOW	5202	3282	*63.1%*	2749	*52.8%*	2188	*34.5%*	1794	*34.5%*	7.9%	4512.0%	-10798.1%
INVESTING CASH FLOW	5202	2565	*49.3%*	1920	*36.9%*	1320	*25.4%*	933	*17.9%*	14.3%	4864.0%	-2091.8%

1 Defined as (last cash flow – first cash flow)/number of years   2 Calculated for firms with <1% (**) Goodness of Fit

The subset of firms with exclusively positive cash flows, was further analysed. Interestingly, a relative high proportion of those firms (71.4% for operating cash flow and 56.1% for investing cash flows) have a good fit (at a 1% significance level) with the exponential and the linear specifications alike.

Discussion

The literature on business growth processes and the results from my empirical cash flow tests suggest that none of the two examined drift functions is a superior specification. The consensus in the literature is that exponential growth is the benchmark process; however more recent papers all point into the direction of a more complex exponential process characterised by a mild form of scale variance (as opposed to a constant instantaneous relative growth rate) and significant long tails. In line with these observations, the empirical findings noted above indicate that both the exponential and the linear growth process are by and large consistent with real-world cash flow data, albeit with a diverging exponential growth for operating cash flows and a converging exponential growth for investing cash flows. Therefore, the combination of two pure processes into a composite drift function with specification $\alpha(C_t, t) = \mu_1 C_t + \mu_0$ is proposed. As will become clear in the remainder of this study, (the stochastic variant) of this composite process has properties that transcend the added properties of the two individual processes.

2.3. Specification of the diffusion function

There are at least four important determinants of a diffusion function for a cash flow process:

1. The development of the instantaneous change in cash flow variance;
2. The specification of the transition (conditional) distribution;
3. The specification of the marginal (unconditional) distribution;

4. The evolution of moments of the cash flow process.

Development of the instantaneous change in cash flow variance

Considering the general cash flow model $dC_t = \alpha(C_t, t)dt + \sqrt{\beta(C_t, t)}dW_t$, the question arises how the diffusion function $\beta(C_t, t)$, also called the the instantaneous change in cash flow variance, develops in relation to cash flow size. In the following section the assumption is made that the diffusion function is *time homogeneous*, i.e. $\beta(C_t)$ does not explicitly depend on t but only implicitly by cash flows themselves changing in time.

On theoretical grounds, it would be expected that a *quadratic* diffusion function in C_t better incorporates the jump-like volatility of most cash flow processes than a *linear* relationship with C_t (for an explanation see Section 1-5). This assumption was tested on cash flow data of 5,202 listed North-American firms by calculating the quarterly change of cash flow variance. Prior to calculating the variance, cash flows were adjusted for a linear cash flow growth trend, i.e. $\alpha(C_t) = \mu_1 C_t + \mu_0$, as proposed at the end of Section 2-2.

As is apparent from the F-tests[18] in Table 2-3, 50% of the total number of examined firms have operating cash flow data that show a good fit (at a 5% significance level) with a linear diffusion specification, whilst this percentage is 64.2% for a quadratic specification. Table 2-4 confirms a similar conclusion pertaining to investing cash flows with 50% for a linear specification respectively 71.5% for a quadratic specification (at a 5% significance level).

Table 2-3 Operating Cash Flows: Fit with Linear and Quadratic Diffusion function

OPERATING CASH FLOWS

TOTAL FIRMS EXAMINED	**5191**	***100.0%***
GOOD FIT WITH A LINEAR DIFFUSION FUNCTION		
F -TEST (*)	2596	*50.0%*
F-TEST (**)	1963	*37.8%*
GOOD FIT WITH A QUADRATIC DIFFUSION FUNCTION		
F -TEST (*)	3333	*64.2%*
F-TEST (**)	2619	*50.5%*

[18] Since the diffusion function is by nature stochastic, reporting goodness of fit measures like the R^2 statistic is not useful.

Table 2-4 Investing Cash Flows: Fit with Linear and Quadratic Diffusion function

INVESTING CASH FLOWS CASH FLOWS		
TOTAL FIRMS EXAMINED	**5197**	***100.0%***
GOOD FIT WITH A LINEAR DIFFUSION FUNCTION		
F -TEST (*)	2597	*50.0%*
F-TEST (**)	1872	*36.0%*
GOOD FIT WITH A QUADRATIC DIFFUSION FUNCTION		
F -TEST (*)	3717	*71.5%*
F-TEST (**)	2805	*54.0%*

At first glance these results may not seem to convincingly suggest a quadratic specification but one ought to take into account that the measured variance data are likely to have been influenced by the occurrence of some significant jumps (see Section 1-6).

To ascertain that the quadratic specification is indeed superior to the linear specification, further tests were performed. For each specification, the AIC ("Akaike Information Criterion") and BIC ("Bayesian Information Criterion") statistics were calculated and compared for all examined firms (Table 2-5).

Table 2-5 Comparison of AIC and BIC for Linear and Quadratic Diffusion functions

	OCF		ICF		TOT	
NUMBER OF FIRMS WITH F-TEST (*) FOR LM AND QM	**2426**	***100.0%***	**2370**	***100.0%***	**4796**	***100.0%***
AIC QM< AIC LM	999	*41.2%*	1108	*46.8%*	2107	*43.9%*
AIC QM>= AIC LM	1427	*58.8%*	1262	*53.2%*	2689	*56.1%*
BIC QM< BIC LM	960	*39.6%*	1076	*45.4%*	2036	*42.5%*
BIC QM>= BIC LM	1466	*60.4%*	1294	*54.6%*	2760	*57.5%*

Note: LM: Linear Model, QM: Quadratic Model, OCF: operating cash flows, ICF: investing cash flows

Of the number of firms that tested positively at a 5% significance level for *both* the Linear Model and the Quadratic Model, in total 43.9% have a lower AIC value for a Quadratic Model than for Linear Model, suggesting that a quadratic diffusion specification is preferred to a linear specification. With respect to BIC this percentage is 42.5%. Between operating and investing cash flows no important variances are identified.

The above results seemingly contradict the usefulness of a quadratic diffusion function. However, it should be noted that the quadratic specification does *include* a linear expression as a *special case*. This can be easily observed from $\beta(c) = \sigma_2 c^2 + \sigma_1 c + \sigma_0$ which reduces to $\beta(c) = \sigma_1 c + \sigma_0$ if parameter σ_2 is not significant. An additional t-test on the significance of each of the parameters σ_2, σ_1 and σ_0 for firms with a significant F-test (at a 5% level) of a quadratic specification, reveals the following results.

Table 2-6 Significance of parameters: results of t-test

	OCF		ICF		TOT	
NUMBER OF FIRMS WITH F-TEST (*) FOR QM	**3333**	***100.0%***	**3717**	***100.0%***	**7050**	***100.0%***
A. ONLY σ_0 SIGNIFICANT (BASED ON ABM)	666	*20.0%*	817	*22.0%*	1483	*21.0%*
B. ONLY σ_1 SIGNIFICANT (BASED ON CIR)	31	*0.9%*	0	*0.0%*	31	*0.4%*
C. ONLY σ_2 SIGNIFICANT (BASED ON GBM)	5	*0.2%*	0	*0.0%*	5	*0.1%*
D. ONLY σ_1 AND σ_0 SIGNIFICANT (BASED ON CIR PLUS CONSTANT)	754	*22.6%*	483	*13.0%*	1237	*17.5%*
E. ONLY σ_2 AND σ_0 SIGNIFICANT (BASED ON MSR)	614	*18.4%*	954	*25.7%*	1568	*22.2%*
F. ONLY σ_2 AND σ_1 SIGNIFICANT	66	*2.0%*	11	*0.3%*	77	*1.1%*
G. ALL THREE PARAMETERS σ_2, σ_1 AND σ_0 SIGNIFICANT	1197	*35.9%*	1452	*39.1%*	2649	*37.6%*

Note 1: OCF: Operating Cash Flows, ICF: Investing Cash Flows

Note 2: Results of t-test on significance of parameters under the condition that F-test agrees with quadratic specification

Note that the basis process reported in Table 2-6, for example ABM, CIR and GBM, is derived from $\sqrt{\beta(C_t)}$ and not from the instantaneous change in variance $\beta(C_t)$. The link to a process refers only to the diffusion process, not to the stochastic process including the drift function.

In Table 2-6 the totals of cases A, B and D, comprising 38.9% of all firms, agree also with a linear specification despite being classified as a quadratic specification. In contrast, cases C, E, F and G are true quadratic expressions and count for 61.1% of all investigated firms. From this analysis, in addition to the AIC and BIC statistics in Table 2-5, it becomes clear that a quadratic diffusion specification, in full or in reduced form, does agree with a significant number (4,294; lines E, F, and G aggregated in Table 2-6) of examined aggregated operating and investing cash flow processes (10,388) of firms examined.

Transition probabilities

The transition (conditional) probabilities are governed by a continuous-time Gaussian distribution which, as explained in Section 1-4, follows directly from the assumption that a cash flow process is continuous in time. Recall from Section 1-4 that the Fokker-Planck equation for transition probabilities in a *very small* discrete time Δt is

$$\frac{\partial p(c_2,t_2 \mid c_1,t_1)}{\partial t} = \frac{-\partial\alpha(c,t)p(c_2,t_2 \mid c_1,t_1)}{\partial c} + \frac{1}{2}\frac{\partial^2\beta(c,t)p(c_2,t_2 \mid c_1,t_1)}{\partial c^2} \tag{2.6}$$

Equation (2.6) is equivalent to:

$$\frac{\partial p(c_2, t+\Delta t \mid c_1, t)}{\partial t} = \frac{-\partial \alpha(c,t) p(c_2, t+\Delta t \mid c_1, t)}{\partial c} + \frac{1}{2}\frac{\partial^2 \beta(c,t) p(c_2, t+\Delta t \mid c_1, t)}{\partial c^2} \tag{2.7}$$

In Gardiner (1985, p. 53), one can find the analytic solution to Equation (2.7):

$$p(c_2, t+\Delta t \mid c_1, t) = \frac{1}{\sqrt{2\pi\beta(c,t)\Delta t}} \exp\left[-\frac{\left(\Delta c - \alpha(c,t)\right)^2}{2\beta(c,t)\Delta t}\right] \tag{2.8}$$

where $\Delta c = c_2 - c_1$.

The question can be raised how the empirical findings reported in Section 1.6. are compatible with the normal distribution governed by Equation (2.8)? Indeed, Q-Q plots show that transitional probabilities measured over a quarterly period are far from normally distributed, although de-scaled cash flows are closer to a normal distribution than unscaled cash flows. The explanation is that Equation (2.8) is valid only in very small time Δt but if Δt is aggregated over a much longer period ΔT where $\Delta T \gg \Delta t$, and Equation (2.13) in Chapter 1 is discretised over $n = \frac{\Delta T}{\Delta t}$ time increments with $\Delta t = t_{i+1} - t_i, \Delta T = t_n - t_0,\ t_0 < t_1 \ldots < t_i \ldots < t_n$, then the process is defined by the iterative relationship (see for instance Iacus (2009))

$$C_{i+1} = C_i + \alpha(C_i, t_i)\Delta t + \sqrt{\beta(C_i, t_i)}\Delta W_i \tag{2.9}$$

Chapter 3 develops a continuous-time cash flow model where $\alpha(C_i, t_i)$ is replaced by a linear equation $\mu_1 C_t + \mu_0$ and $\beta(C_i, t_i)$ by a quadratic function $\sigma_2 C_t^2 + \sigma_1 C_t + \sigma_0$. With this particular parametrisation of Equation (2.9) the CLT is not applicable if the transition probability is aggregated over ΔT. Given that the incremental cash flows $C_{i+1} - C_i$ in very small-time step Δt are normally distributed, the distribution of the aggregated increments $C_n - C_0$ in ΔT will become more leptokurtic as ΔT increases. This is caused by the quadratic term in $\beta(C_i, t_i)$ which represents a multiplicative process. The foregoing provides additional evidence that a cash flow process could well be modelled with a quadratic diffusion function.

In Section 1-5 it was stated that for a continuous stochastic process to describe a jump-like (jagged) cash flow process, it must have a relative large instantaneous variance. Equation (2.8) facilitates more precise expression of this goal. The variance of the normal distribution is described by the term $\beta(c,t)\Delta t$ and hence the instantaneous variance is $\beta(c,t)$. However, to compare firms it is necessary to normalise (de-scale) the variance and divide it by the instantaneous mean of the normal distribution: $\frac{\beta(c,t)}{\alpha(c,t)}$. Consequently, the set of functions $\{\alpha(c,t), \beta(c,t)\}$ that fit the goal are those that have a large $\beta(c,t)$ relative to $\alpha(c,t)$. The inference is that quadratic distribution functions allow for a better description of jump-like behaviour then lower-order distribution functions. This is in agreement with the conclusions from Section 1-5 (where the unconditional instantaneous variance was considered).

Marginal probabilities

The marginal (unconditional) probability density function represents the evolution of the (macroscopic) probability function of cash flows. It is a solution of the corresponding Fokker-Plank PDE. The following theoretical grounds, provide the basis for postulating the properties that the joint probability time function of cash processes should have.

A first requirement is that probability density functions must allow for negative cash flows since these are commonly observed in real-world cash flow processes. As already noted, this condition excludes the GBM (Equation (2.1), Section 2-1) and the square root process (Equation (2.4), Section 2-1) as appropriate cash flow specifications since these equations presuppose non-negative values.

A second expected characteristic is a strong leptokurtic distribution. The vast majority of firms will be modestly profitable. Yet a significant number (relative to a normal distribution) will be successful to extremely successful. Assuming that operating cash flows and investing cash flows are linked to a firms' growth path, this means that small changes happen more frequently (compared to a normal distribution) but that large fluctuations are also more likely, which is consistent with the fat tail frequency distributions found for firm's growth rates (see for example Coad (2009)).

A third requirement of cash flow specifications is right-sided skewness, which is related to the concept of *managed randomness*. This concept can best be explained by comparing it to unmanaged randomness where the outcome of a process is completely random. In that case, it is expected to find a symmetric probability distribution embodying approximate equal chances of positive and negative outcomes. However, it is sensible to assume that managers will strive to capture upside opportunities and limit downside risks: for an explanation of these ideas see, for example, Lafley and Martin (2013). In addition, there is a natural floor to the ability of the firm to fund on-going negative cash flows: financiers are seldom prepared to write a blank cheque. Right-sided skewness is measured by a positive third moment of the cash flow distribution.

As a result of the above properties, a large proportion of firms can be expected to be closely under or above the average cash flow of the population but a significant number of firms will be well above this average.

These assumptions, were tested by examining time-series of cash flows kernel densities to approximate a space-time density function. Figure 2-2 (below) displays the approximated space-time density functions of operating and investing cash flows of the study's 5,202 North America firms followed over a period of 120 quarters (from 1986 Q4 – 2016 Q3). Clearly, the space-time density functions are very peaked around the median cash flow. Further inspection of a cross section of time (in this case 1991 Q1 was selected) affirms how sharp in fact the peak is (Figure 2-3). Right-skewness is only detectable after zooming-in on the density chart (upper and lower right-hand graphs).

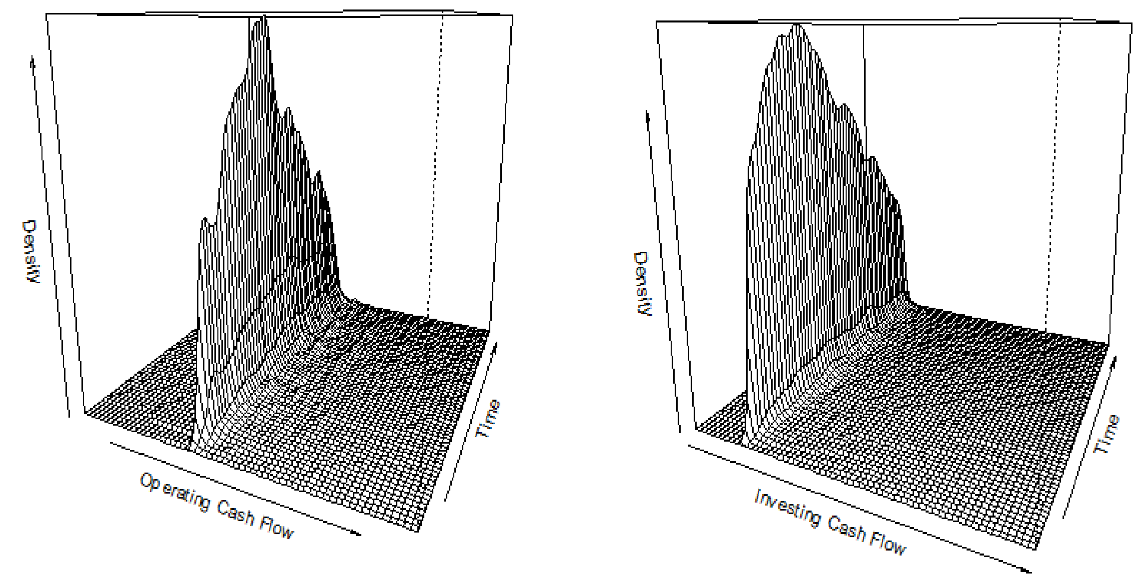


Figure 2-2 Time-density functions of Operating cash flow (L) and Investing cash flow (R)

In addition to the time cross-section of Figure 2-3, Figure 2-4 shows other (enlarged) cross-sections (1998 Q2, 2007 Q3, 2015 Q1) of the two space-time density functions of Figure 2-2.

Not only do the sequences of each cash flow exhibit significant consistency over time, but Figures 2-3 and 2-4 confirm that the space-time density functions of operating cash and investing cash flows also show a great deal of similarity which suggests that the underlying stochastic processes are fairly analogous. This conclusion is interesting and will play an important part in the development of a new cash flow model in Chapter 3. As expected, the tails of each space-time density function tend to become fatter as time progresses, an effect that is more pronounced for the right-side tails than for the left-side.

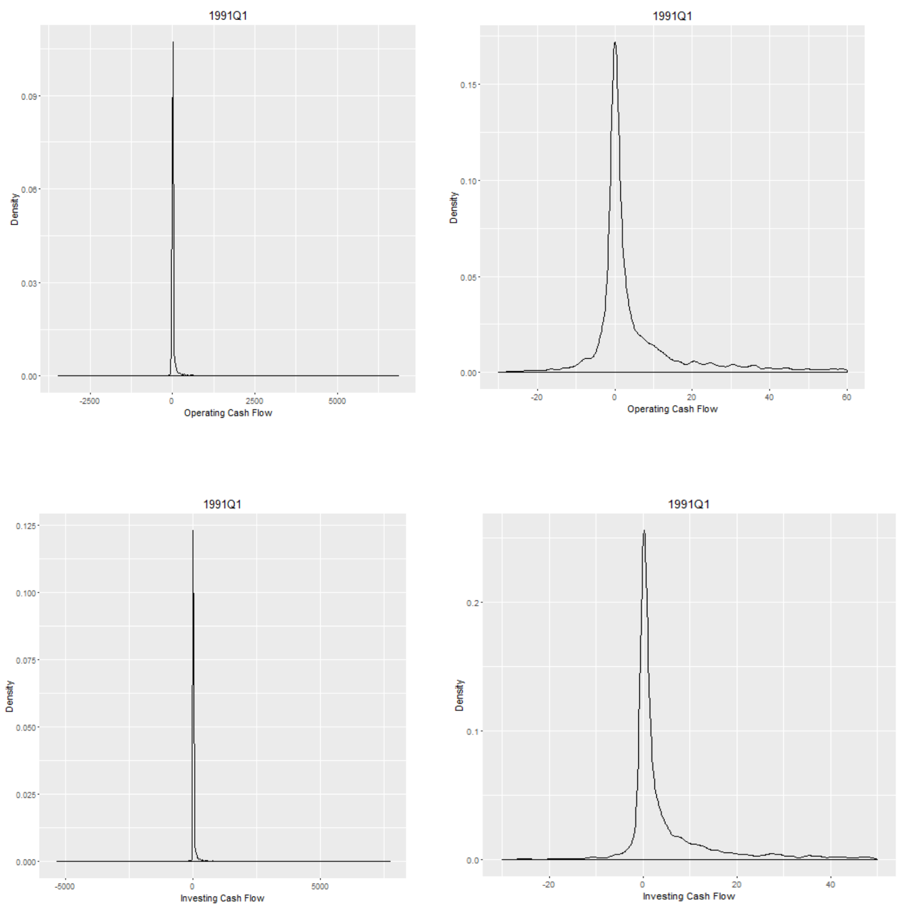


Figure 2-3 Time cross-sections of the time-density functions of Operating and Investing cash flows (left-side graphs are full-scale, right-side graphs are increased-scale)

The resulting next question is: "which of the well-known continuous distributions (if any) does fit the sample real-world cash flow data best?" A closely matching distribution can reveal important information about stochastic properties of cash flow processes. Only a limited number of continuous distributions do admit values in the full range of $\mathbb{R}\ [-\infty;\ \infty]$, (refer to Krishnamoorthy (2016)).

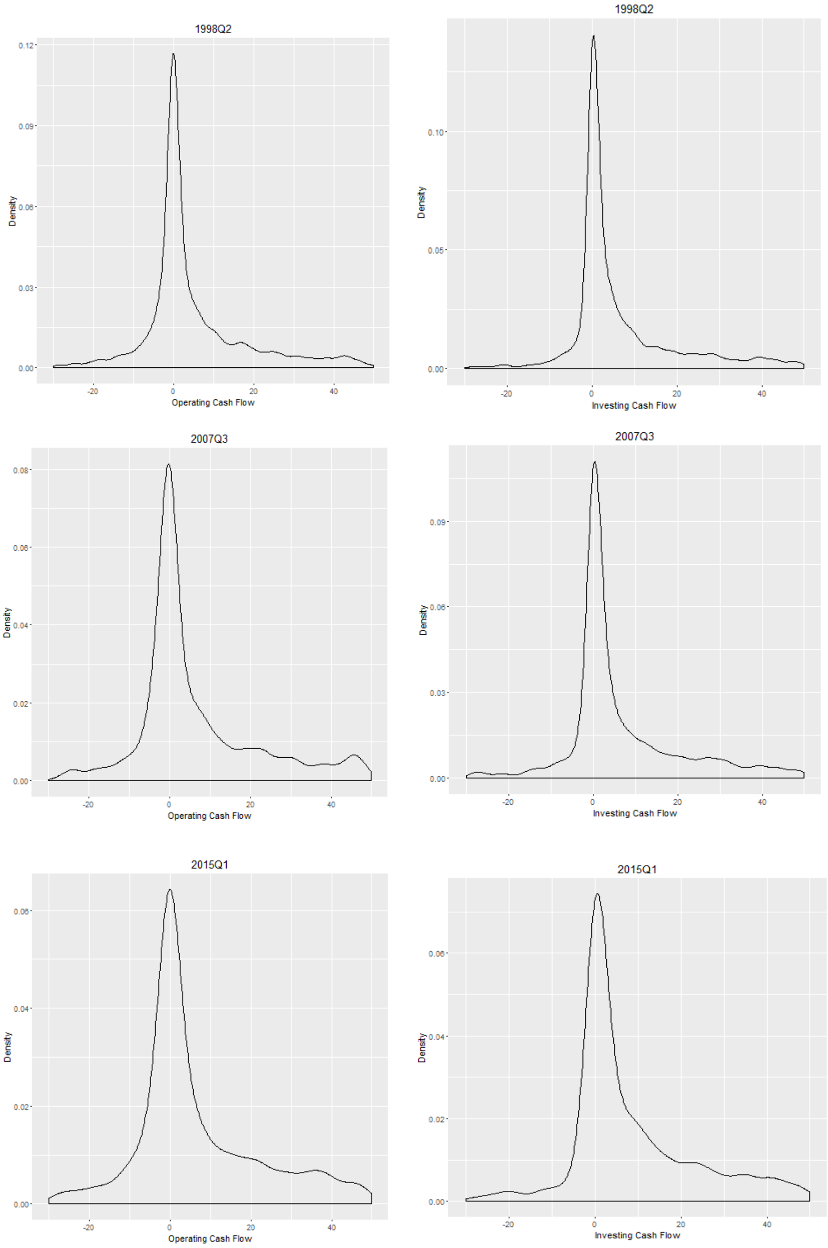


Figure 2-4 Time cross-sections of the time-density functions of operating and investing cash flows (constant scale)

In stochastic models for firm growth[19], for example, Amaral et al. (2001), Bottazzi and Secchi (2003a) and Bottazzi and Secchi (2003b), a good fit with the Laplace distribution is consistently identified. The shape of the distribution is explained from competition amongst firms whose market success is cumulative or self-reinforcing[20], leading to phenomena such as economies of scale, economies of scope, network externalities, and knowledge accumulation. In addition, the Laplace distribution has a strong connection with the Law of Proportionate Effect as discussed in Section 2-2.

The left-hand side graphs of Figure 2-3 seem to coincide with a (two-sided exponential) Laplace distribution with its distinctive convex tent shape and (very) heavy tails explained from strong cumulative or self-reinforcing effects. Closer inspection of the right-hand side graphs (increased scale) in Figure 2-3, however, reveals at least two anomalies with a Laplace distribution: a convex-concave shape, distantly comparable to a bell-shaped distribution, and a mild right-sided skewness (as expected from the concept of managed randomness). Accordingly, the initial approach is to advocate a different set of unconditional distributions that is more appropriate to describe cash flow randomness, namely the family of Pearson distributions. There is a particular reason to use this family of distributions: the general Pearson distribution equation can be directly derived from a Fokker-Planck equation with $\alpha(c,t)$ defined as linear in c and $\beta(c,t)$ as quadratic in c. The combination of a linear drift function and quadratic diffusion function are preferred functions to describe some important stochastic characteristics of cash flow processes.

If a stationary solution to the Fokker-Planck equation exists, it satisfies $\frac{\partial p(c,t)}{\partial t} = 0$ or equivalently the forward equation

$$\frac{-\partial \alpha(c,t) p_{st}(c,t)}{\partial c} - \frac{1}{2}\frac{\partial^2 \beta(c,t) p_{st}(c,t)}{\partial c^2} = 0 \qquad (2.10)$$

Meerschaert and Sikorskii (2012, chapter 7) show that integrating Equation (2.10) once, yields:

$$\frac{d\beta(c,t) p_{st}(c,t)}{dc} - 2\alpha(c,t) p_{st}(c,t) = K \qquad (2.11)$$

where K is an integration constant.

[19] This presupposes that the distribution of growth rates translates into a similar distribution of firm sizes which in turn are related to the levels of operating cash flow and investing cash flow.

[20] Which behaviour can be statically modelled by Polya's urn scheme.

Equation (2.11) reduces to $\frac{\frac{dp_{st}(c,t)}{dc}}{p_{st}(c,t)} =$

$$\frac{2\alpha(c,t) - \frac{d\beta(c,t)}{dc}}{\beta(c,t)} \tag{2.12}$$

for $K = 0$.

Assuming a linear drift function $\alpha(c,t) = \mu_0 + \mu_1 c$ and a quadratic diffusion function $\beta(c,t) = \sigma_2 c^2 + \sigma_1 c + \sigma_0$, Equation (2.12) becomes

$$\frac{p'_{st}}{p_{st}} = \frac{2(\mu_1 - \sigma_2)c + (2\mu_0 - \sigma_1)}{\sigma_2 c^2 + \sigma_1 c + \sigma_0} \tag{2.13}$$

where $p'_{st} = \frac{dp_{st}(c,t)}{dc}$. Equation (2.13) is called the K. Pearson (1893, 1894, 1901, 1916) differential equation and the solutions describe the family of Pearson distributions. The general solution to ODE (13) is

$$p_{st}(c) = K_1 \exp\left[\int \frac{2(\mu_1 - \sigma_2)c + (2\mu_0 - \sigma_1)}{\sigma_2 c^2 + \sigma_1 c + \sigma_0} dc\right] \tag{2.14}$$

where K_1 is a normalisation constant. For cash flow processes two distinctive classes of solutions to Equation (2.14) are important:

1. solutions obeying $D < 0$ with complex roots, and
2. solutions obeying $D \geq 0$ with one or two real roots, where $D = \sigma_1^2 - 4\sigma_0\sigma_2$ is the discriminant of the diffusion function $\beta(c,t)$.

First, examine *case 1* and apply the following transform: $c' = c + \frac{\sigma_1}{2\sigma_2}$. In addition, define a new variable $\lambda = \frac{\sqrt{4\sigma_0\sigma_2 - \sigma_1^2}}{2\sigma_2}$. Note that since $D < 0$, $4\sigma_0\sigma_2 - \sigma_1^2$ will always be greater than zero. Making these substitutions, Equation (2.14) becomes

$$p_{st}(c') = K_1 \exp\left[\int \frac{2(\mu_1 - \sigma_2)c' - \frac{\sigma_1}{\sigma_2}\mu_1 + 2\mu_0}{\sigma_2(c'^2 + \lambda^2)} dc'\right] \tag{2.15}$$

Consequently, the general three-parameter quadratic equation $\sigma_2 c^2 + \sigma_1 c + \sigma_0$ is now mapped to a two-parameter quadratic equation $\sigma_2(c'^2 + \lambda^2)$. Integrating Equation (2.15) leads to the following composite trigonometric function:

$$\mathrm{p_{st}}(c') = \mathrm{K_1}\exp\left[\nu_1 \ln(c'^2+\lambda^2) + \nu_2 \tan^{-1}(\frac{c'}{\lambda}) + \mathrm{K_2}\right] = \mathrm{K_3}(c'^2+\lambda^2)^{\nu_1}\exp\left[\nu_2\tan^{-1}(\frac{c'}{\lambda})\right] \quad (2.16)$$

where $\nu_1 = \frac{\mu_1-\sigma_2}{\sigma_2}$, $\nu_2 = \frac{2\mu_0-\frac{\sigma_1}{\sigma_2}\mu_1}{\sigma_2\lambda}$, $\lambda > 0$ and $\mathrm{K_3} = \mathrm{K_1}\exp(\mathrm{K_2})$ is a normalising constant.

Transforming c' back to c provides the stationary distribution function for c:

$$\mathrm{p_{st}}(c) = \mathrm{K_3}\left[\left(c+\frac{\sigma_1}{2\sigma_2}\right)^2 + \lambda^2\right]^{\nu_1} \exp\left[\nu_2\tan^{-1}\left[\frac{c+\frac{\sigma_1}{2\sigma_2}}{\lambda}\right]\right] \quad (2.17)$$

After normalisation, Equation (2.17) becomes the basis equation for the Pearson Type IV distribution which has support on the full cash flow range $\mathbb{R}$ (for a more detailed description of the Pearson Type IV distribution, see for example Heinrich (2004)).

Case 2 implies that Equation (2.14) can be re-written to:

$$\mathrm{p_{st}}(c) = \mathrm{K_1}\exp\left[\int \frac{2(\mu_1-\sigma_2)c + (2\mu_0-\sigma_1)}{\sigma_2(c-\lambda_1)(c-\lambda_2)}dc\right] \quad (2.18)$$

where $\lambda_{1,2} = \frac{-\sigma_1 \pm \sqrt{\sigma_1^2 - 4\sigma_0\sigma_2}}{2\sigma_2}$ are real roots of the quadratic diffusion function.

Integration of Equation (2.18) results in the following expression:

$$\begin{aligned}\mathrm{p_{st}}(c) &= \mathrm{K_1}\exp[\pi_1\ln(c-\lambda_1) - \pi_2\ln(c-\lambda_2) + \mathrm{K_4}] = \mathrm{K_5}[(c-\lambda_1)^{\pi_1}(c-\lambda_2)^{-\pi_2}] \\ &= \mathrm{K_5}\left[(c-\lambda_1)^{-(a\lambda_1+b)\nu_3}(c-\lambda_{12})^{(a\lambda_2+b)\nu_3}\right] \\ &= \mathrm{K_6}\left[(1-\frac{c}{\lambda_1})^{-(a\lambda_1+b)\nu_3}(1-\frac{c}{\lambda_2})^{(a\lambda_2+b)\nu_3}\right] \end{aligned} \quad (2.19)$$

where $\pi_1 = \frac{-2\lambda_1(\mu_1-\sigma_2)-(2\mu_0-\sigma_1)}{\sigma_2(\lambda_2-\lambda_1)}$, $\pi_2 = \frac{2\lambda_2(\mu_1-\sigma_2)+(2\mu_0-\sigma_1)}{\sigma_2(\lambda_2-\lambda_1)}$, $a = 2(\mu_1-\sigma_2)$, $b = (2\mu_0 - \sigma_1)$, $\nu_3 = \frac{1}{\sigma_2(\lambda_2-\lambda_1)}$, $\lambda_2 \geq \lambda_1$, $\mathrm{K_5} = \mathrm{K_1}\exp(\mathrm{K_4})$ and $\mathrm{K_6} = {\lambda_1}^{(a\lambda_1+b)\nu_3}{\lambda_2}^{-(a\lambda_2+b)\nu_3}\mathrm{K_5}$ and can be interpreted as a normalising constant. In contrast to Case 1, cash flows are defined on a restricted domain. Only if the roots of the diffusion function $\sigma_2c^2 + \sigma_1c + \sigma_0$ are of opposite sign then c is consistently defined on the range between the lower and upper root: $\lambda_1 \leq c \leq \lambda_2$.

After normalisation, Equation (2.19) forms the basis for distributions as versatile as Pearson distribution Type I (generalised beta distribution), II (generalised symmetric beta distribution), III (gamma or chi square distribution), V (inverse gamma distribution) and VI (beta prime distribution or F-distribution), all belonging to the wider family of Pearson distributions, see Crooks (2017), and refer to Figure 2-5 below.

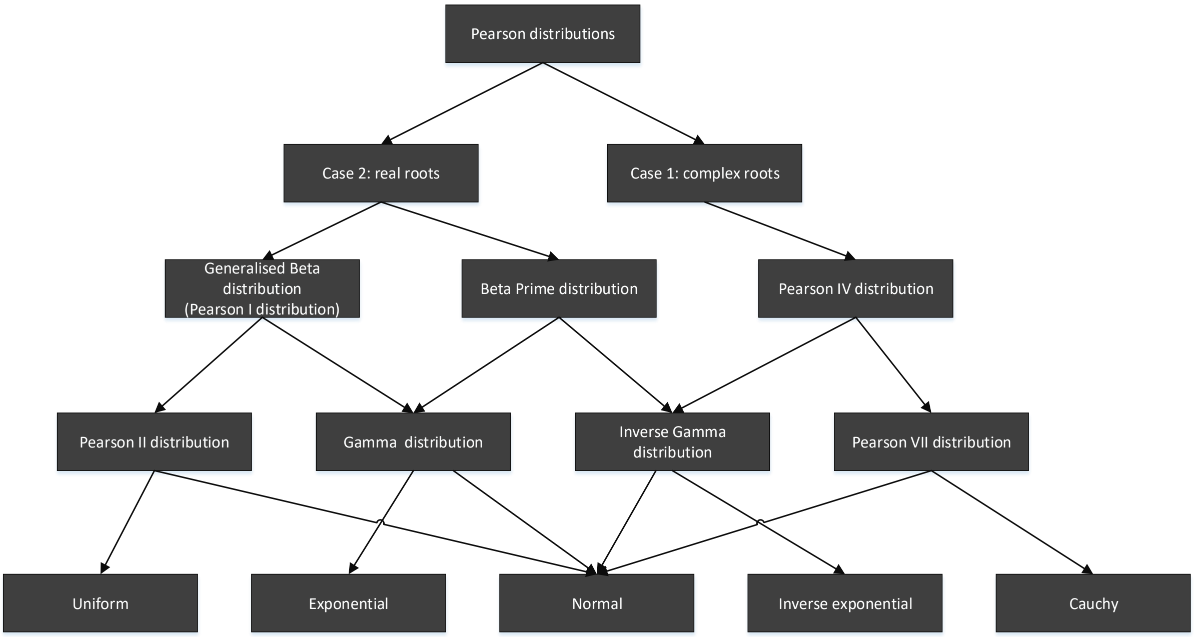


Figure 2-5 Hierarchy of Pearson distributions (lower-ranked distributions are specialties of higher-ranked general distributions)

In order to analyse the occurrence of Case 1 and Case 2 distributions, cash flow data grouped into 114 quarters, was fitted to the general Pearson equation represented by Equation (2.13). The estimation method applied is the Method of Moments; the Pearson distribution has the convenient feature that its parameters can be directly expressed in the first four moments; for the method used, see Andreev et al. (2005). One of the challenges encountered is approximating a stationary distribution for all examined 114 quarters and this was achieved by normalising the cash flow variables. The following transform was applied: $c'_n = \frac{c_n - \mu_n}{\sigma_n}$ where n is the number in the sequence of the quarters, μ_n is the average of cash flows of quarter n, and σ_n is the standard deviation of cash flows of quarter n. Details of the analysis are reported in Appendix S4.

Table 2-7 Results of analysing the roots of the Pearson distribution

	OPERATING CASH FLOW			INVESTING CASH FLOW		
TOTAL N. OF QUARTERS	**114**	**100.0%**		**114**	**100.0%**	
N. OF QUARTERS D < 0	15	13.2%		111	97.4%	
N. OF QUARTERS D>=0	**99**	**86.8%**	**100.0%**	**3**	**2.6%**	**100.0%**
OF WHICH N. OF QUARTERS WITH ROOTS OF OPPOSITE SIGN[21]	99		100.0%	3		100.0%
OF WHICH N. OF QUARTERS WITH ROOTS OF SAME SIGN	0		0.0%	0		0.0%

Table 2-7 describes an interesting contrast between operating and investing cash flows. Almost without exception the roots of investing cash flows are complex roots, typical for a cash flow process that follows a Pearson IV distribution. Unlike investing cash flows, operating cash flows of the majority of quarters are better modelled by a probability distribution defined on a restricted domain.

A possible explanation for this finding is that operating cash flow typically show a diverging exponential growth process whereas investing cash flows tend to follow a converging exponential growth process (Section 2.2). A condition for a stationary probability distribution is that the underlying stochastic process must converge in time. Recall the following conclusion drawn in Section 1.5. For a set of drift and diffusion functions $\{\alpha(C_t, t), \beta(c, t)(C_t, t)\}$ the following inequality must approximately hold for a finite stationary variance : $\frac{1}{2}\beta''(\bar{c}) + 2\alpha'(\bar{c}) < 0$. Applied to a linear drift function and a quadratic diffusion function, in this study used to describe cash flow processes, it can be shown that the variance condition translates to: $\mu_1 < -\frac{1}{2}\sigma_2$. If this condition is combined with Equation (2.13) then the ratio $\frac{p'_{st}}{p_{st}}$ [22] becomes smaller with increasing cash flows, a prerequisite for a stable conditional probability distribution. Hence, there is no reason to place *a priori* domain restrictions on the probability distribution of stationary processes. By implication, diverging processes have no stationary distribution and therefore a *bounded* rather than an *unbounded* diffusion process is more appropriate to describing their intertemporal dynamics. This issue will be analysed more comprehensively in Chapter 4.

[21] Roots of an opposite sign, i.e. $\lambda_1 < 0 < \lambda_2$, are characteristic for the Pearson Type I distribution.

[22] This ratio can be interpreted as the relative change of probability density per unit cash flow.

Evolution of moments

This subparagraph examines the evolution of the first four moments of the time-density functions of operating and investing cash flow. Contingent on prior observations, the following is expected to be found:

- The evolution of the first moment will depend on macro-economic growth and prosperity indicators that reflect the movement of the overall business cycle (for the dataset used in this study: the North American cycle);
- The second moment (here measured as standard deviation) will be proportionally related to the evolution of the mean. This follows from the assumed constant risk (variability) per dollar cash flow;
- A consistently modest positive third moment (here measured as the usual skewness statistic); and
- A significant fourth moment (here measured as the usual kurtosis statistic) in all observed quarters, which follows from the pronounced leptokurtic shape of the time-density functions (Figure 2-2).

Figure 2-6 (below) provides the development of the aggregated first moment for all the study's 5,202 North American firms[23]. Evidently, the graph shows the impact of the GFC (2008Q3) on the trend line of both cash flows; however, the break in the trend is markedly stronger for investing cash flow then for operating cash flow. A plausible explanation is that most firms consider investments largely as discretionary spending and with the prevailing pessimistic outlook after the third quarter of 2008, directors decided to considerably reduce investment levels in the light of an anticipated economic contraction.

Figure 2-7 (below) shows how the GFC has affected cash flow volatility: hardly any impact at all on operating cash flow but a notable jump in volatility of investing cash flow which endured to the close of the period.

[23] The number and composition of firms in each quarter varies; nevertheless, the firms analysed are considered to be a representative sample of the population even though some of the statistics may have been influenced by outliers.

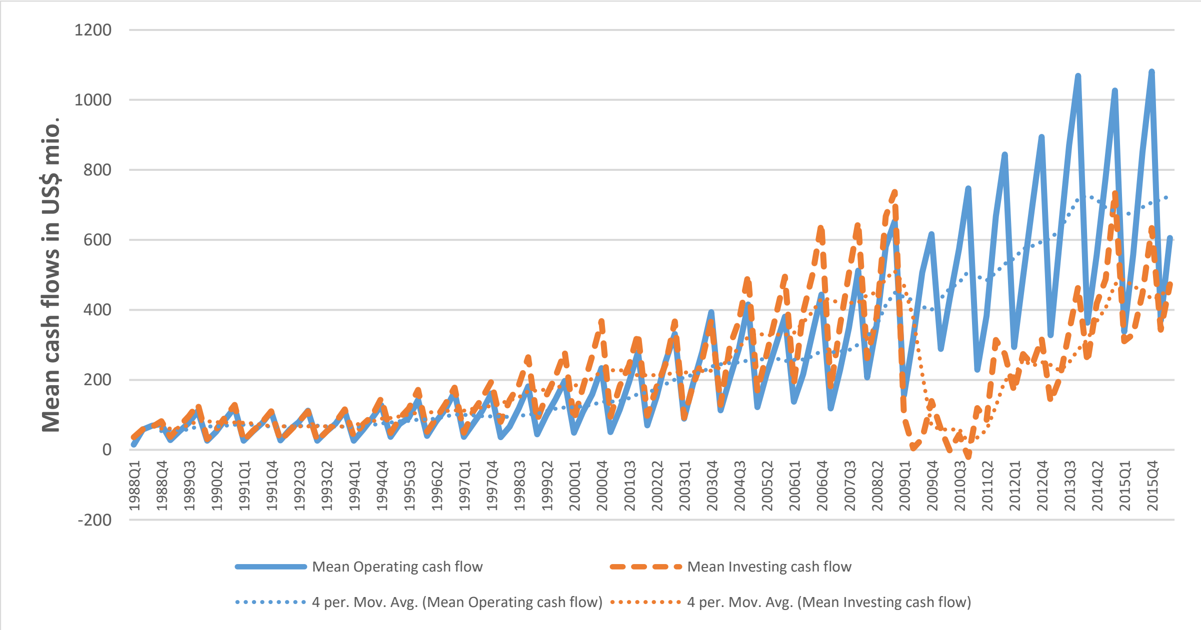


Figure 2-6 Evolution of first moment of operating cash flow and investing cash flow

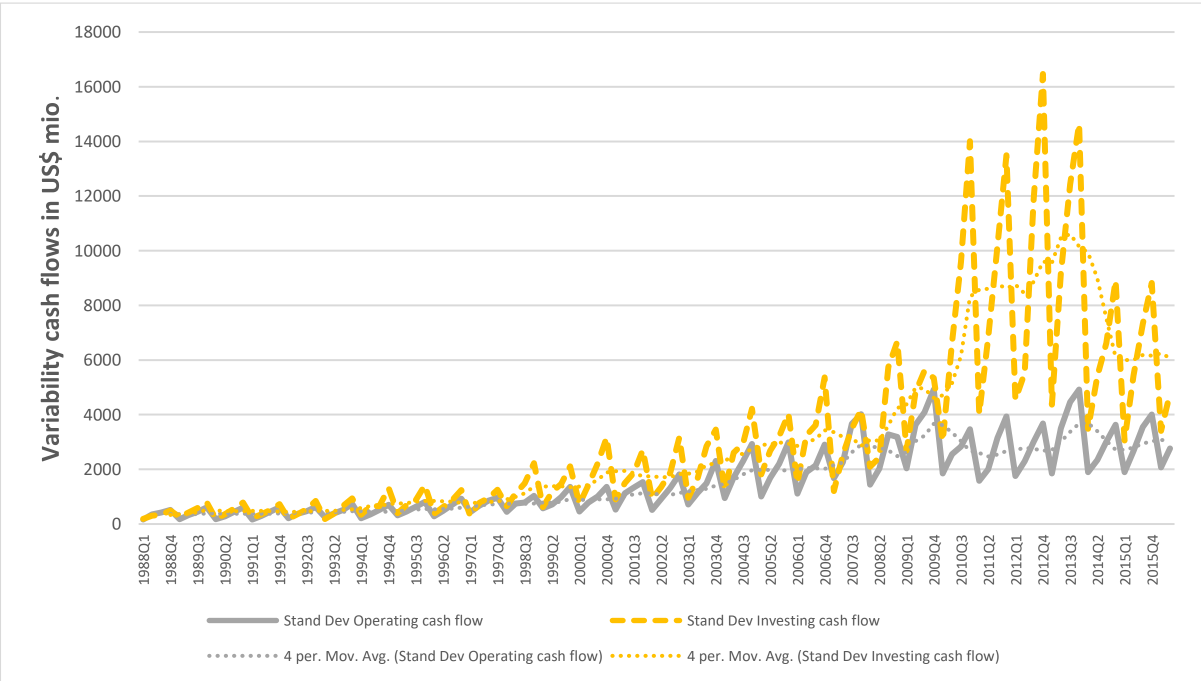


Figure 2-7 Evolution of second moment of operating cash flow and investing cash flow

A regression of the second moment on the first, is compatible with a strong linear relationship for operating cash flow (Table 2-8) but a much weaker linear relationship for investing cash flow (Table 2-9). Interestingly, both regression tables include a statistically significant positive constant (intercept) corresponding to a diminishing cash flow risk as the size of cash flow increases. This suggests that some form of diversification may be at work, contrary to the paradigm that in a perfect and efficient capital market investors are better placed to diversify business risks in financial markets than companies can do in the real economy (refer to Brealey et al. (2011) for an explanation of the value additivity principle). Schlegel (2015) points out that few empirical tests have been conducted to investigate the existence of the value additivity principle. Those tests show mixed results which the author attributes to (i) a lagging arbitrage mechanism (it takes time to merge and demerge company structures), (ii) a natural risk diversification amongst activities of most companies, and (iii) agency costs related to management not strictly aligning their objectives with those of shareholders. The value additivity principle is also contested on other theoretical grounds: see for example Magni (2007).

Table 2-8 Statistics of regression of the second moment (STD) on the first moment of operating cash flow

Regression Statistics	
Multiple R	0.915913
R Square	0.838896
Adjusted R Square	0.837458
Standard Error	498.8484
Observations	114

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	1.45E+08	1.45E+08	583.2042	3.25E-46
Residual	112	27871173	248849.8		
Total	113	1.73E+08			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	386.0707	67.61629	5.70973	9.42E-08	252.0977	520.0437	252.0977	520.0437
STD Operating cash flow	4.405471	0.182424	24.14962	3.25E-46	4.044021	4.76692	4.044021	4.76692

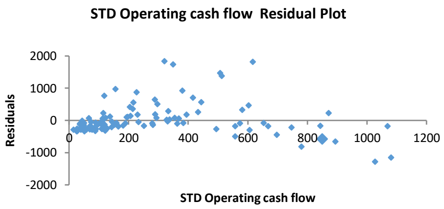


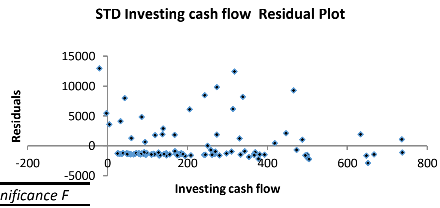
Table 2-9 Statistics of regression of the second moment (STD) on the first moment of investing cash flow

Regression Statistics	
Multiple R	0.449725813
R Square	0.202253307
Adjusted R Square	0.195130568
Standard Error	3102.394316
Observations	114

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	273301888.7	273301888.7	28.39544251	5.17524E-07
Residual	112	1077983255	9624850.491		
Total	113	1351285144			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	1195.575607	459.9653313	2.599273305	0.010599809	284.2132634	2106.93795	284.2132634	2106.93795
STD Investing cash flow	8.964007435	1.682201003	5.328737421	5.17524E-07	5.630941781	12.29707309	5.630941781	12.29707309



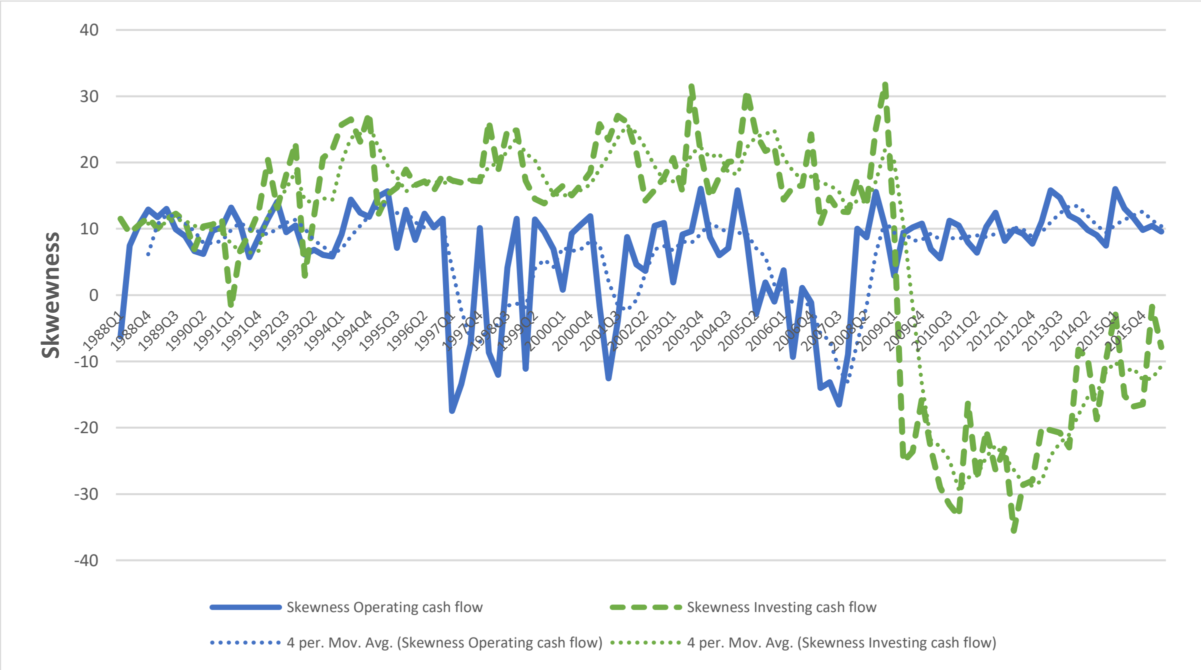


Figure 2-8 Evolution of third moment of operating cash flow and investing cash flow

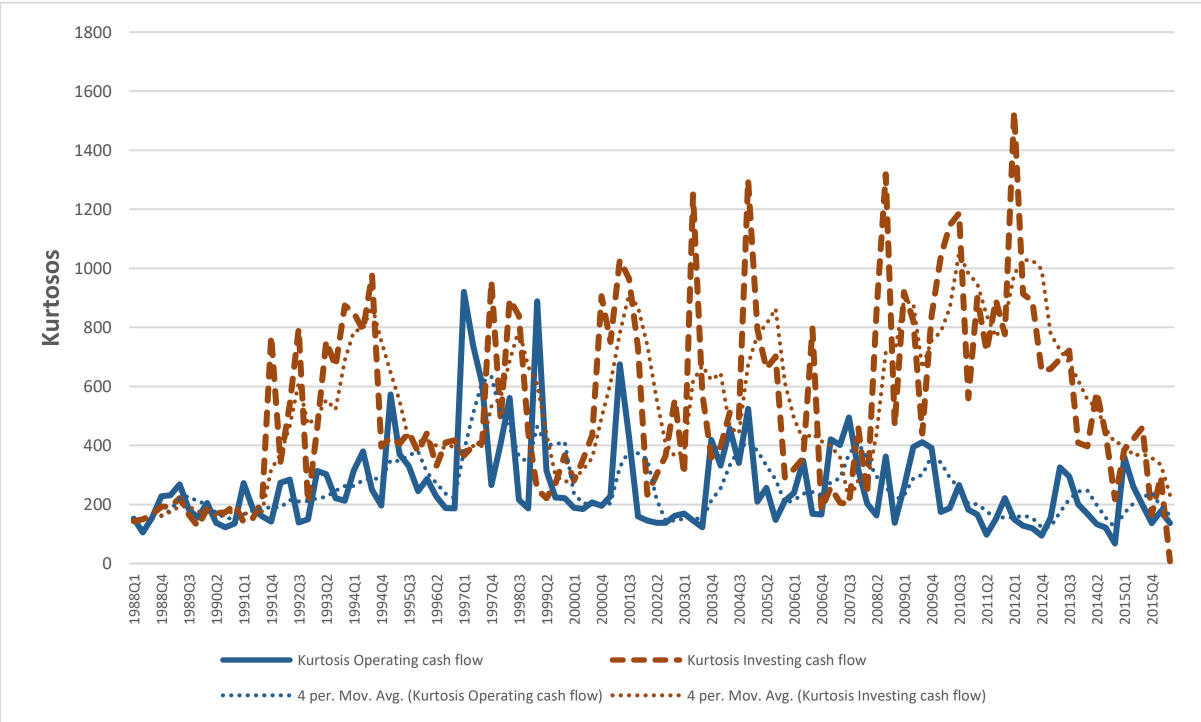


Figure 2-9 Evolution of fourth moment of operating cash flow and investing cash flow

The results of the skewness graph (Figure 2-8) present a bit of a surprise. Despite the predominance of right-sided skewness, there is no consistency, in particular, in the operating cash flow time series. Positive and negative skewness often alternate (operating cash flows) whilst the impact of the GFC is visible in an abrupt and major change from positive to negative skewness (investing cash flow). Negative skewness can be interpreted as a larger than 50% probability of cash flows being biased to the downside[24]. High but varying levels of kurtosis are confirmed in Figure 2-9 notably that of investing cash flow.

Discussion

In this section four important factors that determine a diffusion function for a cash flow process were discussed. The analysis of the instantaneous change in variance suggests that specifications linear and quadratic in cash flow, are both adequate specifications to model the diffusion function of cash flow processes of the majority of examined firms. Of those firms, between about 40% and 50% (depending on operating or investing cash flow process) have a better fit with a quadratic diffusion model than with a linear model. This number, however, increases significantly if it is recognised that a linear specification can be derived from a full quadratic specification as a special case.

As has become evident in the above subparagraphs, each of the first four moments is relevant to characterising the diffusion function of cash flow processes. Visual inspection of the shape of the space-time density functions affirms stochastic similarity between operating cash flow and investing cash flow processes; a more detailed analysis, however, points to a few notable differences. Both cash flow processes are portrayed by a strongly leptokurtic probability density function which is apparent from significant kurtosis. Leptokurtic distributions are often related to self-reinforcing mechanisms ("success breeds success"). In addition, both cash flow processes show a mild right-skewness which can be explained by the concept of managed randomness.

It is suggested that the Pearson family of distributions, with a linear drift function and a quadratic diffusion function, can describe the marginal density function of cash flow processes. Here, operating and investing cash flow part ways. Investing cash flow invariably have a stationary distribution typical for a converging process. Mostly, this is not the case for operating cash flow that is linked to a diverging diffusion process. Consequently, investing cash flow diffusions can be described by a (stable) Pearson Type IV distribution

[24] Which means that a larger number of firms are concentrated at the lower end of the cash flow spectrum. This does not per se contradict growing average cash flows.

whilst the probability dynamics of the operating cash flow diffusions are more complex and require a detailed inspection of the underlying Fokker-Planck equation.

The risk inherent in operating cash flow is strongly correlated to the evolution of mean cash flows; for investing cash flow this relationship is less unequivocal and is likely to be influenced by other variables as well. Generally, investing cash flow risk is significantly higher than operating cash flow risk which could be explained by better diversification of the latter risk compared to the first[25]. Regardless, if a firm grows (and scales-up its operating and investing cash flows) then the analysis shows that the impact of diversification on investing cash flow is larger than on operating cash flow. A diffusion function of cash flow processes should be adept to modelling the occurrence of materially positive as well as negative skewness.

2.4. The relationship between operating and investing cash flow

In order to develop a more substantial foundation for the choice of a cash flow specification, it is proposed to consider operating and investing cash flows as elements of one system of interacting cash flows. This proposition leads to the following important question: are the levels of operating cash flow and investment cash flow correlated, and if so, then what are the determinants of this relationship? A closely associated question, is the issue of how does causality runs? The relationship between operating and investment cash flows can be approached from at least three different angles:

1. A firm may invest more from internally generated cash if (external) funding opportunities are constrained (soft or hard capital rationing) or if the cost of capital of external funding exceeds that of internal funding (related, for instance, to information asymmetries);
2. Managers tend to overspend on internally available funds; and
3. Self-enforcing mechanisms (as described in Section 2.3.) could provide an explanation.

The literature provides evidence under the broad heading of 'investment-cash flow sensitivities', albeit that the conclusions from various studies are rather ambivalent. Kaplan and Zingales (1997) do find significant investment-cash flow sensitivities but conclude that

[25] Commonly firms have wider opportunities to diversify income sources in the external market whilst investment is more idiosyncratic spending.

these sensitivities are not a good measure of financing constraints enforced on the firm. H. Chen and Chen (2012) contend that investment-cash flow sensitivities have declined and even completely disappeared in recent years, for which the authors admit to not having ascertained a satisfying explanation. In contrast, Lewellen and Lewellen (2016) find convincing evidence for those sensitivities, and an even stronger relationship between investments and a firm's expected cash flow.

The second angle, overspending on internally available funds, has been analysed by, for example, Pawlina and Renneboog (2005) and Cadenillas and Clark Steven (2007). In spite of traditional finance theory suggesting that each investment opportunity should be considered on its own merits, independent of how much prior cash the firm has generated, the problem of over or underinvesting is found to exist.

The third angle, probably the least discussed in the literature, is the explanation of stochastic cumulative mechanisms at work in the interaction between operating and investing cash flows. Tails fatter than those defined by a normal distribution are usually an indication that recursive relationships, such as the one examined in this study (increase in investments → higher operating cash flow → more increase of investments, etcetera), are present (see Sornette (2003)).

The above considerations underpin a two-way causality:

1. One that runs from investing cash flow $\mathrm{I_t}$ to operating cash flow $\mathrm{C_t}$, i.e.

   $$\mathrm{C_t} = \mathrm{F}\left(\mathcal{L}(\mathrm{I_t}), \mathrm{W_{C,t}}\right) \tag{2.20}$$

   and

2. One that goes in the opposite direction:

   $$\mathrm{I_t} = \mathrm{G}\left(\mathcal{L}(\mathrm{C_t}), \mathrm{W_{I,t}}\right) \tag{2.21}$$

where $\mathcal{L}(\ )$ is a time-lag operator.

It should be noted that $\mathrm{W_{C,t}}$ and $\mathrm{W_{I,t}}$ are dependent Wiener processes since investing is not a completely autonomous random process: the amount and timing of investments are decided by the firm's management after taking all relevant information, importantly, the generation of operating cash flow, into consideration.

Figure 2-7 demonstrates that at a *macroscopic* level (aggregate – averages for all firms) a modestly strong correlation exists between operating and investing cash flow (with a clear trend break at the start of the GFC). This relationship is further quantified in Appendix S5. From the regression statistics, summarised in Table 2-10, it is obvious that, prior to the GFC, there is a very strong fit between operating cash flows and investing cash flows pertaining to the same quarters. After, and arguably because of the GFC, the fit is much weaker.

Table 2-10 Summary of statistics of regression of operating cash flow on investing cash flow

	FIT - R^2	$\frac{\Delta C}{\Delta I}$	CONSTANT
FULL PERIOD 1988 Q1-2016 Q2	42.9%	0.97	62.0
PRE-GFC PERIOD 1988 Q1-2008 Q2	93.2%	0.74	7.5
POST-GFC PERIOD 2008 Q3-2016 Q2	34.9%	0.40	391.2

Table 2-10 shows that prior to the GFC one dollar of investment translates almost instantaneously in \$0.74 extra operating cash flow (regardless of future additional cash flows) whereas, due to the GFC, the $\frac{\Delta C}{\Delta I}$ ratio drops to 0.40. In addition, there is a notable shift in the constant parameter: in the pre-GCF period the constant is statistically insignificant (suggesting a proportional relationship) but post-GFC the constant becomes relatively large and surely significant. The analysis indicates that as a consequence of the GFC, the operating cash flow- investing cash flow relationship is more variable, coupled with a lower immediate return on investments, i.e. it takes longer for investments to come to fruition and/or investing is less efficient (at least in the short run).

The cash flow relationship at a *microscopic* (individual firm) level is much more complicated and prone to the influence of elevated randomness; Figures 1-3 and 1-4 in Chapter 1 provide a visual testimony of this statement. A stable correlation typical for a macroscopic environment would not be expected to be found. It is supposed that the microscopic relationship is not uniform for all firms but is likely to vary in terms of, for example, the duration of the investment cycle, i.e. (i) the time between investment and its effect on business activity, and (ii) the mathematical specification of the link. As regards the latter, amongst others, Kaplan and Zingales (1997) and Lewellen and Lewellen (2016), suggest that the specification is a non-monotonic function, hence nonlinear. The analysis of the microscopic relationship between operating cash flow and investing cash flow is deferred until this relationship is described in more detail in Chapters 3 and 4.

Discussion

On average (measured over all firms) there is a fairly strong connection between operating cash flow and investing cash flow, albeit that the fit between the two variables has considerably deteriorated after (and presumably from) the GFC.

Unsurprisingly, at an individual firm-level, this connection is much fuzzier due to its stochastic nature. The literature provides good reasons for the relationship between investments and operating cash flow, despite the evidence sometimes being contradictory.

2.5. Conclusions from Chapter 2

From the review of the literature, five commonly used continuous cash flow specifications are identified: Geometric Brownian Motion, Arithmetic Brownian Motion, the Vasicek process, the Cox, Ingersoll and Ross process (CIR process or square root), and the Modified Square Root process (MSR process). These specifications will be used in Chapter 5 as bench mark specifications to test a newly developed cash flow model.

The functional specifications of the drift and diffusion functions are further examined by using the available set of cash flow data from the study's 5,202 North American firms. The particular interest is to identify similarities and discrepancies between operating cash flow processes and investing cash flow processes.

With respect to the drift function, it is found that the choice between an exponential growth process and a linear growth process is too simple; therefore, a composite specification is advocated that supports a much wider range of stochastic cash flow processes than an exponential or a linear drift function in isolation. There are strong indications that, in the long run, operating cash flow processes diverge without reaching a stable level (and a stationary probability density function), as opposed to investing cash flows that do display converging behaviour.

From examining the approximation of cash flows by a continuous process, it is clear that a nonlinear (second order) diffusion function is appropriate to mimic the jump-like behaviour typical of a significant number of cash flow processes. Further analysis of the diffusion function reveals that a quadratic specification is in most cases superior to a linear specification, specifically because a linear diffusion can be seen as a special form of a more inclusive quadratic diffusion.

A quadratic diffusion function also explains the gap between a Gaussian transition density in very small time, and the empirically observed quarterly transition probabilities that predominantly are characterised by fat and long tails. The space-time density functions, constructed of real-time cash flows, provide deeper insight in the unconditional density function. For both operating cash flow and investing cash flow these marginal density functions are (asymmetric) leptokurtic. The leptokurtic shape is often seen in connection with self-reinforcing mechanisms which are suspected of also playing an important role in the interaction between operating and investing cash flows. Asymmetry in the form of mild right-skewness is explained by the concept of managed randomness.

The literature about stochastic business growth processes frequently mentions a (asymmetric) Laplace distribution as an appropriate distribution to describe the related marginal density function. A superficial observation of the marginal density function of cash flow processes seems to indicate such a distribution. However, closer inspection reveals properties different to the Laplace distribution which are better described by the Pearson family of distributions. Pearson distributions imply a linear drift function and a quadratic diffusion function, and in several places in this study are identified as appropriate functions to model cash flow processes. However, by definition, Pearson distributions converge to a stable distribution that requires the linear drift function to be mean-reverting. My empirical research suggests that investing cash flow processes can be adequately described by a Pearson Type IV distribution defined on the full range of cash flows. In contrast, operating cash flow processes converge to a stable Pearson distribution only within a restricted cash flow spectrum. Outside this spectrum, the process diverges. These findings are entirely consistent with the prior conclusion that investing cash flows characterised by a converging drift function and operating cash flows by diverging drift function. Therefore, operating cash flow processes cannot be described by a (stable) Pearson distribution. To examine the stochastic properties of operating cash flow processes, it is necessary to refocus on the intertemporal dynamics of the process rather than on its long-time behaviour.

From Table 2-11, it is obvious that operating and investing cash flow processes have different stochastic properties. Investigation of the evolution of the first four moments of the respective space-time density functions, reveals some important differences. The second moment (representing cash flow risk) is strongly correlated to the evolution of mean operating cash flow; this is less obvious for investing cash flow. The analysis shows a mild form of risk diversification if cash flow grows. This effect is more pronounced for investing cash flow than for operating cash flow.

Review of the literature reinforces the presumption that operating cash flow and investing cash flow are connected and can be considered within one integrated stochastic model. This conclusion, in conjunction with other conclusions from Chapter 2, underpin the development of an integrated hybrid operating and investing cash flow model which is described in detail in Chapter 3.

Table 2-11 Similarities and differences between operating and investing cash flow processes

	OPERATING CASH FLOW PROCESSES	INVESTING CASH FLOW PROCESSES
PROCESS APPROXIMATED BY A CONTINUOUS PROCESS	Fair approximation	Acceptable approximation after re-scaling
DRIFT FUNCTION	Diverging exponential growth process	Converging exponential growth process
	Reasonable fit with a linear growth process	Moderate fit with a linear growth process
	Good fit with a linear drift function	Good fit with a linear drift function
DIFFUSION FUNCTION – TRANSITION PROBABILITIES	Small-time approximation by normal distribution	Small-time approximation by normal distribution
	In discrete time, heavy tail distribution	In discrete time, significant heavy tail distribution
DIFFUSION FUNCTION – MARGINAL PROBABILITIES	Strongly leptokurtic with mild right-sided skewness	Strongly leptokurtic with mild right-sided skewness
	Predominantly a diverging diffusion process	Almost exclusively a converging diffusion process
	Well described by a quadratic diffusion function	Well described by a quadratic diffusion function
	No full-range stationary density distribution	Pearson Type-IV stationary density distribution
DIFFUSION FUNCTION – EVOLUTION OF MOMENTS	Evolution of first and second moment strongly connected	Evolution of first and second moment loosely connected
	Third moment (skewness) alternating positive and negative	Third moment (skewness) alternating positive and negative
	Significant heavy tails (kurtosis) for full time spectrum	Very significant heavy tails (kurtosis)for full time spectrum
RELATIONSHIP	Causal relationship between operating and investing CF	Causal relationship between operating and investing CF

3. A Coupled Linear-Quadratic Cash Flow Model

Chapter 3 lays the foundation for what is called in this study a coupled hybrid linear-quadratic cash flow model. The terms hybrid and coupled will be explained. The major contribution of this chapter is that it shows how a versatile deterministic, and even more sophisticated stochastic, cash flow model can be derived from a few premises that are underpinned by findings from the literature in addition to common finance and business knowledge. A simulation of the model is included to demonstrate its fit with real-world cash flows.

3.1. Introduction

Section 2.1. provided a review of five continuous-time cash flow specifications that commonly feature in the literature. In Chapter 1 strong indications were found that, to a varying degree, these specifications are inadequate to properly model cash flow processes. First, there is the issue of the cash flow domain. Restricted support on $\mathbb{R}^{+}$ already disqualifies the pure Geometric Brownian Motion and the Cox, Ingersoll and Ross (square root) processes. The second issue relates to theoretical and empirically desirable drift and diffusion functions. A linear growth process, underpinning the Arithmetic Brownian Motion, does not agree with the findings from the literature nor is it supported by unequivocal empirical evidence. The results of the empirical research, indicates a skewed, heavy-tailed diffusion process, materially different from a normal distribution. This finding excludes again ABM and Vasicek processes which assume an unconditional normal distribution.

In Chapter 1 repeated suggestions were made that a quadratic diffusion function is a preferable specification to mimic jump like behaviour of cash flow processes. All specifications but the Modified Square Root process have diffusion functions of less than second order. Even the Modified Square Root process, of all five specifications the most appropriate to model cash flows, has its limitations. It assumes that the drift function follows a pure exponential growth process despite the conclusion in Section 2.2. that the drift function of cash flow processes fits better with a more versatile linear specification, combining an exponential and linear growth process in one equation. Also, from Section 2.3. it transpires that, assuming a quadratic diffusion function, there is no reason to a priori reduce the number of parameters from three to two, and thus limit the admissible members of the Pearson family of distributions.

In conclusion, there are good reasons to develop an alternative continuous cash flow specification (alternative to the five well-known specifications) that not only has a strong theoretical foundation but, additionally, is firmly supported by empirical evidence. The cash flow model to be developed in this chapter, is a *coupled* linear-quadratic model linking operating and investing cash flows in one integral model based on a *hybrid* specification, which is a combination of two basic SDEs, the Geometric and Arithmetic Brownian motions.

In the prior chapters, the symbols $\mathrm{C_t}$ and c were used to denote a cash flow or a cash flow process; in the sequent sections of this study a distinction will be made between *operating cash flows* represented by $\mathrm{C_t}$ and *investing cash flows* designated by $\mathrm{I_t}$. The symbol $\mathrm{X_t}$ will denote a general cash flow process including operating and investing cash flows.

3.2. Foundation of the model

In Section 2.4. preliminary evidence was found for a stochastic relationship between operating cash flow and investing cash flow. There are strong indications that this relationship is nonlinear. This section explores the *dynamics* of a supposed stochastic relationship between operating and investing cash flows in detail. The view will be supported that this relationship is *bi-causal*.

The starting point is the supposed time-lag between investing cash flow as an explanatory variable and operating cash flow as a dependent variable (‘*the first fundamental relationship’*), to be followed by the causality between operating cash flow as an explanatory variable and investing cash flow as a dependent variable (‘*the second fundamental relationship’*). In the following, endogenous and exogeneous factors are analysed that, in aggregate, explain the change of operating cash flow in time, and the evolution of the level of investing cash flow.

The first fundamental relationship

From macroeconomics, it is well-known that there is a relationship between the level of investments and the firm’s output - Cobb and Douglas (1928); Rowley (1970); Smith (1961).

Since capital goods generally include tangible assets only[26], this relationship is strong for industries with a high intensity of tangible assets and less relevant to firms that largely create intangible assets. Nevertheless, investments in productive assets are critical to the growth of the firm. Investments can be split into (i) replacement investments, here defined as investments required to maintain the current level of operating cash flows, and (ii) expansion investments, i.e. investments made with the aim of growing future operating cash flows beyond the current level.

In the following a justification will be given for the first fundamental relationship. First, determining factors will be explained in isolation; thereafter it is assumed that these individual factors can be aggregated linearly. Suppose that replacement investments are approximately proportional to the level of operating cash flow to be conserved minus a constant m: $\mathrm{C_t} = \mathrm{k\,I_{r,t}} - \mathrm{m}\ (\mathrm{k} > 0)$ where $\mathrm{I_{r,t}}$ is called the *investment replacement* threshold. Future growth of operating cash flows is considered also approximately proportional to the level of current investments above the investment replacement threshold $\mathrm{I_{r,t}}$: $\mathrm{C_{t+1}} - \mathrm{C_t} = \beta(\mathrm{I_t} - \mathrm{I_{r,t}})$ if $\mathrm{I_t} > \mathrm{I_{r,t}}$, i.e. It is assumed that the proportionality factor β holds for expansion investments only and can be different from k .

Normally, investments will take time before they become fully productive and translate into incremental cash flows. Consequently, it is appropriate to consider a time-lagged response of cash flows to investments. The proportionality parameter β (≥ 0), also called the *investment response* parameter, is assumed to be time invariant. The investment response parameter is thought to be determined by industry characteristics and within an industry by firm-specific characteristics such as the ability of management to successfully turn investments into business growth.

However, the expression $\mathrm{C_t} = \mathrm{k\,I_{r,t}} - \mathrm{m}$ can be reformulated to $\mathrm{I_{r,t}} = \frac{1}{\mathrm{k}}\mathrm{C_t} + \frac{\mathrm{m}}{\mathrm{k}}\ (\mathrm{k}, \mathrm{m} > 0)$. Therefore, the expected future growth of operating cash flow $\mathrm{C_{t+1}}$ will not only depend on the amount of total investments $\mathrm{I_t}$ but it also will be negatively proportional to $\mathrm{C_t}$ as follows: $\mathrm{C_{t+1}} - \mathrm{C_t} = \beta\,\mathrm{I_t} - \frac{\beta \mathrm{m}}{\mathrm{k}} - \frac{\beta}{\mathrm{k}}\mathrm{C_t} = \beta\,\mathrm{I_t} - \tau_{0,\mathrm{t}} - \tau \mathrm{C_t}$ where parameter $\tau_{0,\mathrm{t}} = \frac{\beta \mathrm{m}}{\mathrm{k}}$ and $\tau =$

[26] The definition of investing cash flow used in this study (see Chapter 10, Definitions of Terms Used), includes the value of intangible assets if the firm acquires or sells an (equity instrument in) another entity at which moment an objective market value of those assets can be established.

$\frac{\beta}{k}$ (> 0). The expression $-\tau_{0,t} - \tau C_t$ is called the *cash flow attrition rate* representing a decline in cash flow generating capacity if the firm has an investment level below the investment replacement threshold, i.e. $I_t < I_{r,t}$. If $I_t = 0$ then the attrition rate $-\tau_{0,t}$ is independent of the cash flow level. The cash flow attrition rate is supposed to be largely determined by the technology replacement rate in a specific industry.

Furthermore, the firm's existing capital goods K_t can have the ability to generate additional operating cash flows, for example because those capital goods are not producing at full capacity. The *occupancy rate* ω is here defined as $\omega_t = \frac{C_t}{K_t}$. If the firm succeeds in a better utilisation of its current production capacity K_t, then the additional operating cash flow generated will equate to $C_{t+1} - C_t = (\omega_{t+1} - \omega_t)K_t = \frac{(\omega_{t+1} - \omega_t)}{\omega_t} C_t$.

Combining the two effects, the increase in operating cash flows from better capacity utilisation and the natural operating cash flow deterioration from capital goods attrition, in one equation leads to $C_{t+1} - C_t = \left[\frac{(\omega_{t+1} - \omega_t)}{\omega_t} - \tau\right] C_t = \alpha C_t$ where a new parameter α replaces the expression $\frac{(\omega_{t+1} - \omega_t)}{\omega_t} - \tau$. This parameter α is called the *cash flow growth rate* and is assumed to be constant. Notice that α will be positive if the impact of improved capacity utilisation on operating cash flow, on balance, is stronger than the cash flow deterioration effect, and α will be negative if the opposite is true.

Other variables also affect the growth of future cash flows. These variables, in aggregate represented by parameter n, are exogenous to the model and it is assumed that (in aggregate) n is approximately constant over time. Another reason to include a constant n is that the assumed proportionality between C_{t+1} and I_t could well be diminishing as the size of the firm grows and continues to invest: $\frac{C_{t+1}}{I_t} = \beta + \frac{n}{I_t}$. Hereafter, a parameter δ_t is defined as the sum of $-\tau_0$ and n. It is assumed that δ_t is time-invariant, at least in small time Δt, i.e. $\delta_t = \delta$.

A mathematical summary of the above aggregated factors, prompts the following discrete-time equation with operating cash flow as both a lead and lag term, and investing cash flows as a lag term

$$C_{t+1} - C_t = \alpha C_t + \beta I_t + \delta \tag{3.1a}$$

The second fundamental relationship

This section examines the reverse relationship between operating cash flow as input variable and investing cash flow as output variable. Beside the literature already mentioned in Section 2.4. in support of this relationship, there is other evidence and alternative explanations for the reverse relationship. A good fit is normally found between (i) investment and (ii) current cash flow and Tobin's Q. The latter reflects (market) expectations of future business growth (Abel and Eberly (2011) and ultimately cash flow growth. The significant coefficient in the regression of investment on cash flow, is explained by financing constraints faced by a firm (Abel (2016)), for instance smaller firms having restricted access to capital markets (Kadapakkam et al. (1998)). Another explanation is given by Gilchrist & Himmelberg (1995) which study suggests that cash flow is also a good indicator of the firm's (future) investment opportunities since (current) cash flow is a fundamental (adjusted for the possible impact of financing constraints) predictor of future cash flows. The analysis below follows this reasoning.

The level of investments is assumed proportional to the available cash balance of the firm at the time of investing: $I_t = r\,B_t$. Furthermore, it is assumed that accumulated cash flow is an even better predictor of future cash flows than a cash flow at a single point in time. The other, implicit, assumption is that the firm in aggregate has enough investment projects that meet the firm's hurdle rate to invest at least rB_t each period. The movement in cash balance B_t is the sum of operating cash flow, investing cash flow and financing cash flow (F_t): $B_t = C_t + I_t + F_t + B_{t-1}$. Note that if the firm requires additional funding to externally finance investments, it is already included in the cash balance B_t.

Combining the two expressions, gives I_t as $I_t = \frac{r}{1-r} C_t + \frac{r}{1-r} F_t + \frac{r}{1-r} B_{t-1}$. Now, express $\frac{r}{1-r}$ as γ which turns the prior expression into $I_t = \gamma\, C_t + \gamma\, F_t + \gamma\, B_{t-1}$. Parameter γ ($0 < \gamma \leq 1$) is called the *cash investment rate* and is assumed to be constant over time.

In a next step, consider the expression $\gamma\, F_t + \gamma\, B_{t-1}$ in isolation and label it ε_t. It is assumed that ε_t is approximately time-homogeneous and thus $\varepsilon_t = \varepsilon$. This follows from the assumption that γB_t and F_{t+1} are balancing quantities: if the firm has not accumulated enough cash $\mathrm{B_{t-1}}$ at time t-1 it will need to attract additional funding $\mathrm{F_t}$, and conversely, if the firm has a sufficient cash balance $\mathrm{B_{t-1}}$ it will require less external funding. A large cash surplus allows the firm to pay dividends or to accelerate the down payment of debt, and in either case $\mathrm{F_t}$ becomes negative.

Similar to Equation (3.1a), the second fundamental relationship can be expressed as

$$\mathrm{I_{t+1}} = \gamma \mathrm{C_t} + \varepsilon \tag{3.1b}$$

Here, investing cash flow is the lead term and operating cash flow is the lag term.

Equations (3.1a) and (3.1b) can be written in difference form

$$\Delta \mathrm{C_{t+1}} = \alpha \mathrm{C_t} + \beta \mathrm{I_t} + \delta \tag{3.2a}$$

$$\Delta \mathrm{I_{t+1}} = \gamma \mathrm{C_t} - \mathrm{I_t} + \varepsilon \tag{3.2b}$$

or more conveniently expressed in a matrix notation

$$\Delta \mathbf{u}_{\mathrm{t+1}} = \mathbf{A}\mathbf{u}_{\mathrm{t}} + \mathbf{b} \tag{3.3}$$

where $\Delta \mathbf{u}_{\mathrm{t+1}} = \begin{pmatrix} \Delta \mathrm{C_{t+1}} \\ \Delta \mathrm{I_{t+1}} \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & -1 \end{pmatrix}$, $\mathbf{u}_{\mathrm{t}} = \begin{pmatrix} \mathrm{C_t} \\ \mathrm{I_t} \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \delta \\ \varepsilon \end{pmatrix}$.

Equation (3.3) represents a $\{\mathrm{C_t}, \mathrm{I_t}\}$ coupled system. The dynamics of Equation (3.2a) is based on a *multiplicative* time-series as opposed to the dynamics of Equation (3.2b) which is *additive*. Notice that Equation (3.3) describes a deterministic system, nevertheless, the interaction of the additive and multiplicative dynamics in the coupled system $\{\mathrm{C_t}, \mathrm{I_t}\}$ is already capable of describing interesting cash flow behaviour. In Section 3.4. the deterministic system will be transposed to a random environment in order to model much more realistic stochastic cash flow behaviour as observed in the real world. However, the price paid is increased complexity. The first priority is to investigate solutions to Equation (3.3) in a deterministic environment.

3.3. The cash flow model in a deterministic environment

Studying the properties of a solution to the deterministic cash flow model is conducive to better understanding the dynamics of the model in a stochastic setting. No matter if a solution is deterministic or stochastic, decoupling the system is a condition necessary to finding a solution. Decoupling is achieved by an eigen-decomposition[27] of the system. For a detailed explanation of decomposition techniques refer to Davidson and Tippett (2012, chapter 8) and Geiser (2009).

Decoupling the system

In order to solve the equations, decouple the system $S\,\{\Delta C_{\Delta t}, \Delta I_{\Delta t}\}$ where $\Delta t = [t, t+1]$. Decoupling is achieved by transforming the set of variables $\begin{pmatrix} C_t \\ I_t \end{pmatrix}$ into a new set of variables $\begin{pmatrix} C'_t \\ I'_t \end{pmatrix}$ such that changes of each variable are expressed exclusively in their own variable.

Mathematically this is done by diagonalising matrix $\mathbf{A}$. Define a new set of variables $\mathbf{v}_t = \begin{pmatrix} C'_t \\ I'_t \end{pmatrix} = \mathbf{Q^{-1}}.\mathbf{u}_t$ where $\mathbf{Q^{-1}}$ is the inverse matrix of the eigenvectors of $\mathbf{A}$. The eigenvectors of $\mathbf{A}$ are $\mathbf{Q} = \begin{pmatrix} \frac{2\beta}{-1-\alpha+\omega} & \frac{2\beta}{-1-\alpha-\omega} \\ 1 & 1 \end{pmatrix}$, hence $\mathbf{Q^{-1}} = \begin{pmatrix} \frac{(-1-\alpha+\omega)(1+\alpha+\omega)}{4\beta\omega} & \frac{-1-\alpha+\omega}{2\omega} \\ -\frac{(-1-\alpha+\omega)(1+\alpha+\omega)}{4\beta\omega} & \frac{1+\alpha+\omega}{2\omega} \end{pmatrix}$, and the diagonal matrix of eigenvalues is $\mathbf{\Lambda} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\alpha-1)+\frac{1}{2}\omega & 0 \\ 0 & \frac{1}{2}(\alpha-1)-\frac{1}{2}\omega \end{pmatrix}$

where $\omega = \sqrt{(\alpha-1)^2 + 4(\beta\gamma+\alpha)}$. After some matrix algebra, the decoupled system becomes: $\Delta\mathbf{v}_{t+1} = \mathbf{\Lambda}.\mathbf{v}_t + \mathbf{Q^{-1}}.\mathbf{b}$ or

$$\Delta C'_{\Delta t} = \left(\frac{1}{2}(\alpha-1)+\frac{1}{2}\omega\right)C'_t + \vartheta_1 \tag{3.4a}$$

$$\Delta I'_{\Delta t} = \left(\frac{1}{2}(\alpha-1)-\frac{1}{2}\omega\right)I'_t + \vartheta_2 \tag{3.4b}$$

where for notational convenience $\boldsymbol{\vartheta} = \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix}$ replaces $\mathbf{Q^{-1}}.\mathbf{b}$. Note that the solution $\mathbf{v}_t$ can be transformed back to $\mathbf{u}_t$ by $\mathbf{u}_t = \mathbf{Q}.\mathbf{v}_t$.

The dynamics of the continuous-time system

In order to analyse the dynamics of the deterministic cash flow system, it is useful to express Equations (3.2a) and (3.2b) in their continuous-time variant

[27] Also known as spectral decomposition.

$$\frac{dC_t}{dt} = \alpha C_t + \beta I_t + \delta \tag{3.5a}$$

$$\frac{dI_t}{dt} = \gamma C_t - I_t + \varepsilon \tag{3.5b}$$

Observe that the deterministic system S $\{C_t, I_t\}$ consists of two coupled ODEs.

Similar to Equation (3.3), Equations (3.5a) and (3.5b) can be expressed in matrix form as

$$d\mathbf{u}_t = \mathbf{A}\mathbf{u}_t + \mathbf{b} \tag{3.6}$$

where $d\mathbf{u}_t = \begin{pmatrix} \frac{dC_t}{dt} \\ \frac{dI_t}{dt} \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & -1 \end{pmatrix}$, $\mathbf{u}_t = \begin{pmatrix} C_t \\ I_t \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \delta \\ \varepsilon \end{pmatrix}$.

The following analysis is based on the methods and techniques set out in Perko (2008).

First, transform $\mathbf{u_t}$ to $\mathbf{u}'_\mathbf{t}$: $\mathbf{u}'_\mathbf{t} = \mathbf{u}_t - \mathbf{A}^{-1}\mathbf{b}$ to eliminate vector $\mathbf{b}$**.** The transformed system becomes

$$d\mathbf{u}'_\mathbf{t} = \mathbf{A}\mathbf{u}'_\mathbf{t} \tag{3.7}$$

It is not difficult to see that the equilibrium point of $\mathbf{u}'_\mathbf{t}$ (where the system is stationary in time) is $(C^*, I^*) = (0,0)$[28] if $\alpha + \beta\gamma \neq 0$. The characteristic equation of $\mathbf{u}'_\mathbf{t}$, $\lambda^2 - (\alpha - 1)\lambda - (\alpha + \beta\gamma) = 0$, provides insight in the stability of the system. The roots of the characteristic equation are $\Lambda_{1,2} = \frac{1}{2}(\alpha - 1) \pm \frac{1}{2}\sqrt{(\alpha - 1)^2 + 4(\beta\gamma + \alpha)}$. Since $\beta, \gamma \geq 0$ the expression under the square root is always greater than zero and hence the system has only real roots.

It can be shown that for $C'_t \in (-\infty; \infty)$, the root of the operating cash flow process Λ_1 goes asymptotically from the value -1 to infinity. Likewise, for $I'_t \in (-\infty; \infty)$, the root of the investing cash flow process starts at minus infinity to asymptotically approach the value -1. For all I'_t, values of Λ_2 are *negative* implying an investment process that is always *converging* in time to $I^* = 0$ (conditional on the system $\mathbf{u}'_\mathbf{t}$). Depending on the value of α, Λ_1 can be positive or negative with a transition point at $\alpha = -\beta\gamma$. Therefore, if $\alpha < -\beta\gamma$ then

[28] For the original system $\mathbf{u}_t$ the equilibrium point is $(C^*, I^*) = (\frac{-(\delta+\beta\varepsilon)}{\alpha+\beta\gamma}, \frac{\alpha\varepsilon-\gamma\delta}{\alpha+\beta\gamma})$.

the operating cash flow process becomes *converging* in time to reach $C^* = 0$ (conditional on the system $\mathbf{u}_t'$) for values $\alpha > -\beta\gamma$ the process is *diverging* over time to $C^* =$ infinity.

The *greater* the value of parameter α is, the faster operating cash flow diverges to infinity, and the slower investment cash flow converges to a long-time value (Table 3-1).

The *smaller* the value of parameter α is, the slower operating cash flow diverges to infinity; however, beyond the transition point $\alpha = -\beta\gamma$ it will converge faster to the long-time value. Lower alpha values also translate into faster converging investment cash flow (Table 3-1).

Table 3-1 Values of roots $\Lambda_{1,2}$ with changing parameter α (given $\beta = 1, \gamma = 1$)

α	Λ_1	Λ_2
1000	1000.000999	-1.000999
100	100.0099	-1.00990002
10	10.09016994	-1.090169944
9	9.099019514	-1.099019514
8	8.109772229	-1.109772229
7	7.123105626	-1.123105626
6	6.140054945	-1.140054945
5	5.16227766	-1.16227766
4	4.192582404	-1.192582404
3	3.236067977	-1.236067977
2	2.302775638	-1.302775638
1	1.414213562	-1.414213562
0.5	1	-1.5
0	0.618033989	-1.618033989
-0.5	0.280776406	-1.780776406
-1	0	-2
-2	-0.381966011	-2.618033989
-3	-0.585786438	-3.414213562
-4	-0.697224362	-4.302775638
-5	-0.763932023	-5.236067977
-6	-0.807417596	-6.192582404
-7	-0.83772234	-7.16227766
-8	-0.859945055	-8.140054945
-9	-0.876894374	-9.123105626
-10	-0.890227771	-10.10977223
-100	-0.98990002	-100.0101
-1000	-0.998999	-1000.001001

The above conclusions have interesting practical implications (Table 3-2). To ensure that the operating cash flow process continues to grow, the multiplication of the investment response rate β and the cash investment rate γ must be greater than minus the cash flow growth rate $-\alpha$. However, if $\beta\gamma$ is smaller than -α, then the operating cash flow will asymptotically approach a long-term value. Regardless of the value of parameters α, β, γ, the investment cash flow will always approach a long-term value, i.e. once growth of operating cash flow has taken off by ensuring $\beta\gamma > -\alpha$, only a constant level of investment is required to maintain cash flow growth.

Table 3-2 How cash flow process characteristics depend on parameter values

	OPERATING CASH FLOW PROCESS	INVESTING CASH FLOW PROCESS	FIRM CHARACTERISTICS
$\boldsymbol{\beta\gamma} < -\boldsymbol{\alpha}$	Reverting to a long-time value (converging process)	Reverting to a long-time value (converging process)	Stable free cash flow, typical for static value firms
$\boldsymbol{\beta\gamma} > -\boldsymbol{\alpha}$	Exploding growth (diverging process)	Reverting to a long-time value (converging process)	Increasing free cash flow, typical for growth value firms

Equation (3.6) can be decoupled by setting $\mathbf{v}_t = \begin{pmatrix} C'_t \\ I'_t \end{pmatrix} = \mathbf{Q}^{-1}\mathbf{u}_t$ where $\mathbf{Q}^{-1}$ is the inverse of the eigenvectors of $\mathbf{A}$. Matrix $\mathbf{Q}^{-1} = \begin{pmatrix} \frac{(-1-\alpha+\omega)(1+\alpha+\omega)}{4\beta\omega} & \frac{-1-\alpha+\omega}{2\omega} \\ -\frac{(-1-\alpha+\omega)(1+\alpha+\omega)}{4\beta\omega} & \frac{1+\alpha+\omega}{2\omega} \end{pmatrix}$ and the diagonal matrix of eigenvalues is $\mathbf{\Lambda} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\alpha-1)+\frac{1}{2}\omega & 0 \\ 0 & \frac{1}{2}(\alpha-1)-\frac{1}{2}\omega \end{pmatrix}$ where $\omega = \sqrt{(\alpha-1)^2+4(\beta\gamma+\alpha)}$. The transformed decoupled system is

$$d\mathbf{v}_t = \mathbf{\Lambda}\mathbf{v}_t + \mathbf{Q}^{-1}.\mathbf{b} \tag{3.8}$$

Using $e^{-\mathbf{\Lambda}t}$ as an integrating factor, Equation (3.8) can be turned into the following (decoupled) system of differential equations

$$e^{-\mathbf{\Lambda}t} d\mathbf{v}_t = e^{-\mathbf{\Lambda}t}\mathbf{\Lambda}\mathbf{v}_t + e^{-\mathbf{\Lambda}t}\mathbf{Q}^{-1}.\mathbf{b} \tag{3.9a}$$

with equivalent expression

$$d(e^{-\mathbf{\Lambda}t}\mathbf{v}_t) = e^{-\mathbf{\Lambda}t}\mathbf{Q}^{-1}.\mathbf{b} \tag{3.9b}$$

Notice that the step from Equation (3.9a) to (3.9b) is admissible because the equality $e^{-\mathbf{\Lambda}t}\mathbf{Q}^{-1} = \mathbf{Q}^{-1}e^{-\mathbf{\Lambda}t}$ holds. Integrating both sides gives the general solution to the uncoupled system

$$\mathbf{v}_t = e^{\mathbf{\Lambda}t}\int_0^t e^{-\mathbf{\Lambda}s} ds\, \mathbf{Q}^{-1}\mathbf{b} + e^{\mathbf{\Lambda}t}\mathbf{k} = \left[\widehat{\mathbf{\Lambda}} - e^{-\mathbf{\Lambda}t}\right]\mathbf{Q}^{-1}\mathbf{b} + e^{\mathbf{\Lambda}t}\mathbf{k} \tag{3.10}$$

where $\widehat{\mathbf{\Lambda}} = \begin{pmatrix} \frac{1}{\Lambda_1} & 0 \\ 0 & \frac{1}{\Lambda_2} \end{pmatrix}$ and $\mathbf{k}$ is a vector of integration constants $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$.

Recall that $\mathbf{u}_t = \mathbf{Q}.\mathbf{v}_t$ so that the general solution to the coupled system becomes

$$\mathbf{u}_t = \mathbf{Q}\left[\widehat{\boldsymbol{\Lambda}} - e^{-\boldsymbol{\Lambda} t}\right]\mathbf{Q}^{-1}\mathbf{b} + \mathbf{Q}e^{\boldsymbol{\Lambda} t}\mathbf{k} \tag{3.11}$$

Finally, define $\mathbf{u}_0$ as a vector of initial values at t = 0. Replacing $\mathbf{k}$ by the appropriate expression in $\mathbf{u}_0$ yields

$$\mathbf{u}_t = \mathbf{Q}e^{\boldsymbol{\Lambda} t}\mathbf{Q}^{-1}\mathbf{u}_0 + [\mathbf{I}_2 - e^{-\boldsymbol{\Lambda} t}]\,\widehat{\boldsymbol{\Lambda}}\mathbf{Q}^{-1}\mathbf{b} = e^{\boldsymbol{\Lambda} t}\mathbf{u}_0 + [\mathbf{I}_2 - e^{-\boldsymbol{\Lambda} t}]\,\widehat{\boldsymbol{\Lambda}}\mathbf{Q}^{-1}\mathbf{b} \tag{3.12a}$$

where $\mathbf{I}_2$ is the identity matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

The elements $\{C_t, I_t\}$ of Equation (3.12a) can now be written as

$$C_t = C_0 e^{\frac{1}{2}(\alpha-1)+\frac{1}{2}\omega} + \left(1 - e^{-\left[\frac{1}{2}(\alpha-1)+\frac{1}{2}\omega\right]}\right)\frac{(-1-\alpha+\omega)(1+\alpha+\omega)}{(2\beta\omega(\alpha-1)+2\beta\omega^2)}\delta + \left(1 - e^{-\left[\frac{1}{2}(\alpha-1)-\frac{1}{2}\omega\right]}\right)\frac{(-1-\alpha+\omega)}{(\omega(\alpha-1)+\omega^2)}\varepsilon \tag{3.12b}$$

and as

$$I_t = I_0 e^{\frac{1}{2}(\alpha-1)-\frac{1}{2}\omega} - \left(1 - e^{-\left[\frac{1}{2}(\alpha-1)+\frac{1}{2}\omega\right]}\right)\frac{(-1-\alpha+\omega)(1+\alpha+\omega)}{(2\beta\omega(\alpha-1)+2\beta\omega^2)}\delta + \left(1 - e^{-\left[\frac{1}{2}(\alpha-1)-\frac{1}{2}\omega\right]}\right)\frac{(1+\alpha+\omega)}{(\omega(\alpha-1)+\omega^2)}\varepsilon \tag{3.12c}$$

where $\omega = \sqrt{(\alpha-1)^2 + 4(\beta\gamma+\alpha)}$.

Matrix $e^{-\boldsymbol{\Lambda} t}$ represents the exponential part of the cash flow processes $e^{\frac{1}{2}(\alpha-1)+\frac{1}{2}\omega}$ and $e^{\frac{1}{2}(\alpha-1)-\frac{1}{2}\omega}$ from which the uncoupled (Equation (3.10)) and the coupled (Equation (3.12a)) solutions are derived. In the graphs below the difference between a growth and a stable basic cash flow scenario is explained.

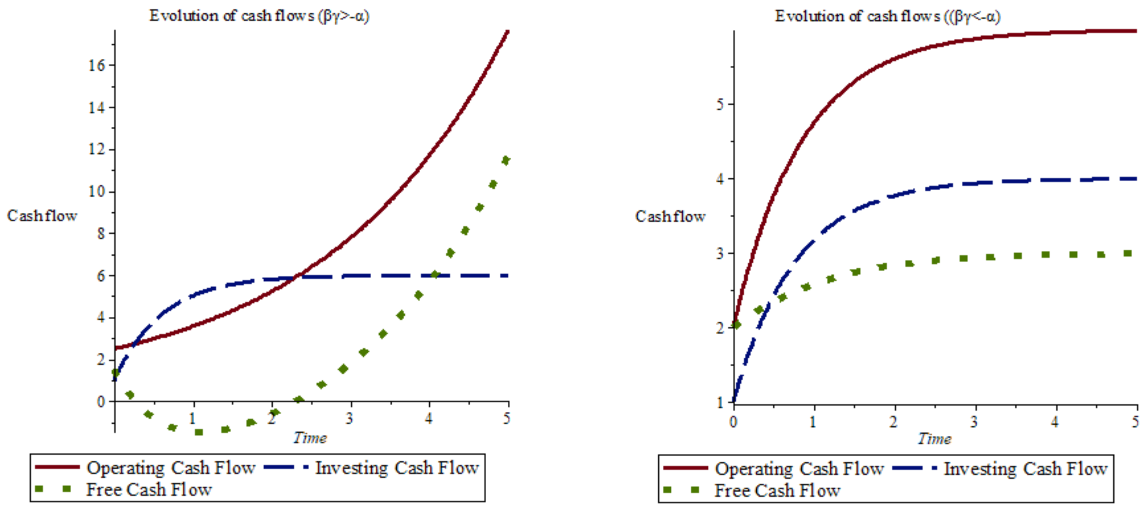


Figure 3-1 Evolution of basic cash flow processes depending on parameter values (stationary values > initial values)

The coupled solutions are (weighted) composite solutions of above uncoupled cash flows (for the stochastic coupled solution see Section 4-4).

If *variable* parameters α, β, γ are admitted then the system is even capable of describing operating cash flow processes where the firm first experiences a growth phase ($\beta\gamma > -\alpha$) followed by a stabilisation phase ($\beta\gamma < -\alpha$).

Similar to Equations (3.5a) and (3.5b) a decoupled system can be defined that is governed by the following equations (for convenience label the transformed variables also C_t and I_t):

$$dC_t = \left(\frac{1}{2}(\alpha - 1) + \frac{1}{2}\omega\right) C_t + \vartheta_1 \tag{3.13a}$$

$$dI_t = \left(\frac{1}{2}(\alpha - 1) - \frac{1}{2}\omega\right) I_t + \vartheta_2 \tag{3.13b}$$

where $\omega = \sqrt{(\alpha - 1)^2 + 4(\beta\gamma + \alpha)}$ and $\begin{pmatrix}\vartheta_1 \\ \vartheta_2\end{pmatrix} = \mathbf{Q}^{-1}\mathbf{b} = \begin{pmatrix} \frac{(\beta\gamma+\alpha)}{\beta\omega} & \frac{1}{2}(1 - \frac{(\alpha-1)}{\omega}) \\ \frac{-(\beta\gamma+\alpha)}{\beta\omega} & \frac{1}{2}(1 - \frac{(\alpha-1)}{\omega}) \end{pmatrix}\begin{pmatrix}\delta \\ \varepsilon\end{pmatrix}$

From (3.13a) and (3.13b) a new equation in which $C_t$and I_t are linked to each other for all t is

$$\frac{dC_t}{dI_t} = \frac{\left(\frac{1}{2}(\alpha-1)+\frac{1}{2}\omega\right)C_t+\vartheta_1}{\left(\frac{1}{2}(\alpha-1)-\frac{1}{2}\omega\right)I_t+\vartheta_2} \rightarrow$$

$$\frac{dC_t}{\left[\left(\frac{1}{2}(\alpha-1)+\frac{1}{2}\omega\right)C_t+\vartheta_1\right]}-\frac{dI_t}{\left[\left(\frac{1}{2}(\alpha-1)-\frac{1}{2}\omega\right)I_t+\vartheta_2\right]}=0 \tag{3.14}$$

Figure 3-2 hereunder depicts a phase portrait of Equation (3.14).

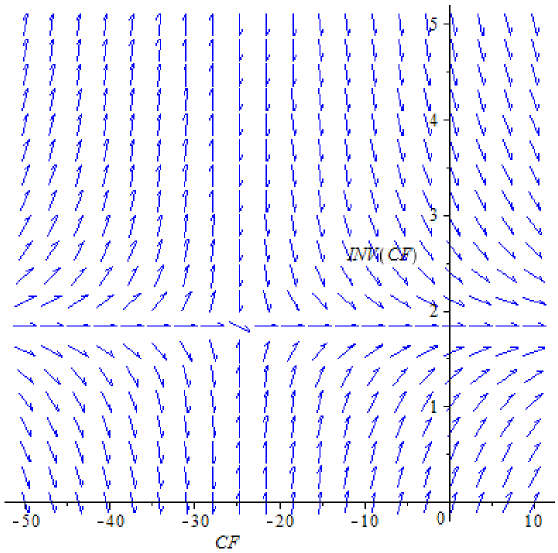


Figure 3-2 Phase portrait of Equation (3.14)

In the upper-right quadrant, firms with a high initial investing cash flow and a low operating cash flow move to a combination of lower investing cash flow and higher operating cash flow. In the lower right-quadrant the same movement is apparent, however the initial investment base is much lower. The firms in those two quadrants are called successful firms since (in a deterministic world) they will all be able to turn investments into growing operating cash flows.

In contrast to the two right quadrants, firms in the two left quadrants are unsuccessful: their operating cash flow hits a ceiling at a relative low level and if cash flow remains negative they will fold. For those firms increasing investments is not effective to break though the cash flow ceiling nor can a reduction of investment spending save them from potentially liquidation.

Solutions to Equation (3.14) are obtained as follows: set $\left(-\frac{1}{2}(\alpha-1)+\frac{1}{2}\omega\right)=\varphi_1$ and $(-\frac{1}{2}(\alpha-1)-\frac{1}{2}\omega)=\varphi_2$, then integrating Equation (3.14) gives

$$C_{t_k}(I_{t_k})=K_1(\varphi_2 I_{t_k}+\vartheta_2)^{\frac{\varphi_1}{\varphi_2}}-\frac{\vartheta_1}{\varphi_1} \tag{3.15a}$$

$$I_{t_k}(C_{t_k}) = K_2(\varphi_1 C_{t_k} + \vartheta_1)^{\frac{\varphi_2}{\varphi_1}} - \frac{\vartheta_2}{\varphi_2} \tag{3.15b}$$

where $K_1$and K_2 integration constants, and $t = t_k$ is a specific time.

3.4. The cash flow model in a stochastic environment

Cash flows as a Markov process

The model presented in this section is inspired by similar approaches from the field of stochastic population models in biological science, found in E. Allen (2010); L. J. S. Allen (2010); Matis and Kiffe (2012).

Define a Markov process as follows. For simplicity, suppose that in a small time Δt changes to the system $S\ \{\Delta C_{\Delta t}, \Delta I_{\Delta t}\}$ can only occur in the following four states:

State 1 describes a change of operating cash flow and consists of a constant in addition to a simultaneous uptick and downtick. The uptick is related to the prior level of investment and the downtick is caused by the deterioration of the firm's cash flow earning capacity due to obsolescence of capital goods (attrition). For simplicity, it is assumed that the uptick and downtick are stochastically indistinguishable and therefore considered one and the same event (with altering upticks and downticks alike).

Similar to the process description of State 1, **State 2** expresses how each of the current operating cash flow (uptick), the prior investment level (downtick) and a constant representing the level of investment that is not directly related to the current amount of cash flow, affect the level of investment. Again, upticks and downticks are stochastically indistinguishable.

State 3, simultaneous changes of C_t and I_t. In discrete time, this can be excluded and consequently **State 3** would not exist. This follows from the *principle of recurrence*, i.e. $C_{t-2} \rightarrow I_{t-1} \rightarrow C_t$ etcetera, which implies that C_t and I_t will not change at exactly the same time and therefore the probability of those simultaneous states occurring is 0. However, in continuous time the principle of recurrence presupposes that operating and investing cash flows are stochastically *independent* processes, a possibility that was already discounted in Section 2.4. Therefore, in an infinitesimal small-time interval dt when a discrete-time

Markov process is replaced by continuous-time Markov process, it is suggested that there is a link between dC_t and dI_t underpinning the interdependency between the two processes. If such connection is defined in continuous time, then it also exists in discrete time. Furthermore, it is suggested that the simultaneous relationship between ΔC_t and ΔI_t is governed by a constant (time-invariant) parameter φ: $\frac{\Delta C_{\Delta t}}{\Delta I_{\Delta t}} = \varphi$.

In **state 4** none of the forgoing changes will happen.

Table 3-2 presents a summary of the above analysis in the form of a Markov State-Change matrix $\{\Delta C_t, \Delta I_t\}$.

Table 3-2 State-Change matrix $\{\Delta C_t, \Delta I_t\}$

STATE	CHANGES IN C_t	CHANGES IN I_t	PROBABILITY
1	$\alpha C_{\Delta t} + \beta I_{\Delta t} + \delta$	0	$p_1 \Delta t$
2	0	$\gamma C_{\Delta t} - I_{\Delta t} + \varepsilon$	$p_2 \Delta t$
3	1	φ	$p_3 \Delta t$
4	0	0	$1 - p_1 \Delta t - p_2 \Delta t - p_3 \Delta t$

The vector of transition rates $\{p_1, p_2, p_3\}$ is usually interpreted as a vector of *rates of change*, i.e. the speed at which changes to C_t and I_t occur in time Δt. Likewise, the parameters $\{p_1 \Delta t, p_2 \Delta t, p_3 \Delta t\}$ are the transition amounts over time Δt which can be normalised as follows: $\sum_i p_i \Delta t = 1$ after which the parameters can be interpreted as probabilities of a change happening to either C_t ,I_t or to both in time Δt.

Supposedly, the transition probability of ΔC_t, $p_1 \Delta t$, is predominantly influenced by external factors including:

- the size and frequency of market opportunities available to the firm (upward movement);
- the speed of technology developments (downward movement).

In contrast, the transition probability of I_t, $p_2 \Delta t$, depends largely on internal factors such as:

- management recognising investment opportunities available to the firm;
- management timely adapting required investment levels to those investment opportunities;
- the firm having sufficient access to required external funding.

The sum of transition probabilities with non-zero changes is $P_g = (p_1 + p_2 + p_3)\Delta t$. It is obvious that P_g determines the overall growth rate of the system and that the breakdown in $\{p_1\Delta t, p_2\Delta t, p_3\Delta t\}$ embodies the different sources of growth.

The dynamics of the stochastic system $\{\Delta C_{\Delta t}, \Delta I_{\Delta t}\}$ are characterised by a set of external model parameters $\{\alpha, \beta, \delta, \varepsilon, p_1\}$ with a corresponding set of internal model parameters $\{\gamma, p_2\}$, together with parameters $\{\varphi, p_3\}$ that describe the dependency between the operating and investing cash flow processes. Note that parameters $\{\alpha, \beta, \delta, \varepsilon, \gamma\}$ also define the deterministic version of the system, and that, in addition, parameters $\{\varphi, p_{1,} p_2, p_3\}$ are required to typify the stochastic component of the system.

The expected value and variance of the decoupled system

In Appendix M2 the expected value vector and the variance-covariance matrix are derived for the decoupled, stochastic system $S'\{\Delta C'_{\Delta t}, \Delta I'_{\Delta t}\}$. It is shown that the expected value vector is *linear* and *diagonal* in variables $\{\Delta C'_{\Delta t}, \Delta I'_{\Delta t}\}$:

$$\mathbb{E}(\Delta \mathbf{v}) = \mathbf{M}.\mathbf{v} + \boldsymbol{\mu} \tag{3.16}$$

where $\Delta \mathbf{v} = \begin{pmatrix} \Delta C'_t \\ \Delta I'_t \end{pmatrix}, \mathbf{v} = \begin{pmatrix} C'_t \\ I'_t \end{pmatrix}, \mathbf{M} = \begin{pmatrix} \mu_{C,1} & 0 \\ 0 & \mu_{I,1} \end{pmatrix}$ is a symmetric diagonal matrix and $\boldsymbol{\mu} = \begin{pmatrix} \mu_{C,2} \\ \mu_{I,2} \end{pmatrix}$ is a vector of constants.

The elements of the vector $\mathbb{E}(\Delta \mathbf{v})$ are

$$\mathbb{E}\Delta C'_{\Delta t} = \mu_{C,1} C'_t \Delta t + \mu_{C,0}\, \Delta t \tag{3.17a}$$

$$\mathbb{E}\Delta I'_{\Delta t} = \mu_{I,1} I'_t \Delta t + \mu_{I,0}\, \Delta t \tag{3.17b}$$

Likewise, the variance-covariance matrix is *quadratic* and *diagonal* in variables $\{\Delta C'_{\Delta t}, \Delta I'_{\Delta t}\}$:

$$\mathbb{V}\Delta \mathrm{v} = \boldsymbol{\Lambda} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \tag{3.18}$$

where

$$\Lambda_{11} = \varsigma_{C,1} {C'_t}^2 \Delta t + \varsigma_{C,2} C'_t \Delta t + \varsigma_{C,3} \Delta t \tag{3.19a}$$

$$\Lambda_{12} = \varsigma_{C,4} {C'_t}^2 \Delta t + \varsigma_{C,5} C'_t \Delta t + \varsigma_{C,6} \Delta t \tag{3.19b}$$

$$\Lambda_{21} = \varsigma_{I,1} {I'_t}^2 \Delta t + \varsigma_{I,2} I'_t \Delta t + \varsigma_{I,3}\, \Delta t \tag{3.19c}$$

$$\Lambda_{22} = \varsigma_{I,4} {I'_t}^2 \Delta t + \varsigma_{I,5} I'_t \Delta t + \varsigma_{I,6}\, \Delta t \tag{3.19d}$$

From the variance-covariance matrix $\boldsymbol{\Lambda}$ the standard deviation matrix $\boldsymbol{\Sigma}'$ can be derived:

$$\boldsymbol{\Sigma}' = \boldsymbol{\Lambda}^{\frac{1}{2}} = \begin{pmatrix} \varsigma_{C,1}{C'_t}^2 + \varsigma_{C,2}C'_t + \varsigma_{C,3} & \varsigma_{C,4}{C'_t}^2 + \varsigma_{C,5}C'_t + \varsigma_{C,6} \\ \varsigma_{I,1}{I'_t}^2 + \varsigma_{I,2}I'_t + \varsigma_{I,3} & \varsigma_{I,4}{I'_t}^2 + \varsigma_{I,5}I'_t + \varsigma_{I,6} \end{pmatrix}^{\frac{1}{2}} \Delta t\,.$$

Since $\mathbb{V}\Delta v$ is the variance-covariance matrix of the *decoupled* system $S'\{\Delta C'_{\Delta t}, \Delta I'_{\Delta t}\}$, the processes $\Delta C'_{\Delta t}$ and $\Delta I'_{\Delta t}$ can be considered to be stochastically *independent* thus:

$$\boldsymbol{\Sigma}' = \boldsymbol{\Lambda}^{\frac{1}{2}} = \begin{pmatrix} \sqrt{\varsigma_{C,1}{C'_t}^2 + \varsigma_{C,2}C'_t + \varsigma_{C,3}} & \sqrt{\varsigma_{C,4}{C'_t}^2 + \varsigma_{C,5}C'_t + \varsigma_{C,6}} \\ \sqrt{\varsigma_{I,1}{I'_t}^2 + \varsigma_{I,2}I'_t + \varsigma_{I,3}} & \sqrt{\varsigma_{I,4}{I'_t}^2 + \varsigma_{I,5}I'_t + \varsigma_{I,6}} \end{pmatrix} \Delta t \tag{3.20}$$

Matrix $\boldsymbol{\Sigma}'$ is the standard deviation of the process in localised time Δt. In order to observe the process over a series of sequential Δt's, it is necessary to derive the *variance process*:

$$\boldsymbol{\Sigma}'.\boldsymbol{\zeta}_{\Delta t} = \begin{pmatrix} \sqrt{\varsigma_{C,1}{C'_t}^2 + \varsigma_{C,2}C'_t + \varsigma_{C,3}} & \sqrt{\varsigma_{C,4}{C'_t}^2 + \varsigma_{C,5}C'_t + \varsigma_{C,6}} \\ \sqrt{\varsigma_{I,1}{I'_t}^2 + \varsigma_{I,2}I'_t + \varsigma_{I,3}} & \sqrt{\varsigma_{I,4}{I'_t}^2 + \varsigma_{I,5}I'_t + \varsigma_{I,6}} \end{pmatrix} \begin{pmatrix} \Delta\zeta_{1,\Delta t} \\ \Delta\zeta_{2,\Delta t} \end{pmatrix} \Delta t \tag{3.21}$$

where $\zeta_{\Delta t} = \begin{pmatrix} \zeta_{1,\Delta t} \\ \zeta_{2,\Delta t} \end{pmatrix}$ is a vector of increments with $\zeta_{\Delta t} \sim \mathcal{N}(0, I)$ for a number of Δt observations, sufficiently large to invoke the CLT. This is the discrete-time equivalent to the derivation of Equation (1.8) in Section 1.4.

Subsequently, replace $\begin{pmatrix} \Delta\zeta_{1,\Delta t} \\ \Delta\zeta_{2,\Delta t} \end{pmatrix} \Delta t$ by a vector of discrete-time Wiener processes $\Delta\mathbf{W}' = \begin{pmatrix} \Delta W_{1,\Delta t} \\ \Delta W_{2,\Delta t} \end{pmatrix}$ and calculate the product $\boldsymbol{\Sigma}'.\boldsymbol{\zeta}_{\Delta t}$:

$$\boldsymbol{\Sigma}'.\boldsymbol{\zeta}_{\Delta t} = \begin{pmatrix} \sqrt{(\varsigma_{C,1} + \varsigma_{C,4}){C'_t}^2 + (\varsigma_{C,2} + \varsigma_{C,5})C'_t + (\varsigma_{C,3} + \varsigma_{C,6})}\Delta W_{C,\Delta t} \\ \sqrt{(\varsigma_{I,1} + \varsigma_{I,4}){I'_t}^2 + (\varsigma_{I,2} + \varsigma_{I,5})I'_t + (\varsigma_{I,3} + \varsigma_{I,6})}\Delta W_{I,\Delta t} \end{pmatrix} \tag{3.22}$$

by applying the formula of the sum of two independent Brownian motions (Wiersema (2008, pp. 89-95)): the combined Brownian motion of $\sigma_1 \Delta W_{1,\Delta t}$ and $\sigma_2 \Delta W_{2,\Delta t}$ is $\sigma \Delta W_{\Delta t}$ where $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$.

Replace $\varsigma_{C,1} + \varsigma_{C,4}$ by $\sigma_{C,2}$, $\varsigma_{C,2} + \varsigma_{C,5}$ by $\sigma_{C,1}$, $\varsigma_{C,3} + \varsigma_{C,6}$ by $\sigma_{C,0}$, $\varsigma_{I,1} + \varsigma_{I,4}$ by $\sigma_{I,2}$, $\varsigma_{I,2} + \varsigma_{C,5}$ by $\sigma_{I,1}$ and $\varsigma_{I,3} + \varsigma_{I,6}$ by $\sigma_{I,0}$ after which Equation (3.22) becomes:

$$\mathbf{\Sigma}'.\boldsymbol{\zeta}_{\Delta t} = \begin{pmatrix} \sqrt{(\sigma_{C,2}{C'_t}^2 + \sigma_{C,1}C'_t + \sigma_{C,0})}\Delta W_{C,\Delta t} \\ \sqrt{(\sigma_{I,2}{I'_t}^2 + \sigma_{I,1}I'_t + \sigma_{I,0})}\Delta W_{I,\Delta t} \end{pmatrix} = \tag{3.23}$$

Combining Equation (3.16) with Equation (3.23) provides the equations for the integral model

$$\Delta C'_{\Delta t} = (\mu_{C,1}C'_t + \mu_{C,0})\Delta t + \sqrt{(\sigma_{C,2}{C'_t}^2 + \sigma_{C,1}C'_t + \sigma_{C,0})}\Delta W_{C,\Delta t} \tag{3.24a}$$

$$\Delta I'_{\Delta t} = \left(\mu_{I,1}I'_t + \mu_{I,0}\right)\Delta t + \sqrt{(\sigma_{I,2}{I'_t}^2 + \sigma_{I,1}I'_t + \sigma_{I,0})}\Delta W_{I,\Delta t} \tag{3.24b}$$

or expressed in matrix notation

$$\Delta\mathbf{v} = (\mathbf{M}.\mathbf{v} + \boldsymbol{\mu})dt + \mathbf{\Sigma}\Delta\mathbf{W} \tag{3.25}$$

where $\Delta\mathbf{v} = \begin{pmatrix}\Delta C'_t \\ \Delta I'_t\end{pmatrix}, \mathbf{v} = \begin{pmatrix}C'_t \\ I'_t\end{pmatrix}, \mathbf{M} = \begin{pmatrix}\mu_{C,1} & 0 \\ 0 & \mu_{I,1}\end{pmatrix}$ is a symmetric diagonal matrix and $\boldsymbol{\mu} = \begin{pmatrix}\mu_{C,2} \\ \mu_{I,2}\end{pmatrix}$ is a vector of constants, $\mathbf{\Sigma} = \begin{pmatrix} \sqrt{(\sigma_{C,2}{C'_t}^2 + \sigma_{C,1}C'_t + \sigma_{C,0})} & 0 \\ 0 & \sqrt{(\sigma_{I,2}{I'_t}^2 + \sigma_{I,1}I'_t + \sigma_{I,0})} \end{pmatrix}$ is the standard deviation matrix of independent processes $\Delta W_{C,\Delta t}$ and $\Delta W_{I,\Delta t}$, and $\Delta\mathbf{W}$ is the corresponding vector.

The continuous-time model

The analysis above provides all the ingredients necessary to describe the stochastic process $S'\{\Delta C'_{\Delta t}, \Delta I'_{\Delta t}\}$ in the limiting case of continuous-time. According to the CLT the process S' is approximately normally distributed with $S \sim \mathcal{N}(\mathbb{E}(\Delta\mathbf{v}), \mathbb{V}(\Delta\mathbf{v}))$ provided of course that there are a sufficient number of $\Delta t'$s.

Taking the time limit $\Delta t \rightarrow 0$ the system $S\,\{C'_t, I'_t\}$ can now be described by the following two SDEs

$$dC'_t = \left(\mu_{C,1}C'_t + \mu_{C,0}\right)dt + \sqrt{(\sigma_{C,2}C'^2_t + \sigma_{C,1}C'_t + \sigma_{C,0})}\, dW_{C,t} \tag{3.26a}$$

$$dI'_t = \left(\mu_{I,1}I'_t + \mu_{I,0}\right)dt + \sqrt{(\sigma_{I,2}I'^2_t + \sigma_{I,1}I'_t + \sigma_{I,0})}\, dW_{I,t} \tag{3.26b}$$

There is some caution warranted when taking the time limit: recall from Sections 1.5. and 1.6. the issues that may arise when a (fairly spiked) discrete process is approximated by a continuous process. One of the solutions to reduce process variability per time unit is to de-scale cash flows by an appropriate proxy for the system size, i.e. $c_t' = \frac{C_t'}{N_1}$ and $i_t' = \frac{I_t'}{N_2}$, where $\{N_1, N_2\}$ are system size proxies and $\{c_t', I_t'\}$ are de-scaled cash flows. The de-scaled system then becomes

$$dc_t' = \left(\mu_{C,1}c_t' + \mu_{C,0}\right)dt + \sqrt{\frac{(\sigma_{C,2}c_t'^2 + \sigma_{C,1}c_t' + \sigma_{C,0})}{N_1}}\ dW_{C,t} \tag{3.27a}$$

$$di_t' = \left(\mu_{I,1}i_t' + \mu_{I,0}\right)dt + \sqrt{\frac{(\sigma_{I,2}i_t'^2 + \sigma_{I,1}i_t' + \sigma_{I,0})}{N_2}}\ dW_{I,t} \tag{3.27b}$$

Specifically for investing cash flow processes, Equation (3.27b) can be expected to provide better fitting statistics than Equations (3.26b).

The above SDEs are called *hybrid* SDEs since they can be considered to be a mix of two basic SDEs. Shaw and Schofield (2015) demonstrate how, for example in this study, the equation $dC_t' = \left(\mu_{C,1}C_t' + \mu_{C,0}\right)dt + \sqrt{(\sigma_{C,2}C_t'^2 + \sigma_{C,1}C_t' + \sigma_{C,0})}\ dW_{C,t}$ can be decomposed into a Geometric Brownian motion and an Arithmetic Brownian motion which are stochastically *dependent* on each other

$$dC_t' = \left(\mu_{C,1}C_t' + \mu_{C,0}\right)dt + \sqrt{(\sigma_{C,2}C_t'^2 + \sigma_{C,1}C_t' + \sigma_{C,0})}\ dW_{C,t} = \left(\mu_{C,1}C_t' + \mu_{C,0}\right)dt + \sqrt{(\sigma_{C,2}C_t'^2 + 2\rho\sqrt{(\sigma_{C,2}\sigma_{C,0}}C_t' + \sigma_{C,0}}\ dW_{C,t} = \left[\mu_{C,1}C_t'dt + \sqrt{\sigma_{C,2}}C_t'dW_{C,2,t}\right]_{GBM} + \left[\mu_{C,2}dt + \sqrt{\sigma_{C,0}}\ dW_{C,0,t}\ \right]_{ABM} \tag{3.28}$$

where $\sigma_{C,1} = 2\rho\sqrt{(\sigma_{C,2}\sigma_{C,0}}$ and ρ is the correlation coefficient between $W_{C,2,t}$ and $W_{C,1,t}$. Notice that an alternative expression for Equations (3.26a) (and similarly for (3.26b)) is:

$$dC_t' = \left(\mu_{C,1}C_t' + \mu_{C,0}\right)dt + \sqrt{(\sigma_{C,2}C_t'^2 + 2\rho\sqrt{\sigma_{C,2}\sigma_{C,0}}C_t' + \sigma_{C,0})}\ dW_{C,t} \tag{3.29}$$

Another example of a hybrid SDE, a combination of an ABM and a CIR process, can be found in Schofield (2015). Interestingly, as will also become clear in Chapter 4, the stochastic properties of hybrid SDEs are much richer than the sum of the properties of the basic SDEs. These SDEs underpin probability distributions that are more versatile and often more complex than those of the underlying basic processes. Here it suffices to cite Shaw and Schofield (2015, p. 977): "Under mean-reverting circumstances we are led naturally to equilibrium fat-tailed return distribution with a Student-t or skew Student form, with the latter defined in the framework of 'Pearson diffusions'….". Indeed, observing Equations (3.13) and (3.14) in Chapter 2, it is not hard to see the connection between the equilibrium Pearson differential equation and Equations (3.26a) and (3.26b). This connection potentially provides access to all members of the Pearson family of probability distributions, especially the Type-IV ones as mentioned by Shaw & Schofield, that were supported by empirical research as a good fit with the investing cash flow process (refer to Section 2-4). Furthermore, Equations (3.26a) and (3.26b) not only describe mean-reverting behaviour as implied by the Pearson diffusion process, but, in addition, diverging behaviour typical for the operating cash flow process. In Chapter 4 solutions to these equations, that admit a wide and diverse range of solutions, will be investigated in more detail.

The issue of the cash flow domain on which Equations (3.26a) and (3.26b) are valid, is now discussed. The linear drift functions, $\mu_{C,1}C'_t + \mu_{C,0}$ and $\mu_{I,1}I'_t + \mu_{I,0}$, warrant no particular domain restrictions. Test results in Chapter 2 show that parameter $\mu_{C,1}$ usually has positive values and hence the domain of operating cash-flow must be unrestricted to permit a diverging process in time. By contrast, values for $\mu_{I,1}$are almost always negative and consequently the process is mean-reversing, implying a restricted investing cash flow domain. The domain of the diffusion function (instantaneous chance of variance) is slightly more complicated. For the square root of to be defined, it must have positive real values. By implication: the diffusion function of investing cash flow must be positive over the full domain and the diffusion function of operating cash flow must be positive over part of the domain. In Chapter 2 evidence is presented that operating cash flows have predominantly two real roots and investing cash flows are characterised by complex roots. First the investing cash flow case: If $\sigma_{I,2}I'^2_t + \sigma_{I,1}I'_t + \sigma_{I,0} \geq 0$ then it follows that $\sigma_{I,2} > 0$. Operating cash flows have two real roots and the domain of the diffusion function is *between the lower and upper root*, as will be substantiated in Chapter 4. Hence, for $\sigma_{C,2}C'^2_t + \sigma_{C,1}C'_t +$

$\sigma_{C,0} \geq 0$ and $\lambda_1 < C'_t < \lambda_2$ (where $\lambda_1; \lambda_2$ are the roots of $\sigma_{C,2}C_t'^2 + \sigma_{C,1}C'_t + \sigma_{C,0}$) to be true, $\sigma_{C,2}$ must be smaller than zero. This poses an apparent contradiction that is resolved by accepting that for the purpose of this study $\sigma_{C,2}C_t'^2 + \sigma_{C,1}C'_t + \sigma_{C,0}$ is equivalent to $-(\sigma_{C,2}C_t'^2 + \sigma_{C,1}C'_t + \sigma_{C,0})$. The same reasoning applies to investing cash flow diffusion functions. A summary of the foregoing is presented in Figure 3-3.

If, however, positive cash flow values on the whole domain $(-\infty;\ +\infty)$ are a prerequisite, then an absolute value of the quadratic function serves the purpose. For instance, some inverse transformations require positive domain values only. Figure 3-4 illustrates this.

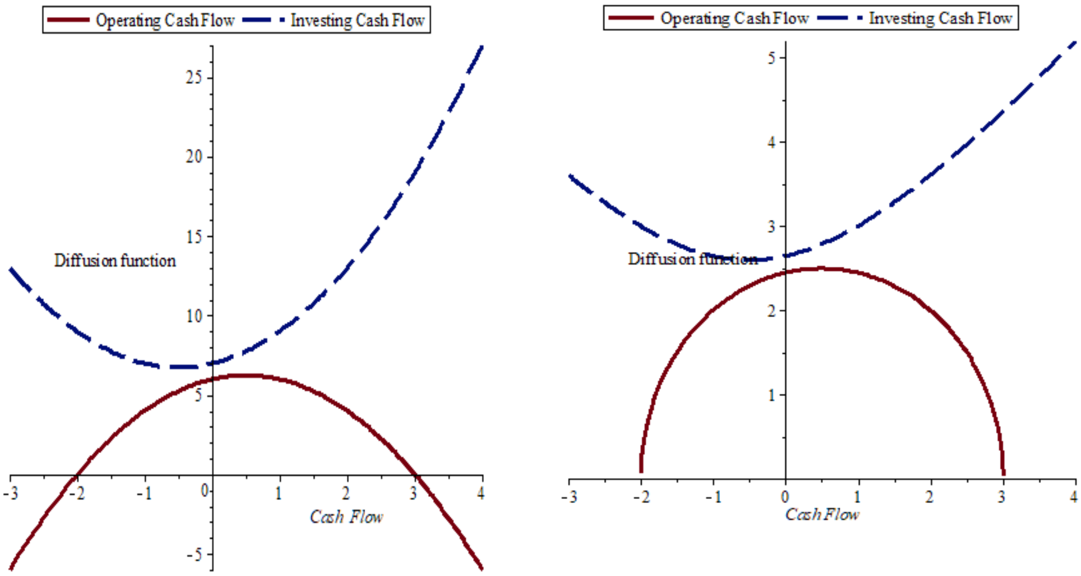


Figure 3-3 Domain of diffusion functions (LHS: quadratic function; RHS: square root of quadratic function)

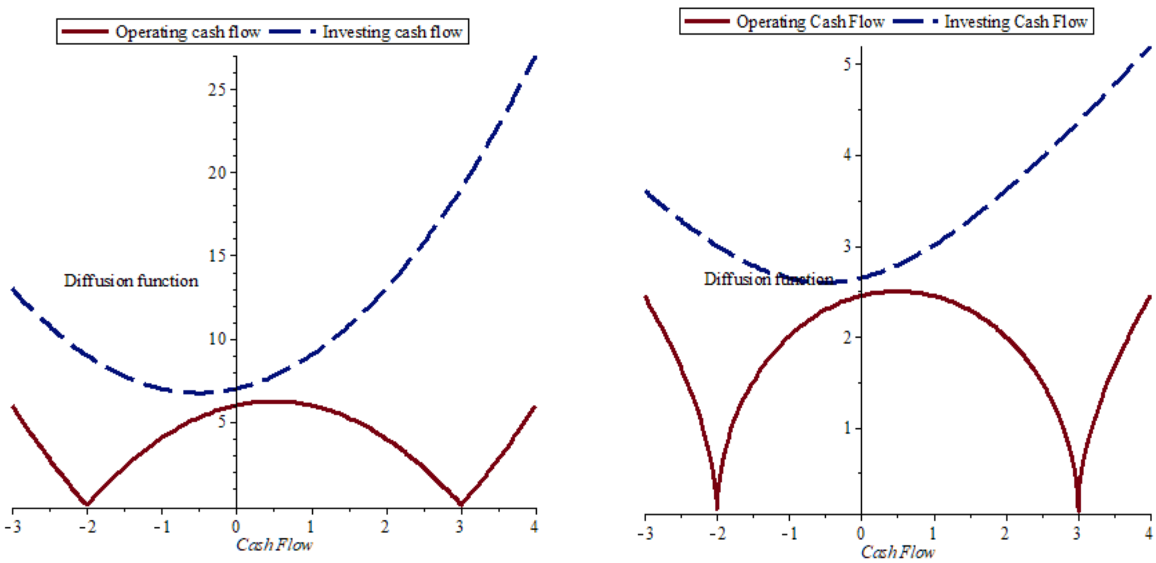


Figure 3-4 Domain of absolute value of diffusion functions (LHS: quadratic function; RHS: square root of quadratic function)

In Chapters 4 and 5 the inverse of the diffusion function will be employed, for instance when applying a Lamperti transformation. From Figure 3-5 it is obvious that the inverse diffusion function of operating cash flows has singularities for each of the (real) roots.

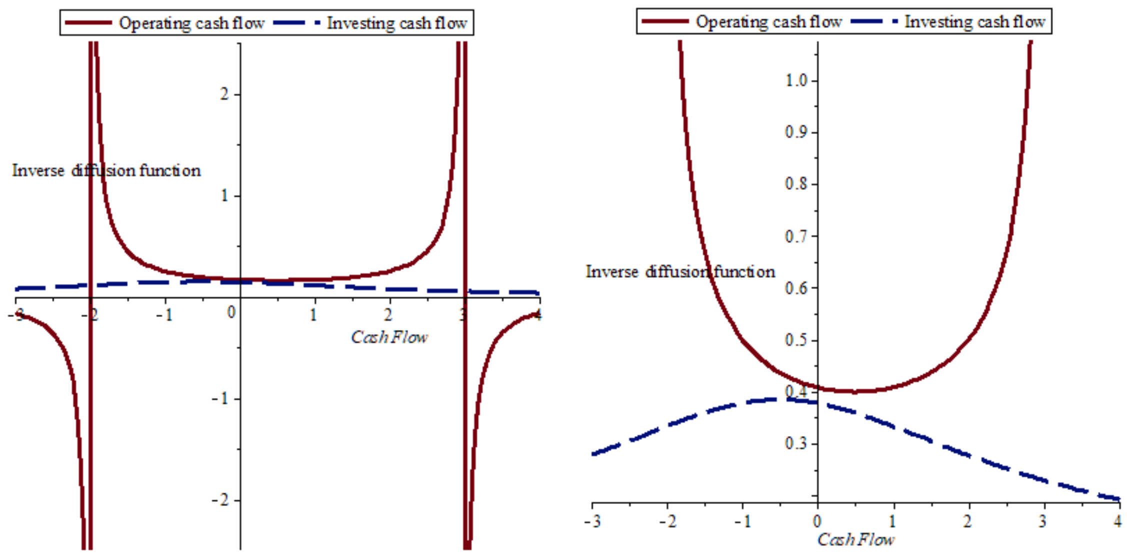


Figure 3-5 Domain of inverse of diffusion functions (LHS: quadratic function; RHS: square root of quadratic function)

One final remark: solutions to Equations (3.26a) and (3.26b) are solutions to the *decoupled* system $S\,\{C'_t, I'_t\}$; solutions to the *coupled* system $S\,\{C_t, I_t\}$ require one extra step $\mathbf{u}_t = \mathbf{Q}\,\mathbf{v}_t$ where $\mathbf{u}_t = \begin{pmatrix} C_t \\ I_t \end{pmatrix}$, $\mathbf{v}_t = \begin{pmatrix} C'_t \\ I'_t \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} \frac{2\beta}{-1-\alpha+\omega} & \frac{2\beta}{-1-\alpha-\omega} \\ 1 & 1 \end{pmatrix}$ where $\omega = \sqrt{(\alpha-1)^2 + 4(\beta\gamma+\alpha)}$. Similar to the solutions of the coupled system in a deterministic environment (see Section 2.3.), these *linear combinations* of solutions to the coupled system augment the stochastic capabilities of the model even further.

3.5. A simulation of the linear-quadratic cash flow model

Before exploring solutions to the linear-quadratic cash flow model, it is useful to perform a reality check. A discrete-time model was constructed that simulates trajectories of stochastic realisations, each representing a fictitious firm, encompassing 100 points of operating and investing cash flow data on a time line. The results of one such simulation are depicted in Figures 3-6 and 3-7 below.

Comparison of the simulated data in Figures 3-6 and 3-7 to the real-world data of Figures 1-3 and 1-4, allows a number of observations to be made:

- It takes considerable time before the growth of cash flows becomes noticeable: this is the exponential growth process, particularly visible in Figure 2-3 (operating cash flow), at work. When making the above comparison, only data-points from roughly time 50 in Figures 2-3 and 2-4 should be considered: all prior data-points, from inception date to time = 50, represent the firm in its infancy state prior to being qualified for a public listing.
- Both operating cash flow graphs clearly agree on a fat tail marginal probability distribution: few firms (which is not exceptionally rare with only 20 realisations) are very successful while the majority of firms experience modest growth. Note that in the simulation model no attempt was made to mimic the seasonal revenue pattern characteristic of real-world quarterly operating cash flow data.

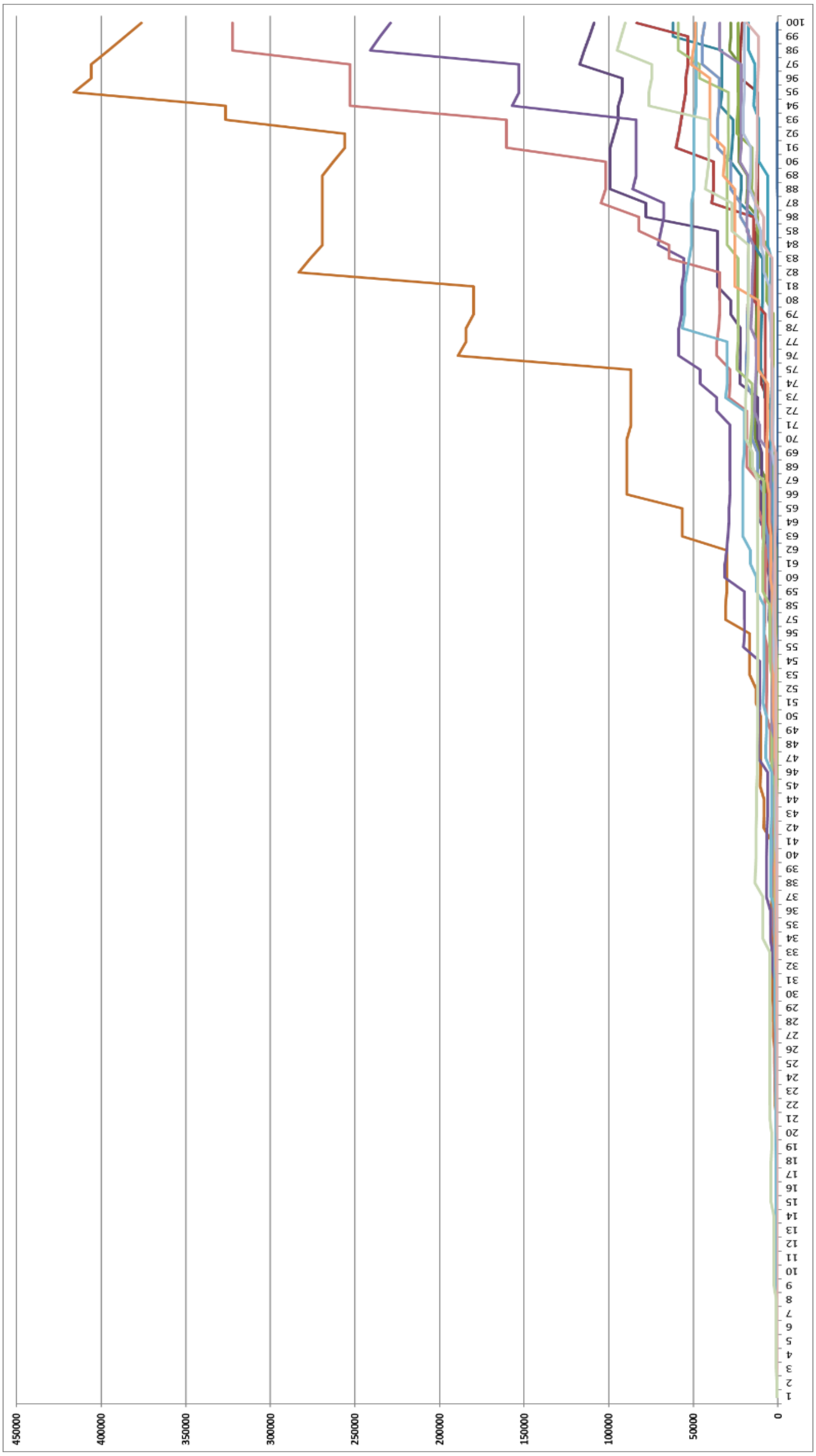


Figure 3-6 Simulated trajectories of Operating Cash Flow - Hybrid Coupled model

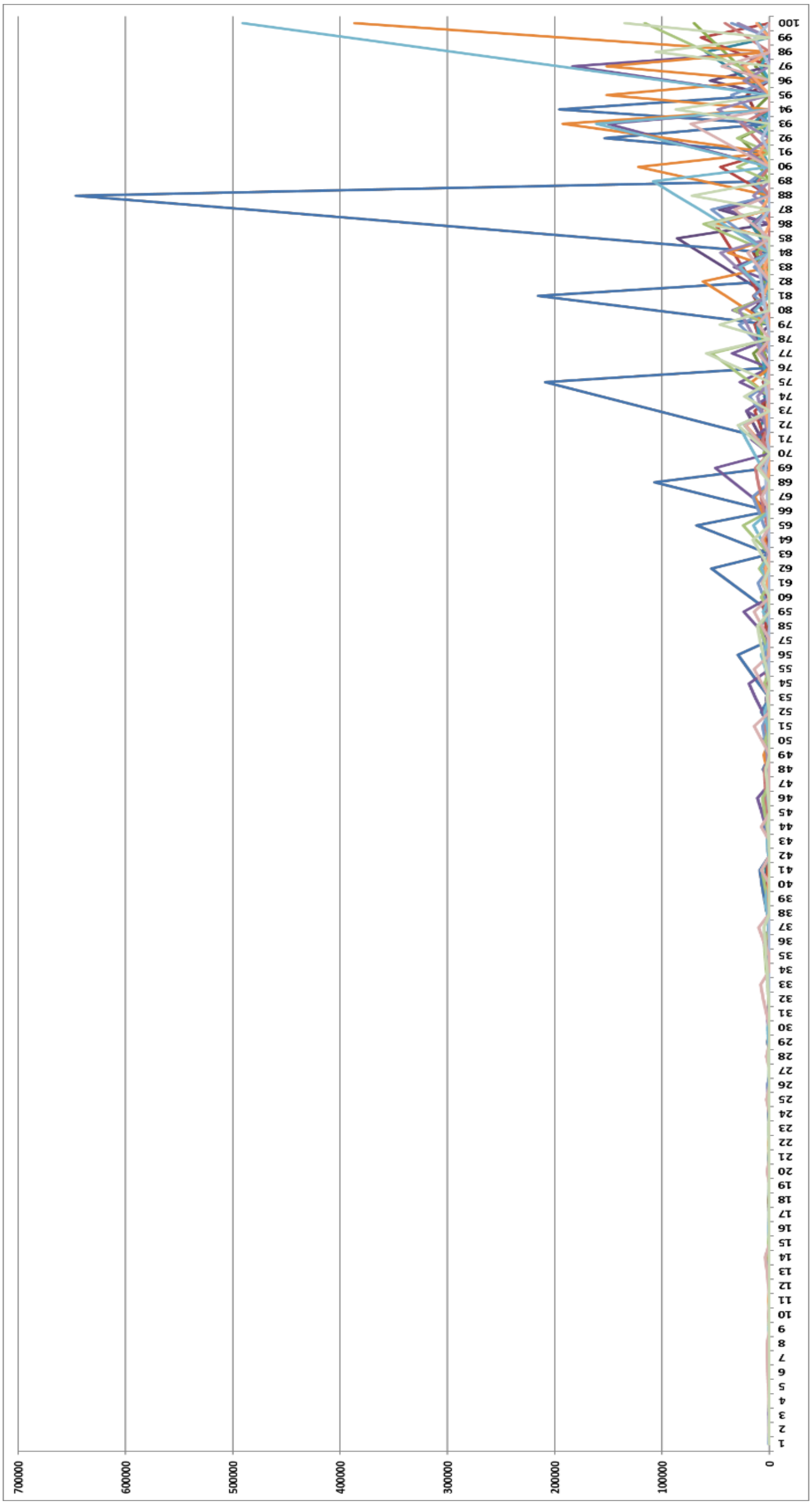


Figure 3-7 Simulated trajectories of Investing Cash Flow - Hybrid Coupled model

3.6. Conclusions from Chapter 3

Five-well-known continuous-time specifications that are frequently encountered in the literature, are all rejected as being adequate, on different grounds and to varying degrees, for modelling cash flow processes. Therefore, there is a need for a new cash flow specification.

It was demonstrated that a relatively simple model consisting of two stochastic equations, is capable of simulating real-world cash flow behaviour. The model is founded on a bi-causal relationship between operating cash flow and investing cash flow, mathematically configured as a coupled model.

Furthermore, it was shown that the discrete Markov version of the model, can be sufficiently well approximated by a continuous-time model consisting of two decoupled and independent Wiener processes, both consisting of a linear drift and a quadratic diffusion function. This model is a hybrid of two basic stochastic processes: a geometric Brownian motion and an arithmetic Brownian motion.

Consistent with observations and theoretical considerations from Chapters 1 and 2, the approximated continuous-time model is expected to be a more versatile and accurate representation of real cash flow processes than the five well-known cash flow specifications from the literature. However, this must still be corroborated by proper benchmark statistical testing which will be performed in Chapter 5.

4. Solutions to the Coupled Linear-Quadratic Cash Flow Model

Chapter 4 deals with solutions to the coupled linear-quadratic cash flow model as described in detail in Chapter 3. Possible solutions to the stochastic differential equation and the corresponding Kolmogorov equations, are analysed and discussed. The emphasis is on techniques to derive particular and limiting solutions if an overarching general solution is not feasible. The last section of the chapter shows how the decoupled solutions to the operating and investing cash flow processes, can be recombined to a coupled solution.

4.1. Solutions to the hybrid cash flow Stochastic Differential Equation

Existence, uniqueness and convergence of strong solutions

In Equation (3.26a) of Chapter 3, the decoupled operating cash flow was derived in this SDE format: $dC'_t = (\mu_{C,1}C'_t + \mu_{C,0})dt + \sqrt{(\sigma_{C,2}C'^2_t + \sigma_{C,1}C'_t + \sigma_{C,0})}\, dW_{C,t}$ (4.1a)

For notational convenience, Equation (4.1a) can be reformulated to a (uncoupled) *general cash flow process*

$$dX_t = (\mu_1 X_t + \mu_0)dt + \sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)}\, dW_t \tag{4.1b}$$

where X_t can be either an operating cash flow C_t or investing cash flow I_t.

Appendix M3 provides mathematical evidence that a general cash flow process is approximately Lipschitz continuous and exactly continuous according to the (less restrictive) Aït-Sahalia conditions. The importance of this test is to show that unique, strong[29] solutions to the SDE do exist.

In addition to continuity, Appendix M3 examines under what conditions a general cash flow process (1b) converges or diverges in time. It was found that convergence (in the mean-square sense) of the diffusion function is entirely dependent on the behaviour of the drift function. If the drift function reverts to the mean, which occurs under the condition that $\mu_1 < 0$ (Equation (4.1b)), then the diffusion process is also converging. If, however, the drift

[29] A strong solution to the SDE has the functional form $X_t(t, W_t)$.

function shows diverging behaviour, $\mu_1 > 0$, then the diffusion process is dominated by the drift process and the combined behaviour of Equation (4.1b) becomes also diverging.

Exact general solutions to the SDE

After having established that there are unique, strong solutions to the (uncoupled) general cash flow process equation, the logical next question is whether solutions are analytically tractable. It is a well-known fact that only a limited number of SDEs have exact (closed-form) solutions, and that most SDEs must be solved approximately by numerical solution techniques. In this section, it is examined if, and under which conditions, Equation (4.1b) has exact solutions.

A very useful test for exact solutions, based on Itô's lemma, is presented in Stepanov (2013, paragraphs 2.4 and 2.7). For a general Itô process $dX_t = \alpha(X_t, t)dt + \sqrt{\beta(X_t, t)}dW_t$ the so-called compatibility condition can be expressed in the following equality

$$\frac{1}{s(t)}\frac{\partial}{\partial t}\left\{\frac{s(t)}{\sqrt{\beta(X_t, t)}}\right\} = \frac{1}{2}\frac{\partial^2\sqrt{\beta(X_t, t)}}{\partial x^2} - \frac{\partial}{\partial x}\left\{\frac{\alpha(X_t, t)}{\sqrt{\beta(X_t, t)}}\right\} \tag{4.2}$$

where $F_t(X_t)$ is a transformation function, accomplishing that the transformed process becomes an Itô integrable process

$$dF_t = f(t)dt + s(t)dW_t \tag{4.3}$$

with $f(t) = \frac{\partial F_t}{\partial t} + \alpha(X_t, t)\frac{\partial F_t}{\partial x} + \frac{\beta(X_t,t)}{2}\frac{\partial^2 F_t}{\partial x^2}$ and $s(t) = \sqrt{\beta(X_t, t)}\ \frac{\partial F_t}{\partial x}$.

For time-invariant functions $\left\{\alpha(X_t), \sqrt{\beta(X_t)}\right\}$, assumed to exist for the general cash flow process, the compatibility condition (2) can be written in the following form

$$\frac{\frac{\partial s(t)}{\partial t}}{s(t)} = \frac{1}{2}\ \sqrt{\beta(X_t)}\ \frac{\partial^2\sqrt{\beta(X_t)}}{\partial x^2} - \frac{\partial\left[\frac{\alpha(X_t)}{\sqrt{\beta(X_t)}}\right]}{\partial x} \tag{4.4}$$

Since the LHS of Equation (4.4) depends on t only, and the RHS on X_t only, both sides can be equated to a constant λ.

Integrating Equation (4.4) yields the following expression in which the drift function (LHS) is related to the diffusion function (LHS)

$$\alpha(X_t) = \frac{1}{2}\frac{\partial\beta(X_t)}{\partial x} + K\sqrt{\beta(X_t)} - \lambda\sqrt{\beta(X_t)}\int\frac{dx}{\sqrt{\beta(X_t)}} \tag{4.5}$$

where K is an integration constant.

Substituting the drift and diffusion functions of Equation (4.1b) into Equation (4.5), results in the following specific compatibility condition

$$\mu_1 X_t + \mu_0 = \frac{1}{2}(2\sigma_2 X_t + \sigma_1) + K\sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)} -$$

$$\lambda \frac{\sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)}}{\sqrt{\sigma_2}} \int \frac{dx}{\sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)}} \tag{4.6}$$

Hence, Equation (4.1b) has exact solutions for all X_t and all parameters $\{\mu_0, \mu_1, \sigma_0, \sigma_1, \sigma_2\}$, only if the equality in Equation (4.6) is obeyed. Since the LHS of Equation (4.6) is linear in X_t, the three terms of the RHS will need to be also linear in X_t, so that, with arbitrary values for K and λ, the equation meets the prior conditions of a general solution.

It is clear that the second term of the RHS of Equation (4.6) is nonlinear (for $K \neq 0$, $\{\sigma_0, \sigma_1, \sigma_2\} \neq 0$) which is, generally, also true for the third term (for $\lambda \neq 0$, $\{\sigma_0, \sigma_1, \sigma_2\} \neq 0$). Jeffrey (2004, section 4.3.4) provides the following (multiple) solutions to the integral in Equation (4.6)

$\frac{1}{\sqrt{\sigma_2}} \ln |2\sqrt{\sigma_2}\sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)} + 2\sigma_2 X_t + \sigma_1|$ if $\sigma_2 > 0$ which expression evolves to

(i) $\frac{1}{\sqrt{\sigma_2}} \sinh^{-1}\left[\frac{2\sigma_2 X_t + \sigma_1}{\sqrt{\Delta}}\right]$ if $\Delta > 0$, or

(ii) $\frac{1}{\sqrt{\sigma_2}} \ln |\, 2\sigma_2 X_t + \sigma_1|$ if $\Delta = 0$, or

(iii) $\frac{-1}{\sqrt{-\sigma_2}} \sin^{-1}\left[\frac{2\sigma_2 X_t + \sigma_1}{\sqrt{-\Delta}}\right]$ if $\sigma_2 < 0$ and $\Delta < 0$,

where $\Delta = 4\sigma_2\sigma_0 - \sigma_1^2$.

Evidently, there is no general linear equation[30] with parameters $\{\mu_0, \mu_1, \sigma_0, \sigma_1, \sigma_2, K, \lambda\}$ that can replace the second and third terms on the RHS of Equation (4.6) and therefore the conclusion is that SDE (1b) admits no *general* exact solution.

Nevertheless, if Equation (4.1b) is considered a hybrid of the two underlying basic SDEs in the form of a GBM and an ABM process, comparable to the decomposition of Equation (3.28) in Chapter 3, an exact *fundamental* solution can indeed be derived (see Shaw and Schofield (2015, p. 984) for an outline of this method).

[30] Other than a linear approximation of nonlinear functions.

Equation (4.1b) written in decomposed format is

$$dX_t = \mu_1 X_t dt + \sqrt{\sigma_2} X_t dW_{1,t} + \mu_0 dt + \sqrt{\sigma_0} dW_{2,t} \tag{4.7}$$

Define the stochastic integrating factor as

$$I_t = \exp(-\sqrt{\sigma_2} W_{1,t} + (\frac{1}{2}\sigma_2 - \mu_1)t) \tag{4.8}$$

which implies that

$$dI_t = (\sigma_2 - \mu_1) I_t dt - \sqrt{\sigma_2} I_t dW_{1,t} \tag{4.9}$$

Furthermore, define $Q_t = I_t X_t$. Then $dQ_t = d(I_t X_t) = I_t dX_t + X_t dI_t + dI_t dX_t$ and observe that $dW_{1,t} dW_{2,t} = \rho dt$ where ρ is the correlation coefficient between W_1 and W_2.

Then $dQ_t = I_t\left[(\mu_0 - \rho\sqrt{\sigma_0}\sqrt{\sigma_2})dt + \sqrt{\sigma_0} dW_{2,t}\right]$ (4.10)

and $X_t = I_t^{-1} \int I_s\left[(\mu_0 - \rho\sqrt{\sigma_0}\sqrt{\sigma_2})ds + \sqrt{\sigma_0} dW_{2,s}\right]$ (4.11)

Since t is the reference time, Equation (4.11) can be re-written to:

$$X_t = \int I_t^{-1} I_s\left[(\mu_0 - \rho\sqrt{\sigma_0}\sqrt{\sigma_2})ds + \sqrt{\sigma_0} dW_{2,s}\right] \tag{4.12}$$

The expression $I_t^{-1} I_s$ can be interpreted as a time shift function which evaluates to:

$$I_t^{-1} I_s = \exp\left[\sqrt{\sigma_2}\widetilde{W}_{1,u} + (\tfrac{1}{2}\sigma_2 - \mu_1)u\right] \tag{4.13}$$

where $\widetilde{W}_{1,u} = W_{1,t} - W_{1,s}$ and $u = s - t$

Now, Equation (4.22) becomes the following functional expression:

$$X_t = \exp\left[\sqrt{\sigma_2}\widetilde{W}_{1,u} + (\frac{1}{2}\sigma_2 - \mu_1)u\right]\int\left[(\mu_0 - \rho\sqrt{\sigma_0}\sqrt{\sigma_2})ds + \sqrt{\sigma_0} dW_{2,s}\right] \tag{4.14}$$

Equation (4.14) provides an insight into the *functional form* of general solutions to the cash flow process: these can be defined as the product of an exponential (time-shifted) Brownian motion and the integral of another linear Brownian motion, correlated to the first motion.

Exact particular solutions to the SDE

In the prior subsection, it was inferred that there are no general solutions to the cash flow process. Hence, one has to revert to specific solutions such as particular solutions and approximated solutions. In this subsection, some particular solutions to Equation (4.1b) are analysed.

Here, *particular solutions* are defined as exact solutions that can be derived from imposing restrictions on the set of model parameters $\{\mu_0, \mu_1, \sigma_0, \sigma_1, \sigma_2\}$. Particular solutions can, for example, be extracted from Equation (4.6) by setting one or more of the parameters $\{K, \lambda\}$ to zero. The obvious first choice is parameter $\lambda = 0$ which restriction implies that function $s(t)$ in Equation (4.2) is a constant. Then Equation (4.6) is reduced to

$$(\mu_1 - \sigma_2)X_t + (\mu_0 - \frac{1}{2}\sigma_1) = K\sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)} \tag{4.15}$$

The first special case can be deduced from Equation (4.15).

Particular case one:

The term $K\sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)}$ can be made linear by equating parameter σ_1 to $2\sqrt{\sigma_0\sigma_2}$. This restriction brings about a linear diffusion process as a reduced form of the usual quadratic diffusion function. Now, Equation (4.15) becomes

$$(\mu_1 - \sigma_2)X_t + (\mu_0 - \frac{1}{2}\sigma_1) = K(\sqrt{\sigma_2}\,X_t + \sqrt{\sigma_0}) \tag{4.16}$$

which equation is solved for $(\mu_1 - \sigma_2) = K\sqrt{\sigma_2}$ and $(\sigma_2 - \frac{1}{2}\sigma_1) = K\sqrt{\sigma_0}$.

The solution to this equation implies that $\mu_1 = \sigma_2 + K\sqrt{\sigma_2}$ and $\mu_0 = \frac{1}{2}\sigma_1 + K\sqrt{\sigma_0}$, i.e. for $K \in \mathbb{R}$, the set of parameters $\{\mu_0, \mu_1\}$ can take any real value. As a result, Equation (4.1b) is now reduced to a *linear* SDE

$$dX_t = (\mu_1 X_t + \mu_0)dt + K(\sqrt{\sigma_2}\,X_t + \sqrt{\sigma_0})dW_t \tag{4.17}$$

Solutions to a linear SDE are well known: Kloeden and Platen (2011, section 4.2)

$$X_{t,K} = I_t\left[X_o + (\mu_0 - K^2\sqrt{\sigma_0\sigma_2})\int I_s^{-1}\,ds + K\sqrt{\sigma_2}\int I_s^{-1}dW_s\right] \tag{4.18}$$

where I_t is an integration factor and $I_t = \exp\left[(\mu_1 - \frac{1}{2}K^2\sigma_2)t + K\sqrt{\sigma_2}W_t\right]$ and X_o is the initial cash flow at $t = 0$.

After evaluating the two integrals, Equation (4.18) yields the following expression

$$X_{t,K} = X_o \exp\left[(\mu_1 - \tfrac{1}{2}K^2\sigma_2)t + K\sqrt{\sigma_2}W_t\right] + \left[\left(\mu_0 - K^2\sqrt{\sigma_0\sigma_2}\,\right)t + K\sqrt{\sigma_2}\,W_t\right] \tag{4.19}$$

Define a new parameter $\sigma_3 = K\sqrt{\sigma_2}$ and re-write Equation (4.19) to a three-parameter equation that is more generic in X_t

$$X_t = X_o \exp\left[(\mu_1 - \tfrac{1}{2}\sigma_3^2)t + \sigma_3 W_t\right] + \left[\left(\mu_0 - \frac{\sqrt{\sigma_0}}{\sqrt{\sigma_2}}\sigma_3^2\right)t + \sigma_3 W_t\right] \tag{4.20}$$

Interestingly, Equation (4.20) is the sum of a multiplicative Brownian motion and an additive Brownian motion. Recall that the fundamental solution in Equation (4.14) also includes a multiplicative and an additive component. Comparing the two equations, one can observe that linearization of the diffusion equation alters the functional form to the sum of two Brownian motions instead of two multiplied Brownian motions where different Wiener processes in the functional solution are transformed into one common Wiener process after linearization.

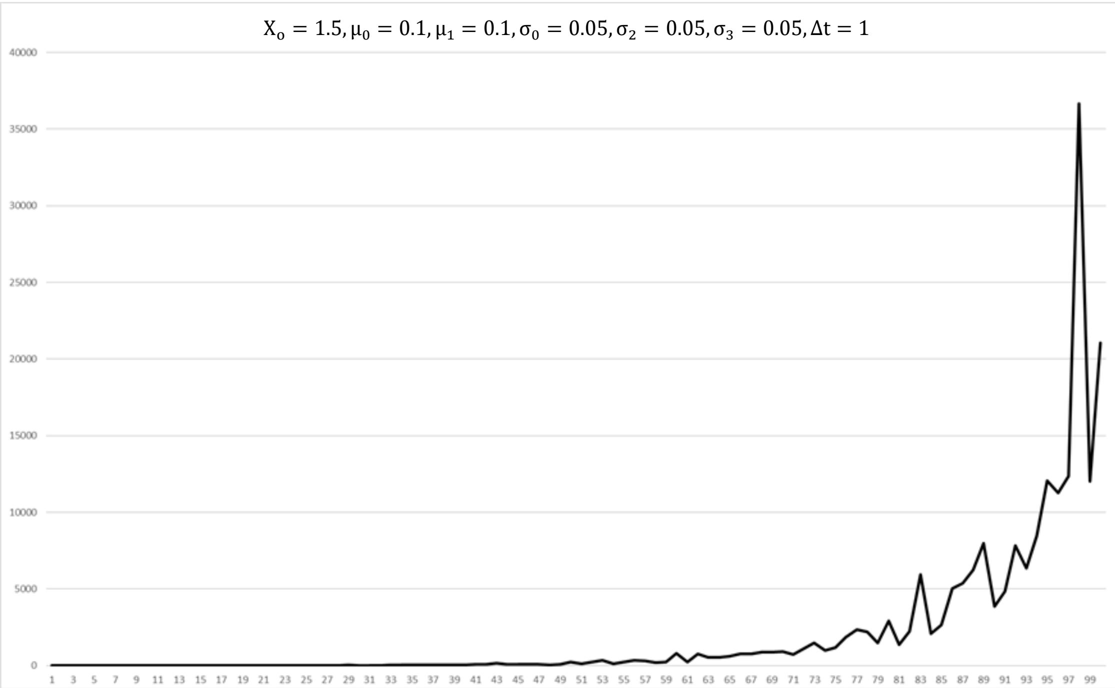


Figure 4-1 Example realisation of particular case one

As visible in Figure 4-1, the process of particular case one is dominated by an exponential random growth process if $\mu_1 > \frac{1}{2}\sigma_3^2$ and by a linear random process if $\mu_1 < \frac{1}{2}\sigma_3^2$. With the appropriate choice of parameters, in particular setting $\mu_0 = 0$ and selecting a very small parameter μ_1, the process can be adapted to simulate behaviour that is mean-reverting: see Figure 4-2.

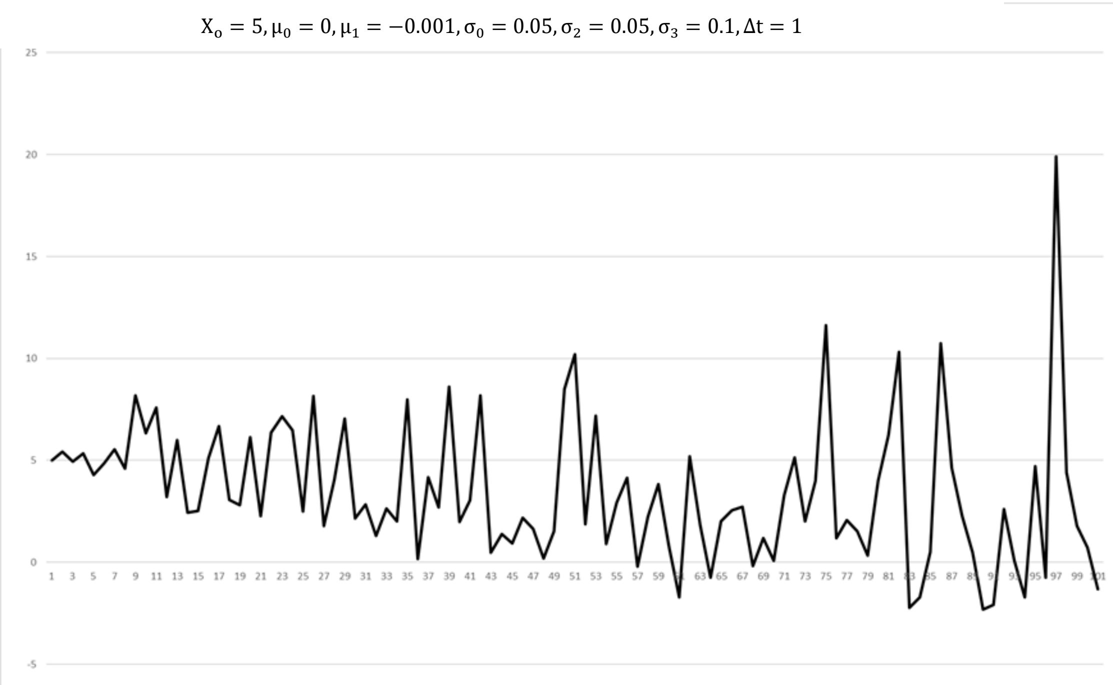


Figure 4-2 Example realisation of particular case one

Particular case two:

A second special case can be derived from Equation (4.15) by imposing a further condition $K = 0$ which reduces the equation to

$$(\mu_1 - \sigma_2)X_t + (\mu_0 - \frac{1}{2}\sigma_1) = 0 \tag{4.21}$$

Equation (4.21) is solved for $\mu_1 = \sigma_2$ and $\mu_0 = \frac{1}{2}\sigma_1$. Solutions are now restricted by parameters of the diffusion function bounded to those of the drift function.

This is one of the cases of reducible SDEs mentioned in Kloeden and Platen (2011, p. 120) where the general SDE $dX_t = \alpha(X_t, t)dt + \sqrt{\beta(X_t, t)}dW_t$ is reduced to a solvable SDE: $dX_t = \frac{1}{2}\sqrt{\beta(X_t, t)}\frac{\partial\sqrt{\beta(X_t,t)}}{\partial x}dt + \sqrt{\beta(X_t, t)}dW_t$.

Applying this reduction technique, the general cash flow equation obeys the following three-parameter equation

$$dX_t = \left(\sigma_2 X_t + \frac{1}{2}\sigma_1\right)dt + \sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)}dW_t \tag{4.22}$$

There are multiple, conditional solutions to Equation (4.22), Jeffrey (2004, section 4.3.4)

$$X_t = \frac{\sqrt{\Delta}}{2\sigma_2}\sinh\left(\sqrt{\sigma_2}\,(W_t + H_0)\right) - \frac{\sigma_1}{2\sigma_2} \tag{4.23a}$$

If $\sigma_2 > 0$ and $\Delta > 0$, where $\Delta = 4\sigma_2\sigma_0 - \sigma_1^2$ and $H_0 = \frac{1}{\sqrt{\sigma_2}}\sinh^{-1}\left[\frac{2\sigma_2 X_0 + \sigma_1}{\sqrt{\Delta}}\right]$

$$X_t = \frac{1}{2\sigma_2}e^{\sqrt{\sigma_2}(W_t + H_0)} - \frac{\sigma_1}{2\sigma_2} \tag{4.23b}$$

If $\sigma_2 > 0$ and $\Delta = 0$, where $\Delta = 4\sigma_2\sigma_0 - \sigma_1^2$ and $H_0 = \frac{1}{\sqrt{\sigma_2}}\ln|\,2\sigma_2 X_0 + \sigma_1|$

$$X_t = \frac{\sqrt{-\Delta}}{2\sigma_2}\sin\left(-\sqrt{-\sigma_2}\,(W_t + H_0)\right) - \frac{\sigma_1}{2\sigma_2} \tag{4.23c}$$

If $\sigma_2 < 0$ and $\Delta < 0$, where $\Delta = 4\sigma_2\sigma_0 - \sigma_1^2$ and $H_0 = \frac{-1}{\sqrt{-\sigma_2}}\sin^{-1}\left[\frac{2\sigma_2 X_0 + \sigma_1}{\sqrt{-\Delta}}\right]$.

Note that X_o is the initial cash flow at $t = 0$.

The solutions to particular case two are depicted in Figure 4-3. To display the three graphs in one chart, a time unit of 0.01 had to be used. Measured on this time scale, the volatility of the Sinh-solution (Equation (4.23a)) is subdued to those of the Exp-solution (Equation (4.23b)) and the Sin-solution (Equation (4.23c)). All three solutions, however, show an increasing periodic randomness.

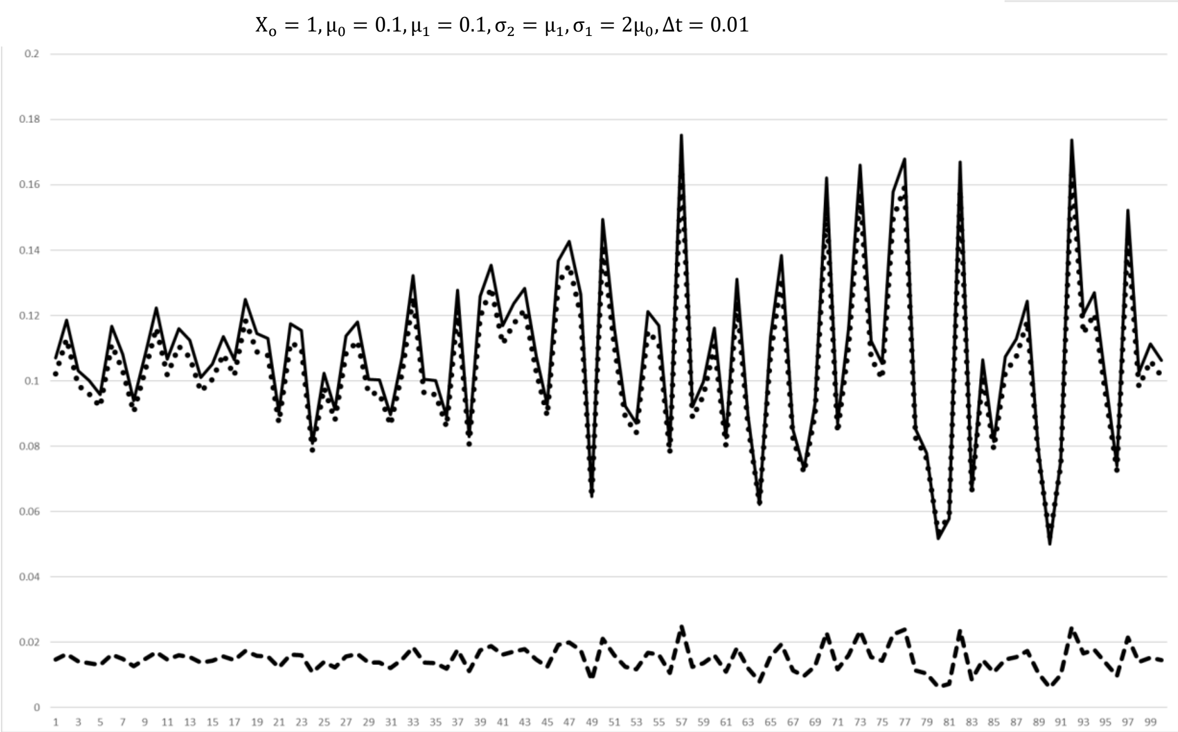


Figure 4-3 Example realisation of particular case two

If the solutions are observed on a much longer time-scale like the ones of Figures 4-1 and 4-2, periodicity remains but occasionally significant positive and negative spikes are seen.

Observe that since $\mu_1 = \sigma_2$, a negative value of σ_2 will force the drift function to become mean-reverting.

Particular case three:

It is well-known, Pierre et al. (2005), Møller (2011)), that the Lamperti transformation $Z_t = \psi(X_t) = \int \frac{dx}{\sqrt{\beta(X_t)}}$ converts a process, in this study a general cash flow, with time-independent parameters $dX_t = \alpha(X_t)dt + \sqrt{\beta(X_t)}dW_t$ into a process with a unit instantaneous change of variance

$$dZ_t = \left[\frac{\alpha(X_t)}{\sqrt{\beta(X_t)}} + \frac{1}{2}\frac{d\sqrt{\beta(X_t)}}{dx}\right]dt + dW_t \tag{4.24}$$

Setting $\alpha(X_t) = (\mu_1 X_t + \mu_0)$ and $\sqrt{\beta(X_t)} = \sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)}$, the corresponding Lamperti transfer becomes

$$Z_t = \psi(X_t) = \frac{1}{\sqrt{\sigma_2}} \ln \left| \sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)} + \frac{2\sigma_2 X_t + \sigma_1}{2\sqrt{\sigma_2}} \right| \tag{4.25}$$

Refer to Equations (4.23a) – (4.23c) for the evaluation of this equation under different conditions of parameters σ_2 and $\Delta = 4\sigma_2\sigma_0 - \sigma_1^2$.

The associated transformed process dZ_t is represented by this equation

$$dZ_t = \left[\frac{\left(\mu_1 - \frac{1}{2}\sigma_2\right)X_t + \left(\mu_0 - \frac{1}{4}\sigma_1\right)}{\sqrt{\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0}}\right]dt + dW_t = \left[\frac{\left(\mu_1 - \frac{1}{2}\sigma_2\right)\psi^{-1}(Z_t) + \left(\mu_0 - \frac{1}{4}\sigma_1\right)}{\sqrt{\sigma_2[\psi^{-1}(Z_t)]^2 + \sigma_1\psi^{-1}(Z_t) + \sigma_0}}\right]dt + dW_t \quad (4.26)$$

The Lamperti transformation has a number of attractive properties. Importantly, the transformation is reversible. Another attractive property of the Lamperti transformation is that it can be used to transform self-similar processes[31] into strongly stationary processes and vice versa (Lee et al. (2016); Viitasaari (2016)). Nevertheless, from Equation (4.1b) it is obvious that the cash flow process under study, is not self-similar due its quadratic expression and additive constants[32].

Here, the Lamperti transformation will be applied for another reason: to derive a particular linear solution of the form $Z_t = \lambda t + W_t$ to Equation (4.28) where λ is a parameter independent of X_t. This solution can be subsequently transformed back to X_t by using the inverse of Equation (4.25).

Setting the Lamperti transformed drift function to a constant, that is λ, gives

$$\left[\left(\mu_1 - \frac{1}{2}\sigma_2\right)X_t + \left(\mu_0 - \frac{1}{4}\sigma_1\right)\right]^2 = \lambda^2(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0) \quad (4.27)$$

Equating the parameters on both sides of Equation (4.27), yields three expressions for λ in different combinations of parameters $\{\mu_0, \mu_1, \sigma_0, \sigma_1, \sigma_2\}$.

$$\lambda = \frac{\mu_1 - \frac{1}{2}\sigma_2}{\sqrt{\sigma_2}},\ \lambda = \frac{\sqrt{2\left(\mu_1 - \frac{1}{2}\sigma_2\right)\left(\mu_0 - \frac{1}{4}\sigma_1\right)}}{\sqrt{\sigma_1}} \text{ and } \lambda = \frac{\mu_0 - \frac{1}{4}\sigma_1}{\sqrt{\sigma_0}} \quad (4.28)$$

where $\sigma_2, \sigma_1, \sigma_0 > 0$.

[31] Self-similar processes are stochastic processes that are invariant in distribution under a suitable scaling of time and space.

[32] However, Equation (4.1b) is functionally invariant under translations and scale-transformation as explained in Schmidt (2008, p. 21)

Solving the system of Equations (4.28) effects the reduction of a five-parameter SDE to a four-parameter system, a mild restriction of the original SDE. Furthermore, integrating expression $dZ_t = \lambda dt + dW_t$, gives the following linear function in Z_t:

$$Z_t = Z_0 + \lambda t + W_t \tag{4.29}$$

where $Z_o = \psi(X_0) = \frac{1}{\sqrt{\sigma_2}} \ln |2\sqrt{\sigma_2}\sqrt{(\sigma_2 X_0^2 + \sigma_1 X_0 + \sigma_0)} + 2\sigma_2 X_0 + \sigma_1|$.

The inverse of Equation (4.25) is

$$X_t = \psi^{-1}(Z_t) = \frac{1}{2\sqrt{\sigma_2}} e^{\sqrt{\sigma_2} Z_t} + \left(\frac{{\sigma_1}^2}{8{\sigma_2}^{\frac{3}{2}}} - \frac{\sigma_0}{2\sqrt{\sigma_2}} \right) e^{-\sqrt{\sigma_2} Z_t} - \frac{\sigma_1}{2\sigma_2} \tag{4.30}$$

Combined Equations (4.29) and (4.30) will eliminate Z_t from expression (30)

$$X_t = \psi^{-1}(Z_t) = \frac{1}{2\sqrt{\sigma_2}} e^{\sqrt{\sigma_2}\lambda(Z_0+t+W_t)} + \left(\frac{{\sigma_1}^2}{8{\sigma_2}^{\frac{3}{2}}} - \frac{\sigma_0}{2\sqrt{\sigma_2}} \right) e^{-\sqrt{\sigma_2}\lambda(Z_0+t+W_t)} - \frac{\sigma_1}{2\sigma_2} =$$

$$\frac{1}{2\sqrt{\sigma_2}} k_1 e^{\sqrt{\sigma_2}\lambda(t+W_t)} + \left(\frac{{\sigma_1}^2}{8{\sigma_2}^{\frac{3}{2}}} - \frac{\sigma_0}{2\sqrt{\sigma_2}} \right) k_1 e^{-\sqrt{\sigma_2}\lambda(t+W_t)} - \frac{\sigma_1}{2\sigma_2} \tag{4.31}$$

where $k_1 = e^{\sqrt{\sigma_2}\lambda Z_0}$ and $k_2 = e^{-\sqrt{\sigma_2}\lambda Z_0}$. Figure 4-4 below, gives an impression of a possible realisation of Equation (4.31).

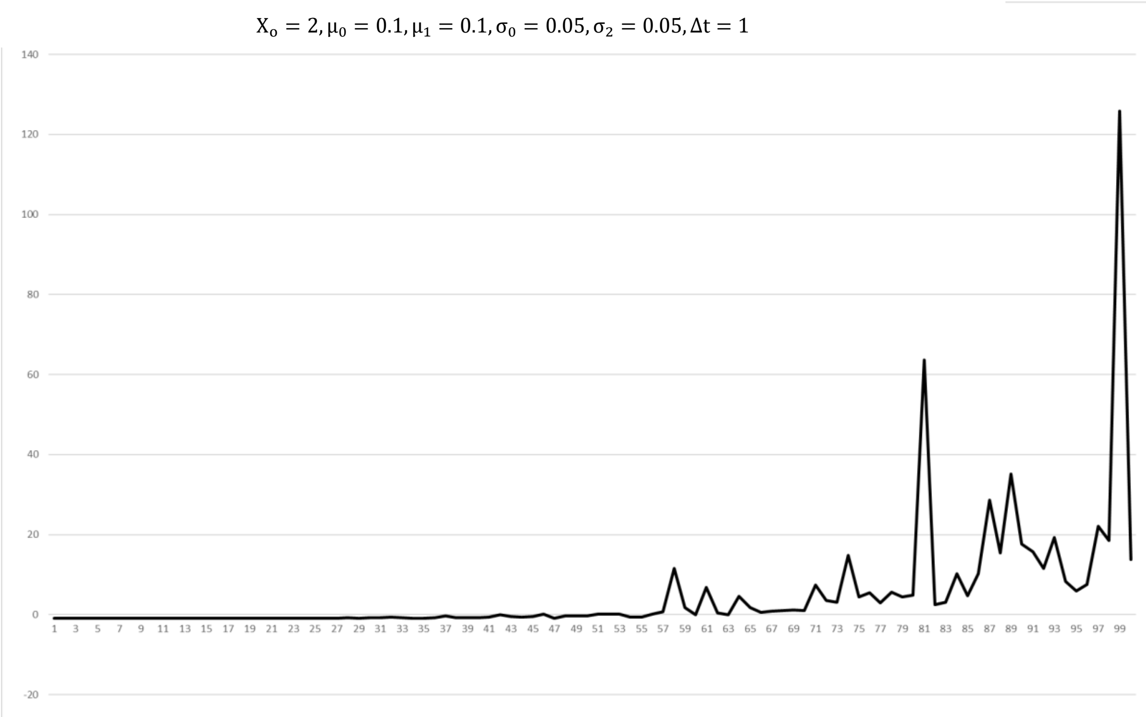


Figure 4-4 Example realisation of particular case three

Notice that if the Lamperti transformed drift coefficient is positive, $\lambda > 0$, then the first term of the RHS of Equation (4.31) will dominate the process, whereas a negative drift coefficient $\lambda < 0$ makes the second term the controlling term. However, unlike particular solution one it is not possible to turn the process into an exclusively mean-reverting process: if σ_2 is made smaller to reduce the growth rate in the exponential terms then consequently parameter λ will increase and so will the coefficients in front of the exponential terms.

Approximated solutions

Approximated solutions are solutions that are derived from the original SDE by replacing one or more non-linear terms by a linear approximation. As a consequence, such approximation is usually only sufficiently accurate in a limited domain of the cash flow variable X_t.

Approximation method one:

First, function $\sqrt{\beta(X_t, t)}$ can be expressed as $\sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)} = \sqrt{\sigma_2} X_t \sqrt{(1 + \frac{\sigma_1}{\sigma_2} X_t^{-1} + \frac{\sigma_0}{\sigma_2} X_t^{-2})}$ for $\sigma_2 > 0$. Provided that σ_2 is not very small relative to σ_1

and σ_0, the square root term can be approximated by a Taylor expansion[33]:

$\sqrt{(1+\frac{\sigma_1}{\sigma_2}X_t^{-1}+\frac{\sigma_0}{\sigma_2}X_t^{-2})} \approx 1+\frac{\sigma_1}{2\sigma_2}X_t^{-1}+\mathcal{O}(X_t^{-2})$. Therefore, the following linear function serves as an approximation of the diffusion function:

(i) if $\frac{\sigma_1}{2\sqrt{\sigma_2}} > 0$ then $\sqrt{(\sigma_2 X_t^2+\sigma_1 X_t+\sigma_0)} \approx \sqrt{\sigma_2}X_t+\frac{\sigma_1}{2\sqrt{\sigma_2}}$, or

(ii) if $\frac{\sigma_1}{2\sqrt{\sigma_2}} < 0$ then $\sqrt{(\sigma_2 X_t^2+\sigma_1 X_t+\sigma_0)} \approx |\sqrt{\sigma_2}X_t+\frac{\sigma_1}{2\sqrt{\sigma_2}}|$.

The approximation can be shown to be increasingly accurate, relative to X_t, as X_t becomes greater. Since cash flows are indeed frequently sizable amounts, it can be ascertained that the approximated SDE is a reasonable replacement for the original one.

Table 4-1 Accuracy of linear approximation for different cash flow amounts

$\lvert X_t \rvert$	Quadratic diffusion term	Linear approximation	Variance	in %
1	0.3873	0.3354	0.05189	13.397%
10	2.2583	2.3479	-0.08955	-3.965%
100	22.3629	22.4725	-0.10957	-0.490%
1000	223.6070	223.7186	-0.11158	-0.050%
10000	2236.0680	2236.1798	-0.11178	-0.005%
100000	22360.6798	22360.7916	-0.11180	0.000%

$\sigma_0 = 0.05, \sigma_1 = 0.05, \sigma_2 = 0.05$

Applying the suggested approximation, Equation (4.1b) can be transformed into the following solvable linear SDE:

$$dX_t = (\mu_1 X_t+\mu_0)dt+\left(\sqrt{\sigma_2}\,X_t+\frac{\sigma_1}{2\sqrt{\sigma_2}}\right)dW_t \qquad (4.32)$$

The solution is very similar to the solution in particular case one:

$$X_t = X_o \exp\left[(\mu_1-\tfrac{1}{2}\sigma_2)t+2\sqrt{\sigma_2}W_t\right]+[(\mu_0-\sigma_1)t+2\sqrt{\sigma_2}\,W_t] \qquad (4.33)$$

for $\sigma_2 > 0$. Parameter values are estimated by using for instance the method of moments.

Approximation method two:

This method combines the linear approximation from Approximation method one with the Lamperti transformation described under Particular case three.

Substituting the aforementioned linear approximation into Equation (4.26), the approximated Lamperti transformation can be cast as follows

[33] $\sqrt{1+Z} \approx 1+\frac{Z}{2}+\mathcal{O}(Z^2)$ where $Z=\frac{\sigma_1}{\sigma_2}X_t^{-1}$. The smaller Z is, the better the approximation. In this case Z is indeed small: cash flows X_t are normally sizeable amounts, and it was assumed that σ_2 is not very small relative to σ_1.

$$dZ_t = \left[\frac{\left(\mu_1 - \frac{1}{2}\sigma_2\right)X_t + \left(\mu_0 - \frac{1}{4}\sigma_1\right)}{\sqrt{\sigma_2}\,|X_t| + \frac{\sigma_1}{2\sqrt{\sigma_2}}}\right] dt + dW_t = \left[\frac{\left(\mu_1 - \frac{1}{2}\sigma_2\right)\psi^{-1}(Z_t) + \left(\mu_0 - \frac{1}{4}\sigma_1\right)}{\sqrt{\sigma_2}\,)\psi^{-1}(Z_t) + \frac{\sigma_1}{2\sqrt{\sigma_2}}}\right] dt + dW_t \tag{4.34}$$

The aim is to approximate the Lamperti transformed cash flow process by the first term only, i.e. the deterministic drift function in isolation, since the unit diffusion term (or more precisely: $\mathcal{N}(0,1)$ after adjusting for dt) becomes very small compared the size of the drift term. For cash flow processes this approximation will be achieved if the ratio $\frac{\left(\mu_1 - \frac{1}{2}\sigma_2\right)X_t + \left(\mu_0 - \frac{1}{4}\sigma_1\right)}{\sqrt{\sigma_2}\,|X_t| + \frac{\sigma_1}{2\sqrt{\sigma_2}}}$ increases as X_t becomes greater which equates to the condition $\mu_1 > \frac{1}{2}\sigma_2 + 2\sqrt{\sigma_2}$.

Table 4-2 Values of the Lamperti-transformed drift function for different cash flow amounts

$\lvert X_t \rvert$	Lamperti-transformed drift function	2 sigma dW_t	dW_t /drift
0.01	0.1864	2	10.729
0.1	0.1987	2	10.066
1	0.3107	2	6.438
10	0.9845	2	2.031
100	3.3198	2	0.602
1,000	10.5953	2	0.189
10,000	33.5374	2	0.060
100,000	106.0649	2	0.019
1,000,000	335.4098	2	0.006
10,000,000	1060.6601	2	0.002
100,000,000	3354.1019	2	0.001

$\mu_0 = 0.1, \mu_1 = 0.1, \sigma_0 = 0.05, \sigma_1 = 0.05, \sigma_2 = 0.05$

In Table 4-2 it can be seen that the approximation, calculated for the specific parameters used, is significantly (~ 50 times) greater than $\mathcal{N}(0,1)$, measured on a probability interval of $\pm 2\sigma$, for cash flows roughly above 100,000 and below -100,000.

Now, applying the chain rule, Equation (4.34) can be re-written to an ODE (in variable X_t):

$$\frac{dX_t}{dt} = \frac{dX_t}{dZ_t}\left[\frac{\left(\mu_1 - \frac{1}{2}\sigma_2\right)X_t + \left(\mu_0 - \frac{1}{4}\sigma_1\right)}{\sqrt{\sigma_2}\,|X_t| + \frac{\sigma_1}{2\sqrt{\sigma_2}}}\right] dt \tag{4.35}$$

where $\frac{dX_t}{dZ_t} = \sqrt{\sigma_2}\,|X_t| + \frac{\sigma_1}{2\sqrt{\sigma_2}}$ (which follows from to the approximated Lamperti-transformation, Equation (4.35)). The resulting linear ODE is easy to solve:

$$\frac{dX_t}{dt} = \left(\mu_1 - \frac{1}{2}\sigma_2\right)X_t + \left(\mu_0 - \frac{1}{4}\sigma_1\right) \tag{4.36}$$

with solution:

$$X_t = Ke^{\left(\mu_1 - \frac{1}{2}\sigma_2\right)t} - \frac{\left(\mu_0 - \frac{1}{4}\sigma_1\right)}{\left(\mu_1 - \frac{1}{2}\sigma_2\right)} \tag{4.37}$$

where K is an integration constant. If Equation (4.37) is compared to the pure deterministic process $X_t = Ke^{\mu_1 t} - \frac{\mu_0}{\mu_1}$, see Section 3.3, then the equation can be interpreted as a 'risk adjusted' deterministic approximation of the stochastic cash flow process.

Approximation method three:

This method considers a transformation of the cash flow variable X_t into a new variable $\widetilde{X}_t = \frac{qX_t + r}{s}$ where $q = \sqrt{\sigma_2}$ for $\sigma_2 \geq 0$, $r = \frac{\sigma_1}{2\sqrt{\sigma_2}}$ and $s^2 = \frac{\sigma_1^2}{4\sigma_2} - \sigma_0$. Then, Equation (4.1b) takes the following specification

$$d\widetilde{X}_t = (\mu_1 \widetilde{X}_t + \mu_0')dt + s\sqrt{\left(\widetilde{X}_t^2 + 1\right)}\, dW_t \tag{4.38}$$

where $\mu_0' = \frac{q\mu_0 - r}{s}$. Since $s = \frac{D}{2\sqrt{\sigma_2}}$ and the discriminant D is defined as $D = \sqrt{\sigma_1^2 - 4\sigma_2\sigma_0}$, it turns out that μ_0' expressed in basis parameters is $\mu_0' = \frac{2\sigma_2\mu_0 - \sigma_1}{D}$.

Comparable to Particular case three and Approximation method two, a second transformation is proposed

$$Z_t = \int \frac{s\, d\varepsilon}{\sqrt{2}\sqrt{\varepsilon_t^2 + 1}} = \frac{s}{\sqrt{2}} \sinh^{-1}\left(\widetilde{X}_t\right) \tag{4.39a}$$

and subsequent re-scaling of Z_t to Z_t'

$$Z_t' = \frac{\sqrt{2}}{s} Z_t \tag{4.39b}$$

which changes Equation (4.38) into

$$dZ_t' = \frac{\sqrt{2}}{s}\left[\frac{(2\mu_1 + q)\widetilde{X}_t + 2\mu_0'}{2q\sqrt{\widetilde{X}_t^2 + 1}}\right]dt + \sqrt{2}\, dW_t \tag{4.40a}$$

Using Equation (4.39) $\widetilde{X}_t = \sinh(Z_t')$ and the hyperbolic trigonometric identity $\sqrt{\sinh^2(\frac{\sqrt{2}}{s}Z_t') + 1)} = \cosh(\frac{\sqrt{2}}{s}Z_t')$, Equation (4.40a) can be re-expressed as

$$dZ_t' = \frac{\sqrt{2}}{s}\left[\frac{(2\mu_1 + q)}{2q}\tanh(Z_t') + \frac{\mu_0'}{q}\operatorname{sech}(Z_t')\right]dt + \sqrt{2}\, dW_t \tag{4.40b}$$

Since $|Z_t'| < 1$, the two hyperbolic functions in Equation (4.40b) are approximated by a series expansion

$$\tanh(Z_t') = Z_t' + \mathcal{O}({Z_t'}^3), \qquad \operatorname{sech}(Z_t') = Z_t' + \mathcal{O}({Z_t'}^2) \tag{4.41}$$

so that Equation (4.40b) is replaced by an approximated version

$$dZ_t' = \left[\frac{(2\mu_1 + q)}{2q}\frac{\sqrt{2}}{s} Z_t' + \frac{\mu_0'}{q}\frac{\sqrt{2}}{s}\right] dt + \sqrt{2}\, dW_t \tag{4.42a}$$

or after substituting $q = \sqrt{\sigma_2}$, $\mu_0' = \frac{2\sigma_2\mu_0 - \sigma_1}{D}$, $s = \frac{D}{2\sqrt{\sigma_2}}$, where $D = \sqrt{\sigma_1^2 - 4\sigma_2\sigma_0}$

$$dZ_t' = \left[\frac{(2\mu_1 + \sqrt{\sigma_2})\sqrt{2}}{D} Z_t' + \frac{(4\sigma_2\mu_0 - 2\sigma_1)\sqrt{2}}{D^2}\right] dt + \sqrt{2}\, dW_t \tag{4.42b}$$

or, re-expressed as

$$dZ_t' = -\frac{(2\mu_1 + \sqrt{\sigma_2})\sqrt{2}}{D}\left[-\frac{(4\sigma_2\mu_0 - 2\sigma_1)}{(2\mu_1 + \sqrt{\sigma_2})D} - Z_t'\right] dt + \sqrt{2}\, dW_t \tag{4.42c}$$

In its converging form, Equation (4.42b) is similar to the well-known Vasicek process (Appendix O1) with long-time average of $-\frac{(4\sigma_2\mu_0 - 2\sigma_1)}{(2\mu_1+\sqrt{\sigma_2})D}$, a speed of reversion $\frac{-(2\mu_1+\sqrt{\sigma_2})\sqrt{2}}{D}$, and an instantaneous change of variance equal to constant 2. The associated inhomogeneous linear SDE with constant coefficients is solvable in closed-form (Kloeden and Platen (2011, p. 118)). The general solution is

$$Z_t' = -\frac{4\sigma_2\mu_0 - 2\sigma_1}{(2\mu_1+\sqrt{\sigma_2})D} + \left[Z_0' + \frac{4\sigma_2\mu_0 - 2\sigma_1}{(2\mu_1+\sqrt{\sigma_2})D}\right] e^{\left[\frac{(2\mu_1+\sqrt{\sigma_2})\sqrt{2}}{D}\right]t} + \sqrt{2}\int_0^t e^{\left[\frac{(2\mu_1+\sqrt{\sigma_2})\sqrt{2}}{D}\right]t'} dW_{t'} \tag{4.43}$$

where Z_0 is an integration constant representing the value of Z at $t = 0$. Given that $\sigma_2 > 0$ the process is converging or diverging in time, depending on the following conditions

	$\boldsymbol{\mu_1 < -\frac{\sqrt{\sigma_2}}{2}}$	$\boldsymbol{\mu_1 > -\frac{\sqrt{\sigma_2}}{2}}$
$\mathbf{D > 0}$	converging	diverging
$\mathbf{D < 0}$	diverging	converging

A converging process goes to a stationary (long-time) value ($\lim t \to \infty$) equal to $-\frac{4\sigma_2\mu_0 - 2\sigma_1}{(2\mu_1+\sqrt{\sigma_2})D}$. The conditional density function of Z_t is Gaussian: $Z_t \sim \mathcal{N}(\mu_t, \sigma_t^2)$ where

$\left[\mu_t = -\frac{4\sigma_2\mu_0-2\sigma_1}{(2\mu_1+\sqrt{\sigma_2})D} + \left[Z_0' + \frac{4\sigma_2\mu_0-2\sigma_1}{(2\mu_1+\sqrt{\sigma_2})D}\right] e^{\left[\frac{(2\mu_1+\sqrt{\sigma_2})\sqrt{2}}{D}\right]t}\right]$, and $\sigma_t^2 = -\frac{(2\mu_1+\sqrt{\sigma_2})D}{2\sigma_2\mu_0-\sigma_1}\left[1 - e^{\left[\frac{(2\mu_1+\sqrt{\sigma_2})\sqrt{2}}{2D}\right]t}\right]$. The evolution of the first moment is described by the following equation

$$\mathbb{E}(Z_t') = -\frac{4\sigma_2\mu_0 - 2\sigma_1}{(2\mu_1 + \sqrt{\sigma_2})D} + \left[Z_0' + \frac{4\sigma_2\mu_0 - 2\sigma_1}{(2\mu_1 + \sqrt{\sigma_2})D}\right] e^{\left[\frac{(2\mu_1+\sqrt{\sigma_2})\sqrt{2}}{D}\right]t} \tag{4.44}$$

and the evolution of the variance by

$$\mathbb{V}(Z_t') = -\frac{(2\mu_1 + \sqrt{\sigma_2})D}{2\sigma_2\mu_0 - \sigma_1}\left[1 - e^{2\left[\frac{(2\mu_1+\sqrt{\sigma_2})\sqrt{2}}{D}\right]t}\right] \tag{4.45a}$$

if the process is converging, or

$$\mathbb{V}(Z_t') = \frac{(2\mu_1 + \sqrt{\sigma_2})D}{2\sigma_2\mu_0 - \sigma_1}\left[e^{2\left[\frac{(2\mu_1+\sqrt{\sigma_2})\sqrt{2}}{D}\right]t} - 1\right] \tag{4.45b}$$

if the process is diverging.

Equations (4.44) and (4.45b) can be re-parametrised to

$$\mathbb{E}(Z_t') = -\Theta_1 + [Z_0 + \Theta_1]e^{\Theta_2 t} \tag{4.46}$$

$$\mathbb{V}(Z_t') = \Theta_3(e^{2\Theta_2 t} - 1) \tag{4.47}$$

where $\Theta_1 = \frac{4\sigma_2\mu_0-2\sigma_1}{(2\mu_1+\sqrt{\sigma_2})D}$, $\Theta_2 = \frac{(2\mu_1+\sqrt{\sigma_2})\sqrt{2}}{D}$ and $\Theta_3 = \pm\frac{(2\mu_1+\sqrt{\sigma_2})D}{2\sigma_2\mu_0-\sigma_1}$.

The set of parameters $\{\widehat{\Theta}_1, \widehat{\Theta}_2, \widehat{\Theta}_3\}$ are estimated from Equations (4.46) and (4.47). Nevertheless, the complete parameter system $\{\mu_0, \mu_1, \sigma_0, \sigma_1, \sigma_2\}$ is underspecified and without at least two restrictions on the parameters, it is seemingly not solvable. In such cases it is sometimes possible to derive a set of ODEs, sufficient in number to define a completely specified system of estimated parameters. For a discussion of evolution of moment ODEs and associated closure techniques refer to, for example, Kuehn (2016) and Nasell (2017). Despite the system being closed and thus admitting recursive solutions, mathematically it can be demonstrated that equations describing the evolution of higher moments ($n \geq 2$) are all defined by a composite expression of the same parameter set $\{\widehat{\Theta}_1, \widehat{\Theta}_2, \widehat{\Theta}_3\}$ and thus the under specification-problem is not removed. Under-specification is caused by the first variable transform (as per Equation (4.38)) that requires knowledge of parameters $\{\sigma_0, \sigma_1, \sigma_2\}$; however, these will follow from the final estimation.

Discussion

Figures 4-1 and 4-3 lead to the tentative conclusion that particular solutions one and three are suitable equations to model operating cash flow processes. Both equations include a randomised exponential growth process commonly observed in operating cash flow processes. Nevertheless, particular solution one can only under very restrictive parameter conditions be used to mimic the behaviour of investing cash flow processes. Comparing particular solution one to the functional form of a general solution explains why: linearization of the SDE transforms a multiplicative exponential growth specification into an additive specification. A similar conclusion can be drawn for particular solution three. This solution is the sum of an exponential growth process and a mean-reverting exponential process but there are no parameter values that can turn the equation into an exclusively mean-reverting process.

Particular solution two exhibits randomised growing periodicity, behaviour that is typically seen in investing cash flow processes. However, this behaviour is caused by bounded drift and diffusion functions, which is only observed in relatively rare cases.

In this section, two approximate solution methods were suggested. For both methods, larger cash flow values, as usually observed in financial datasets, result in a better approximation. Approximation method one leads to almost the same functional specification as Particular solution one. Therefore, the method is equally suitable to model operating cash flows but not investing cash flows. Approximation method two, links the corresponding deterministic process to stochastic solutions by including a risk-adjustment factor in the deterministic solution equation, an approach that generally works well for significantly large positive and negative cash flows (amounts in the order of at least \$100,000). Method three transforms a linear-quadratic process into a Vasicek process. The corresponding Gaussian conditional density function seems attractive; however, since the evolution of higher moments is defined by the same three parameters only, it does not accommodate the estimation of the full parameter set (five parameters) of the linear-quadratic equation.

In the absence of an exact general solution to the cash flow process under study, and no particular and approximate solutions that satisfy operating and investing cash flows alike, or provide acceptable solutions over the whole range of cash flows, other approaches should be tried. One obvious approach is to examine weak[34] solutions to the SDE by analysing the associated Fokker-Planck (or more general: Kolmogorov) equation(s). This will be the topic of Section 4.2 (investing cash flow) and Section 4.3 (operating cash flow).

4.2. Solutions to the investing cash flow Fokker Planck Equation

In this section, solutions to the investing cash flow Fokker-Planck equation are examined in more detail. Investing cash flows are analysed first since their solutions are considered less complicated than solutions for operating cash flow processes.

The Fokker-Planck equation describes the full stochastic dynamics of the (uncoupled) cash flow process at a mesoscopic level:

$$\frac{\partial p(I_t,t)}{\partial t} = \frac{-\partial(\mu_1 I_t + \mu_0)p(I_t,t)}{\partial I_t} + \frac{1}{2}\frac{\partial^2(\sigma_2 I_t^2 + \sigma_2 I_t + \sigma_0)p(I_t,t)}{\partial I_t^2} \tag{4.48}$$

with initial condition $p(I_t, 0) = \delta(c - c_0)$ where $p(I_t, t)$ is the conditional investing cash flow probability density function and $\delta()$ is Dirac's delta function.

The stationary solution $p_{st}(I)$ is found by expressing Equation (4.48) as $\frac{\partial p(I_t,t)}{\partial t} = \frac{-\partial J(I_t,t)}{\partial I_t}$[35], where $J(I_t, t)$ represents the probability current $\left[(\mu_1 I_t + \mu_0) + \frac{1}{2}\frac{\partial(\sigma_2 I_t^2 + \sigma_1 I_t + \sigma_0)}{\partial I_t}\right] p(I_t, t)$, and subsequently setting $\frac{\partial J(I_t,t)}{\partial I_t}$ to zero, gives

$$p_{st}(I) = K\, e^{-\Phi(I)} \tag{4.49}$$

[34] Weak solutions, as opposed to strong solutions (see Section 4.1), are defined as solutions that share a common probability density function.

[35] The representation of the Fokker-Planck equation in probability current format, explicates the conservation of probability in a spatial density system as an essential property of the equation.

where $\Phi(\mathrm{I}) = \ln\left(\frac{\sigma_2\mathrm{I}^2+\sigma_2\mathrm{I}+\sigma_0}{2}\right) - \int \frac{2(\mu_1\xi+\mu_0)}{(\sigma_2\xi^2+\sigma_1\xi+\sigma_0)}\mathrm{d}\xi$[36] and K is a normalisation constant. Equation (4.49) is derived in for instance Risken and Frank (2012, p. 98). Equation (2.17) in Section 2-3 provided an alternative expression for $\mathrm{p_{st}}(\mathrm{I})$: $\mathrm{K}'\left[(\mathrm{I}+\frac{\sigma_1}{2\sigma_2})^2 + \lambda^2\right]^{\nu_1}\exp\left[\nu_2\tan^{-1}\left[\frac{\mathrm{I}+\frac{\sigma_1}{2\sigma_2}}{\lambda}\right]\right]$, where $\nu_1 = \frac{\mu_1-\sigma_2}{\sigma_2}$, $\nu_2 = \frac{2\mu_0-\frac{\sigma_1}{\sigma_2}\mu_1}{\sigma_2\lambda}$, $\lambda > 0$ and K' is a normalisation constant. From $\lambda = \frac{\sqrt{4\sigma_0\sigma_2-\sigma_1^2}}{2\sigma_2}$ in conjunction with $\lambda > 0$[37] and $4\sigma_0\sigma_2 - \sigma_1^2 > 0$, it follows that σ_2 must be positive and therefore the sign of $\nu_1$depends on the value of μ_1. If $\mu_1 < \sigma_2$ then ν_1 is smaller than zero. Since $\sigma_2 > 0$, the condition $\mu_1 < 0$ must also hold for any value of σ_2. Consequently, $\mathrm{p_{st}}(\mathrm{I}) \downarrow 0$ as $\mathrm{I} \to +\infty$ or $-\infty$. Notice that the term $(\mathrm{I}+\frac{\sigma_1}{2\sigma_2})^2 + \lambda^2$ always has positive value and the term $\nu_2\tan^{-1}\left[\frac{\mathrm{I}+\frac{\sigma_1}{2\sigma_2}}{\lambda}\right]$ asymptotically approaches $+\frac{\nu_2}{2\pi}$ as $\mathrm{I} \to +\infty$ and $-\frac{\nu_2}{2\pi}$ as $\mathrm{I} \to -\infty$. Since investing cash flow is defined on the whole range $\mathbb{R}$, $-\infty$ and $+\infty$ act here as natural boundaries; hence there is no need to impose additional boundary conditions.

Supported by empirical evidence, Section 2-3 shows that investing cash flow processes converge to a steady state governed by the class Pearson type IV distribution which follows from a quadratic diffusion function characterised by complex roots (and a negative discriminant). The stochastic properties of investment cash flow processes can therefore be studied from their stationary distributions. Stochastic processes based on Equation (4.1b) that have a stationary probability density function to which the process converges as $t \to \infty$, are called *Pearson diffusions*. Since stationary density functions are generally observed for investment cash flow processes, this section will briefly discuss Pearson diffusions. Pearson diffusion processes are widely studied in the literature: see for instance E. Wong, The construction of a class of stationary Markov processes, in: Bellman and Society (1964, pp. 264–276), Forman and Sorensen (2008), G. M. Leonenko and Phillips (2012), Shaw and Schofield (2015).

[36] Often the term $\ln\left(\frac{\sigma_2\mathrm{I}^2+\sigma_2\mathrm{I}+\sigma_0}{2}\right)$ is omitted from the potential function $\Phi(\ )$.

[37] Heinrich (2004, p. 1) states that the probability density function $\mathrm{p_{st}}$ is invariant under a simultaneous transformation $\sigma_2 \to -\sigma_2$ and $\nu_2 \to -\nu_2$ and therefore the parameter σ_2 is taken positively by convention.

The Pearson diffusion equation is commonly expressed as

$$dX_t = \Theta(\mu - X_t)dt + \sqrt{2\Theta(\sigma_2' X_t^2 + \sigma_1' X_t + \sigma_0')}\, dW_t \tag{4.50}$$

where X_t is some converging cash flow process and parameter $\Theta > 0$ represents the speed of the diffusion. Parameters μ, σ_2', σ_1' and σ_0' determine the state-space of the diffusion and the shape of the corresponding density function (Forman (2007, p. 9)). Note that the mapping to the parameters of Equation (4.1b) is the following: $\mu_1 = -\Theta$, $\mu_0 = \Theta\mu$, $\sigma_2 = 2\Theta\sigma_2'$, $\sigma_1 = 2\Theta\sigma_1'$ and $\sigma_0 = 2\Theta\sigma_0'$.

For six particular (or limiting) cases[38], Equation (4.50) is known to have explicit (analytical) stationary solutions in the form of polynomial eigenfunctions (G. M. Leonenko and Phillips (2012)). Particular solutions are defined by (i) the degree of the quadratic equation d (0, 1, or 2), (ii) the sign of parameter σ_2 (positive or negative) and (iii) the value of the discriminant $D = \sigma_1^2 - 4\sigma_2\sigma_0$ (positive, negative, zero) of the quadratic diffusion function. Only two of the six cases are (unconditionally) defined on the full cash flow spectrum $\mathbb{R}$: the Ornstein-Uhlenbeck diffusion (d=0) and the Student diffusion (d=2, $\sigma_2 > 0$, D < 0) (G. M. Leonenko and Phillips (2012)) and hence they are possible equations to describe investment cash flow processes. Since the vast majority of investing cash flow processes are of degree 1 or 2 (see Section 2-3), the Student diffusion process is the process of choice. Indeed, Steinbrecher and Shaw (2008) and Shaw and Schofield (2015, Section 4), see the Student diffusion as a natural candidate for a mean-reverting GBM-ABM hybrid process as described by Equation (4.1b).

Traditionally Student diffusion processes are parametrised as

$$dX_t = -\Theta(X_t - \mu)dt + \sqrt{\alpha\Theta[(X_t - \beta)^2 + \gamma^2]}\, dW_t \tag{4.51}$$

where, compared to Equation (4.1b), the parameters are: $\mu_1 = -\Theta$, $\mu_0 = \mu\Theta$, $\sigma_2 = \alpha\Theta$, $\sigma_1 = -2\alpha\beta\Theta$ and $\sigma_0 = \alpha(\beta^2 + \gamma^2)\Theta$.

[38] The six cases are: Ornstein-Uhlenbeck diffusion, CIR diffusion, Jacobi diffusion, Reciprocal gamma diffusion, Student diffusion and Fisher-Snedecor diffusion.

Interestingly, using the hybrid decomposition of Equation (3.28) in Section 3-4, $dX_t = (\mu_1 X_t + \mu_0)dt + \sqrt{\sigma_2} X_t dW_{1,t} + \sqrt{\sigma_0}\, dW_{2,t}$, Equation (4.51) can be reformulated, see Avram et al. (2013, p. 13), as

$$dX_t = -\Theta(X_t - \mu)dt + \sqrt{\sigma_2\left[X_t + \rho\frac{\sqrt{\sigma_0}}{\sqrt{\sigma_2}}\right]^2 + (1-\rho^2)\frac{\sigma_0}{\sigma_2}}\; dW_t \tag{4.52}$$

where ρ is the correlation coefficient between the Brownian motions $W_{1,t}$ and $W_{2,t}$.

Substituting the parameters of Equation (4.52) into Equation (4.48) leads to the following specific Fokker-Planck equation for investment cash flow processes

$$\frac{\partial p(I_t,t)}{\partial t} = \frac{\Theta\, \partial((I_t-\mu))p(I_t,t)}{\partial I_t} + \frac{\alpha\Theta}{2}\frac{\partial^2[(I_t-\beta)^2+\gamma^2]p(I_t,t)}{\partial I_t^2} \tag{4.53}$$

The stationary (time-invariant) equation can be derived from Equation (4.53) by setting $\frac{\partial p(I_t,t)}{\partial t} = 0$ with the resulting specific Pearson equation

$$\frac{\frac{dp(I)}{dI}}{p(I)} = \frac{(\mu + 2\alpha\beta) - (1+2\alpha)I}{\alpha[(I_t-\beta)^2+\gamma^2]} \tag{4.54}$$

The Student diffusion process has been studied extensively in the literature (E. Wong, The construction of a class of stationary Markov processes, in: Bellman and Society (1964, pp. 264–276), Grigelionis (2013), Avram et al. (2013)), and solutions to Equation (4.54) can be found for instance in Meerschaert and Sikorskii (2012, pp. 226-227) and G. M. Leonenko and Phillips (2012, pp. 2864-2867). The (asymmetric) invariant density function is

$$p(I) = \eta(\mu,\alpha,\beta,\gamma)\frac{\exp\left[\frac{\mu-\beta}{\alpha\gamma}\tan^{-1}\left[\frac{I_t-\beta}{\gamma}\right]\right]}{\left[1+\left[\frac{I_t-\beta}{\gamma}\right]^2\right]^{\frac{\nu+1}{2}}} \tag{4.55}$$

where $I \epsilon \mathbb{R}$, $\alpha > -1$, $\eta(\mu,\alpha,\beta,\gamma) = \frac{\Gamma(\frac{\nu+1}{2})}{\gamma\sqrt{\pi}\,\Gamma(\frac{\nu}{2})}\prod_{k=0}^{\infty}\left[1+\left(\frac{\frac{\mu-\beta}{\alpha\gamma}}{\frac{\nu+1}{2}+k}\right)^2\right]^{-1}$, $\nu = 1+\frac{1}{\alpha}$ representing the degrees of freedom.

Equation (4.55) can be considerably simplified if the density distribution is *symmetric*, i.e. $\mu = \beta$ in which case the equation becomes

$$p(\mathrm{I}) = \eta(\mu, \alpha, \beta, \gamma) \frac{1}{\left[1 + \left[\frac{\mathrm{I_t} - \beta}{\gamma}\right]^2\right]^{\frac{\nu+1}{2}}} \tag{4.56}$$

where $\mathrm{I} \in \mathbb{R}$, $\alpha > -1$, $\eta(\mu, \alpha, \beta, \gamma) = \frac{\Gamma(\frac{\nu+1}{2})}{\gamma\sqrt{\pi}\ \Gamma(\frac{\nu}{2})}$, $\nu = 1 + \frac{1}{\alpha}$ representing the degrees of freedom. Equation (4.56) is a pre-eminent function to describe processes that evolve to heavy-tailed distributions such as the ones empirically found in Section 2-3 for investing cash flow processes.

The next section deals with the dynamics of operating cash flows and gives a method to analyse the full (intertemporal) dynamics of the investing cash flow Fokker-Planck equation.

4.3. Solutions to the operating cash flow Fokker Planck Equation

In contrast to investing cash flows, (uncoupled) operating cash flow processes appear not to be converging to a stationary distribution (see Section 2-3), barring a few exceptions. From the conclusion in Section 4-1, that is, the convergence of cash flow processes is uniquely determined (dominated) by the drift function, and the drift function $\alpha(c) = \mu_1 c + \mu_0$ of operating cash flow is defined by $\mu_1 > 0$, the inference must be that operating cash flow processes do not have a stable probability distribution as $t \to \infty$. Therefore, the Pearson diffusion process is, excluding a few exceptions, not a suitable equation to describe operating cash flows. A wider net has to be cast as will become clear in the remainder of this section.

There exist a large number of (combinations of) methods to solve the Fokker-Planck equation. Figure 4-5 outlines the methods considered in this study. *Mutatis mutandis*, the methods are applicable to the forward and backward Kolmogorov equations alike.

Of all main solution techniques, the emphasis is on the method of separation of variables for reasons set-out in the following subsection. Since operating cash flows have arbitrary boundary values $[\lambda_1; \lambda_2]$, standardising these values enhances the chances of finding solutions. Two normalisations are further examined: the transformation to boundary values $[-1; 1]$ and to boundary values $[0; 1]$. Additionally, recasting the problem to a Sturm-Liouville problem, provides access to a range of well-known solutions. Once formulated as a Sturm-Liouville problem, other standard techniques can be applied. In this study the Jacobi, Hermitian and Schrödinger transformations are identified as convenient and commonly applied second-step solution techniques. Notice that not all these techniques will necessarily lead to a general solution; some special and approximated solutions will have to be considered as is the case with the linear-quadratic SDE (see Section 4-1).

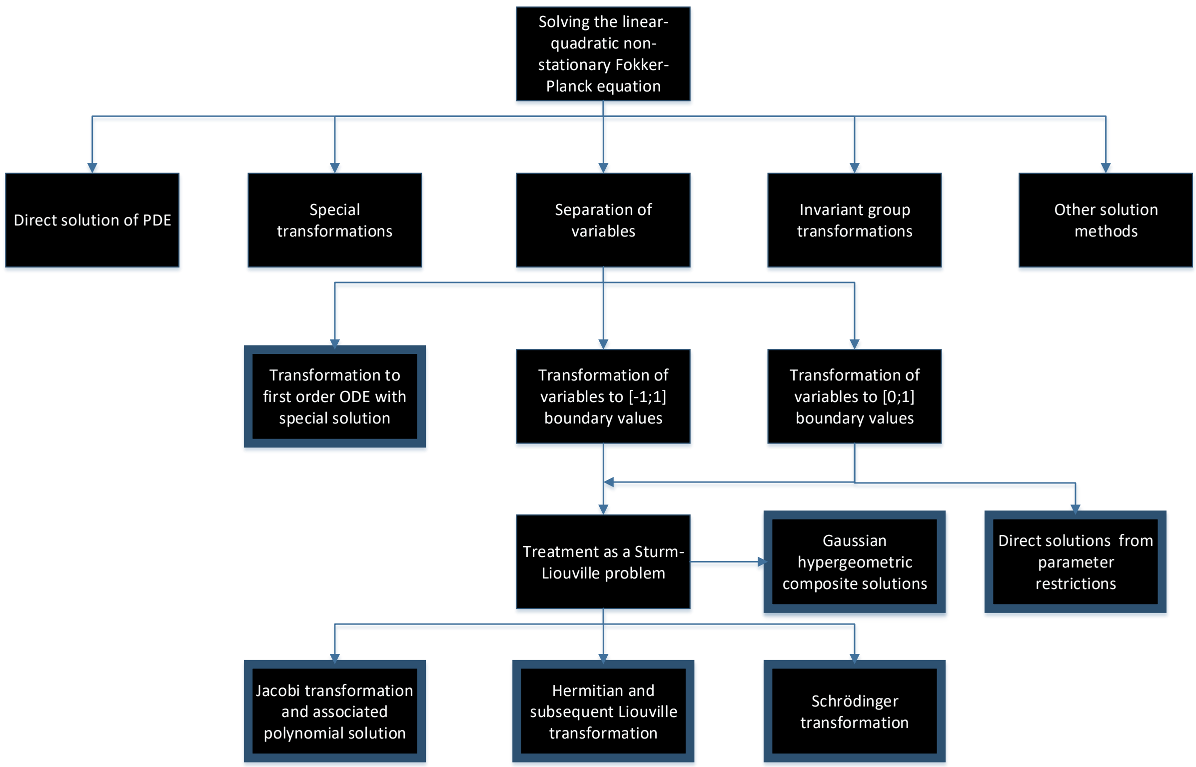


Figure 4-5 Selection of Fokker-Planck solution techniques

Note: in this section solutions are discussed that are marked as bold rectangles in Figure 4-5.

The forward Kolmogorov diffusion equation

The forward Kolmogorov diffusion equation, i.e. the Fokker-Planck equation, corresponding to the SDE of operating cash flow processes is

$$\frac{\partial p(C_t,t)}{\partial t} = \frac{-\partial(\mu_1 C_t + \mu_0)p(C_t,t)}{\partial C_t} + \frac{1}{2}\frac{\partial^2(\sigma_2 C_t^2 + \sigma_1 C_t + \sigma_0)p(C_t,t)}{\partial C_t^2} \tag{4.57a}$$

with initial condition $p(C_t, 0) = \delta(c - c_0)$ where $p(C, t)$ is the conditional operating cash flow probability density function and $\delta()$ is Dirac's delta function. In non-divergence form the equation can be written as

$$\frac{\partial p(C_t,t)}{\partial t} = (\sigma_2 - \mu_1)p(C_t, t) + [(2\sigma_2 - \mu_1)C_t + (\sigma_1 - \mu_0)]\frac{\partial p(C_t,t)}{\partial C_t} + \frac{1}{2}(\sigma_2 C_t^2 + \sigma_1 C_t + \sigma_0)\frac{\partial^2 p(C_t,t)}{\partial C_t^2} \tag{4.57b}$$

To examine the inter-temporal dynamics of a space-time density function, solving the Fokker-Planck equation is commonly considered the preferred method since there are no constraints placed on the evolution of the density function in the form of an end condition. The Fokker-Planck equation is accompanied by an initial condition, often expressed as Dirac's delta function that admits a wide variety of possible density specifications as the process evolves (over time).

Boundary conditions

One of the conclusions of the prior section is that an investing cash flow process has natural boundary conditions $-\infty$ and $+\infty$ attached to it. The question raises whether the same boundary conditions pertain to operating cash flows or that more stringent boundaries conditions are required. The latter is indeed the case as will now be set-out.

Consider the density function $p_{lt}(C) = K\, e^{-\Phi(C)}$, similar to the stationary density function $p_{st}(I)$ derived in Equation (4.49) for investing cash flows. It is clear that since an operating cash flow process does not converge to a stationary solution as $t \to \infty$, no stationary density function $p_{st}(C)$ exists and therefore a long-time density function $p_{lt}(C)$, where the probability current *approximates* a constant, needs to be defined[39]. Referring to Equations (18) and (19) in Section 4-3, it can be shown that, in analogy to Pearson's Case 2, the long-time density function can be evaluated to

$$p_{lt}(c) = K\,(c - \lambda_1)^{-\nu_1}(c - \lambda_2)^{-\nu_2} \tag{4.58}$$

[39] An alternative approach is to require that the space-time density function approaches zero sufficiently fast as the cash flow range expands over time, Pavliotis (2014, pp. 88-89).

where λ_1, λ_2 are the (real) roots of $\beta(c) = \sigma_2 c^2 + \sigma_1 c + \sigma_0$, $\alpha(c) = \mu_1 c + \mu_0$, $\nu_1 = \frac{-(\mu_1\lambda_1+\mu_0)}{\sigma_2(\lambda_2-\lambda_1)}$, $\nu_2 = \frac{(\mu_1\lambda_2+\mu_0)}{\sigma_2(\lambda_2-\lambda_1)}$, $K > 0$ is a normalisation constant, $\lambda_1 < c < \lambda_2$ and $\lambda_1 < 0 < \lambda_2$. For the following calculations, it is assumed that $\nu_1, \nu_2 \geq -1$.

Notice that for notational convenience, and for consistency with the remainder of this chapter, C is now replaced by c.

From the fact that probability numbers must be greater than or equal to zero, it follows that (i) $(c-\lambda_1)^{\nu_1}(c-\lambda_2)^{\nu_2} \geq 0$ if $\sigma_2 > 0$ or (ii) $(c-\lambda_1)^{\nu_1}(c-\lambda_2)^{\nu_2} \leq 0$ if $\sigma_2 < 0$.
Premise (i) can only be true if $(c-\lambda_1)^{\nu_1} \leq 0$ and $(c-\lambda_2)^{\nu_2} \leq 0$, *or*, $(c-\lambda_1)^{\nu_1} \geq 0$ and $(c-\lambda_2)^{\nu_2} \geq 0$. Both cases have to be ignored since cash flows can take any value, *excluding* $\lambda_1 \leq c \leq \lambda_2$ which is not logical.
Premise (ii) can only be true if $(c-\lambda_1)^{\nu_1} \leq 0$ and $(c-\lambda_2)^{\nu_2} \geq 0$, *or*, $(c-\lambda_1)^{\nu_1} \geq 0$ and $(c-\lambda_2)^{\nu_2} \leq 0$. The first case has to be dismissed because c cannot be smaller than λ_1 and greater than λ_2 at the same time. Hence, c must be between the lower-boundary λ_1 and the upper-boundary λ_2 on a finite cash flow range

$$\lambda_1 \leq c \leq \lambda_2 \tag{4.59a}$$

Extended to the full space-time density function, these boundary conditions become time-variant

$$\lambda_{1,t} \leq c \leq \lambda_{2,t} \tag{4.59b}$$

Besides the mathematical logic of a constrained cash flow process, financial-economic rationales can also be mentioned. Businesses that incur consistent negative cash flows without the prospect of a likely turnaround, will sooner or later fail. Practically, bankruptcy risk puts a natural lower limit on the size of negative operating cash flows. Hence, a lower-boundary condition is justified. Comparable reasoning applies to upper-boundary conditions. No business, regardless of how successful it is, can maintain unlimited growth: there are natural restrictions to the physical size of organisations.

Two boundary conditions are relevant when dealing with probability functions: (i) absorbing boundary conditions, and (ii) reflective boundary conditions.

Reflective boundary conditions impose a *change of probability current* that is equal to zero at the upper and lower cash flow boundaries. The probability current of the forward Kolmogorov equation can be found by expressing the equation as $\frac{\partial p(c,t)}{\partial t} = \frac{-\partial J(C_t,t)}{\partial c}$ where $J(c,t)$ is the probability current $(\mu_1 c + \mu_0) + \frac{1}{2}\frac{\partial(\sigma_2 c^2+\sigma_1 c+\sigma_0 p(c,t))}{\partial c}$. At the boundaries, the reflective boundary conditions produce

$$(\mu_1 c + \mu_0) + \frac{1}{2}\frac{\partial(\sigma_2 c^2 + \sigma_1 c + \sigma_0 p(c,t))}{\partial c} = 0 \qquad (4.60)$$

for $c = \lambda_1, \lambda_2$.

Reflective boundary conditions admit a wide range of intermediate distribution functions, not only those with probability values approaching zero at both boundaries, i.e. $p(c,t) = 0$ for $c = \lambda_1, \lambda_2$. The latter restriction is typical for absorbing boundary conditions, hence associated probability density functions are a subset of those of reflective boundary conditions. In the following part of this section, conscious that less general solutions are likely to be obtained but benefiting from often lesser complicated mathematics, absorbing boundary conditions instead of reflective boundary conditions, will be applied to find solutions to the Kolmogorov equations.

Finally, following Linetsky (2004, p. 348), closed boundary conditions $[\lambda_1;\ \lambda_2]$ lead to a discrete spectrum with eigenvalues that are always non-oscillatory[40].

Discussion of solution methods

The forward Kolmogorov equation was introduced above. From the literature it is apparent that finding general solutions to the Fokker-Planck equation is complicated and usually only possible for special drift and diffusion functions (Risken and Frank (2012, p. 99), Araujo and Drigo Filho (2012)). Therefore, the backward Kolmogorov equation is considered to be the next best option for finding a closed-form specification of the space-time density function. At the end of this section, the backward Kolmogorov equation, in particular its connection to the forward equation, will be discussed. First, however, methods for solving both Kolmogorov equations are to be discussed.

[40] Meaning that the solution to the ODE has a finite number of roots.

Presuming that the Kolmogorov equations are not directly solvable, which they are only in rare cases with very rudimentary drift and diffusion functions, there exist at least two well-known methods to solving the Kolmogorov equations under boundary restrictions:

(i) Transforming one or more variables such that the resulting equation is easier to solve; and

(ii) The method of separation of variables.

The success of the first solution method, transformation of variable(s), depends on the specific form of the Kolmogorov equation. The Laplace or Fourier transform are frequently used to reduce the Kolmogorov PDE to an ODE in cash flow, respectively, to an ODE in time. Despite the attractiveness of this reduction technique, the Laplace and Fourier transform are often considered intractable because, except for a few very simple specifications of the Kolmogorov equations, an inverse transform cannot be found (in closed form).

Potentially more productive methods transform all variables, independent and dependent alike, into a new PDE that is *invariant* relative to the original one. These invariant group transformation methods are explained in Bluman and Cole (1974) and specifically for the Fokker-Planck equation in Nariboli (1977). For instance, Sachdev (2000, Chapter 3) and Meleshko (2006, Chapter 5) provide a more recent treatment of the method including newer developments. In many cases, however, invariant group transformations result in an even more complex system of PDEs. Accordingly, success of the method hinges on finding the right set of transformations that significantly reduce the equation's complexity. Furthermore, applied to PDEs the method leads to a reduction in the number of variables rather than order, and hence solutions of the transformed (system of) PDE(s) often become a particular solution of the untransformed PDE. Nevertheless, the method may be conducive to discovering hitherto unknown solutions to the linear-quadratic Kolmogorov equations.

The method of separation of variables, considers the unconditional density function $\mathrm{p}(\mathrm{c}(\mathrm{t}),\mathrm{t})$ as the product of two separate functions, one in c and the other in t: $\mathrm{p}(\mathrm{c}(\mathrm{t}),\mathrm{t}) = \mathrm{p}_\mathrm{c}(\mathrm{c})\mathrm{p}_\mathrm{t}(\mathrm{t})$, see Cain and Meyer (2005); Logan (2014, Chapter 4)). There are strong indications that the method of separation of variables and invariant group transformations (also known as infinitesimal symmetry methods) are related to each other; however, the exact nature of the relationship is still unresolved. For the supposed connection between the two methods, see e.g. Miller (1977) and Olver (2000). It is a matter of fact that in physics the method of separation of variables customarily does produce plausible results. Irrespective, there is no guarantee that the method leads to a *general solution* that enumerates all possible solutions.

The question can be raised if separation of variables can be used to solve the Kolmogorov Equations (4.57a), (4.57b) and (4.58). Since these equations are second order linear parabolic and homogeneous PDEs, there is in principle no technical objection. In this regard, Weinberger (1995, p. 68), provides three conditions for the method to be applicable:

(i) the differential operator must be separable, that is, there must be a function $\phi(c, t)$ such that $\frac{\mathcal{L}(p_c(c)p_t(t))}{\phi(c,t)p_c(c)p_t(t)} = f(c) + g(t)$,

(ii) all initial and boundary conditions must be on lines c = constant and $p_c(c)$ = constant, and

(iii) the linear operators defining the boundary conditions at c = constant must not have partial derivatives of $p(c(t), t)$ with respect to variable t, and those defining boundary conditions at $p_c(c)$ = constant must not have partial derivatives of $p(c(t), t)$ with respect to variable c.

The PDEs analysed in this study, all obey these three conditions.

Separation of variables prompts solutions that have the following functional specification

$$p(c(t), t) = e^{-\kappa t} p_c(c) \tag{4.61}$$

where κ is a constant that can be positive or negative, and $p_c(c)$ is the solution of an ODE, which specification is determined by the specific drift and difussion functions. The derivation of this equation is explained in the next subsection.

Since Kolmogorov equations deal with probabilities, the associated normalisation of the space-time density function, for all time t, may be regarded as a complication. In order to warrant the normalisation condition (that is, the cumulative density function equates to 1 at all times), solutions are expected to show a widening density function on the cash flow axis coupled with a shrinking probability height as time progresses. Reduction of probability height follows from the time function $e^{-\kappa t}$ if $\kappa > 0$ in the forward Kolmogorov equation, and, if $\kappa < 0$ in the backward Kolmogorov equation. The cash flow component of the solution, $p_c(c)$, seems static at first glance which is incompatible with the normalisation condition. The issue can be dealt with if $e^{-\kappa t}$ is split into three functions $e^{-\varrho t}$, $f_p(t)$ and $f_e(t)$ and $p_c(c)$ is further factorised as $p_p(c)p_e(c)$, where $p_p(c)$ is a (composite) power function and is $p_e(c)$ is a (composite) exponential function. Commonly, power and/or exponential functions form part of many well-known continuous probability distrbution functions.

$$p(c(t),t) = e^{-\kappa t} p_c(c) = e^{-\varrho t}\, f_p(t) f_e(t) p_p(c) p_e(c) = e^{-\varrho t}\, p_p\big(f_p(t)c\big) p_e\big(c + f_e(t)\big) \qquad (4.62)$$

Now, $f_p(t)$ has become a (time-dependent) *scaling factor* and $f_e(t)$ a (time-dependent) *translation factor*. In fact, the equality $f_p(t)p_p(c) = p_p\big(f_p(t)c\big)$ is valid for all *self-similar functions* of which power functions are only a sub-set. The self-similar property was already encountered in Section 4-1, Equation (4.27), where the Lamperti-transformation was discussed.

In summary, the implied transformation of the space-time density function into the product of a (non-linear) factor $e^{-\kappa t}$ and a time-independent density function $p_c(c)$ is only allowed if $p_c(c)$ is translated and scaled on a time-line, i.e. if, heuristically speaking, the shape of the density function is not fundamentally changed with a more complex transformation. Checking Figure 2-4 in Chapter 2, reveals that the shapes of the density function, observed at different points in time, show a close resemblence. The tentative conclusion must be that for both operating and investing cash flows, the method of separation of variables is a justifiable solution method.

Consequently, in the remainder of this section focus will be on the method of separation of variables as a general strategy to solving the polynomial (linear-quadratic) Fokker-Planck equation.

Separation of variables and the forward Kolmogorov equation

Equation (4.57a) may be written in operator form, Risken and Frank (2012, pp. 101-108)

$$\frac{\partial p(c,t)}{\partial t} = \mathcal{L}_{FP}\, p(c,t) \qquad (4.63a)$$

The applicable (absorbing) boundary conditions are

$$p\left(\lambda_{1,t}, t\right) = 0, \qquad p\left(\lambda_{2,t}, t\right) = 0 \tag{4.63b}$$

The Fokker-Planck operator is defined as follows

$$\mathcal{L}_{FP} = -\frac{\partial\, \alpha(c)}{\partial c} + \frac{\partial^2 \beta(c)}{\partial c^2} = \frac{\partial}{\partial c}\left[\beta(c) e^{-\Phi(c)} \frac{\partial}{\partial c} e^{\Phi(c)}\right] \tag{4.64}$$

where $\Phi(c) = -\int \frac{\alpha(\xi)}{\beta(\xi)} d\xi$. For a non-stationary process, $\Phi(c)$ represents a long-time density function for which the probability current becomes approximately constant (that is, asymptotically approaches a stable value). In case of the linear-quadratic specification, $\alpha(c) = (2\sigma_2 - \mu_1)c + (\sigma_1 - \mu_0)$, and $\beta(c) = \frac{1}{2}[\sigma_2 c^2 + \sigma_1 c + \sigma_0]$.

After separation of the variables, Equation (4.63a) is transformed into a system of two ODEs

$$\frac{dp_t(t)}{dt} = -\kappa\, p_t(t) \tag{4.65a}$$

$$\mathcal{L}_{FP} p_c(c) = -\kappa\, p_c(c) \tag{4.65b}$$

where κ is an arbitrary constant, by convention set to a negative value.

The solution to Equation (4.65a) is straightforward

$$p_t(t) = K_0 e^{-\kappa t} \tag{4.66}$$

where K_0 is an integration constant. This result was already used in the previous subsection when discussing the validity of the method of separation of variables.

Solving Equation (4.65b) requires a more involved solution strategy. The full differential specification of the ODE is

$$\beta(c) \frac{d^2 p_c(c)}{dc^2} + \alpha(c) \frac{dp_c(c)}{dc} + K\, p_c(c) \tag{4.67a}$$

with associated boundary conditions

$$p_c(\lambda_1) = 0, \qquad p_c(\lambda_2) = 0 \tag{4.67b}$$

In the case of the linear-quadratic specification

$$\frac{1}{2}[\sigma_2 c^2 + \sigma_1 c + \sigma_0] \frac{d^2 p_c(c)}{dc^2} + [2\sigma_2 - \mu_1)c + (\sigma_1 - \mu_0)] \frac{dp_c(c)}{dc} + \sigma_2 - \mu_1 + \kappa p_c(c) = 0 \tag{4.67c}$$

where $\alpha(c) = (2\sigma_2 - \mu_1)c + (\sigma_1 - \mu_0)$, $\beta(c) = \frac{1}{2}[\sigma_2 c^2 + \sigma_1 c + \sigma_0] = \frac{1}{2}\sigma_2(c - \lambda_1)(c - \lambda_2)$, with λ_1, λ_2 being the (real) roots of $\beta(c)$, and $K = \sigma_2 - \mu_1 + \kappa$.

Equation (4.67c) is of the general hypergeometric type. No known general exact solutions exist (G. M. Leonenko and Phillips (2012, p. 2855)). Regardless, spectral and pseudo-spectral methods based on (classical and non-classical) polynomials are used to solve the Fokker-Planck and related Schrödinger equations (Shizgal (2015, Chapter 6)). Some of these methods will be applied in the remainder of this section. Essentially, they transform Equation (4.67a) into differential equations that have known solutions. There are two ways to accomplish this:

(i) The general hypergeometric differential equation can be directly cast to a polynomial differential equation by restricting the number of parameters $\{\mu_0, \mu_1, \sigma_0, \sigma_1, \sigma_2\}$; or
(ii) The general hypergeometric differential equation can be transformed into a general *Sturm-Liouville problem* for which a wide range of standard solutions occur.

The first technique is often used in the literature. For a criterion to derive differential equations with polynomial solutions, and an extensive overview of examples, refer to Nasser et al. (2006). All six limiting cases known to be solutions of the Pearson diffusion Equation (4.50), Section 4.2, have polynomial solutions (G. M. Leonenko and Phillips (2012)). In the case of the Pearson diffusion, however, Equation (4.67a) is derived directly as a stationary ODE which presupposes a mean-reverting process. The six cases with known solutions are: Hermite polynomials (OU process), Laguerre polynomials (CIR process), Jacobi polynomials (Jacobi process), Bessel polynomials (reciprocal gamma process), Romanovski polynomials (Student process), and Fisher-Snedecor polynomials (Fisher-Snedecor process).

Recall from Section 4.2 that the Student diffusion was considered a particularly useful process to describe investing cash flow processes since it is one of the few particular solutions that admit values in the whole cash flow range $(-\infty; \infty)$. Noticeably, the Student diffusion is regarded as a special case of the more general hypergeometric diffusion (Linetsky (2004)). If the method of separation of variables is applied to examine the inter-temporal dynamics of *investing cash flow processes* (which was suggested in Section 4.2) then the corresponding space-time density function can be expressed as $p(c,t) = e^{-\kappa t} p_c(c)$ where, in this case, $p_c(c)$ is a solution consisting of Romanovski polynomials[41].

Whilst the Student diffusion process is a convenient process to describe investing cash flow (for the reasons set-out above), it has no bearing on operating cash flows. Hence a different approach ought to be considered. Below, the general hypergeometric differential Equation (4.69a) is transformed into a solvable polynomial differential equation.

Transformation of the general hypergeometric equation

Two transformations that turn Equation (4.67c) into a solvable polynomial equation, have been selected on the basis that they do not compromise generality. One transformation pertains to hypergeometric polynomials, a speciality of the general hypergeometric equation; the other to Jacobi polynomials. The two classes of polynomials are related, Kristensson (2010, Chapter 5).

The *first transformation* (for a detailed explanation refer to Zaitsev and Polyanin (2002)) of the cash flow variable is the linear transformation $c' = \frac{c-\lambda_1}{\lambda_2-\lambda_1}$ where λ_1, λ_2 are the (real) roots of the diffusion function $\sigma_2 c^2 + \sigma_2 c + \sigma_0$. This transformation changes the boundary values from $[\lambda_1; \lambda_2]$ into $[0; 1]$. The transformed differential equation becomes

$$\frac{1}{2}\sigma_2 c'(1-c')\frac{d^2 p_{c'}(c')}{dc'^2} + \left[(2\sigma_2-\mu_1)c' + \frac{(2\sigma_2-\mu_1)\lambda_1+\sigma_1-\mu_0}{(\lambda_2-\lambda_1)}\right]\frac{dp_{c'}(c')}{dc'} + (\sigma_2-\mu_1+\kappa)p_{c'}(c') = 0 \qquad (4.68)$$

Equation (4.68) brings about a wide range of solutions all related to Gaussian hypergeometric functions (Agarwal and O'Regan (2008, chapter 10); Aomoto et al. (2011); Schoutens (2012); J. Pearson (2009); Plastino and Rocca (2016)).

The hypergeometric differential Equation (4.68) has exact solutions that can be found for instance in Zaitsev and Polyanin (2002, chapter 2, section 1, equations 171 and 179). If γ is not an integer, solutions can be expressed as

[41] Romanovski polynomials $R_n(c;\ \alpha,\beta)$ solve the following special version of the hypergeometric differential equation: $(1+c^2)R_n''(c;\ \alpha,\beta) + (2\beta c+\alpha)R_n'(c;\ \alpha,\beta) - n(2\beta+n-1)R_n(c;\ \alpha,\beta) = 0$.

$$p_{c'} = \mathcal{F}_{2,1}(\alpha,\beta,\gamma,c') = C_1\mathcal{F}_{2,1}(\alpha,\beta,\gamma,c') + C_2 c'^{1-\gamma}\mathcal{F}_{2,1}(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,c') \quad (4.69a)$$

and for $\gamma \neq 0,-1,-2,-3,\ldots\ldots.,$ solutions are

$$p_{c'} = \mathcal{F}_{2,1}(\alpha,\beta,\gamma,c') = 1 + \sum_{k=1}^{\infty}\frac{\alpha_k\beta_k c'^k}{\gamma_k\,k!} \quad (4.69b)$$

where $\mathcal{F}_{2,1}(\alpha,\beta,\gamma,c')$ is a Gaussian hypergeometric (2,1) function and the Pochhammer symbol k is defined as $\alpha_k = \alpha(\alpha+1)\ldots.(\alpha+k-1)$.

In both cases $\{\alpha,\beta,\gamma\}$ must be solved from the following (nonlinear) system of equations:

$$\left[\alpha+\beta+1 = (2\sigma_2-\mu_1), \gamma = \frac{-(2\sigma_2-\mu_1)\lambda_1+\sigma_1-\mu_0}{(\lambda_2-\lambda_1)}, \alpha\beta = (\sigma_2-\mu_1+\kappa)\right] \quad (4.69c)$$

Recall that κ is an arbitrary constant. From Equation (4.69c) it follows that parameters α and β are dependent on κ which implies that for every κ there is a specific density function $p_{c',\kappa}$ and consequently there are uncountably many solutions to Equation (4.68).

If κ is set to a predetermined value, solutions to the hypergeometric differential equation are usually derived as a Gaussian hypergeometric function that, depending on particular values of $\{\alpha,\beta,\gamma\}$, includes many other special functions as specific or limiting cases. It should be noted that there is no known system for organising all of the identities but a number of algorithms are available that generate different series of identities. Important specific or limiting cases are: Kummer's function (confluent hypergeometric function), Bessel functions, Legendre functions, incomplete beta functions and a variety of other polynomial functions.

The conclusion is that the suggested transformation has the benefit of encapsulating a large number of different solution classes belonging to the general solution but its inherent weakness is that it does not provide a consistent framework to enumerate and describe these solutions.

The *second transformation* changes Equation (4.67c) into one that has orthogonal polynomial solutions to arrive at easier solvable ODEs: Bochner (1929); Dunn and Stein (1961); W. Norrie Everitt (2005); Stein and Klopfenstein (1963). Special cases have been extensively studied in the literature, for example Koekoek et al. (2010); Kristensson (2010). For an overview of solutions to the respective reduced ODEs see Zaitsev and Polyanin (2002, Section 2.1.2). Of all ODEs with polynomial solutions, a subset has *orthogonal* polynomial solutions. Bochner (1929) proves that, up to a linear change of variables, within the class of orthogonal polynomials, only the Hermite, Laguerre and Jacobi polynomials (also known as the classical orthogonal polynomials), satisfy Equation (4.67c). This theorem is explicitly (in a Sturm-Liouville setting) confirmed by Stein and Klopfenstein (1963). The Hermite and Laguerre polynomial differential equations lack generality in their linear terms (belonging to the first order derivate) as opposed to the Jacobi polynomial equation. Therefore, the Jacobi differential equation will now be derived from Equation (4.67c). Express the term $\frac{1}{2}[\sigma_2 c^2 + \sigma_1 c + \sigma_0]$ in this format: $\frac{1}{2}s^2\left[\left[\frac{qc+r}{s}\right]^2 - 1\right]$, where the transformed variable $c' = \frac{qc+r}{s}$, $q = \sqrt{\sigma_2}$, $r = \frac{\sigma_1}{2\sqrt{\sigma_2}}$ and $s^2 = \frac{\sigma_1^2}{4\sigma_2} - \sigma_0$. This transformation was discussed in Section 4-1 under Approximation method three. The transformed differential equation yields

$$\frac{1}{2}\sigma_2(c'+1)(c'-1)\frac{d^2 p_{c'}(c')}{dc'^2} + \left[(2\sigma_2 - \mu_1)c' + \frac{(\sigma_1\mu_1 - 2\sigma_2\mu_0)}{\sqrt{\Delta}}\right]\frac{dp_{c'}(c')}{dc'} + (\sigma_2 - \mu_1 + \kappa)p_{c'}(c') = 0 \tag{4.70}$$

where the discriminant $\Delta = \sigma_1^2 - 4\sigma_2\sigma_0$. Recall that since the two roots of $\sigma_2 c^2 + \sigma_1 c + \sigma_0$ are real, the discriminant is always positive. The transformation brings about a boundary value change from $[\lambda_1; \lambda_2]$ into $[-1; 1]$. This transformation is a useful normalisation which enhances the possibilities of finding standard closed-form solutions to the ODE. Therefore, the normalised Ordinary Differential Equation (4.70) will be used in the Jacobi, Hermitian and Schrödinger transformation to follow.

The general solution as a Sturm-Liouville problem

Transforming the Fokker-Planck equation into a Sturm-Liouville problem, facilitates the general solution strategy considerably as outlined in, for example, Mathew (2015). The class of particular polynomial solutions $p_c(c) = Q_n(c)$, can be found by setting $\kappa = \kappa_n$ as eigenvalues where Q_n is a polynomial eigenfunction in c of degree at most n. The Sturm-Liouville solution strategy includes the following steps. First, the stationary density function (also called the Sturm-Liouville weight function) is established, then eigenvalues (for closed boundaries: discrete) and eigenfunctions of the ODE are found by applying appropriate standard techniques, and lastly, the normalisation constant that satisfies the initial condition is calculated. Now, solutions take the following general format

$$p(c,t|c_0) = p_{st}(c)\sum_{\kappa}\left[e^{-\kappa t}\rho_\kappa p_\kappa(c)p_\kappa(c_0)\right] \quad (4.71a)$$

where $p_{st}(c)$ is the stationary density function as $t \to \infty$, c_0 is the initial cash flow, κ are eigenvalues, $\rho_\kappa = \frac{1}{\int_{-1}^{1}\omega(\varepsilon)p_\kappa^2(\varepsilon)d\varepsilon}$ are Fourier coefficients that serve also as normalisation constants, $\omega(c)$ is the Sturm-Liouville weight function, $p_\kappa(c)$ and $p_\kappa(c_0)$ are corresponding (orthogonal) eigenfunctions. Commonly, Dirac's delta function $\delta(c - c_0)$ is adopted as an initial condition[42], that is, the initial probability density function at $t = 0$.

Equation (4.71a) shows how the stationary solution, as examined in the prior section for investing cash flows, can be turned into a dynamic solution. In this study, it was nevertheless found that operating cash flows rarely have stationary solutions. Therefore, it was suggested to replace the stationary density function by an approximate long-time density function; see the discussion leading to Equation (4.58). Thus, for operating cash flows Equation (4.71a) can be expressed as

$$p(c,t|c_0) = p_{lt}(c)\sum_{\kappa}\left[e^{-\kappa t}\rho_\kappa p_\kappa(c)p_\kappa(c_0)\right] \quad (4.71b)$$

where $p_{lt}(c) = K\,(c - \lambda_1)^{-\nu_1}(c - \lambda_2)^{-\nu_2}$ (see Equation (4.58) for restrictions on validity).

The Sturm-Liouville theory states (Atkinson and Mingarelli (2010, Chapter 1)) that solutions to the problem consist of an ordered sequence of eigenvalues $\kappa_1 < \kappa_2 < \kappa_3 \dots \dots \kappa_n$, with corresponding to each eigenvalue, an eigenfunction $p_{c,n}(c)$, if it exists. The eigenfunctions, i.e. the terms in the brackets $\rho_\kappa p_\kappa(c)p_\kappa(c_0)$ of Equation (4.71b), form together a Fourier series that defines a complete set of solutions $p(c) = \sum_{i=1}^{i=n}\rho_i p_{c,i}(c)$ to the cash flow ODE. Fourier coefficients ρ_i are calculated from the property that eigenfunctions are mutually orthogonal[43]. There exist a considerable body of literature with a wide range of specific solutions to the Sturm-Liouville problem (for instance Al-Gwaiz (2008); Amrein et al. (2005)). The most important transformations in respect of Sturm-Liouville problems are discussed in W. Norrie Everitt (2005) and Avram et al. (2013, section 3.3).

[42] It can be shown that if the original cash flow with boundaries $[\lambda_1; \lambda_2]$ at time zero has a Dirac delta function to describe the initial probability density then also the transformed cash flow with boundaries $[-1; 1]$ will have a Dirac delta function.

[43] Orthogonality is defined as $\int_a^b p_{c,m}(c)p_{c,n}(c)\omega(c)dc = 0$, where $m \neq n$ and $\omega(c)$ is called a weight function.

In the remainder of this subsection, three standard solution techniques will be discussed with respect to the $[-1;1]$-transformed linear-quadratic cash flow model outlined in Equation (4.60). These techniques are: (i) Jacobi transformation, (ii) Hermitian transformation, and (ii) Schrödinger transformation. For each of these techniques, the general hypergeometric equation will be first developed into an appropriate Sturm-Liouville problem.

For notational convenience Equation (4.70) will be written in the sequel as

$$(c+1)(c-1)p'' + [\mu_1' c + \mu_0']p' + \kappa' p = 0 \tag{4.72a}$$

where $p'' = \frac{d^2 p_{c'}(c')}{dc'^2}$, $p' = \frac{dp_{c'}(c')}{dc'}$, $p = p_{c'}(c')$, $\mu_1' = \frac{(2\sigma_2 - \mu_1)}{\frac{1}{2}\sigma_2}$, $\mu_0' = \frac{(\sigma_1\mu_1 - 2\sigma_2\mu_0)}{\frac{1}{2}\sigma_2\sqrt{\Delta}}$ and $\kappa' = \frac{(\sigma_2 - \mu_1 + \kappa)}{\frac{1}{2}\sigma_2}$. Notice that c' is replaced by c despite being the same variable.

Equivalent expression of Equation (4.72a) are

$$(c^2-1)p'' + [\mu_1' c + \mu_0']p' + \kappa' p = 0 \tag{4.72b}$$

and

$$(1-c^2)p'' - [\mu_1' c + \mu_0']p' - \kappa' p = 0 \tag{4.72c}$$

The latter equation safeguards that $1 - c^2$ is always non-negative on the domain $[-1;1]$.

Associated boundary conditions that apply to all three equations, are

$$p(-1) = 0, \qquad p(1) = 0 \tag{4.73}$$

Jacobi transformation:

If solutions (eigenfunctions) to Equation (4.74) are transformed into Jacobi polynomials then the Jacobi operator $\mathcal{L}_{J,\alpha,\beta}$ links these eigenfunctions to related eigenvalues $\kappa_{n,\alpha,\beta}$

$$\mathcal{L}_{J,\alpha,\beta} p = -\kappa_{n,\alpha,\beta}\, p \tag{4.74}$$

where $p = J_n^{\alpha,\beta}(c) = \frac{\Gamma(n+\alpha+1)}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{(n+\alpha+\beta+1,k)}{\Gamma(\alpha+k+1)}\left[\frac{c-1}{2}\right]^k$, $\alpha, \beta > -1$, and n represents the degree of the polynomial. The Jacobi operator is defined as

$$\mathcal{L}_{J,\alpha,\beta} = -(1-c)^{-\alpha}(1+c)^{-\beta}\frac{d}{dc}(1-c)^{-\alpha}(1+c)^{-\beta}\frac{d}{dc} = (c+1)(c-1)\frac{d^2}{dc^2} + [(\alpha+\beta+2)c + ((\alpha-\beta)]\frac{d}{dc} \qquad (4.75)$$

Eigenvalues are calculated from the following formula

$$\kappa_{n,\alpha,\beta} = n(n+\alpha+\beta+1) \qquad (4.76)$$

For a derivation refer to Shen et al. (2011, Section 3.2). Similar to the hypergeometric differential equation, parameters α, β, n must be solved from the following system of equations

$$[(\alpha+\beta+2) = \mu_1',\ \ (\alpha-\beta) = \mu_0',\ \ n(n+\alpha+\beta+1) = \kappa'\ ] \qquad (4.77a)$$

The parameters of ODE (72) are now directly related to the Jacobi polynomial $J_n^{\alpha,\beta}$ as a solution to the ODE. Expressed in explicit form, parameters α, β, n become

$\alpha = \frac{1}{2}(\mu_1' + \mu_0') - 1);\ \beta = \frac{1}{2}(\mu_1' - \mu_0') - 1)$ and n are the roots of the following quadratic equation in Z

$$Z^2 + (\mu_1' - 1)Z - \kappa' \qquad (4.77b)$$

Written in the usual Sturm-Liouville format, Equation (4.74) is equal to

$$\frac{d}{dc}\left[(1-c)^{\alpha+1}(1+c)^{\beta+1}\frac{dp}{dc}\right] = \kappa_{n,\alpha,\beta}\, p \qquad (4.78)$$

The orthogonal property requires that for any two particular solutions $J_m^{\alpha,\beta}$ and $J_n^{\alpha,\beta}$ $(m \neq n)$ the following relation holds

$$\int_{-1}^{1} J_m^{\alpha,\beta}(c) J_n^{\alpha,\beta}(c)\, \omega(c) = \kappa_{n,\alpha,\beta}\delta_{mn} \qquad (4.79)$$

where the weight function $\omega(c) = (1-c)^{\alpha}(1+c)^{\beta}$, and δ_{mn} is the Kronecker delta.

Importantly, the weight function, sometimes called integration factor, is equal to the stationary density function $\Phi_{st}(\,)$, Herman (2008, Chapter 6), and for a more detailed derivation: Cain and Meyer (2005, p. 57). Recall that in the beginning of this section, it was advocated that for operating cash flows the stationary density could be approximated by a long-time density function $\Phi_{lt}(c)$

$$\Phi_{lt}(c) = -\int \frac{\alpha(\xi)}{\beta(\xi)} d\xi = -\int \frac{[(\alpha+\beta+2)\xi + ((\alpha-\beta)]}{(\xi+1)(\xi-1)} d\xi = (1-c)^{\alpha}(1+c)^{\beta} \qquad (4.80a)$$

Using the substitution $1 - c = 2z$, it is easy to see that $\Phi_{lt}(z) = 2z^{\alpha}(1-z)^{\beta}$ is a Beta function (to be normalised) with parameters $\{\alpha+1, \beta+1\}$ and cash flow range z: $[0; 1]$. If the transformation that led to Equation (4.70) is reversed then after some algebra it can be shown that

$$\Phi_{lt}(c) = (c-\lambda_1)^{\alpha}(c-\lambda_2)^{\beta} \qquad (4.80b)$$

where λ_1, λ_2 are the (real) roots of the diffusion function $\sigma_2 c^2 + \sigma_2 c + \sigma_0$. Equation (4.80b) is equivalent to Pearson's Case 2, Equation (4.19) in Section 2-3, and admits a wide range of possible density function amongst which is the (generalised) Beta function.

Recall that Equation (4.74) is subject to boundary conditions (73). At each of the boundary values, the Jacobi function $J_n^{\alpha,\beta}(c)$ evaluates to (Doha (2002, p. 3469)

$$J_n^{\alpha,\beta}(1) = \frac{\Gamma(n+\alpha+1)}{n!\ \Gamma(\alpha+1)} = 0 \qquad (4.81a)$$

$$J_n^{\alpha,\beta}(-1) = \frac{(-1)^n\Gamma(n+\beta+1)}{n!\ \Gamma(\beta+1)} = 0 \qquad (4.81b)$$

For known parameters α, β, Equations (4.81a) and (4.81b) facilitate the calculation of the degrees of the polynomial at which two of the equation's zeros lie exactly at the upper and lower boundary values. Observe that the equations must be solved numerically. Once the values for n are found, corresponding eigenvalues are directly calculated from Equation (4.76).

The *general solution* follows from Equations (4.71), (4.74) and (4.76)

$$p(c,t|c_0) = (1-c)^{\alpha}(1+c)^{\beta} \sum_{k=0}^{n} \left\{ \rho_n\, e^{-n(n+\alpha+\beta+1)t} \left[\frac{\Gamma(n+\alpha+1)}{n!} \binom{n}{k} \frac{(n+\alpha+\beta+1,k)}{\Gamma(\alpha+k+1)}\right]^2 \left[\frac{(c_0-1)(c-1)}{2}\right]^k \right\} \qquad (4.82)$$

where $\rho_n = \frac{1}{\int_{-1}^{1} \omega(\varepsilon) p_K^2(\varepsilon) d\varepsilon} = \frac{\Gamma(2n+\alpha+\beta+2)}{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}$.

<u>Hermitian transformation</u>:

The Fokker-Planck operator of Equation (4.64) can be turned into a (self-adjoint) Hermitian operator $\mathcal{L}_H$[44] that has some useful properties

$$\mathcal{L}_H = \mathrm{e}^{\frac{\Phi(c)}{2}} \mathcal{L}_{FP} \mathrm{e}^{\frac{-\Phi(c)}{2}} \tag{4.83}$$

where $\Phi(c) = -\int \frac{\alpha(\xi)}{\beta(\xi)} d\xi$.

Risken and Frank (2012, p. 106) show that Equation (4.83) can be re-written to

$$\mathcal{L}_\mathrm{H} = \frac{d}{dc}\beta(c)\frac{d}{dc} - V \tag{4.84}$$

where $V(c) = \frac{\left(\beta'(c)-\alpha(c)\right)^2}{4\beta(c)} + \frac{(\alpha'(c)-\beta''(c))}{2}$.

After above operator transformation, Equation (4.67a) satisfies the ODE

$$\beta(c)q'' + \beta'(c)q' - \left[\frac{\left(\beta'(c)-\alpha(c)\right)^2}{4\beta(c)} + \frac{\left(\beta''(c)-\alpha'(c)\right)}{2}\right] q = -\kappa'\, q \tag{4.85a}$$

or, specifically for the linear-quadratic specification in Equation (4.67c), transformed into $[-1; 1]$ boundaries in Equation (4.70)

$$(c^2-1)q'' + 2c\, q' - \left[\frac{\left[(2-\mu_1')c-\mu_0'\right)]^2}{4(c^2-1)} - \frac{1}{2}\mu_1' + 1\right] q = -\kappa'\, q \tag{4.85b}$$

where $q = p'_{c'}(c') = e^{\frac{\Phi(c)}{2}} p_{c'}(c')$, $\mu_1' = \frac{(2\sigma_2-\mu_1)}{\frac{1}{2}\sigma_2}$, $\mu_0' = \frac{(\sigma_1\mu_1-2\sigma_2\mu_0)}{\frac{1}{2}\sigma_2\sqrt{\Delta}}$ and $\kappa' = \frac{(\sigma_2-\mu_1+\kappa)}{\frac{1}{2}\sigma_2}$.

The advantage of this operator transformation and associated probability density transform, is that solutions to Equation (4.85b) take the general format of Equation (4.71b). Since the operator is Hermitian, eigenvalues must be *positive* (for finite boundaries) and pairwise orthogonal. Importantly, the eigenvalues of q are equal to those of the non-Hermitian probability density function $p_{c'}(c')$.

Equation (4.85b) is often expressed in Sturm-Liouville notation

[44] Also called Sturm-Liouville operator.

$$[(c^2-1)q']' - \left[\frac{[(2-\mu_1')c-\mu_0')]^2}{4(c^2-1)} - \frac{1}{2}\mu_1' + 1\right]q = -\kappa' q \tag{4.85c}$$

or more appropriately

$$[(1-c^2)q']' + \left[\frac{[(2-\mu_1')c-\mu_0')]^2}{4(1-c^2)} - \frac{1}{2}\mu_1' + 1\right]q = \kappa' q \tag{4.85d}$$

Equation (4.85d) represents a *singular* Sturm-Liouville problem because the equation is singular at the boundary values $[-1, 1]$.

Since Equation (4.85c) is mathematically not tractable, a further transformation is required with the aim of yielding a (hopefully) simpler and better solvable Sturm-Liouville form. A class of transformations that can achieve this purpose (if solutions do exist), is the Liouville normal transformation, described in for instance, William Norrie Everitt (1982, Section 4.3.), Zwillinger (1998, Section 31), W. Norrie Everitt (2005, Section 7) and Atkinson and Mingarelli (2010, Section 1.3).

The suggested Liouville transformation is explained in W. Norrie Everitt (2005, Section 7). It comprises of a simultaneous transformation of the cash flow variable c to $C(c)$ and the probability density variable from q to $Q(q)$. Specifically, the cash flow transformation is

$$C(c) = \int \frac{1}{\sqrt{1-\varepsilon^2}} d\varepsilon = \ln\left[|\sqrt{(1-c^2} + c|\right] = \sin^{-1}\left[\frac{1}{c}\right] \tag{4.86a}$$

whilst the proposed probability density transformation is given by

$$Q(q) = (1-c^2)^{\frac{1}{4}} q(c) \tag{4.86b}$$

Equation (4.85c) can then be re-cast to

$$-Q'' + \left[\frac{(\chi_2 c^2 + \chi_1 c + \chi_0)}{(1-c^2)}\right] Q = \kappa' Q \tag{4.87a}$$

where $c = \frac{1}{\sin(C)}$, $\chi_2 = (\frac{1}{2} - \frac{1}{2}\mu_1')^2$, $\chi_1 = \frac{1}{2}(2-\mu_1')\mu_0'$ and $\chi_0 = \frac{1}{4}\mu_0'^2 - \frac{1}{2}\mu_1' + \frac{1}{2}$.

Unfortunately, Equation (4.87a) is also not tractable; however, its stochastic behaviour can be approximately analysed for parts of the cash flow range $[-1, 1]$. The suggested method divides the total cash flow range into five sub-ranges: $(-1.0; -0.7]$, $(-0.7; -0.4]$, $(-0.4; 0.4)$, $[0.4; 0.7)$ and $[0.7; 1.0)$. Since $\frac{(\chi_2 c^2 + \chi_1 c + \chi_0)}{(1-c^2)}$ is symmetric at $c = 0$, the function is approximated by three different constants $c = \psi_j$, $j = 1,2,3$. Each constant is linked to the (set of) sub-ranges $\{(-1.0; -0.7], [0.7; 1.0)\}$, $\{(-0.7; -0.4], [0.4; 0.7)\}$ and $(-0.4; 0.4)$ as is apparent from Figure 4-6. The optimum value of the respective constants ψ_j can be determined by employing numerical optimisation techniques (in Figure 4-6, for example, by minimising the sum of the squared error terms for each set of subranges. The error is defined as calculated minus approximated cash flows).

Figure 4-6 shows the differences between the calculated and approximated values of the middle term of Equation (4.87a), and measures these differences in one number: an average absolute error of 15.3%. Since this number is significantly influenced by large deviations close to the boundaries, for a large part of the cash flow spectrum the specification $c = \psi_j$, $j = 1, 2, 3$ (the three dotted lines in Figure 4-6) is nevertheless a reasonable approximation. However, in order to achieve a more accurate approximation, the proposed solution technique can easily be expanded to a much larger j coinciding with an increasing granular division of the cash flow spectrum. As will become clear below, the whole set of approximate constants have the same eigenfunctions.

The proposed approximation is an uncomplicated example of the Perturbation theory that uses approximation schemes to describe a complex system in terms of a simpler one ((Nayfeh (2011), Skinner (2011)). Perturbation theory includes more sophisticated approximations than the above, for example a series of linear approximations, or considering the second order ODE as a first order ODE with a second order perturbation term.

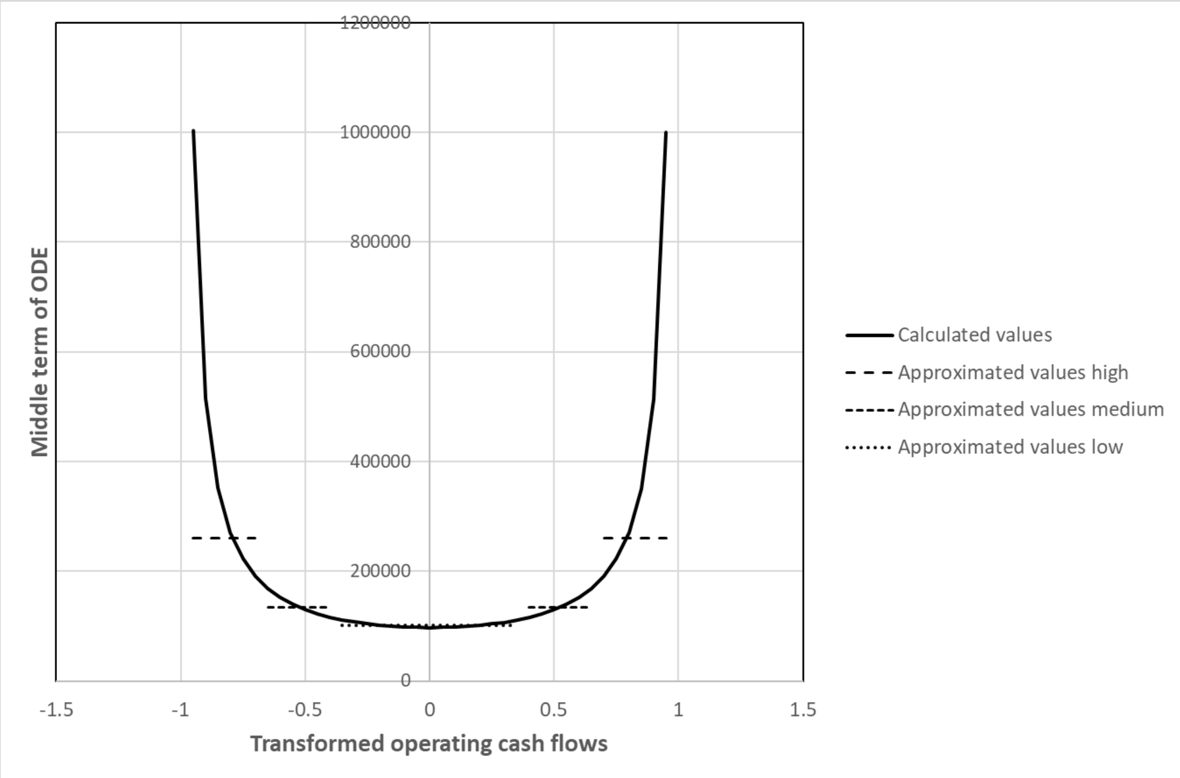
The proposed approximation reduces the complexity of Equation (4.87a) considerably

$$-Q'' + \psi_j Q = \kappa' Q \tag{4.87b}$$

For $\kappa' > 0$ the solution to Equation (4.87b) is

$$Q_j(c) = K_1 \sin(\phi_j c) + K_2 \cos(\phi_j c) \tag{4.88a}$$

where K_1 and K_2 are normalisation constants and, since $\kappa' - \psi_j$ is assumed to be non-negative, $\phi_j^2 = \kappa' - \psi_j$.



CASH FLOW RANGE	CALCULATED VALUES	APPROXIMATED VALUES	VARIANCE	ABSOLUTE VARIANCE AS % OF CALCULATED VALUES
-1				
-0.95	1002544	259699.3	742844.5	74.1%
-0.9	514422.1	259699.3	254722.8	49.5%
-0.85	352188.8	259699.3	92489.5	26.3%
-0.8	271457.1	259699.3	11757.9	4.3%
-0.75	223352.6	259699.3	-36346.7	16.3%
-0.7	191586.2	259699.3	-68113.1	35.6%
-0.65	169179.4	134001.2	35178.2	20.8%
-0.6	152645.8	134001.2	18644.6	12.2%
-0.55	140050.9	134001.2	6049.7	4.3%
-0.5	130236.9	134001.2	-3764.3	2.9%
-0.45	122470.1	134001.2	-11531.2	9.4%
-0.4	116264.4	134001.2	-17736.9	15.3%
-0.35	111286.9	102183.9	9103.0	8.2%
-0.3	107303.8	102183.9	5119.9	4.8%
-0.25	104147.9	102183.9	1964.0	1.9%
-0.2	101698.8	102183.9	-485.2	0.5%
-0.15	99870.07	102183.9	-2313.8	2.3%
-0.1	98601.2	102183.9	-3582.7	3.6%
-0.05	97852	102183.9	-4331.9	4.4%
0	97599.56	102183.9	-4584.3	4.7%
0.05	97836.34	102183.9	-4347.6	4.4%
0.1	98569.64	102183.9	-3614.3	3.7%
0.15	99822.13	102183.9	-2361.8	2.4%
0.2	101633.7	102183.9	-550.2	0.5%
0.25	104064.5	102183.9	1880.6	1.8%
0.3	107200.8	102183.9	5016.9	4.7%
0.35	111162.3	102183.9	8978.4	8.1%
0.4	116115.6	134001.2	-17885.7	15.4%
0.45	122293.8	134001.2	-11707.5	9.6%
0.5	130028.6	134001.2	-3972.6	3.1%
0.55	139804.5	134001.2	5803.3	4.2%
0.6	152352.9	134001.2	18351.7	12.0%
0.65	168827.8	134001.2	34826.6	20.6%
0.7	191157.4	259699.3	-68541.9	35.9%
0.75	222817	259699.3	-36882.3	16.6%
0.8	270762.9	259699.3	11063.6	4.1%
0.85	351231.8	259699.3	91532.5	26.1%
0.9	512942.3	259699.3	253243.0	49.4%
0.95	999499.7	259699.3	739800.4	74.0%
1				
AVERAGE				**15.3%**

Note: $\mu_0 = 10, \mu_1 = 5, \sigma_0 = -1, \sigma_1 = 0.5, \sigma_2 = 4$

Figure 4-6 Values of the middle term of equation (77a)

Equation (4.86b) can be used to revert transform Q to q (note: Equation (4.88a) is already expressed in c)

$$q_j(c) = (1-c^2)^{-\frac{1}{4}}\left[K_1 \sin(\phi_j c) + K_2 \cos(\phi_j c)\right] \tag{4.88b}$$

In the following step transform q(c) back to $p_{c,j}(c)$ by $p_{c,j}(c) = e^{\frac{-\Phi(c)}{2}} q_j(c)$

$$p_{c,j}(c) = e^{\frac{-\Phi(c)}{2}}(1-c^2)^{-\frac{1}{4}}\left[K_1 \sin(\phi_j c) + K_2 \cos(\phi_j c)\right] \tag{4.88c}$$

where $-\frac{\Phi(c)}{2} = -\frac{1}{2}\int \frac{\mu_1'\xi+\mu_0'}{\xi^2-1} d\xi = \ln\left[(1-c)^{\frac{1}{4}(\mu_0'+\mu_1')}(1+c)^{\frac{1}{4}(\mu_0'-\mu_1')}\right]$.

The first two terms can be combined to

$$p_{c,j}(c) = (1-c)^{\nu_1}(1+c)^{\nu_2}\left[K_1 \sin(\phi_j c) + K_2 \cos(\phi_j c)\right] \tag{4.88d}$$

where $\nu_1 = \frac{1}{4}(\mu_0' + \mu_1' - 1)$, $\nu_2 = \frac{1}{4}(\mu_0' - \mu_1' - 1)$, and $\nu_1, \nu_2 \geq -1$.

Now, the boundary conditions of Equation (4.83) can be applied to Equation (4.88c), beginning with the upper-boundary value $c = 1$

$$(0)^{\nu_1}(2)^{\nu_2}\left[K_1 \sin(\phi_j) + K_2 \cos(\phi_j)\right] = 0 \tag{4.89a}$$

Since the first term $(0)^{\nu_1}(2)^{\nu_2}$ is always zero, the second term can either be *zero* or *non-zero*. These two cases will be separately discussed.
First, the non-zero case

$$K_1 \sin(\phi_j) + K_2 \cos(\phi_j) = (K_1 - K_2)\cos(\phi_j) \tag{4.89b}$$

The RHS of Equation (4.89b) is an arbitrary constant that can be set equal to $(K_1 - K_2)\cos(\phi_j)$. Notice that K_1 and K_2 ($K_1 \neq K_2$) are also arbitrary constants, however one of the two is determined by the normalisation condition.

After some algebra, Equation (4.89b) turns into

$$\tan(\phi_j) = 1 \tag{4.89c}$$

for which parameters ϕ_j are

$$\phi_j = \frac{1}{4}\pi + \pi n \tag{4.89d}$$

where $n = 0, 1, 2 \ldots\ldots \infty$.

Next, the zero case where the second term is also set to zero

$$K_1 \sin(\phi_j) + K_2 \cos(\phi_j) = 0 \tag{4.90a}$$

For the cases $\{K_1 \neq 0, K_2 = 0\}$ and $\{K_1 = 0, K_2 \neq 0\}$, Equation (4.90a) has two mutual exclusive solutions that are well-known (Schiff (1955, p. 35), and Logan (2014, pp. 160-161) for the heat equation, i.e. the sine-function only)

$$\sin(\phi_j) = 0 \ , \ \cos(\phi_j) = 0 \tag{4.90b}$$

with parameters ϕ_j

$$\phi_j = \frac{1}{2}\pi n \tag{4.90c}$$

where $n = 0, 1, 2 \ldots\ldots \infty$ and n is even for $\sin(\phi_j)$ and n is odd for $\cos(\phi_j)$.

The foregoing step can be repeated for the lower boundary value $c = -1$. In this case the expression $(-2)^{\nu_1}(0)^{\nu_2}$ is also zero. The associated boundary equation yields for the non-zero case

$$\tan(\phi_j) = -1 \tag{4.91a}$$

with parameters ϕ_j

$$\phi_j = \frac{3}{4}\pi + \pi n \tag{4.91b}$$

and for the zero case the parameters ϕ_j are identical to the ones of Equation (4.90c).

Eigenvalues can be calculated from $\kappa'_{n,j} = \phi_j^2 + \psi_j$, and corresponding eigenfunctions are found by substituting parameters ϕ_j values from Equations (4.89d) and (4.91b) into Equation (4.88d), respectively, the parameter values from Equation (4.90c) into Equation (4.88d). The result is the following *three* equations

$$p_c(c) = (1-c)^{\nu_1}(1+c)^{\nu_2}\left[K_1 \sin\left(\frac{1}{4}n_1\pi c\right) + K_2 \cos\left(\frac{1}{4}n_1\pi c\right)\right] \tag{4.92a}$$

$$p_c(c) = (1-c)^{\nu_1}(1+c)^{\nu_2}\left[K_1 \sin\left(\frac{1}{2}n_2\pi c\right)\right] \tag{4.92b}$$

$$p_c(c) = (1-c)^{\nu_1}(1+c)^{\nu_2}\left[K_1 \cos\left(\frac{1}{2}n_3\pi c\right)\right] \quad (4.92c)$$

where K_1 and K_2 are normalisation constants, $\nu_1 = \frac{1}{4}(\mu_0' + \mu_1' - 1)$, $\nu_2 = \frac{1}{4}(\mu_0' - \mu_1' - 1)$, $\nu_1, \nu_2 \geq -1$, $n_1, n_2, n_3 = 0, 1, 2 \ldots\ldots \infty$, n_1 and n_3 are *odd* numbers, and n_2 are *even* numbers. Notice that all three ψ_j $(j = 1, 2, 3)$ have the same parameter ϕ_j values but *not* the same eigenvalues.

All ingredients are now available to arrive at *approximated general solutions* to the Fokker-Planck equation. By superposition, the equation

$$p(c,t) = K'\,(c-1)^{\nu_1}(c+1)^{\nu_2}\sum_{n=0}^{n=\infty}\sum_{j=1}^{j=k}\left[e^{-\left[\frac{1}{16}n^2\pi^2-\psi_j\right]t}\left[a_n \sin\left(\frac{1}{4}n\pi c\right) + b_n \cos\left(\frac{1}{4}n\pi c\right)\right]\right] = K\,(c-1)^{\nu_1}(c+1)^{\nu_2}\,e^t\sum_{n=0}^{n=\infty}\left[e^{-\left[\frac{1}{16}n^2\pi^2-\psi_j\right]t}\left[a_n \sin\left(\frac{1}{4}n\pi c\right) + b_n \cos\left(\frac{1}{4}n\pi c\right)\right]\right] \quad (4.93a)$$

satisfies the Fokker-Planck equation and applicable boundary conditions. First notice that Equation (4.93) includes all of the solutions (92a) – (92c) if $n = 0, 1, 2, 3, 4 \ldots\ldots.\infty$. Parameter K is a remaining normalisation constant ensuring that $\int_{-1}^{1} p(c,t)\,dc = 1$. Constant K is defined as: $K = K'\sum_{j=1}^{j=k} e^{-\psi_j}$ where parameter k represents the number of constants used in approximating the middle term of Equation (4.87a).

Equation (4.93a) can be written in the form of

$$p(c,t) = K\,(c-1)^{\nu_1}(c+1)^{\nu_2}\,e^{-t}\left\{a_0 + \sum_{n=1}^{n=\infty}\left[e^{-\frac{1}{16}n^2\pi^2 t}\left[a_n \sin\left(\frac{1}{4}n\pi c\right) + b_n \cos\left(\frac{1}{4}n\pi c\right)\right]\right]\right\} \quad (4.93b)$$

The summation of the term $a_0 + \sum_{n=1}^{n=\infty}\left[e^{-\frac{1}{16}n^2\pi^2 t}\left[a_n \sin\left(\frac{1}{4}n\pi c\right) + b_n \cos\left(\frac{1}{4}n\pi c\right)\right]\right]$ is a Fourier series. Given that Dirac's delta function $\delta(c - c_0)$ is adopted as an initial condition, Fourier coefficients a_0, a_n and b_n can be calculated from

$$a_0 = \frac{1}{2}\int_{-1}^{1} \delta(c - c_0)dc = 1 \tag{4.93c}$$

$$a_n = \int_{-1}^{1} \sin(\frac{1}{4}n\pi c)\,\delta(c - c_0)dc = \sin(\frac{1}{4}n\pi c_0) \tag{4.93d}$$

$$b_n = \int_{-1}^{1} \cos(\frac{1}{4}n\pi c)\,\delta(c - c_0)dc = \cos(\frac{1}{4}n\pi c_0) \tag{4.93e}$$

Now, Equation (4.93b) becomes

$$p(c,t) = K\,(c-1)^{\nu_1}(c+1)^{\nu_2}\,e^{-t}\left\{a_0 + \sum_{n=1}^{n=\infty}\left[e^{-\frac{1}{16}n^2\pi^2 t}\left[a_n \sin\left(\frac{1}{4}n\pi c\right) + b_n \cos\left(\frac{1}{4}n\pi c\right)\right]\right]\right\} \tag{4.94}$$

where c_0 is the initial cash flow at $t = 0$, $\nu_1 = \frac{1}{4}(\mu'_0 + \mu'_1 - 1)$ and $\nu_2 = \frac{1}{4}(\mu'_0 - \mu'_1 - 1)$, and $\nu_1, \nu_2 \geq -1$. It is easy to see that Equation (4.94) also satisfies the long-time end-condition

$$p(c,T) = p_{lt}(c) = \widetilde{K}(c-1)^{\nu_1}(c+1)^{\nu_2}e^{-T} = K^*(c-1)^{\nu_1}(c+1)^{\nu_2} \tag{4.95}$$

where $e^{-(1+\frac{1}{16}n^2\pi^2)t} \downarrow 0$ faster than $e^{-t} \downarrow 0$ as $t \to \infty$ for $n > 0$, and $\widetilde{K} = Ke^{-T}a_0$ is a new normalisation constant. Indeed, Equation (4.95) has the same functional specification as Equation (4.58), admitting Pearson's Case 1 family of distributions.

Notice that variable c is still the $[-1, 1]$-transformed cash flow variable but can easily be transformed back to the original cash flow variable c with $[\lambda_1, \lambda_2]$ boundary values.

In addition to the preceding approximated general solution, Equation (4.87a) produces a number of interesting special cases of which six are analytically solvable. These are shown in Table 4-3 below.

Table 4-3 Special cases of the solution to Equation (4.87a)

PARAMETER RESTRICTIONS	SPECIFICATIONS	SOLUTIONS
$\chi_2 = 0$	$-Q'' + \left[\frac{\chi_1 c + \chi_0)}{(1-c^2)}\right] Q = -\kappa' Q$	Composite Heun confluent function, Hortacsu (2011)
$\chi_1 = 0$	$-Q'' + \left[\frac{(\chi_2 c^2 + \chi_0)}{(1-c^2)}\right] Q = -\kappa' Q$	Composite Heun confluent function, Hortacsu (2011)
$\chi_0 = 0$	$-Q'' + \left[\frac{\chi_0}{(1-c^2)}\right] Q = -\kappa' Q$	Composite Heun confluent function, Hortacsu (2011)
$\chi_1 = 0, \chi_2 = -\chi_0$	$-Q'' + \chi_2 Q = \kappa' Q$	Equation of free oscillations, Zaitsev and Polyanin (2002, Section 2.1.2). Equation 2.1.2-1.1
$\chi_1 = \chi_2 + \chi_0$	$-Q'' + \left[\frac{\chi_2 c + \chi_0}{1-c}\right] Q = \kappa' Q$	Combined Whittaker functions, Lebedev and Silverman (2012, Section 9.13)
$\chi_1 = -(\chi_2 + \chi_0)$	$-Q'' + \left[\frac{\chi_2 c - \chi_0}{1+c}\right] Q = \kappa' Q$	Combined Whittaker functions, Lebedev and Silverman (2012, Section 9.13)

Schrödinger transformation:

The Schrödinger transformation, and its connection to the Hermitian transformation, is explained in Pavliotis (2014, Section 4.9). The transformation itself is identical to the cash flow transformation of Equation (4.88a). It changes Equation (4.74) into an ODE with a unit diffusion coefficient and a transformed drift function

$$\widehat{\alpha}(C) = \frac{1}{\sqrt{(1-c^2)}}\left[(\mu_1' - 1)c + \mu_0'\right] \tag{4.96}$$

The corresponding Fokker-Planck operator now becomes the Schrödinger operator, defined as

$$\mathcal{L}_S = -\frac{d\,\widehat{\alpha}(C)}{dC} + \frac{d^2}{dC^2} \tag{4.97}$$

with $p_C(C) = \sqrt{(1-c^2)}\, p_c(c)$. The derivation is found in Risken and Frank (2012, p. 97). Equation (4.96) is almost identical to the Lamperti transform described in Equation (4.24); only a few parameters vary. Importantly, the operator $\mathcal{L}_S$ has the same eigenvalue problem as the Hermitian operator in Equation (4.85): Risken and Frank (2012, p. 107). Hence, $p_C'(C) = e^{\frac{\Phi(c)}{2}} p_C(C)$. In the remainder of this subsection, $p_C'(C)$ will be replaced by the symbol p.

Similar to Equation (4.86), the Schrödinger operator $\mathcal{L}_S$ can be re-written to

$$\mathcal{L}_S = \frac{d^2}{dc^2} - W \tag{4.98}$$

where $W(c) = \frac{1}{4}\hat{\alpha}(c)^2 - \frac{1}{2}\frac{d\hat{\alpha}(c)}{dc} = \frac{1}{4}\frac{[(\mu_1'-1)c+\mu_0']^2}{(1-c^2)} - \frac{1}{2}\frac{[2(\mu_1'-1)c^2+2\mu_0'c)]}{(1-c^2)^{\frac{3}{2}}} - \frac{1}{2}\frac{[(\mu_1'(\mu_1'-1)c+(\mu_1'\mu_0')]}{(1-c^2)^{\frac{1}{2}}}$.

Then, the associated ODE becomes

$$-p'' + \left[\frac{1}{4}\frac{[(\mu_1'-1)c+\mu_0']^2}{(1-c^2)} - \frac{1}{2}\frac{[2(\mu_1'-1)c^2+2\mu_0'c)]}{(1-c^2)^{\frac{3}{2}}} - \frac{1}{2}\frac{[(\mu_1'(\mu_1'-1)c+(\mu_1'\mu_0')]}{(1-c^2)^{\frac{1}{2}}}\right] p = \kappa' p \tag{4.99a}$$

To solve this equation, a Taylor approximation of the term $(1-c^2)^{\frac{1}{2}}$ is used: $\sqrt{(1-c^2)} \approx 1 - \frac{1}{2}c^2$ (see Figure 4-7).

This approximation allows Equation (4.96a) to be turned into an equation with a middle term of a quotient of two fourth-degree polynomial functions

$$-p'' + \left[\frac{\chi_4 c^4 + \chi_3 c^3 + \chi_2 c^2 + \chi_1 c + \chi_0)}{-\frac{1}{2}c^4 + \frac{3}{2}c^2 - 1}\right] p = \kappa' p \tag{4.99b}$$

where $\chi_4 = -\frac{1}{8}(\mu_1'-1)^2$, $\chi_3 = -\frac{1}{2}(\mu_1'-1)(\frac{1}{2}\mu_0' + \mu_1')$, $\chi_2 = -(\frac{1}{8}\mu_0'^2 + (\mu_1'-1) + \frac{1}{2}\mu_1'\mu_0')$, $\chi_1 = \frac{1}{2}(\mu_1'-1)\mu_0' - \mu_0' + \frac{1}{2}\mu_1'(\mu_1'-1)\mu_0'$ and $\chi_0 = \frac{1}{4}\mu_0'^2 + \frac{1}{2}\mu_1'\mu_0'$.

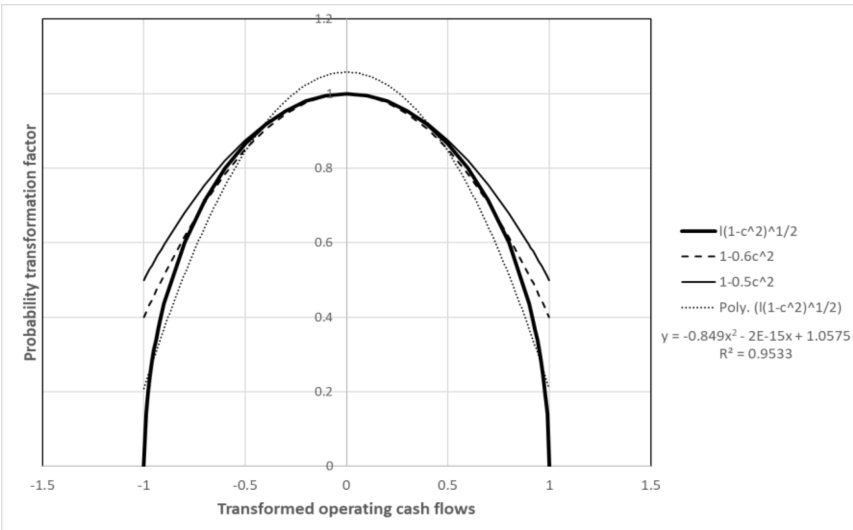


Figure 4-7 Approximation of $\sqrt{(c^2-1)}$ by a Taylor expansion, adapted Taylor expansion and least-square quadratic fit

Figure 4-8 below shows that constants $c = \widehat{\psi}_j$ (for convenience here only the case $j = 1$ is displayed) predominantly are a reasonable approximation of the middle term of Equation (4.99b). Comparing Figure 4-8 to Figure 4-6, it is evident that the quotient of two fourth degree polynomials exhibits a closer resemblance to a straight line then the quotients of two quadratic polynomials, at least for cash flow values not too close to the boundaries $[-1, 1]$.

After replacing the middle-term by a constant $\widehat{\psi}_j$, the resulting equation is very similar to Equation (4.87b)

$$-p'' + \widehat{\psi}_j p = \kappa' p \tag{4.99c}$$

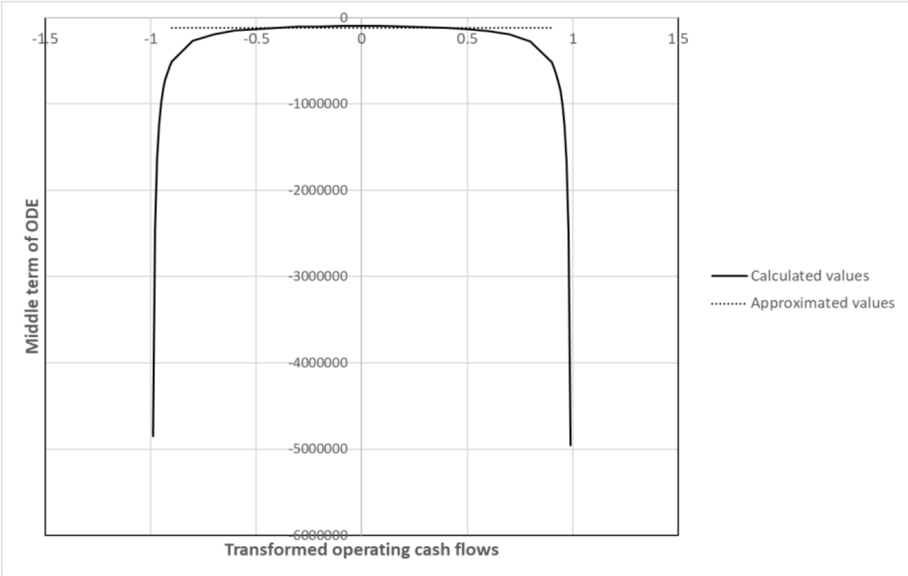
Solutions to Equation (4.99c) are derived by using the same solution techniques as were applied to the Hermitian transformed equation. The Schrödinger equivalent of the Hermitian solution presented in Equation (4.94), is

$$p(c,t) = K(c-1)^{\frac{1}{2}}(c+1)^{\frac{1}{2}}\, e^{t}\left\{a_0 + \sum_{n=1}^{n=\infty}\left[e^{-\frac{1}{16}n^2\pi^2 t}\left[a_n \sin\left(\frac{1}{4}n\pi c\right) + b_n \cos\left(\frac{1}{4}n\pi c\right)\right]\right]\right\} \tag{4.100}$$

Notice that Equation (4.100) is a particular case of Equation (4.94) with $\nu_1 = \nu_2 = \frac{1}{2}$, admitting the same family of probability distributions, that is the Pearson's Case 2 family of distributions, with special parameters. The next subsection will discuss the interpretation of the results obtained as solutions to the Jacobi, Hermitian and Schrödinger transformations.

Discussion of the Jacobi, Hermitian and Schrödinger transformations

All three solution techniques demonstrate how challenging it is to find general, tractable solutions to the linear-quadratic Fokker-Planck equation. Only the Jacobi transformation provides an exact solution, albeit with a mix of combinatorial and polynomial terms that are difficult to interpret in a practical sense. The other two transformations have to include approximated terms to arrive at a closed-form general solution. Perturbation theory can be helpful in finding approximations that meet the accuracy requirements of most practical applications.



$\mu_0 = 10, \mu_1 = 5, \sigma_0 = -1, \sigma_1 = 0.5, \sigma_2 = 4$

Figure 4-8 Values of the middle term of equation (87a)

The remaining challenge is to derive a temporal (family) of probability density distribution(s) from the solutions expressed in Equations (4.94) and (4.100). Inspection of these two equations reveals that, whilst probability density functions are defined on a continuous cash flow spectrum, the solutions include a summation of discrete probability components. Under some mathematical restrictions (Zettl (2005, Section 3.5)), Equations (4.94) and (4.100) can be considered approximately continuous in the number of eigenvalues (represented by parameter n).

In its continuous form and in the limit $n \rightarrow \infty$, the approximation for the term $\left\{a_0 + \sum_{n=1}^{n=\infty}\left[e^{-\frac{1}{16}n^2\pi^2 t}\left[a_n \sin\left(\frac{1}{4}n\pi c\right) + b_n \cos\left(\frac{1}{4}n\pi c\right)\right]\right]\right\}$ is the integral

$$\int_0^\infty \left[e^{-\frac{1}{16}n^2\pi^2 t} \cos\left[\frac{1}{4}n\pi(c-c_0)\right]\right] dn \tag{4.101}$$

which evaluates to

$$\lim_{n\to\infty} \frac{1}{\sqrt{\pi t}} e^{-\frac{1}{4}\frac{(c-c_0)^2}{t}}\left[\operatorname{erf}\left[\frac{-\pi t n + 2i(c-c_0)}{4\sqrt{t}}\right] + \operatorname{erf}\left[\frac{-\pi t n - 2i(c-c_0)}{4\sqrt{t}}\right]\right] \tag{4.102a}$$

where erf ( ) is the error function $\frac{2}{\sqrt{\pi}}\int_0^c e^{-\varepsilon^2}\,d\varepsilon$. Although the term $\mathrm{erf}\left[\frac{-\pi tn+2i(c-c_0)}{4\sqrt{t}}\right]+\mathrm{erf}\left[\frac{-\pi tn-2i(c-c_0)}{4\sqrt{t}}\right]$ includes imaginary parts ($i=\sqrt{-1}$), it can be shown that the expression is equal to $2\mathcal{R}\,(\frac{-\pi tn}{4\sqrt{t}}+\frac{2(c-c_0)}{4\sqrt{t}}i)$ where $\mathcal{R}(\,)$ denotes the real part of a complex function $b+ai$ with $b=\frac{-\pi tn}{4\sqrt{t}}$ and $a=\frac{2(c-c_0)}{4\sqrt{t}}$. Since the limit of $\lim_{n\to\infty}\mathrm{erf}(-n)=-1$, Equation (4.102a) becomes

$$\frac{-2}{\sqrt{\pi t}}e^{-\frac{1}{4}\frac{(c-c_0)^2}{t}} \tag{4.102b}$$

Now, Equations (4.94) and (4.100) take the following forms

$$p(c,t)=\overline{K}(c-1)^{\nu_1}(c+1)^{\nu_2}e^{-\frac{1}{2}t}\frac{2}{\sqrt{2\pi t}}e^{-\frac{(c-c_0)^2}{2t}} \tag{4.103a}$$

and

$$p(c,t)=\overline{K}(c-1)^{\frac{1}{2}}(c+1)^{\frac{1}{2}}e^{-\frac{1}{2}t}\frac{2}{\sqrt{2\pi t}}e^{-\frac{(c-c_0)^2}{2t}} \tag{4.103b}$$

where $\overline{K}=-4K$, time is scaled by a factor $\frac{1}{2}$, and $\nu_1=\frac{1}{4}(\mu_0'+\mu_1'-1)$, $\nu_2=\frac{1}{4}(\mu_0'-\mu_1'-1)$. For a comparable solution technique applied to different potentials (the middle term in Equations (4.87a) and (4.99b)), see Araujo and Drigo Filho (2012); Araujo and Filho (2015); Brics et al. (2013).

The result displayed in Equations (4.103a) and (4.103b), is significant. The space-time density function of operating cash flow processes can be constructed by multiplying two (independent) time-variant probability distributions: (i) the stationary (in case of operating cash flows: the approximate long-time distribution), and (ii) the evolution of a standard normal distribution. Notice that the term $e^{-\frac{1}{2}t}$ in Equation (4.103b) ensures that the height of the stationary distribution function decreases exponentially over time, whilst the time-component of the normal distribution underpins a widening distribution (on the cash flow axis) as time progresses (a condition analysed in the above subsection *Discussion of solution methods*).

An alternative approach is to view Equations (4.94) and (4.100) in the light of the Fourier transform. It can be demonstrated that there is a connection with probability functions via their characteristic functions (Brémaud (2014)). The first step of this approach is to express function $\cos\left[\frac{1}{4}\mathrm{n}\pi(\mathrm{c}-\mathrm{c}_0)\right]$ as a complex term together with its conjugate: $\mathrm{e}^{\frac{1}{4}\mathrm{n}i\pi(\mathrm{c}-\mathrm{c}_0)} + \mathrm{e}^{-\frac{1}{4}\mathrm{n}i\pi(\mathrm{c}-\mathrm{c}_0)}$, to be followed by turning this expression into (an) appropriate characteristic function(s). Unfortunately, a robust exploration of the interconnection between eigenfunctions, Fourier series and space-time density functions, falls outside the scope of this study.

Finally, an observation about the practical meaning of the above solution. From physics, it is known that the general solution to Schrödinger's equation is a wave function for the whole system, assembled from wave functions of individual states; each giving the relative importance of that state to the whole system. Squared wave functions measure the height of a probability distribution at a given point in space (for a comprehensive explanation refer to Gao (2017)). Analogically: solutions to Schrödinger's cash flow equation are composed of uncountable or countable many probability distributions (eigenfunctions), each belonging to a particular state of the world (eigenvalue) with its own growth path $\mathrm{e}^{-\kappa t}$. At a macroscopic level, they form a space-time density function for the ensemble of firms.

Transformation of the general hypergeometric equation to a first order ODE

The preceding subsections dealt with (general, particular and approximated) solutions to the Fokker-Planck Equation (4.63a) in conjunction with boundary conditions (4.63b). This subsection will explore another transformation where the general hypergeometric second order differential equation (equal to Equation (4.67c))

$$\frac{(\sigma_2 c^2+\sigma_1 c+\sigma_0)}{2}p'' + (2\sigma_2 - \mu_1)c + (\sigma_1 - \mu_0))p' + \kappa p = 0 \tag{4.104}$$

where $p'' = \frac{d^2 p_c(c)}{dc^2}$, $p' = \frac{dp_c(c)}{dc}$ and $p = p_c(c)$,

is transformed into a first order ODE; however, with original boundary values. It will be shown that a *particular* solution resembles the Pearson Type IV family of distributions.

The transformed specification is

$$[\mathrm{r(c)p}' + \mathrm{s(c)p}]' = 0 \tag{4.105}$$

where $p = p_c(c)$, $r(c)$ and $s(c)$ are functions yet to be determined, and the accent denotes the first derivative in respect of c.

Equating expression (4.105) to Equation (4.104) leads to the following system of equations:

$$r(c) = \frac{(\sigma_2 c^2 + \sigma_1 c + \sigma_0)}{2} \tag{4.106a}$$

$$r'(c) + s(c) = (2\sigma_2 - \mu_1)c + (\sigma_1 - \mu_0) \tag{4.106b}$$

$$s'(c) = \kappa \tag{4.106c}$$

From (106c) it follows that $g(c) = \kappa c + K_1$ where K_1 is an integration constant. Substituting this result into Equation (4.106b) yields the following identity:

$$(\sigma_2 + \kappa)c + \left(\frac{1}{2}\sigma_1 + K_1\right) = (2\sigma_2 - \mu_1)c + (\sigma_1 - \mu_0) \tag{4.107}$$

Since κ and K_1 are arbitrary constants, this equation is valid for all values of parameters $\{\mu_0, \mu_1, \sigma_1, \sigma_2\}$. Therefore, after integration, Equation (4.105) becomes a first order ODE:

$$\frac{(\sigma_2 c^2 + \sigma_1 c + \sigma_0)}{2} p' + (\kappa c + K_1)p = K_2 \tag{4.108}$$

where K_1 and K_2 are integration constants.

If K_2 is set equal to the particular value zero[45] then Equation (4.108) has the following solution:

$$p_c(c) = K_3\,(\sigma_2 c^2 + \sigma_1 c + \sigma_0)^{\frac{-\kappa}{2\sigma_2}} \exp\left[\frac{2(\kappa\sigma_1 - 2K_1\sigma_2)}{\sigma_2\sqrt{\Delta}} \tanh^{-1}\left(\frac{2\sigma_2 c + \sigma_1}{\sqrt{\Delta}}\right)\right] \tag{4.109}$$

where K_1 and K_3 are integration and normalisation constants, and $\Delta = 4\sigma_2\sigma_0 - \sigma_1{}^2$.

After normalisation, Equation (4.109) admits a Pearson Type IV distribution similar to one derived from the Pearson's Case 1 stationary distribution in Equation (2.17), Section 2-3.

[45] It can be shown that for $K_2 \neq 0$ the solution includes an undefined integral.

Setting Equation (4.109) to zero for both boundary values $[\lambda_1; \lambda_2]$, results in the following equations:

$$\frac{2(\kappa\sigma_1 - 2K_1\sigma_2)}{\sigma_2\sqrt{\Delta}}\tanh^{-1}\left(\frac{2\sigma_2\lambda_1 + \sigma_1}{\sqrt{\Delta}}\right) = 1 \quad (4.110a)$$

$$\frac{2(\kappa\sigma_1 - 2K_1\sigma_2)}{\sigma_2\sqrt{\Delta}}\tanh^{-1}\left(\frac{2\sigma_2\lambda_2 + \sigma_1}{\sqrt{\Delta}}\right) = 1 \quad (4.110b)$$

where K_1 is an integration and normalisation constant, and $\Delta = 4\sigma_2\sigma_0 - \sigma_1{}^2$.

Solving Equations (4.110a) and (4.110b), yields the following expression for κ and K_1:

$$\kappa = \frac{\sigma_2\sqrt{\Delta}}{4\sigma_1}\left\{\left[\tanh^{-1}\left(\frac{2\sigma_2\lambda_1 + \sigma_1}{\sqrt{\Delta}}\right)\right]^{-1} + \left[\tanh^{-1}\left(\frac{2\sigma_2\lambda_2 + \sigma_1}{\sqrt{\Delta}}\right)\right]^{-1}\right\} \quad (4.111a)$$

$$K_1 = \frac{-\sigma_2\sqrt{\Delta}}{8\sigma_1}\left[\tanh^{-1}\left(\frac{2\sigma_2\lambda_1 + \sigma_1}{\sqrt{\Delta}}\right)\right]^{-1} + \frac{\sigma_2\sqrt{\Delta}}{4\sigma_1}\left[\tanh^{-1}\left(\frac{2\sigma_2\lambda_2 + \sigma_1}{\sqrt{\Delta}}\right)\right]^{-1} \quad (4.111b)$$

Substituting expressions (4.111a) and (4.111b) into Equation (4.109), provides an analytic solution to the general hypergeometric differential equation, albeit in limiting cases. Constant K_3 remains as a normalisation factor to ensure that the integral of the density function is 1.

If this result is combined with the time function $p_t(t) = K_0 e^{-\kappa t}$, then the intertemporal dynamics of the time-probability density function $p(c, t)$ can be further analysed.

Solutions to the backward Kolmogorov equation

In the remainder of this section, the focus of attention will be on solving the *backward* Kolmogorov equation. As is apparent from its name, the backward Kolmogorov equation considers the space-time density function *backwards* in time, starting from a long-time probability distribution $p(C_t, T) = P_{lt}(C_t)$ with $t \ll T$ to observe the process dynamics in reversed time. From the equivalence of the forward and the backward Kolmogorov diffusion equation (Ghosh et al. (2010)), it follows that the backward equivalent of Equation (4.57a) can be written as follows

$$\frac{-\partial p(C_t, t)}{\partial t} = (\mu_1 C_t + \mu_0)\frac{\partial p(C_t, t)}{\partial C_t} + \frac{(\sigma_2 C_t^2 + \sigma_1 C_t + \sigma_0)}{2}\frac{\partial^2 p(C_t, t)}{\partial C_t^2} \quad (4.112a)$$

with end-condition $\mathrm{p}(\mathrm{C_t},\mathrm{T}) = \mathrm{P_{lt}}(\mathrm{C_t})$. A similar backward equation is found in Meerschaert and Sikorskii (2012, equation 7.35).

In the case of a non-stationary process, and despite having to impose an end-condition, this approach has nevertheless some benefits compared to the Fokker-Planck equation: (i) no probability distribution exists if $t \rightarrow \infty$; the density function gradually transforms into a an infinite flat line, (ii) if a long-time density function is known or can be postulated, then it can serve as an end-condition of the backward Kolmogorov equation, and (ii) commonly the mathematics of the backward equation is simpler because drift and diffusion function appear outside of the derivatives (Meerschaert and Sikorskii (2012, p. 218)). The requirement of a known long-term-density function can be partially mitigated by assuming a general specification that includes a wide range of specialised density functions. As will become clear in the following, the Pearson Case 2 family of distributions (see Figure 2-5 in Section 4-3), is proposed as a suitable specification of an end-condition.

Compared to the Fokker-Planck equation, the backward Kolmogorov equation requires a different expression of the probability current $\left[\mu_1 \mathrm{C_t} + \mu_0 + \frac{1}{2}\frac{\partial(\sigma_2 \mathrm{C_t^2} + \sigma_1 \mathrm{C_t} + \sigma_0)}{\partial \mathrm{C_t}}\frac{\partial}{\partial \mathrm{C_t}}\right]\frac{\partial \mathrm{p}(\mathrm{C_t},\mathrm{t})}{\partial \mathrm{C_t}}$ (Mahnke et al. (2009, pp. 119-120)). Therefore, the reflective boundary condition, in case of the backward Kolmogorov equation, is

$$\left[\mu_1 \mathrm{c} + \mu_0 + \frac{1}{2}\frac{\partial(\sigma_2 \mathrm{c}^2 + \sigma_1 \mathrm{c} + \sigma_0)}{\partial \mathrm{c}}\frac{\partial}{\partial \mathrm{c}}\right]\frac{\partial \mathrm{p}(\mathrm{c},\mathrm{t})}{\partial \mathrm{c}} = 0 \tag{4.112b}$$

or equivalently,

$$\frac{\partial \mathrm{p}(c,\mathrm{t})}{\partial \mathrm{c}} = 0 \tag{4.112c}$$

for $\mathrm{c} = \lambda_1, \lambda_2$.

Similar to the Fokker-Planck equation, the backward Kolmogorov equation can be expressed in the following format

$$\frac{-\partial \mathrm{p}(\mathrm{c},\mathrm{t})}{\partial \mathrm{t}} = \mathcal{L}_{BK}\, \mathrm{p}(\mathrm{c},\mathrm{t}) \tag{4.113a}$$

with the backward Kolmogorov operator $\mathcal{L}_{BK}$ is defined as follows

$$\mathcal{L}_{BK} = \alpha(c)\frac{d}{dc} + \beta(c)\frac{d^2}{dc^2} \tag{4.113b}$$

The applicable (absorbing) boundary conditions are

$$p(\lambda_{1,t}, t) = 0, \qquad p(\lambda_{2,t}, t) = 0 \tag{4.113c}$$

Duplicating the steps in Equations (4.65a) and (4.65b), the two ODEs corresponding to the backward Kolmogorov equation become

$$\frac{-dp_t(t)}{dt} = -\kappa\, p_t(t) \tag{4.114a}$$

$$\mathcal{L}_{BK} p_c(c) = -\kappa\, p_c(c) \tag{4.114b}$$

Equation (4.114a) produces the following solution

$$p_t(t) = K_0 e^{\kappa t} \tag{4.115}$$

where K_0 is an integration constant. Notice that since the process moves *backwards* in time, the size of $p_t(t)$ will increase.

For a linear-quadratic specification, $\alpha(c) = \mu_1 c + \mu_0$ and $\beta(c) = \frac{1}{2}[\sigma_2 c^2 + \sigma_1 c + \sigma_0]$, Equation (4.114b) can be re-written to

$$\frac{(\sigma_2 c^2 + \sigma_1 c + \sigma_0)}{2}\frac{\partial^2 p_c(c) p_t(t)}{\partial c^2} + \left(\mu_1 c + \mu_0\right)\frac{\partial p_c(c) p_t(t)}{\partial c} = -\kappa\, p_c(c) \tag{4.116a}$$

with approximate end-condition $p(c, T) = p_{lt}(c) = K\, e^{-\Phi(c)}$, and (absorbing) boundary conditions

$$p_c(\lambda_1) = 0, \qquad p_c(\lambda_2) = 0 \tag{4.116b}$$

Notice that $\Phi(c) = -\int \frac{\alpha(\xi)}{\beta(\xi)} d\xi$ where $\alpha(c) = \mu_1 c + \mu_0$ and $\beta(c) = \frac{1}{2}[\sigma_2 c^2 + \sigma_1 c + \sigma_0]$.

It can be shown that the Schrödinger operator described in Equation (4.97), is the nexus between the forward and backward Kolmogorov equation. After applying the Schrödinger transformation, as defined by Equation (4.86a), the function $\Phi(C)$ of the approximate end-condition $p_{lt}(C) = K_1 e^{-\Phi(C)}$ becomes $\Phi(C) = -\int \hat{\alpha}(C)\, d\xi$ where $\hat{\alpha}(C)$ is described in Equation (4.96). If $p(C, t)$ is defined as $p(C, t) = \Phi(C)\, q(C, t)$ then the Fokker-Planck Equation (4.63a) assumes the following form

$$\frac{\partial q(C,t)}{\partial t} = \Phi'(C)\frac{\partial q(C,t)}{\partial C} + \frac{\partial^2 q(C,t)}{\partial C^2} = \widehat{\alpha}(C)\frac{\partial q(C,t)}{\partial C} + \frac{\partial^2 q(C,t)}{\partial C^2} \qquad (4.117a)$$

with $q_{It}(C) = p_{It}(C)\Phi^{-1}(C) = K\frac{1}{\int \widehat{\alpha}(C)d\xi}\, e^{\int \widehat{\alpha}(C)d\xi}$. For a derivation of Equation (4.117a) see Pavliotis (2014, Section 4.5). Since ∂t is the forward movement time, it must be expressed as $-\partial t$ in the backward equation.

$$-\frac{\partial q(C,t)}{\partial t} = \Phi'(C)\frac{\partial q(C,t)}{\partial C} + \frac{\partial^2 q(C,t)}{\partial C^2} = \widehat{\alpha}(C)\frac{\partial q(C,t)}{\partial C} + \frac{\partial^2 q(C,t)}{\partial C^2} \qquad (4.117b)$$

This result is important: Equation (4.117b) is a backward Kolmogorov equation with stochastic properties that are directly derivable from the Schrödinger equation and thus, after the appropriate transformation, also from the related forward Kolmogorov equation. Consequently, if a Schrödinger transformation is applied to the linear-quadratic Fokker-Planck specification, and the resulting equation can be written in the form of equation (117b), then from the solution to this backward equation, the stochastic properties of the solution to the associated Fokker-Planck can be understood without solving the Fokker-Planck equation itself. This approach is salient if the backward equation is markedly easier to solve than the comparable forward equation.

4.4. Solutions to the coupled system

In section 3.4 it was observed that solutions to the *coupled* system S $\{C_t, I_t\}$ are linear combinations of solutions to the *decoupled* system $S\{C_t', I_t'\}$ where the weights are represented by the following matrix $\mathbf{Q} = \begin{pmatrix} \frac{2\beta}{-1-\alpha+\omega} & \frac{2\beta}{-1-\alpha-\omega} \\ 1 & 1 \end{pmatrix}$ with $\omega = \sqrt{(\alpha-1)^2 + 4(\beta\gamma+\alpha)}$. This follows from the cash flow variable transformation $\mathbf{u}_t = \mathbf{Q}\,\mathbf{v}_t$ which is applicable to the system observed in a deterministic environment (Section 3-3) as well as in a stochastic environment (Appendix M2). Recall that parameter α is the *cash flow growth rate*, parameter β (≥ 0) is the *investment response* parameter, and parameter γ $(0 < \gamma \leq 1)$ is called the *cash investment rate* (see section 3.2).

In the preceding sections solutions to the decoupled system were discussed, hence the final step is to combine these to solutions of the coupled system:

$$C_t = w_1 C_t' + w_2 I_t' \qquad (4.118a)$$

$$\mathrm{I_t} = \mathrm{C'_t} + \mathrm{I'_t} \qquad (4.118b)$$

where $w_1 = \frac{2\beta}{-1-\alpha+\omega}$ and $w_2 = \frac{2\beta}{-1-\alpha-\omega}$. Weights $w_1 > 0$ and $w_2 < 0$ which follows from $\beta > 0$ and $\beta\gamma > 0$.

Since $S\{C'_t, I'_t\}$ is a decoupled, that is, a stochastically independent system, it is not hard to see that after applying Itô's lemma to the bi-dimensional system, the following equalities hold:

$$\mathrm{dC_t} = w_1 \mathrm{dC'_t} + w_2 \mathrm{dI'_t} \qquad (4.119a)$$

$$\mathrm{dI_t} = \mathrm{dC'_t} + \mathrm{dI'_t} \qquad (4.119b)$$

Recall from Section 3.4, Equations (3.26a) and (3.26b), the specifications for the decoupled operating and investing cash flows:

$$\mathrm{dC'_t} = \left(\mu_{C,1}\mathrm{C'_t} + \mu_{C,2}\right)\mathrm{dt} + \sqrt{(\sigma_{C,1}\mathrm{C'^2_t} + \sigma_{C,2}\mathrm{C'_t} + \sigma_{C,3})}\ \mathrm{dW_{C',t}} \qquad (4.120a)$$

$$\mathrm{dI'_t} = \left(\mu_{I,1}\mathrm{I'_t} + \mu_{I,2}\right)\mathrm{dt} + \sqrt{(\sigma_{I,1}\mathrm{I'^2_t} + \sigma_{I,2}\mathrm{I'_t} + \sigma_{I,3})}\ \mathrm{dW_{I',t}} \qquad (4.120b)$$

Combining (119a) and (119b) with (120a) and (120b) leads to the expressions below:

$$\mathrm{dC_t} = \left[(w_1\mu_{C,1}\mathrm{C'_t} + w_2\mu_{I,1}\mathrm{I'_t}) + (w_1\mu_{C,2} + w_2\mu_{I,2})\right]\mathrm{dt} +$$
$$\sqrt{(w_1^2\sigma_{C,1}\mathrm{C'^2_t} + w_2^2\sigma_{I,1}\mathrm{I'^2_t}) + (w_1^2\sigma_{C,2}\mathrm{C'_t} + w_2^2\sigma_{I,2}\mathrm{I'_t}) + w_1^2\sigma_{C,3} + w_2^2\sigma_{I,3})}\mathrm{dW_{C,t}} \qquad (4.121a)$$

$$\mathrm{dI_t} = \left[(\mu_{C,1}\mathrm{C'_t} + \mu_{I,1}\mathrm{I'_t}) + (\mu_{C,2} + \mu_{I,2})\right]\mathrm{dt} +$$
$$\sqrt{(\sigma_{C,1}\mathrm{C'^2_t} + \sigma_{I,1}\mathrm{I'^2_t}) + (\sigma_{C,2}\mathrm{C'_t} + \sigma_{I,2}\mathrm{I'_t}) + \sigma_{C,3} + \sigma_{I,3})}\mathrm{dW_{I,t}} \qquad (4.121b)$$

where $w_1 = \frac{2\beta}{-1-\alpha+\omega}$ and $w_2 = \frac{2\beta}{-1-\alpha-\omega}$ with $\omega = \sqrt{(\alpha+1)^2 + 4(\beta\gamma-\alpha)}$.

Note that $W_{C',t}$ and $W_{I',t}$ are independent Brownian motions with correlation coefficient $\rho = 0$, that are combined to coupled Brownian motions $W_{C,t}$ and $W_{I,t}$ respectively.

From Equations (4.121a) and (4.121b) the following conclusions can be drawn:

(i) The coupled operating cash flow and investing cash flow processes are stochastically very similar to their decoupled specifications; only their parameter values differ;

(ii) The diffusion processes of coupled operating cash flow processes and investing cash flow processes are identical;

(iii) Drift parameter values and diffusion parameter values of the coupled cash flow processes are weighted averages of the respective parameters of those of the decoupled processes;

(iv) Weights are determined by the three parameters that are fundamental to cash flow processes: the cash flow attrition rate α, the investment response parameter β, and the cash investment rate γ .

Interestingly, summed diffusion functions, like the one for the coupled operating and investing cash flow processes, are considered a flexible class of stochastic models that fit heavy-tailed, sharply peaked distributions (in this study typically found for cash flow processes), particularly well (Bibby et al. (2005), Forman (2007, section 2.2)).

4.5. Conclusions from Chapter 4

This chapter examines solutions to a general hybrid cash flow process specified as $\mathrm{dX_t} = (\mu_1 \mathrm{X_t} + \mu_0)\mathrm{dt} + \sqrt{(\sigma_2 \mathrm{X_t^2} + \sigma_1 \mathrm{X_t} + \sigma_0)}\mathrm{dW_t}$, where $\mathrm{X_t}$ can either be an operating cash flow or an investing cash flow.

Solutions to the above stochastic process do exist which follows from approximately meeting solutions meeting Lipschitz existence and continuity conditions, as well as exactly obeying Aït-Sahalia’s less restrictive conditions for the existence of strong solutions.

It was shown that the diffusion function of the above specification always converges. However, whether the process as a whole converges or diverges in time, depends on the sign of parameter μ_1 in the drift function. If the sign is positive, as typically found for operating cash flow processes, the process diverges, whereas a negative sign, characteristic for investing cash flow processes, implies a converging (that is, mean-reversion) process.

Unfortunately, no *general* analytic solutions to the above specification exist, only *particular* and *limiting* solutions. Nevertheless, in the literature an exact *formal* solution to the hybrid form, consisting of the two constituting ABM and GBM processes, was found. In addition, this study explores six particular and limiting solutions in greater detail. In the first two solutions, parameter values are chosen contingent on transforming the specification into either a reducible SDE with a known solution, or a linear SDE that is solvable. Solutions are

known in the literature for both specifications. The third solution is a Lamperti-transform of the specification, followed by setting a parameter to a specific value.
Furthermore, the fourth, fifth and sixth solutions are approximated solutions. The first approximation is valid for large X_t which cash flows generally are. The second one is based again on a Lamperti transform and the conditions under which the resulting unit Wiener process approximately vanishes, thus solving the remaining equation as an ODE. The third approximation transforms the linear-quadratic specification into a reduced form Vasicek specification. Nevertheless, (a) further parameter restriction(s) is required to estimate the complete linear-quadratic parameter set from the Vasicek equation.

Associated with the above general cash flow process is a unique Fokker-Planck equation that describes the evolution of the transitional probability density function. Unsurprisingly, the solutions to this equation is very different for each of uncoupled operating cash flows and uncoupled investing cash flows. Converging (uncoupled) investing cash flow processes have a stationary distribution function. This process, also known as a Pearson diffusion process, has been studied in the literature for some time. Analytic solutions to six particular cases are comprehensively analysed and documented. On theoretical grounds, it was found that one of these processes, the (asymmetric) Student diffusion process, is particularly well suited to describe investing cash flow processes.

(Uncoupled) operating cash flows are governed by a diverging general cash flow process with no stable density function. These processes are not well understood and only few examples of possible solutions to the corresponding Fokker-Planck equation can be found in the literature. It can be shown that solutions pertaining to operating cash flow processes must be bounded by two boundary values: a lower and an upper boundary that take the values of the roots of the quadratic diffusion function. This result agrees with Pearson's Case 2 family of density distributions that are, in an untransformed specification, only valid on a limited cash flow range $[\lambda_1, \lambda_2]$.

Often, the Fokker-Planck equation is solved by the technique of separation of variables, an approach that this study advocates on theoretical and empirical grounds, after taking other solution methods into account. Separation of variables leads to a generic hypergeometric differential equation for which there are no known general analytic solutions; however, a transformation of the cash flow variable provides a pathway to solutions. This study considers two such transformations: transforming the boundaries (via the cash flow variable) from $[\lambda_1, \lambda_2]$ to $[0, 1]$ and $[-1, 1]$ respectively. The first transformation results in a composite Gaussian hypergeometric function that, depending on particular parameter values, includes a great number of special functions as specific or limiting cases. Irrespectively, this solution is considered too generic to be of much practical use.

The second boundary transformation is potentially more fruitful, particularly if the derived equation is formulated as an analytically solvable Sturm-Liouville problem. The Sturm-Liouville theory offers a broad range of elegant and well-analysed solution techniques. This study applies Sturm-Liouville techniques in the context of three well-known variable transformations: (i) the Jacobi, (ii) the Hermitian, and (iii) the Schrödinger transformations. Of the three transformations, only the Jacobi transformation provides an exact solution, albeit with a mix of combinatorial and polynomial terms that are difficult to interpret in a practical sense. The other two transformations have to include approximated terms to arrive at a closed-form general solution. Both transformations lead to a composite space-time density function of operating cash flow processes that can be constructed as the multiplication of two (independent) time-variant probability distributions: (i) the stationary (in the case of operating cash flows: the approximate long-time distribution), and (ii) the evolution of a standard normal distribution. It should be noted that the Schrödinger transformation produces a less general solution than the Hermitian transformation: a possible explanation is that two approximations (simplifications) had to be used instead of one approximation in the case of the Hermitian transformation.

Additionally, the transformation of the generic hypergeometric second order ODE into a first order ODE is analysed. This solution technique yields a particular solution only. The form of the solution is akin to the stationary Pearson Type IV distribution, commonly found for investing cash flows. This conclusion is consistent with empirical results reported in Chapter 2 where operating cash flows overwhelmingly follow a Pearson's Case 2 stationary probability distribution but in exceptional cases are better described by a stationary Pearson Type IV distribution.

In some instances, the backward Kolmogorov equation is better adapted to modelling operating cash flows than the Fokker-Planck (forward Kolmogorov) equation. The mathematics is considered less complicated and a possible known or postulated long-space-time density function (comparable to a stationary density function for investing cash flows) could be used as an end condition of the PDE. This study explains a method to transform the linear-quadratic Fokker-Planck equation into an equivalent backward Kolmogorov equation.

The final paragraph deals with converting solutions to the decoupled cash flow system back to solutions of the coupled system. It was found that the coupled processes are stochastically very similar to the decoupled ones and only differ in their parameter values (with parameter values of the coupled equations being a weighted sum of the uncoupled). Both coupled operating and investing cash flow processes appear to have an identical diffusion function. The coupled process can be considered a summed diffusion process that is especially adapted to model heavy-tailed, sharply peaked distributions, as typically found for cash flow processes in this study, underpinning an even richer set of potential solutions.

5. Statistical Estimation of the Linear-Quadratic Cash Flow Model

Chapter 5 has bearing on statistical and econometric parameter estimation to maintain the empirical robustness of the coupled linear-quadratic cash flow model developed in Chapter 3. First, the fundamental relationships and their parameter set $\alpha, \beta, \gamma, \delta$ and ε will be tested. Using these estimated values, decoupled cash flows can be calculated as input to testing the linear-quadratic specifications of the operating and investing cash flows respectively. Estimation results are benchmarked against the cash flow specifications from the literature as reported in Section 2.1. The data employed to estimate and test the relationships, are described in detail in Appendix S1.

5.1. Estimating the fundamental relationships

Introduction

The goal of this section is to derive a set of estimated parameters $\{\hat{\alpha}; \hat{\beta}; \hat{\gamma}; \hat{\delta}; \hat{\varepsilon}\}$ associated with the deterministic model (Section 3-3). Recall that the deterministic system is expanded to a stochastic system (Section 3-4) by introducing a set of probability transition rates $\{p_1, p_2, p_3\}$. After some mathematical manipulations (Appendix M2), it is shown that the uncoupled system, in continuous-time can be approximately described by two independent linear-quadratic SDEs each completely defined by a set of estimated parameters $\{\hat{\mu}_0, \hat{\mu}_1, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$. Using estimated parameters $\{\hat{\alpha}; \hat{\beta}; \hat{\gamma}; \hat{\delta}; \hat{\varepsilon}\}$, the uncoupled system is transformed back to a coupled system (Section 4-4).

Obviously, the first challenge of this chapter is to properly estimate the deterministic parameter set $\{\hat{\alpha}; \hat{\beta}; \hat{\gamma}; \hat{\delta}; \hat{\varepsilon}\}$. This is important to define the transformation matrix $\mathbf{Q} = \begin{pmatrix} \frac{2\beta}{-1-\alpha+\omega} & \frac{2\beta}{-1-\alpha-\omega} \\ 1 & 1 \end{pmatrix}$ with $\omega = \sqrt{(\alpha-1)^2 + 4(\beta\gamma+\alpha)}$. As explained in Section 3-3, matrix $\mathbf{Q}$ serves to uncouple or couple the system by transforming the set of variables $S\{C_t, I_t\}$ to $S\{C_t', I_t'\}$ and vice versa. Finding a deterministic environment from which the parameter set $\{\hat{\alpha}; \hat{\beta}; \hat{\gamma}; \hat{\delta}; \hat{\varepsilon}\}$ is inferred, seems arduous if not impossible. The closest approximation, however, is to examine the whole population by analysing a sufficient large and representative sample. Recall from Figure 1-7 in Section 1-4 that the analysis is then performed at a macroscopic level (with regard to all firms), and hence the outcome represents an average (deterministic) cash flow process. Before the estimation results are presented and discussed, the two fundamental relationships will be summarised.

The two fundamental relationships

In Section 3-2, it was explained that the first fundamental relationship is characterised by the following equation

$$\Delta C_t = \alpha C_{t-1} + \beta I_{t-1} + \delta \tag{5.1}$$

Variable C represents operating cash flow and variable I designates investing cash flow. Parameter α is called the *cash flow growth rate* that is influenced by two opposite forces. The first force is generating additional operating cash flow from better utilisation of existing capital goods, and the second is what is called the cash flow attrition rate. The latter represents a natural decline in cash flow generating capacity if the firm does not invest at all.

Furthermore, the *investment response parameter* is denoted by parameter β (≥ 0). It signifies how much extra dollars future operating cash flow are spawned from one dollar investing cash flow. The investment response parameter is thought to be determined by industry characteristics and within an industry by firm-specific characteristics such as the ability of management to successfully turn investments into business growth.

In aggregate, parameter δ captures all other variables that may affect the movement of operating cash flow, beside the level of operating cash flow itself and investing cash flow. In this study, all three parameters α, β and δ are presumed to be approximately time-invariant; admittedly, a more realistic but significantly more complicated model will allow time-variant parameters.

The second fundamental relationship is embodied in this specification

$$I_t = \gamma C_{t-1} + \varepsilon \tag{5.2}$$

where the level of future investment is determined by current operating cash flow and parameter ε is representing other variables. Parameter γ $(0 < \gamma \leq 1)$ is called the *cash investment rate* and is assumed to be constant over time.

Parameter ε, set equal to the expression $\gamma\, F_t + \gamma B_{t-1}$, is predominantly affected by the firm's cash and financing policy. B_{t-1} represents the prior cash balance, i.e. the amount of cash that the firm usually holds, and F_t stands for the net financing cash flow including items like borrowings, repayments and dividend payments. It is assumed that ε is approximately time-homogeneous, at least for mature firms with stable cash flow patterns.

Both relationships are considered linear in specification. In a continuous-time model, with very small Δt, this should not pose a problem. Nevertheless, the (discrete) data used in the tests are recorded on a quarterly basis and may not fit a linear relationship in the time period observed. Therefore, a linear specification test ought to be included in the post estimation tests performed.

Modelling the fundamental relationships

Since the overall model consisting of Equations (5.1) and (5.2) is recursive, that is, coupled, the Two Stage Least Square (2SLS) estimation method is used (Wooldridge (2015, Chapter 15)). The method is implemented in the statistical software package STATA-15 as part of Statistical Equation Modelling (Hancock and Mueller (2006); Schumacker and Lomax (2015)). The STATA procedure applies MLE (Maximum Likelihood) estimation. It turns out that the MLE estimator is more robust (i.e. converging and efficient) if the independent variable in Equation (5.1) is stated as an undifferenced cash flow amount regressed on time-lagged variants of the same variable. Instead of diagonalising the system matrix, the cash flow system can also be decoupled by transforming Equations (5.1) and (5.2) into second order difference equations. Then, Equation (5.1) produces the following AR(2) autoregressive specification for operating cash flows

$$C_t = (\alpha + 1)C_{t-1} + \beta\gamma C_{t-2} + \beta\varepsilon + \delta \tag{5.3}$$

The process is stable if all of the following conditions hold

$$\beta\gamma < \alpha + 2 \tag{5.4a}$$

$$\beta\gamma < -\alpha \tag{5.4b}$$

$$\beta\gamma > -1 \tag{5.4c}$$

Recall from Section 3-3 that condition (4b) also implies a diverging cash flow process. In order to achieve a consistent parameter mapping, Equation (5.2) is left unaltered.

Expressing the model in the above specification, avoids the complications of a usual 2SLS regression. Nevertheless, the error terms of both equations are assumed to be correlated. The model regresses the operating cash flow variable OCFL on the explanatory variables OCFL1 and OCFL2 where the last two letters '-LX' stands for the time lag X expressed in calendar quarters. This step is embodied in Equation (5.3). The second part regresses the investing cash flow variable ICF on the one-quarter lagged variable OCFL1.

Figure 5-1 provides an overview of the structure (variables and equations) of the estimation model. Recall from Appendix S1 that the data used are from an unbalanced panel dataset with a 'firm' and a 'time' dimension. Empirical evidence reported in Section 2-2 underpins the idea that the trend of cash flow data is a nonlinear function of time, and can best be described by a combination of a linear trend and an exponential growth trend. This is done by including in both regression steps the variables Quarter, Quarter2 (Quarter square) and Quarter3 (Quarter to the power 3). If statistical testing supports a significant unit root in cash flow time series then the effect of the trend must be eliminated from estimated parameters.

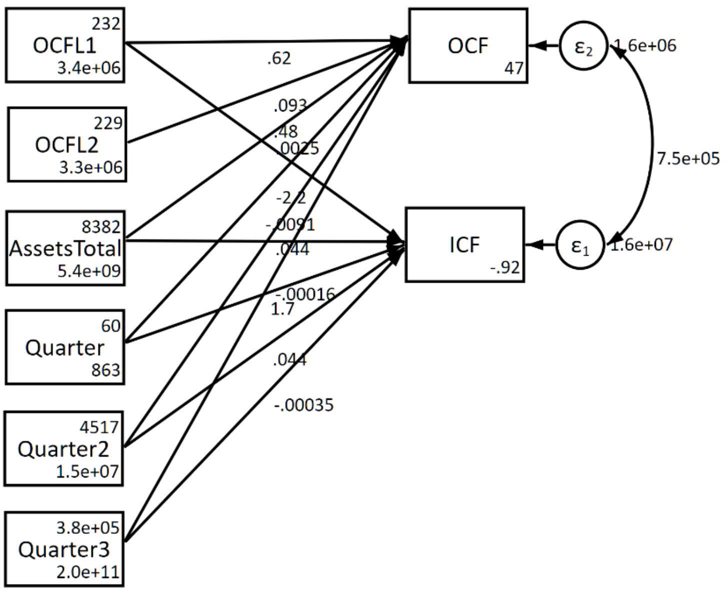


Figure 5-1 Components of the 2SLS regression model used to test the fundamental equations

Preliminary examination of the data set maintains significant heteroscedasticity (see Figure 5-2) in the error-terms. The cause of heteroscedasticity is largely attributed to a non-Gaussian spread and frequency of cash flows associated with individual firms (for a graphic impression see Section 2-3; Figures 2-2, 2-3 and 2-4).

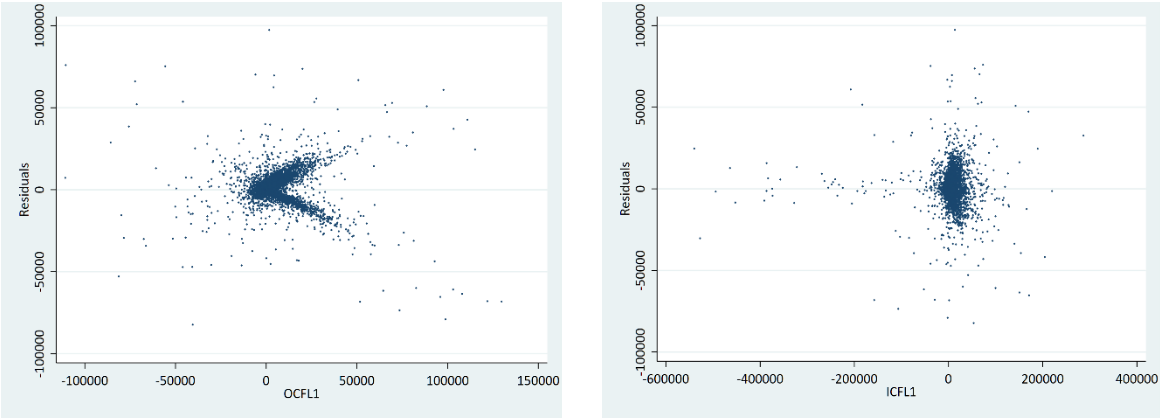


Figure 5-2 Operating cash flow (OCF1) and Investing cash flow (ICF1) plotted against residuals

This effect can be mitigated by including an instrumental variable in the regression model. The total amount of assets, AssetsTotal, was found to be a universally applicable proxy for the size of firms. Effectively, the inclusion of the variable AssetsTotal rescales the values of the estimated parameter set $\{\widehat{\alpha}; \widehat{\beta}; \widehat{\gamma}; \widehat{\delta}; \widehat{\varepsilon}\}$.

Estimating the fundamental relationships

Initially, the model was estimated by the default MLE method as implemented in STATA-15. Tables 5-1 to 5-4 show the test results. For Equation (5.1), all variables but Quarter3 (displayed in Table 5-1 under the heading ‘OCF’), are statistically significant at a 5% confidence level. This justifies the assumption that operating cash flows have a nonlinear trend over time (approximately described by a quadratic time-function). Also, the dynamics of the operating cash flow process depend on the firm’s size, i.e. a positive scale-effect is established. By contrast, for Equation (5.2) only a limited number of variables are confirmed statistically significant at a 5% confidence level. The constant ε appears insignificant, suggesting a purely proportional relationship between variables ICF and OCFL1. Additionally, no significant time-trend is detected which is not surprisingly giving the predominantly periodic character of investing cash flows (Section 1-2); however, scale-dependency is also an issue with investing cash flows. In contrast to operating cash flows, there is a negative scale effect. Irrespectively, coefficient values suggest a relatively mild scale-effect for both operating and investing cash flows.

The Wald test supports that variables in both equations are jointly significant. Overall the (blocked-error) R^2-statistics (Bentler and Raykov (2000)) indicate a good fit, taking loops and correlated residuals into account as commonly found in structural equation models. The overall fit of the model is overwhelmingly determined by Equation (5.3), much less by Equation (5.1).

Table 5-1 Test statistics of the fundamental model – variable statistics

Variable	Mean	Std. Dev.	Min	Max
ICF	201.2967	4088.289	-540050	286346
OCF	234.7685	1839.713	-110807	129731
OCFL1	231.9059	1833.249	-110807	129731
Quarter	60.44838	29.37776	1	121
Quarter2	4517.057	3850.12	1	14641
Quarter3	383425	441879.3	1	1771561
OCFL2	229.3224	1824.91	-110807	129731
AssetsTotal	8381.812	73321	-106.969	3643585

Table 5-2 Test statistics of the fundamental model - parameter estimates

```
Endogenous variables

Observed:  ICF OCF

Exogenous variables

Observed:  OCFL1 Quarter Quarter2 Quarter3 OCFL2 AssetsTotal

Fitting target model:

Iteration 0:   log likelihood =  -23954649
Iteration 1:   log likelihood =  -23954623
Iteration 2:   log likelihood =  -23954623

Structural equation model                      Number of obs     =    329,271
Estimation method  = ml
Log likelihood     =  -23954623
```

	Coef.	OIM Std. Err.	z	P>|z|	[95% Conf.	Interval]
Structural						
ICF						
OCFL1	.4806212	.0040041	120.03	0.000	.4727734	.488469
Quarter	1.683428	3.35077	0.50	0.615	-4.883962	8.250817
Quarter2	.043865	.0575223	0.76	0.446	-.0688767	.1566067
Quarter3	-.0003464	.0002943	-1.18	0.239	-.0009232	.0002304
AssetsTotal	-.0091039	.0001001	-90.93	0.000	-.0093001	-.0089077
_cons	-.9236416	56.59626	-0.02	0.987	-111.8503	110.003
OCF						
OCFL1	.6181012	.0017314	357.00	0.000	.6147077	.6214946
Quarter	-2.235588	1.063029	-2.10	0.035	-4.319087	-.1520893
Quarter2	.0444372	.018249	2.44	0.015	.0086699	.0802045
Quarter3	-.0001647	.0000934	-1.76	0.078	-.0003477	.0000183
OCFL2	.092521	.0017275	53.56	0.000	.0891351	.0959069
AssetsTotal	.0025215	.000032	78.79	0.000	.0024588	.0025843
_cons	46.62436	17.95503	2.60	0.009	11.43316	81.81556
var(e.ICF)	1.59e+07	39082.04			1.58e+07	1.59e+07
var(e.OCF)	1596007	3933.844			1588315	1603736
cov(e.ICF,e.OCF)	746848	8883.227	84.07	0.000	729437.2	764258.8

```
Note: The LR test of model vs. saturated is not reported because the fitted
      model is not full rank.
```

Table 5-3 Test statistics of the fundamental model – goodness of fit

Equation-level goodness of fit

depvars	Variance fitted	Variance predicted	Variance residual	R-squared	mc	mc2
observed						
ICF	1.67e+07	856401.3	1.59e+07	.0512384	.226359	.0512384
OCF	3381532	1785525	1596007	.5280225	.7266515	.5280225
overall				.5434777		

mc = correlation between depvar and its prediction
mc2 = mc^2 is the Bentler-Raykov squared multiple correlation coefficient

Table 5-4 Test statistics of the fundamental model – Wald test

Wald tests for equations

	chi2	df	p
observed			
ICF	17782.46	5	0.0000
OCF	3.7e+05	6	0.0000

The results of the overall Wald test (comparable to the F-test of a single equation), provide evidence that the tested equations are overall sufficiently specified; however, another regression analysis was performed, this time including a variable OCFL1SQ (the squared value of OCFL1). The purpose is to test whether the linearity assumption holds. The test results are displayed in Table 5-5.

Evidently, for quarterly time-intervals a nonlinear specification is highly likely as can be interfered from the high z-scores of variable OCFL1SQ in both equations. Regardless, the impact of nonlinearity on the estimated parameter values is minor, in the order of 10% deviation. Therefore, the parameter estimates of the original, linear model will be analysed.

Table 5-5 Test statistics of the fundamental model - parameter estimates

```
Endogenous variables

Observed:  ICF OCF

Exogenous variables

Observed:  OCFL1 Quarter Quarter2 Quarter3 OCFL2 AssetsTotal OCFL1SQ

Fitting target model:

Iteration 0:   log likelihood =  -30428245
Iteration 1:   log likelihood =  -30428214
Iteration 2:   log likelihood =  -30428214

Structural equation model                       Number of obs     =    329,271
Estimation method  = ml
Log likelihood     =  -30428214

------------------------------------------------------------------------------
                 |                 OIM
                 |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-----------------+------------------------------------------------------------
Structural       |
  ICF            |
           OCFL1 |   .4242028   .0042739    99.25   0.000     .4158261    .4325796
         Quarter |   2.155848   3.343778     0.64   0.519    -4.397836    8.709533
        Quarter2 |   .0346358   .0574024     0.60   0.546    -.0778708    .1471424
        Quarter3 |  -.0002741   .0002937    -0.93   0.351    -.0008497    .0003015
     AssetsTotal |  -.0104202    .000106   -98.31   0.000     -.010628   -.0102125
         OCFL1SQ |   3.04e-06   8.16e-08    37.19   0.000     2.88e-06    3.20e-06
           _cons |  -1.774931   56.47775    -0.03   0.975    -112.4693    108.9194
-----------------+------------------------------------------------------------
  OCF            |
           OCFL1 |   .6622078    .001801   367.69   0.000     .6586779    .6657377
         Quarter |  -2.528199   1.053426    -2.40   0.016    -4.592877   -.4635222
        Quarter2 |   .0501321   .0180841     2.77   0.006     .0146878    .0855763
        Quarter3 |  -.0002092   .0000925    -2.26   0.024    -.0003905   -.0000278
           OCFL2 |    .082714   .0017128    48.29   0.000      .079357     .086071
     AssetsTotal |   .0034169   .0000337   101.38   0.000     .0033509     .003483
         OCFL1SQ |  -2.01e-06   2.58e-08   -78.17   0.000    -2.06e-06   -1.96e-06
           _cons |   47.05143   17.79271     2.64   0.008     12.17836    81.92451
-----------------+------------------------------------------------------------
      var(e.ICF) |   1.58e+07   38918.54                       1.57e+07    1.59e+07
      var(e.OCF) |    1567282   3863.161                        1559728     1574872
-----------------+------------------------------------------------------------
cov(e.ICF,e.OCF) |     794238   8801.745    90.24   0.000     776986.9    811489.1
------------------------------------------------------------------------------
Note: The LR test of model vs. saturated is not reported because the fitted
      model is not full rank.
.
```

Note: including nonlinear cash flow term

From the estimated parameter values in Table 5-2, the set of parameters $\{\hat{\alpha}; \hat{\beta}; \hat{\gamma}; \hat{\delta}; \hat{\varepsilon}\}$ and related parameter values can be calculated. These calculated parameter values are reported in Table 5-6 below.

Table 5-6 Calculated parameters of the fundamental equations (default MLE)

PARAMETERS	CALCULATED PARAMETER VALUES	95%-LL	95%-UL
$\hat{\alpha}$	-0.382	-0.385	-0.379
$\hat{\beta}$	0.192	0.188	0.200
$\hat{\gamma}$	0.481	0.473	0.480
$\hat{\delta}$	46.6	11.4	81.8
$\hat{\varepsilon}$	0	0	0
$\hat{\omega}$	0.867	0.857	0.878
$\hat{\Lambda}_1$	-0.257	-0.264	-0.250
$\hat{\Lambda}_2$	-1.125	-1.121	-1.128

In case the assumptions underpinning the default MLE estimator are violated, the regression analysis is redone by using a robust ML estimator. The results are displayed in Table 5-7: other than substantially increasing the 95%-confidence interval of parameters $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$, robust regression affects the estimated values of all parameters only moderately. Hence, the results of the robust regression will be used in the conclusions below.

Table 5-7 Calculated parameters of the fundamental equations (robust MLE)

PARAMETERS	CALCULATED PARAMETER VALUES	95%-LL	95%-UL
$\hat{\alpha}$	-0.372	-0.452	-0.312
$\hat{\beta}$	0.192	0.098	0.244
$\hat{\gamma}$	0.481	0.340	0.621
$\hat{\delta}$	46.6	21.3	72.0
$\hat{\varepsilon}$	0	0	0
$\hat{\omega}$	0.874	0.658	1.040
$\hat{\Lambda}_1$	-0.249	-0.397	-0.136
$\hat{\Lambda}_2$	-1.123	-1.055	-1.176

Based on above estimated parameters, and using the same model as in Section 3-5, a number of trajectories are simulated. The results are shown in the two graphs on the following page.

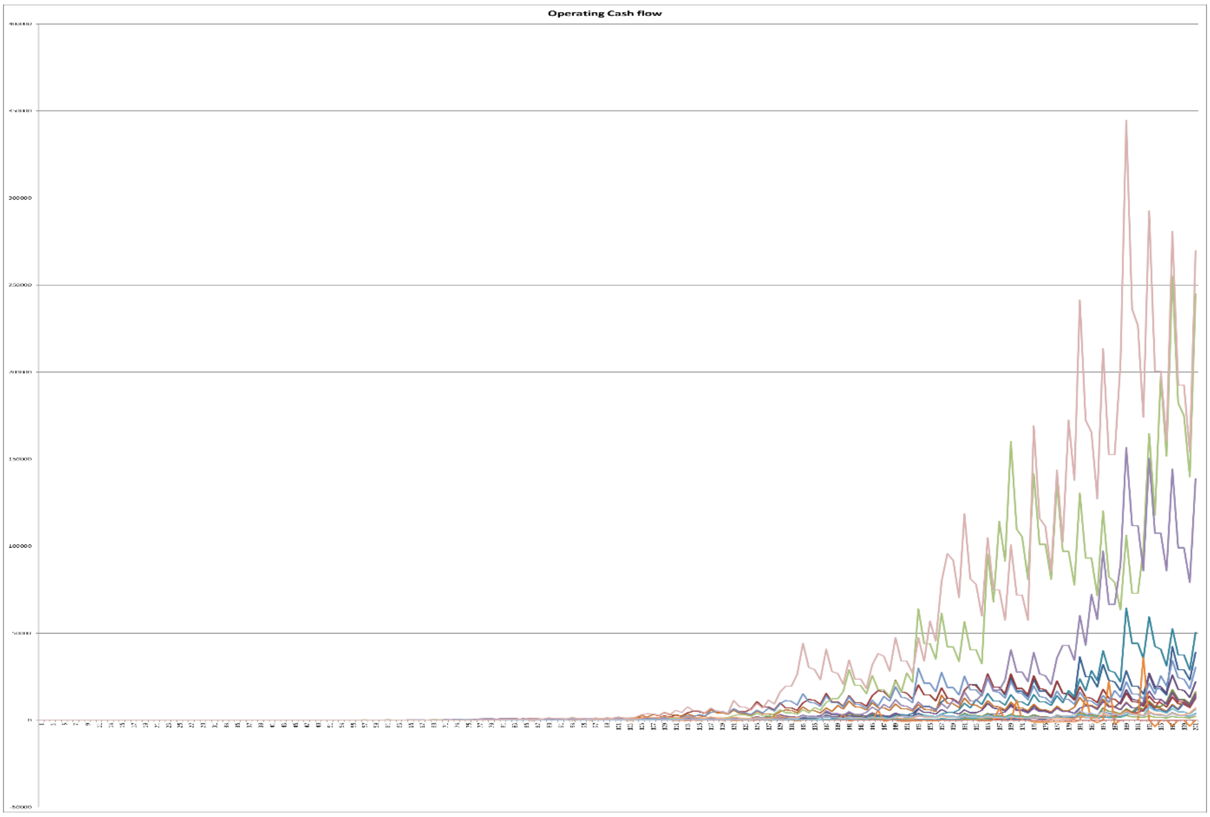


Figure 5-3 Simulated trajectories of Operating Cash Flow – Estimated parameters used

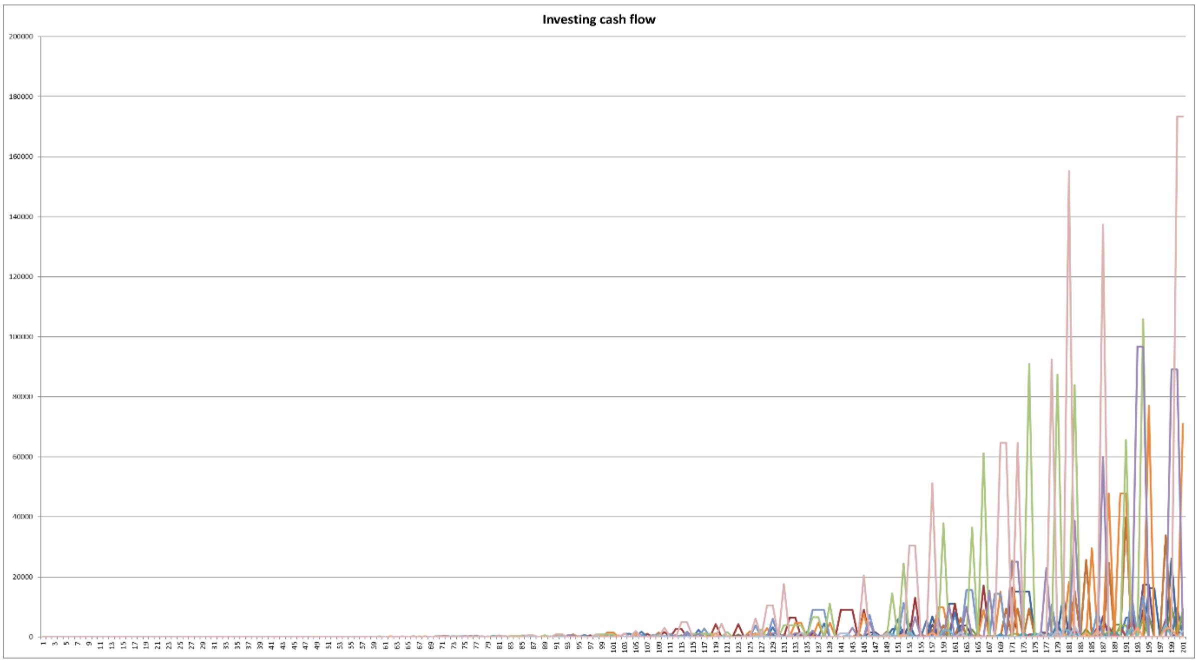


Figure 5-4 Simulated trajectories of Investing Cash Flow – Estimated parameters used

Discussion

Reiterating what was stated in the introduction to the prior subsection, the following conclusions are only valid (on average, in a deterministic sense) for the population of all firms (assuming that the sample examined is representative of all firms[46]). Checking stability conditions expressed in Equations (5.4a) to (5.4c), it can be ascertained that the AR(2) process described by Equation (5.3) is stable. A closer observation of the parameter values, reveals interesting information. In Equation (5.1), $\Delta C_t = \alpha C_{t-1} + \beta I_{t-1} + \delta$, parameter α (the cash flow growth rate), is negative. This suggests a significant cash flow attrition rate (negative) dominating the effect of an improved utilisation rate (positive), or perhaps, a utilisation rate that also deteriorates with time. Furthermore, in the absence of any investments made, changes to operating cash flow declines by 37.2% each quarter until a stable long-term value of 46.6 (million US$, average for all examined firms) is reached. The quarterly cash flow growth rate equates to a continues-time rate of - 0.465. Abstracting out (natural) attrition, a one-dollar investment spending increases operating cash flow by $0.192 in the following quarter. Further analysis shows that in the coupled model the investment response parameter strongly declines in subsequent quarters, also because new capital goods, like existing, become prone to usual obsolesce.

Observing Equation (5.2), $I_t = \gamma C_{t-1} + \varepsilon$, it transpires that of every dollar operating cash flow generated, one quarter later $0.481 is invested in capital goods. Again, this is a pure proportional relationship since parameter ε is not significantly different from zero.

To examine the operating and investing cash flow processes in isolation, one has to observe parameters $\widehat{\omega}$, $\widehat{\Lambda}_1$and $\widehat{\Lambda}_2$ that pertain to the uncoupled cash flow model described in Section 3-3, Equations (5.4a) and (4b). Since $\widehat{\alpha} < -\widehat{\beta}\widehat{\gamma}$, both parameters $\widehat{\Lambda}_1$and $\widehat{\Lambda}_2$ are negative. This implies that the uncoupled operating and investing cash flow processes are converging to long-time constant values, typical for stable cash flows in a static growth environment (depicted by the RHS graph in Figure 3-1, Section 3-3). As expected, the coupled processes are also converging in time (see Figure 5-5 below). Indeed, the population of firms is a

[46] Since the selected firms are all public companies that are required to report their cash flow data regularly, the sample is likely to be biased in favour of larger, probably more professionally managed firms, excluding medium and small sized businesses which nevertheless constitute an important part of overall business activity. This may be a limitation on the application of the results of the study, but only if there are indications that cash flow processes of smaller, private businesses are fundamentally different from those of the sample.

dynamic mix, each firm with their own success story. Some of the new market entrants become hugely successful; however, most experience modest business growth over their life time. Other, less successful firms struggle and eventually fold. In aggregate these individual movements will (about) cancel each other, so that, for the whole population excluding individual firm randomness, average cash flow is believed to be stable to modestly growing in line with economic growth measured by changes to GDP.

The level of operating cash flow approaches a stationary value of 166.86 (million US$, average for all examined firms), whilst the long-time values for investing cash flow is 80.19 and the resulting free cash flow is 86.67.

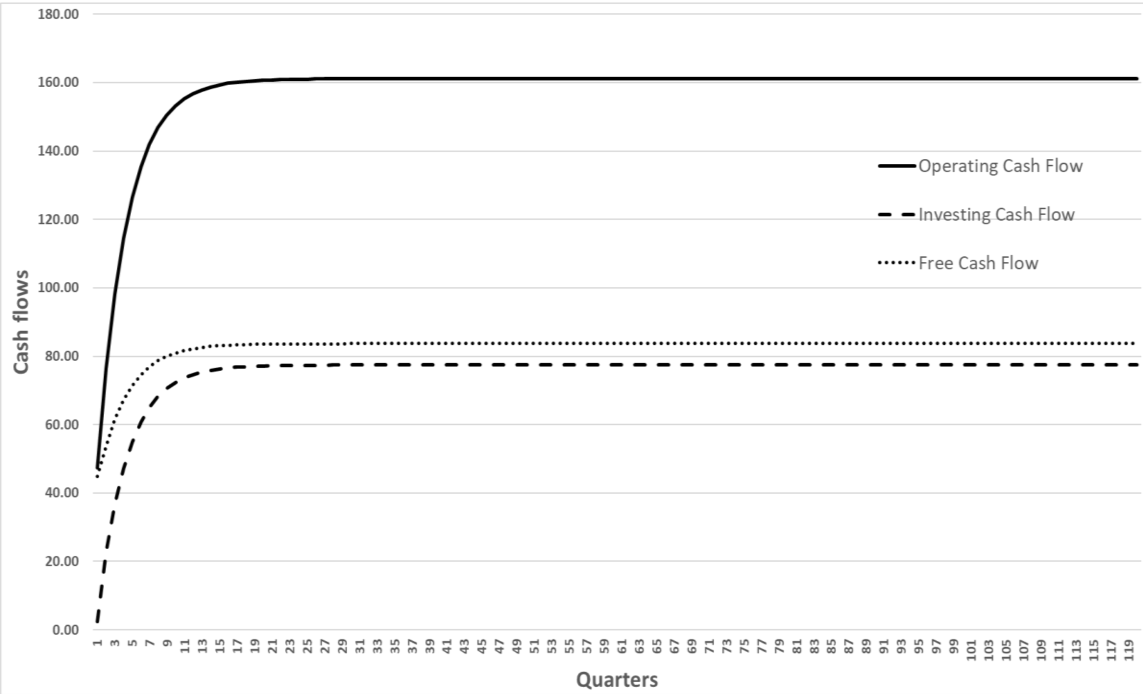


Figure 5-5 Deterministic, coupled cash flow processes based on estimated parameters

From the estimated and calculated parameters in Tables 5-6 and 5-7, the estimated matrixes $\widehat{\mathbf{Q}} = \begin{pmatrix} \frac{2\beta}{-1-\alpha+\omega} & \frac{2\beta}{-1-\alpha-\omega} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1.563 & -0.256 \\ 1 & 1 \end{pmatrix}$ and $\widehat{\mathbf{Q}}^{-1} = \begin{pmatrix} 0.550 & 0.141 \\ -0.550 & 0.859 \end{pmatrix}$ are quantified.

Lastly, the simulated stochastic trajectories portrayed in Figures 5-3 and 5-4, shows a close resemblance with the theoretically derived ones in Section 3-5, and real-world cash flows described in Section 1-2.

5.2. Estimating the linear-quadratic model

Introduction

In this section, the Equations (3.26a) and (3.27b) that were obtained in Section 3-4, will be statistically tested. As a reminder, the decoupled operating and investing cash flow processes respectively, are represented in continuous-time by the following specifications

$$dC'_t = \left(\mu_{C,1}C'_t + \mu_{C,0}\right)dt + \sqrt{\left(\sigma_{C,2}C'^2_t + \sigma_{C,1}C'_t + \sigma_{C,0}\right)}\, dW_{C,t} \tag{5.5a}$$

$$dI'_t = \left(\mu_{I,1}I'_t + \mu_{I,0}\right)dt + \sqrt{\left(\sigma_{I,2}I'^2_t + \sigma_{I,1}I'_t + \sigma_{I,0}\right)}\, dW_{I,t} \tag{5.5b}$$

The estimation results of the linear-quadratic specification will be compared to those of the five specifications that figure commonly in the literature. Recall from Section 2-1 that these specifications (in this instance described for operating cash flows) are

(i) Geometric Brownian Motion (GBM); $dC_t = \mu C_t dt + \sigma C_t\, dW_t$ (5.6a)

(ii) Arithmetic Brownian Motion (ABM); $dC_t = \mu dt + \sigma dW_t$ (5.6b)

(iii) Mean-reverting Vasicek process (Vasicek model); $dC_t = \alpha(m - C_t)dt + \sigma dW_t$ (5.6c)

(iv) Mean-reverting Cox, Ingersoll and Ross process (square root or CIR model); $dC_t = \alpha(m - C_t)dt + \sigma\sqrt{C_t}dW_t$ (5.6d)

(v) Modified Square Root process (MSR model); $dC_t = \mu C_t dt + \sqrt{k_1^2 + k_2^2 C_t^2}dW_t$ (5.6e)

In Section 2-3, it is shown that each of the above specifications, is a special form of the complete linear-quadratic Equation: (5.5a) for operating cash flows and (5.5b) for investing cash flows. The results of preliminary specification analysis that are reported in Tables 2-3 and 2-4, suggest that a quadratic diffusion function is preferred to a linear diffusion function. This follows from the cash flow processes of a significant number of firms agreeing, in full or in reduced form, with a quadratic diffusion function. In this section, more specific and rigorous estimations will be performed on the *complete* linear-quadratic specification.

The estimation in this section centers on validating Equations (5.5a) and (5.5b) for cases examined by testing the hypothesis whether all estimated parameters $\{\hat{\mu}_0, \hat{\mu}_1, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$ are significantly different from zero. If not, the linear-quadratic model must be rejected to the benefit of a more specialised specification, amongst the ones found in the literature (Equations (5.6a) – (5.6e)). Table 5-8 provides the connection between the full parameter set $\{\mu_0, \mu_1, \sigma_0, \sigma_1, \sigma_2\}$ applicable to the linear-quadratic model, and the subsets of those parameters corresponding to the aforementioned particular models.

Table 5-8 Parameters of the estimated specifications

	DRIFT FUNCTION		DIFFUSION FUNCTION		
	μ_0	μ_1	σ_0	σ_1	σ_2
LINEAR-QUADRATIC	√	√	√	√	√
GBM		√			√
ABM	√		√		
VASICEK	√	√	√		
SR	√	√		√	
MSR		√	√		√

Estimation methodology

As noted already in Section 1-3, estimating continuous-time processes with discrete (low-frequency) data poses major challenges. These challenges have been extensively documented in the literature: for example, Ait-Sahalia (2006); Ait-Sahalia and Mykland (2003); B. Chen and Hong (2010); Duffie and Glynn (2004); Florens-Zmirou (1993); Fuchs (2013, Chapter 6); F. Li (2007); Sorensen (2002).

In the hypothetical case that data were to be sampled continuously, it is straightforward to calculate the instantaneous change in variance from the quadratic variation of the cash flow process under consideration. Once the diffusion function is completely determined, for instance the uncoupled diffusion function of an operating cash flow process $\sqrt{(\hat{\sigma}_{C,2} C_t'^2 + \hat{\sigma}_{C,1} C_t' + \hat{\sigma}_{C,0})}$, then the corresponding drift function $\hat{\mu}_{C,1} C_t' + \hat{\mu}_{C,0}$ is estimated from a likelihood estimator based on a transformed, driftless process (Pavliotis (2014, Section 3.6)). Typically, the Radon-Nikodym derivative from Girsanov's formula is used to accomplish a change of probability measure so that the drift function is eliminated, turning the process into a martingale (Fuchs (2013, subsection 6.1.1.)).

Discrete observations, however, require an entirely different approach to estimate parameters of the diffusion function. Recall from Section 1-3 that a cash flow variable in a continuous-time setting, is a *state variable*, sometimes called cash flow intensity. A state variable cannot be measured over a time-interval, except in some special cases (Ait-Sahalia (1996, p. 389)). Furthermore, and related to the foregoing, in Section 2-3 it is explained why a Gaussian transition density function applicable to a very small-time interval δt, becomes significantly non-Gaussian as the observed discrete time-interval ΔT increases. This creates additional challenges when estimating parameters. Even a Lamperti transform (discussed in Section 4-1) where the diffusion function is changed to unit variance and associated Wiener process, does not deal with the issue (entirely). Indeed, in case of a linear-quadratic specification, there is still a material cumulated effect of the transformed (nonlinear) drift function on the observed transition probability density, leading to a progressively non-Normal distribution. Some of these issues can be mitigated if high-frequent sampled data are available but in case of low-frequency discrete data, for instance quarterly data used in this study, the problem of estimating parameters becomes significantly more difficult. In particular, there is no evident approximation scheme that can efficiently compute or mimic the continuous ML-estimator. Likewise, the usual nonparametric kernel estimators, based on differencing, do not provide consistent parameter estimates (Gobet et al. (2004)).

Broadly, two important directions for addressing the problem of estimating parameters from discrete (low-frequency) observations, could be distinguished:

(i) use a (approximated) closed-form solution of the SDE to directly estimate parameters; and
(ii) apply the (approximated) transition density function of the SDE to indirectly estimate parameters.

The first method assumes that a general closed-form solution exists or, at least, an accurate approximation of the exact solution is derivable. In other words, a solution function can be found with the following general specification

$$X_t = F(t, W_t, \boldsymbol{\theta}) \tag{5.7a}$$

where X_t is some cash flow process, and $\boldsymbol{\theta}$ is the set of parameters $\{\mu_0, \mu_1, \sigma_0, \sigma_1, \sigma_2\}$ to be estimated. To eliminate variable W_t, one commonly resorts to the method of the moments by turning Equation (5.7a) into

$$M_{t,X}^n = M_n(t, \boldsymbol{\theta}) \tag{5.7b}$$

where $M_{t,X}^n$ is the evolution of the n[th] moment of X_t. Note that usually $M_n(t, \boldsymbol{\theta})$ is a nonlinear function of t, requiring a nonlinear parameter estimation technique.

The second method is related to the forward Kolmogorov equation that describes the behaviour of the transition probability density function *at any time t*. The transition probability density $p(X_t, t)$ is the solution of

$$\frac{\partial p(X_t, t)}{\partial t} = \frac{-\partial \alpha(X_t, \boldsymbol{\theta}) p(X_t, t)}{\partial X_t} + \frac{1}{2}\frac{\partial^2 \beta(X_t, \boldsymbol{\theta}) p(X_t, t)}{\partial X_t^2} \tag{5.8}$$

where X_t is some cash flow process, $p(X_t, t)$ is the transition probability density function, $\alpha(X_t, \boldsymbol{\theta})$ is the estimated linear drift function, $\sqrt{\beta(X_t, \boldsymbol{\theta})}$ is the estimated diffusion function, and $\boldsymbol{\theta}$ is the complete set of parameters to be estimated. Even though the probability density function follows from Equation (5.8), assuming it exists in analytical form, usually no direct link can be made between the set of functions $\left\{\alpha(X_t, \boldsymbol{\theta}), \sqrt{\beta(X_t, \boldsymbol{\theta})}\right\}$ and $p(X_t, t)$. Obviously, the Fokker-Planck identity, Equation (5.8), connects the two but, the LHS of Equation (5.8) is only valid in small-time ∂t. One way around this issue (Ait-Sahalia (1996)) is to observe Equation (5.8) under a stationary restriction by setting $\frac{\partial p(X_t, t)}{\partial t} = 0$. As already explained in Sections 4-2 and 4-3, then the general solution to Equation (5.8) becomes

$$p_{st}(X) = \frac{n(\boldsymbol{\theta})}{\sqrt{\beta(u, \boldsymbol{\theta})}} \exp\left[\int \frac{2\alpha(u, \boldsymbol{\theta})}{\sqrt{\beta(u, \boldsymbol{\theta})}} du\right] \tag{5.9}$$

where $n(\boldsymbol{\theta})$ is a normalisation constant that depends on the parameter vector $\boldsymbol{\theta}$. Equation (5.9) governs the connection between probability density, and the parameterisation of the drift and diffusion functions. Now, if a relationship between $p_{st}(X)$ and $p(X_t, t)$ can be established, then, in principle, all information encapsulated in the space-time density function is available to estimate the function set $\left\{\alpha(X_t, \boldsymbol{\theta}), \sqrt{\beta(X_t, \boldsymbol{\theta})}\right\}$. In Chapter 4 such relationships are found for the linear-quadratic specification of operating and investing cash flow processes. Conveniently, these will be utilised to estimate parameter vector $\boldsymbol{\theta}$ later in this section. Another work-around is described in Hansen et al. (1998) and Mathieu Kessler and Sorensen (1999). They propose a martingale estimation function based on eigenfunctions for the generator of the diffusion model. Mathieu Kessler and Sorensen

(1999, p. 300) purport that the route via spectral theory gives more explicit estimating equations, not necessarily limited to the usual polynomial expansions.

Regardless of the particular specification, the Markov property allows the parameter set to be estimated from the corresponding maximum likelihood estimator. Assuming that discrete observations are sampled over a constant time-interval Δ, at each time $T_i = \sum_1^i \Delta_i$ the probability density is conditional upon the density at T_{i-1} where $i = 1 \,..\, n$. In that instance, the set of estimated parameters arises from maximising the following log-likelihood function with respect to $\boldsymbol{\theta}$

$$\ell_n = \sum_{i=1}^{n} \ln\left[p\left(X_{T_i} | X_{T_{i-1}}; \boldsymbol{\theta}\right)\right] \tag{5.10a}$$

Consequently, the estimated parameter vector observes

$$\hat{\boldsymbol{\theta}} = \arg\max \ell_n(\boldsymbol{\theta}) \tag{5.10b}$$

Notice that Equation (5.10a) is based on the Chapman-Kolmogorov equation and effectively develops a series of consecutive transition density functions into a marginal probability distribution. This is conducive to solving the problem of relating the function set $\left\{\alpha(X_t, \boldsymbol{\theta}), \sqrt{\beta(X_t, \boldsymbol{\theta})}\right\}$ to discretely observed data. However, the vast majority of stochastic processes have no known analytical probability density function $p(X_t, t)$. Hence, parameter values must be obtained from pseudo-likelihood and approximated likelihood methods. For a general overview see M. Kessler et al. (2012); Rao (2014), and for available estimation methods refer to Fuchs (2013); Iacus (2009). Inference methods specifically applied to the linear-quadratic model are analysed and discussed, amongst others, in Forman and Sorensen (2008); Schmidt (2008).

If exact likelihood inference is not an option because no analytical transition density function exists, then parameter estimation based on pseudo-likelihood methods are an alternative. Pseudo-likelihood methods approximate the (transformed) path of the stochastic process such that the corresponding transformed transition density function is suitable to apply maximum likelihood estimation. Therefore, it is no surprise that (local) Gaussian density function approximations are a popular choice (refer for example to Sorensen (2002) and Wei et al. (2016)). However, as pointed out in Iacus (2009, p. 122), pseudo-likelihood methods provide good approximations only if the time between discrete observations, Δ, is sufficiently small. In this study low-frequency (quarterly) cash flow data are used that are considered too granular to apply pseudo-likelihood methods successfully. Consequently, the focus ought to be on approximated likelihood methods that are, as will become clear in the following subsections, more flexible in dealing with low-frequency data.

Both methods, (i) approximated close-form solutions, and (ii) approximated likelihood estimation, will now be further examined in the context of statistical testing of a linear-quadratic cash flow model.

Approximated closed-form solutions

Section 4-1 shows that a linear-quadratic SDE has no known closed-form solutions; however, two approximations will be used in conjunction to estimate the full set of parameters $\widehat{\boldsymbol{\theta}}\colon \{\hat{\mu}_0, \hat{\mu}_1, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$. These two approximations are analysed in Section 4-1: (i) Approximation one, Equation (4.33), and (ii) Approximation three, Equation (4.43). Note that Approximation method two, Equation (4.37), is not taken into consideration because results are too inaccurate for smaller positive and negative cash flow values (roughly between +\$100,000 and -\$100,000; see Section 4-1).

Method one is considered a good approximation for an operating cash flow process provided that cash flows are not too small (greater than \$100 and smaller than -\$100; see Section 4-1), and parameter σ_2 is not very small relative to parameters σ_1 and σ_0. The method yields the following solution

$$X_t = Z_o \exp\left[(\mu_1 - \tfrac{1}{2}\sigma_2)t + \sqrt{\sigma_2}W_t\right] + [(\mu_0 - \sigma_1)t + \sqrt{\sigma_2}\, W_t] \tag{5.11}$$

from which the expected value of X_t is calculated as

$$\mathbb{E}(X_t) = X_o \exp[\mu_1 t] + (\mu_0 - \sigma_1)t \tag{5.12a}$$

or, in parametrised form

$$\mathbb{E}(X_t) = X_o \exp[\theta_1 t] + \theta_2 t \tag{5.12b}$$

where $\theta_1 = \mu_1$, $\theta_2 = \mu_0 - \sigma_1$ and X_o is set to the arbitrary value of 1.

Approximation method three transforms the cash flow variable X_t first to $\widetilde{X}_t = \frac{qX_t+r}{s}$ where $q = \sqrt{\sigma_2}$ for $\sigma_2 \geq 0$, $r = \frac{\sigma_1}{2\sqrt{\sigma_2}}$ and $s = \frac{D}{2\sqrt{\sigma_2}}$ with $D = \sqrt{\sigma_1^2 - 4\sigma_2\sigma_0}$. A second transformation follows in which variable $\widetilde{X}_t$ is transformed to Z_t: $Z_t = \frac{s}{\sqrt{2}}\sinh^{-1}(\widetilde{X}_t)$. Finally, Z_t is re-scaled to $Z_t' = \frac{\sqrt{2}}{s}Z_t$. The expected value of variable Z_t' is already given in Equation (4.44), Section 4-1

$$\mathbb{E}(Z_t') = -\frac{4\sigma_2\mu_0 - 2\sigma_1}{(2\mu_1 + \sqrt{\sigma_2})D} + \left[Z_0' + \frac{4\sigma_2\mu_0 - 2\sigma_1}{(2\mu_1 + \sqrt{\sigma_2})D}\right] e^{\left[\frac{(2\mu_1+\sqrt{\sigma_2})\sqrt{2}}{D}\right]t} \tag{5.13}$$

For estimation purposes, Equation (5.13) needs to be expressed in $\mathbb{E}(X_t)$ by applying this expected value conversion

$$\mathbb{E}(X_t) = \frac{s}{q}\mathbb{E}\left[\frac{\sqrt{2}}{s}\sinh\left(\frac{s}{\sqrt{2}}Z_t'\right)\right] - \frac{r}{q} \tag{5.14a}$$

Since $|Z_t'| < 1$, $\sinh\left(\frac{s}{\sqrt{2}}Z_t'\right)$ is approximated by a series expansion

$$\sinh\left(\frac{s}{\sqrt{2}}Z_t'\right) = \frac{s}{\sqrt{2}}Z_t' + \mathcal{O}({Z_t'}^3) \tag{5.14b}$$

so that Equation (5.14a) is re-written to

$$\mathbb{E}(X_t) = \frac{s}{q}\mathbb{E}[Z_t'] - \frac{r}{q} \tag{5.14c}$$

Substituting Equation (5.13) into Equation (5.14c) gives

$$\mathbb{E}(X_t) = -\frac{4\sigma_2\mu_0 - 2\sigma_1 + \sigma_1\sqrt{\sigma_2} + 2\sigma_1\mu_1}{2\sigma_2(2\mu_1+\sqrt{\sigma_2})} + \left[\frac{D}{2\sigma_2}\sinh^{-1}\left(\frac{2\sigma_2+\sigma_1}{D}\right) + \frac{2\sigma_2\mu_0 - \sigma_1}{\sigma_2(2\mu_1+\sqrt{\sigma_2})}\right] e^{\left[\frac{(2\mu_1+\sqrt{\sigma_2})\sqrt{2}}{D}\right]t} \tag{5.15a}$$

where Z_0' is replaced by $\sinh^{-1}\left(\frac{2\sigma_2+\sigma_1}{D}\right)$ with $X_0 = 1$.

Parametrised, Equation (5.15a) leads to this structural equation

$$\mathbb{E}(X_t) = \theta_3 + \theta_4 e^{\theta_5 t} \tag{5.15b}$$

where $\theta_3 = -\frac{4\sigma_2\mu_0 - 2\sigma_1 + \sigma_1\sqrt{\sigma_2} + 2\sigma_1\mu_1}{2\sigma_2(2\mu_1 + \sqrt{\sigma_2})}$, $\theta_4 = \left[\frac{D}{2\sigma_2}\sinh^{-1}\left(\frac{2\sigma_2 + \sigma_1}{D}\right) + \frac{2\sigma_2\mu_0 - \sigma_1}{\sigma_2(2\mu_1 + \sqrt{\sigma_2})}\right]$, $\theta_5 = \left[\frac{(2\mu_1 + \sqrt{\sigma_2})\sqrt{2}}{D}\right]$.

Together Equations (5.12b) and (5.15b) contain five estimated parameters $\hat{\boldsymbol{\theta}}$: $\{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5\}$, all to be estimated from the evolution of the expected value of cash flow. These five parameters each represent a formula from which the parameter set $\{\hat{\mu}_0, \hat{\mu}_1, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$ can be solved. In other words: the system is exactly specified.

Estimation results for operating cash flow

A nonlinear regression with expected values of operating cash flow as a dependent variable, produces the following estimation results for Equation (5.12b)

Table 5-9 Estimation results for θ_1 and θ_2 – Operating Cash Flows

```
Nonlinear regression                           Number of obs =        120
                                               R-squared     =     0.8412
                                               Adj R-squared =     0.8385
                                               Root MSE      =   208.7845
                                               Res. dev.     =   1620.441
```

Mean	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
/theta2	4.141726	.4214576	9.83	0.000	3.307125	4.976327
/theta1	.0551186	.0019325	28.52	0.000	.0512918	.0589454

It turns out that Equation (5.15b) yields plausible estimation results only if parameter θ_3 is constrained to zero. Then, the statistics for the remaining parameters are

Table 5-10 Estimation results for θ_4 and θ_5 – Operating Cash Flows

```
Nonlinear regression                           Number of obs =        120
                                               R-squared     =     0.8537
                                               Adj R-squared =     0.8512
                                               Root MSE      =   200.4204
                                               Res. dev.     =   1610.628
```

Mean	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
/theta4	48.24198	10.1545	4.75	0.000	28.1333	68.35066
/theta5	.0267981	.0023852	11.24	0.000	.0220748	.0315213

A mapping of the above parameter set $\hat{\boldsymbol{\theta}}$: $\{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5\}$ to the calculated parameters $\{\hat{\mu}_0, \hat{\mu}_1, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$ is shown in Table 5-11. Notice that there are three solution sets for calculated parameters but the second solution has no practical meaning. Parameter $\hat{\sigma}_0$ is calculated from $\hat{\sigma}_1$, $\hat{\sigma}_2$ and D.

Table 5-11 Mapping of parameter sets - Operating Cash Flows

			SOLUTION SET 1	SOLUTION SET 2	SOLUTION SET 3
$\hat{\theta}_1$	0.0552	$\hat{\mu}_0$	-141.69	4.41417	-89.76
$\hat{\theta}_2$	4.1417	$\hat{\mu}_1$	0.0552	0.0552	0.0552
$\hat{\theta}_3$	0	$\hat{\sigma}_0$	6684.79	-	6000.10
$\hat{\theta}_4$	48.24	$\hat{\sigma}_1$	-145.83	0	-93.90
$\hat{\theta}_5$	0.0268	$\hat{\sigma}_2$	0.7018	0	0.3114
		D	50.03	0	36.66
		$\hat{\lambda}_1$	68.25		91.91
		$\hat{\lambda}_2$	139.54		209.63

Estimation results for investing cash flow

A similar regression is performed on investing cash flow data. The results obtained, are given in Table 5-12 below.

Table 5-12 Estimation results for θ_1 and θ_2 – Investing Cash Flows

```
Nonlinear regression                              Number of obs =        120
                                                  R-squared     =     0.4301
                                                  Adj R-squared =     0.4204
                                                  Root MSE      =    469.572
                                                  Res. dev.     =   1814.966

Bootstrap results
```

Mean	Observed Coef.	Bootstrap Std. Err.	z	P>|z|	Normal-based [95% Conf.	Interval]
/theta2	.1342978	.0120227	11.17	0.000	.1107337	.1578619
/theta1	2.15e-10	2.24e-10	0.96	0.339	-2.25e-10	6.55e-10

Clearly, parameter $\hat{\theta}_1 = \hat{\mu}_1$ is not significantly different from zero. Comparable to the estimation for operating cash flows, here parameter θ_3 is also constrained to zero in order to obtain robust results in the second regression

Table 5-13 Estimation results for θ_4 and θ_5 – Investing Cash Flows

```
Nonlinear regression                             Number of obs =        120
                                                 R-squared     =     0.6053
                                                 Adj R-squared =     0.5986
                                                 Root MSE      =   390.7959
                                                 Res. dev.     =   1770.893
```

Mean	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
/theta4	717.2247	191.8807	3.74	0.000	337.2487	1097.201
/theta5	-.0001508	.0000852	-1.77	0.079	-.0003194	.0000179

Finally, a mapping similar to Table 5-11, this time for investing cash flows, is depicted in Table 5-14. Here too, only solution sets 1 and 3 are useful.

Table 5-14 Mapping of parameter sets - Investing Cash Flows

			SOLUTION SET 1	SOLUTION SET 2	SOLUTION SET 3
$\hat{\theta}_1$	0	$\hat{\mu}_0$	-8176.24	0.1343	-4994.5
$\hat{\theta}_2$	0.1343	$\hat{\mu}_1$	0	0	-89.76
$\hat{\theta}_3$	0	$\hat{\sigma}_0$	45495150	0	39708423
$\hat{\theta}_4$	717.23	$\hat{\sigma}_1$	-8176.37	0	-4994.5
$\hat{\theta}_5$	-0.00015	$\hat{\sigma}_2$	0.7108	0	0.3517
		D	-7906.45	0	-5561.92

Even though parameter $\hat{\theta}_5$ is not different from zero at a 5% significance level, it is decided to include the estimated parameter value in the mapping calculations. A value of zero generates unstable (that is, non-converging) results for the parameter set $\{\hat{\mu}_0, \hat{\mu}_1, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$. Nonetheless, the conclusion is that in Equation (5.12b) the relative weight of exponential growth is insignificant compared to that of the linear growth.

Discussion of estimation results

The estimation results are encouraging with consistently high R^2-statistics (used as a goodness of fit measure in nonlinear estimations) for both cash flow processes. Approximation one produces exact results under the condition that parameter σ_2 is not very small relative to parameters σ_1 and σ_0. Observing the results in Table 5-11 and 5-14, this condition is obviously not met; however, approximation three does not have the same restriction. Since the parameter set $\{\hat{\mu}_0, \hat{\mu}_1, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$ is calculated from both equations, one can assume that inaccurateness introduced by method one is (partially) off-set by a much closer approximation inherent with method three.

It is important to note that all parameters pertaining to the operating cash flow process are significantly different from zero, underpinning the hypothesis that on an aggregated firm-level, the full linear-quadratic specification is superior to all particular specifications included in Table 5-8. This conclusion can be drawn also for investing cash flows but the results must be interpreted with much more caution since parameter $\hat{\mu}_1$ and $\hat{\theta}_5$ are statistically insignificant, the latter parameter affecting the calculations of the diffusion parameters $\{\hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$. It should be noted that these results are (as a sample) representative for the whole population of firms; a more restricted sample, for instance at industry level or at individual firm level, may well lead to a different conclusion.

In respect of investing cash flows, the conclusions are slightly more ambiguous. The full-quadratic specification appears also valid but with the observation that the diffusion function is dominated by the constant in the quadratic expression, represented by parameter $\widehat{\sigma}_0$.

The reported values of D (the square root of the discriminant) are in conformity with expectations articulated in Section 2-3. For operating cash flows these are shown to be positive whilst the discriminants of investing cash flows invariably have large negative values. This contrast brings about important implications. The diffusion of investing cash flows (i) has complex roots, typical for a process that follows a Pearson-IV distribution (Pearson's Case 1, Section 2-3, Figure 2-5), (ii) is defined on the full cash flow range $\mathbb{R}$, and (iii) mostly converges to a stationary probability distribution. In contrast, operating cash flows are better modelled by probability distributions belonging to Pearson's Case 2. Since the process is generally not converging, it must be defined on a restricted domain to ensure finding meaningful solutions to the Kolmogorov equations (refer to Chapter 4).

Introduction to approximated likelihood estimation

As mentioned in the beginning of this chapter, an alternative way to obtain parameter values, is to maximise the log-likelihood function of the transition density function with respect to parameter vector $\boldsymbol{\theta}$. Most SDEs have no known analytical probability density function $p(X_t, t)$ which, in Chapter 4 is corroborated to be the case also for the linear-quadratic model. Hence, an approximation method will be required to conjecture a transition density function (at discrete times) from available data.

The question arises which (family) of distributions the approximated transition density of the linear-quadratic model resembles. Only in special cases will a transition density function in discrete time be governed by a normal distribution; usually the underlying transition density is significantly non-Gaussian. The general linear-quadratic cash flow model is no exception. In Chapter 4, two general approximations for the linear-quadratic specification, Equations (4.55) and (4.103a), were found. For investing cash flows there exists a stationary Student probability density function that has the following specification

$$p(I) = \eta(\mu,\alpha,\beta,\gamma)\frac{\exp\left[\frac{\mu-\beta}{\alpha\gamma}\tan^{-1}\left[\frac{I_t-\beta}{\gamma}\right]\right]}{\left[1+\left[\frac{I_t-\beta}{\gamma}\right]^2\right]^{\frac{\nu+1}{2}}} \tag{5.16}$$

where $I \in \mathbb{R}$, $\alpha > -1$, $\eta(\mu,\alpha,\beta,\gamma) = \frac{\Gamma(\frac{\nu+1}{2})}{\gamma\sqrt{\pi}\ \Gamma(\frac{\nu}{2})}\prod_{k=0}^{\infty}\left[1+\left(\frac{\frac{\mu-\beta}{\alpha\gamma}}{\frac{\nu+1}{2}+k}\right)^2\right]^{-1}$, $\nu = 1+\frac{1}{\alpha}$

and for operating cash flows, the full dynamics of the process is described by the space-time density function

$$p(c,t) = \overline{K}(c-1)^{\nu_1}(c+1)^{\nu_2}e^{-\frac{1}{2}t}\frac{2}{\sqrt{2\pi t}}e^{-\frac{(c-c_0)^2}{2t}} \tag{5.17}$$

where ν_1 and ν_2 are composite parameters derived from the parameter set $\{\mu_0,\mu_1,\sigma_0,\sigma_1,\sigma_2\}$. Especially the density function of operating cash flows is generic and therefore includes a wide range of more specialised density functions (for an overview see Figure 2-5 in Section 2-3). In the absence of known parameter values, it is nearly impossible to ascertain a specific density function that serves as a close enough approximation of the true density. Admittedly, the inference of a stationary Student diffusion, in this study found to be adequate to describe an investing cash flow process, has a specific density function (Equation(5.16)) and a related closed-form estimating function based on a Routh–Romanovski finite polynomial expansion (N. N. Leonenko and Šuvak (2010)). Nevertheless, if possible, the preference is to determine an overarching density function that fits both operating and investing cash flow processes.

In addition, there is the issue of non-normality of the transition density if discretely sampled from low-frequency data. In this instance, the inferred transition density will almost always be non-Gaussian, save in the exceptional case that the stochastic process itself is normal. Clearly, Equation (5.16) is non-Gaussian and Equation (5.17), albeit a composite function of a Gaussian and Pearson's Case 2 long-time density function family, is generally non-Gaussian as well[47]. If non-Gaussian density functions are used in parameter estimation, and they are part of a series approximation method, for instance an Edgeworth or a Gram-

[47] Only in the exceptional case that a particular solution to Pearson's Case 2 density function is normally distributed, the composite function will be Gaussian.

Charlier series expansion, then the results tend to be inaccurate. Often, the tail(s) of the distribution are too thick to ensure that the expansion is (asymptotically) converging rapidly enough (Iacus (2009, p. 138)). This shortcoming is absent in a normal distribution due to its sufficiently thin tails.

It is obvious that the transitional density belonging to the linear-quadratic equation ought to be derived from a generic and widely applicable statistical inference method, capable of dealing with approximated probability functions. The class of Approximate Maximum Likelihood Estimators (AMLE), that is the focus of this section, encapsulates a variety of methods, including but not limited to: Gaussian approximation of the transition density based on the Euler discretisation method (Mathieu Kessler (1997), approximation of continuous-time integrals (between discrete observations) by Itô and Riemann summations (Bibby and Sørensen (1995), Yoshida (1992)), approximation of the density function by series expansion (Ait-Sahalia (1996), Aït-Sahalia (2002a), Chang and Chen (2011), Chen-Charpentier and Stanescu (2014)), and adaptive maximum likelihood estimation (Uchida and Yoshida (2012).

Ait-Sahalia's approximation method

In this study a variant of the series expansion method developed by Ait-Sahalia (Aït-Sahalia (1999, 2002a)) is developed and applied. The Ait-Sahalia-method is chosen because it is well-established in the literature, it is supported by a wide range of different stochastic processes, it converges fast to an (unobservable) continuous-time solution, it is agreeable to low-frequency sampled data and it is applicable to stationary and non-stationary processes alike. The Ait-Sahalia- method approximates an analytical log-likelihood function (refer to Equation (5.10)) by first transforming the transition density function into one that is close to normal, followed by expanding the transformed density using Hermite polynomials. The method is explained, applied and further developed in a range of publications, amongst which are Chang and Chen (2011); Iacus (2009, Sec 3.3.3); M. Kessler et al. (2012, Chapter 1); C. Li (2013); Singer (2006); Poulson (1999). The main characteristic of the Ait-Sahalia-method can be summarised as follows: a transformation of the (conditional) non-Gaussian density function into a Gaussian, with the aim of removing serious obstacles, discussed above, that are related to statistical inference from low-frequent, significantly non-normal data.

A transformed approximated conditional density function is pivotal to the Ait-Sahalia-method

$$\mathrm{P}_{\mathrm{Z}}^{\mathrm{J}}(\Delta, \mathrm{z}|\mathrm{y}_0; \boldsymbol{\theta}) = \mathcal{N}(0,1) \sum_{\mathrm{j}=0}^{\mathrm{J}} \eta_{\mathrm{Z}}^{\mathrm{j}} (\Delta, \mathrm{z}|\mathrm{y}_0; \boldsymbol{\theta}) \mathcal{H}_{\mathrm{j}}(\mathrm{z}) \qquad (5.18)$$

The first term of the RHS of Equation (5.18) is a standard normal distribution, the second term is a Hermitian series-expansion that encapsulates all *non-Gaussian characteristics* of the transformed density distribution. Equation (5.18) includes two important parameters that ensure a broad application of the method:

(i) parameter J represents how close the approximation is to the true transformed density function $\mathrm{P}_{\mathrm{Z}}(\Delta, \mathrm{z}|\mathrm{y}_0; \boldsymbol{\theta})$. Obviously, $\mathrm{P}_{\mathrm{Z}}(\Delta, \mathrm{z}|\mathrm{y}_0; \boldsymbol{\theta}) = \lim_{\mathrm{J}\to\infty} \mathrm{P}_{\mathrm{Z}}^{\mathrm{J}}(\Delta, \mathrm{z}|\mathrm{y}_0; \boldsymbol{\theta})$; however, empirical tests, for example Ait-Sahalia (1996, pp. 399-412), show that the expansion is strongly convergent. Very accurate approximations are feasible with an expansion of up to five or six terms;
(ii) parameter Δ which constitutes the time-interval over which discrete observations are measured. The closer Δ is to zero, the more time-continuous the process becomes, resulting in a transitional density distribution that, with increasing accuracy, resembles a normal distribution.

The two parameters are compensating levers: if Δ becomes too large to underpin an accurate approximation then parameter J can be increased to warrant the desired precision of the approximation results. In this study, a five-term approximation will be used, that is, $\mathrm{J} = 4$; nevertheless, if greater accuracy is required, then J can be increased but at the expense of much more complex and lengthier calculations. Since cash flows are discretely sampled over an equal quarterly time-interval, defined by $\Delta = 1$, t is set to $\mathrm{T_i} = \sum_1^{\mathrm{i}} \Delta_{\mathrm{i}}$ where $\mathrm{i} = 1 \mathrel{..} \mathrm{n}$. For low-frequency sampling, as is clearly the case in this study, significantly non-Gaussian conditional density functions are expected to be found.

More specifically, the method employed to estimate the parameters of the linear-quadratic conditional density function, comprises the following steps:

1. The general linear-quadratic cash flow SDE $dX_t = (\mu_1 X_t + \mu_0)dt + \sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)}\, dW_t$ is first turned into the following equation $dY_t = \tilde{\mu}(Y_t)dt + dW_t$, an SDE with a unit instantaneous change of variance. A Lamperti transformation (discussed in Sections 4-1 and 4-3) achieves this purpose. The transformation function is given by

$$Y_t = F(X_t) = \int \frac{d\xi}{\sqrt{|\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0|}}$$
$$= \frac{1}{\sqrt{\sigma_2}} \ln |2\sqrt{\sigma_2}\ \sqrt{|\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0|} + (\sigma_1 + 2\sigma_2 X_t)| \tag{5.19}$$

where $\sigma_2 > 0$. Notice that absolute values of the quadratic expression and the ln-form are taken to ensure that Y_t is defined on a complete X_t-spectrum.

The transformed drift function satisfies

$$\tilde{\mu}(Y_t) = \frac{\left((\mu_1 - \frac{1}{2}\sigma_2)F^{-1}(X_t) + (\mu_0 - \frac{1}{4}\sigma_1)\right)}{\sqrt{|\sigma_2[F^{-1}(X_t)]^2 + \sigma_1 F^{-1}(X_t) + \sigma_0|}} \tag{5.20}$$

Using the discriminant D of $\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0$, defined as $D = \sigma_1^2 - 4\sigma_2\sigma_0$, Equation (5.19) evaluates to the following more particular forms (refer to Jeffrey (2004, section 4.3.4))

$$Y_t = F(X_t) = \frac{1}{\sqrt{\sigma_2}} \sinh^{-1}\left[\frac{(\sigma_1 + 2\sigma_2 X_t)}{\sqrt{D}}\right] \tag{5.21a}$$

if $D > 0$ and $\sigma_2 > 0$, applicable to most operating cash flow processes,

$$Y_t = F(X_t) = \frac{1}{\sqrt{\sigma_2}} \ln |(\sigma_1 + 2\sigma_2 X_t)| \tag{5.21b}$$

if $D = 0$ and $\sigma_2 > 0$

$$Y_t = F(X_t) = \frac{-1}{\sqrt{-\sigma_2}} \sin^{-1}\left[\frac{(\sigma_1 + 2\sigma_2 X_t)}{\sqrt{-D}}\right] \tag{5.21c}$$

if $D < 0$ and $\sigma_2 < 0$, typical for almost all investing cash flow processes. Notice that since the function $\sin^{-1}(\,)$ is only defined on $-1 \leq \sin^{-1}(\,) \leq 1$, cash flows X_t must be in the range of $\frac{-1}{\sqrt{-\sigma_2}}\varsigma_1 \leq X_t \leq \frac{-1}{\sqrt{-\sigma_2}}\varsigma_2$ where $\varsigma_{1,2} = \frac{-\sigma_1 \pm \sqrt{-D}}{2\sigma_2}$.

2. A second transformation makes the process more suitable to a Hermite polynomial expansion by introducing the pseudo-normalised increment (Iacus (2009, p. 141)) as

$$\mathrm{Z_t} = \frac{\mathrm{Y_t} - \mathrm{Y_0}}{\sqrt{\Delta}} \tag{5.22}$$

where Δ is a discrete, constant time-interval over which observations are sampled and $\mathrm{Z_t}$ is a nearly normal process. The vector of parameters to be estimated is defined as $\boldsymbol{\theta}$: $[\mu_0, \mu_1, \sigma_2, \sigma_1, \sigma_0]$.

3. Next, density $\mathrm{P_Z}(\Delta, \mathrm{z}|\mathrm{y_0}; \boldsymbol{\theta})$ is approximated by Hermitian expansion

$$\mathrm{P_Z^J}(\Delta, \mathrm{z}|\mathrm{y_0}; \boldsymbol{\theta}) = \mathcal{N}(0,1) \textstyle\sum_{\mathrm{j}=0}^{\mathrm{J}} \eta_{\mathrm{Z}}^{\mathrm{j}} (\Delta, \mathrm{z}|\mathrm{y_0}; \boldsymbol{\theta}) \mathcal{H}_{\mathrm{j}}(\mathrm{z}) \tag{5.23}$$

where $\mathcal{N}(0,1)$ is a standard normal function, $\mathcal{H}_j(z)$ are Hermite polynomials defined as $(-1)^{\mathrm{n}} \mathrm{e}^{\frac{\mathrm{z}^2}{2}} \frac{\mathrm{d}^{\mathrm{n}} \mathrm{e}^{-\frac{\mathrm{z}^2}{2}}}{\mathrm{dz}^{\mathrm{n}}}$ for $\mathrm{n} = 0..\mathrm{J}$, parameter J denotes the number of terms included in the expansion, Fourier coefficients $\eta_{\mathrm{Z}}^{\mathrm{j}}(\Delta, \mathrm{z}|\mathrm{y_0}, \boldsymbol{\theta})$ are derived from the orthonormal property of Hermite polynomials

$$\begin{aligned}\eta_{\mathrm{Z}}^{\mathrm{j}}(\Delta, \mathrm{z}|\mathrm{y_0}; \boldsymbol{\theta}) &= \frac{1}{\mathrm{j}!} \int_{-\infty}^{\infty} \mathcal{H}_{\mathrm{j}}(\mathrm{z}) \mathrm{P_Z^J}(\Delta, \mathrm{z}|\mathrm{y_0}, \boldsymbol{\theta}) \mathrm{dz} = \frac{1}{\mathrm{j}!} \int_{-\infty}^{\infty} \mathcal{H}_{\mathrm{j}} \left(\frac{\mathrm{Y_t} - \mathrm{Y_0}}{\sqrt{\Delta}} \right) \mathrm{P_Z^J}(\Delta, \mathrm{z}|\mathrm{y_0}, \boldsymbol{\theta}) \mathrm{dz} \\ &= \frac{1}{\mathrm{j}!} \mathbb{E}_{\boldsymbol{\theta}} \left[\mathcal{H}_{\mathrm{j}} \left(\frac{\mathrm{Y_t} - \mathrm{Y_0}}{\sqrt{\Delta}} \right) |\mathrm{Y_t} = \mathrm{y_0} \right] \end{aligned} \tag{5.24}$$

Note that the integral evaluates to a conditional first moment. The Hermite polynomials are found relatively easy, as opposed to the Fourier coefficients. The latter are the first moments of the terms of the Hermitian expansion. In his landmark paper (Ait-Sahalia (1996, Appendix)), Ait-Sahalia replaces the first conditional moment by a Taylor expansion of (a recursively applied) infinitesimal generator of process $\mathrm{Y_t}$. He then proves that the Fourier coefficients can be described in closed-form as a polynomial expression of the transformed drift coefficient $\tilde{\mu}(\mathrm{Y_t})$ and its derivatives (in increasing order)[48]. However, I will adopt a slightly different approach. Following Singer (2006) and Jeisman

[48] The main reason not to follow Ait-Sahalia's method to calculate Fourier coefficients, is the complexity of the derivatives of $\tilde{\mu}(\mathrm{Y_t})$ for the linear-quadratic model.

(2006, Chapter 4), the Fourier coefficients are derived from a higher-moment approximation.

4. To compute the Fourier coefficients $\eta_Z^j(\Delta, z|y_0; \boldsymbol{\theta})$, a higher-moment expansion is used including terms $\mathbb{E}\left[\mathcal{H}_j(z)\right]$ where $j > 3$. It turns out that the first and second moment vanish from the calculation. From a closed system of ODEs that represent the evolution of the respective higher moments, these higher moments are then solved recursively and in an approximated sense. Solutions take the form $\mathbb{E}_n(t; \boldsymbol{\theta})$.

5. The approximated density function $P_Z^J(\Delta, z|y_0; \boldsymbol{\theta})$ is first transformed back to $P_Y^J(\Delta, y|y_0, \boldsymbol{\theta}) = \sqrt{\Delta}\, P_Z^J\left(\Delta, \frac{Y_t - Y_0}{\sqrt{\Delta}}\middle|y_0; \boldsymbol{\theta}\right)$, and in a final retro-transformation to:

$$P_X^J(\Delta, x|x_0; \boldsymbol{\theta}) = \frac{1}{\sqrt{|\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0|}} P_Y^J(\Delta, F(X_t)|F(X_0); \boldsymbol{\theta}) \tag{5.25}$$

where $Y_t = F(X_t) = \int \frac{d\xi}{\sqrt{|\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0|}} = \frac{1}{\sqrt{\sigma_2}} \ln|2\sqrt{\sigma_2}\ \sqrt{|\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0|} + (\sigma_1 + 2\sigma_2 X_t)|$.

6. Since cash flows are discretely sampled over an equal time-interval $\Delta = 1$, continuous-time t is set to $T_i = \sum_1^i \Delta_i$ where $i = 1\,..\,n$. Thus, in the discrete version of Equation (5.25), variable t is replaced by variable T_i. Given the set of discrete observations $\{X_0, X_1, X_2 \,....\ X_T\}$, the approximated density function, conditional on the first observation X_0, becomes

$$P_X^J\left(\Delta, x_{T_i}\middle|x_0; \boldsymbol{\theta}\right) = \frac{1}{\sqrt{|\sigma_2 x_{T_i}^2 + \sigma_1 x_{T_i} + \sigma_0|}} P_Y^J\left(\Delta\,, F(x_{T_i})\middle|F(x_0); \boldsymbol{\theta}\right) \tag{5.26}$$

7. The conditional probability density function $\widetilde{P}_X^J(\Delta, x_t|x_{t-1}; \boldsymbol{\theta})$ is set equal to $P_X^J\left(\Delta = 1, x_{T_i}\middle|x_0; \boldsymbol{\theta}\right)$ which arises from the Chapman-Kolmogorov identity applied to a homogeneous Markovian setting. Following Aït-Sahalia (2002a, p. 233), the result is obtained by ignoring the marginal initial density $P_X^J(\Delta = 1, x_0; \boldsymbol{\theta})$. In deriving the likelihood estimator, the initial marginal density is dominated by the series of transition densities (alternatively formulated: one observation compared to n observations). Hence,

$$\widetilde{P}_X^J(\Delta, X_t|X_{t-1};\boldsymbol{\theta}) = P_X^J\left(\Delta = 1, x_{T_i}\middle|x_0;\boldsymbol{\theta}\right) = \frac{1}{\sqrt{|\sigma_2 x_{T_i}^2 + \sigma_1 x_{T_i} + \sigma_0|}} P_Y^J\left(\Delta = 1, F(x_{T_i})\middle|F(x_0);\boldsymbol{\theta}\right) \quad (5.27)$$

8. The conditional density function $\widetilde{P}_X^J$ is used in the derivation of the Approximated Maximum Likelihood Estimator (AMLE). Since the process is Markovian, the likelihood estimator is $\mathcal{L}(\boldsymbol{\theta}) = \prod_{i=1}^{n} \widetilde{P}_X^J\left(\Delta = 1, x_{T_i}\middle|x_{T_i-1};\boldsymbol{\theta}\right)$, and the associated log-likelihood estimator is $\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log \widetilde{P}_X^J\left(\Delta = 1, x_{T_i}\middle|x_{T_i-1};\boldsymbol{\theta}\right)$. Commonly, estimates are calculated by setting the first derivative of ℓ to zero under the condition that there exists a single critical vector $\boldsymbol{\theta}_{\max}$ that is indeed a global maximum[49].

In Appendix M4-A it is shown that steps 1 - 4 lead to a z-transformed approximated transitional density function P_Z^J for $J = 4$, satisfying

$$P_Z^4(\Delta, z|y_0;\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \left[1 - \frac{1}{8}(z^4 - 6z^2 + 3) + \frac{1}{6}(z^3 - 3z)\mathbb{E}(z^3,\boldsymbol{\theta}) + \frac{1}{24}(z^4 - 6z^2 + 3)\mathbb{E}(z^4,\boldsymbol{\theta})\right] \quad (5.28)$$

Equation (5.28) consists of a leading Gaussian term and a fourth-degree polynomial term including a third and a fourth moment of the transformed random variable z. If the transition density is non-Gaussian, the last term introduces excess skewness and kurtosis to the normal z-distribution. For a true Gaussian random variable $P_Z^4(\Delta, z|y_0,\boldsymbol{\theta})$, skewness ($\mathbb{E}(z^3)$) and kurtosis ($\mathbb{E}(z^4) - 3$) are zero and the last term takes the value one, reducing $P_Z^4(\Delta, z|y_0;\boldsymbol{\theta})$ to $\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ (Singer (2006, p. 388).

In step 5, the first retro-transformation yields the transitional density function in random variable $\check{y} = y_t - y_0$

$$P_Y^4(\Delta, y|y_0;\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\check{y}^2\Delta}{2}} \Delta^{\frac{1}{2}} \left[1 - \frac{1}{8}(\Delta^2\check{y}^4 - 6\Delta\check{y}^2 + 3) + \frac{1}{6}(\Delta^{\frac{3}{2}}\check{y}^3 - 3\Delta^{\frac{1}{2}}\check{y})\Delta^{\frac{3}{2}}\mathbb{E}(\check{y}^3;\boldsymbol{\theta}) + \frac{1}{24}(\Delta^2\check{y}^4 - 6\Delta\check{y}^2 + 3)\Delta^2\mathbb{E}(\check{y}^4;\boldsymbol{\theta})\right] \quad (5.29a)$$

where $\check{y} = y - y_0$.

[49] Or, if applicable, a local maximum under some parameter restrictions.

Setting Δ equal to one, that is, taking a time-interval of one quarter in a year, gives a more workable version of Equation (5.29a)

$$\mathrm{P}_{\mathrm{Y}}^{4}(\Delta,\mathrm{y}|\mathrm{y}_0;\boldsymbol{\theta})=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\check{\mathrm{y}}^2}{2}}\Big[1-\frac{1}{8}(\check{\mathrm{y}}^4-6\check{\mathrm{y}}^2+3)+\frac{1}{6}(\check{\mathrm{y}}^3-3\check{\mathrm{y}}\,)\mathbb{E}(\check{\mathrm{y}}^3;\boldsymbol{\theta})+\frac{1}{24}(\check{\mathrm{y}}^4-6\check{\mathrm{y}}^2+3)\mathbb{E}(\check{\mathrm{y}}^4;\boldsymbol{\theta})\,\Big] \tag{5.29b}$$

Importantly, under the condition that $\tilde{\mu}(\mathrm{Y}_t)$ is approximated by a *linear Taylor expansion* around y_0 (a condition not imposed by Jeisman (2006, Chapter 4) in an almost identical approximation method), additional to a few other mild approximations, it is shown in Appendix M4-B that $\mathbb{E}(\check{\mathrm{y}}^3;\boldsymbol{\theta})$ and $\mathbb{E}(\check{\mathrm{y}}^4;\boldsymbol{\theta})$ can be solved from a closed system of ODEs representing the evolution of the first four central moments. Indicative calculations shown in Appendix M4-B, substantiate the claim that a linear Taylor expansion serves as a sufficiently accurate approximation of the Lamperti transformed drift function $\tilde{\mu}(\mathrm{Y}_t)$ for expected parameter values.

Note that higher moments $\mathbb{E}(\check{\mathrm{y}}^n;\boldsymbol{\theta})$, expressed in $\boldsymbol{\theta}$ only, are calculated from the stationary probability distribution of $\check{\mathrm{y}}$. Note that the condition for stationarity is $\pi_1(\boldsymbol{\theta})<0$ (Appendix M4-B). Consequently, the marginal probability density admits the following mixed normal-polynomial form

$$\mathrm{P}_{\mathrm{Y}}^{4}(\Delta=1,\mathrm{y}|\mathrm{y}_0;\boldsymbol{\theta}))=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\check{\mathrm{y}}^2}{2}}[1+\vartheta_0+\vartheta_1\check{\mathrm{y}}+\vartheta_2\,\check{\mathrm{y}}^2+\vartheta_3\,\check{\mathrm{y}}^3+\vartheta_4\,\check{\mathrm{y}}^4] \tag{5.30}$$

where $\vartheta_0=\left\{-\frac{3}{8}-\frac{1}{8}\alpha^4(\boldsymbol{\theta})-\frac{3}{16}\alpha^2(\boldsymbol{\theta})\beta(\boldsymbol{\theta})-\frac{3}{32}\beta^2(\boldsymbol{\theta})\right\}$, $\vartheta_1=\left\{\frac{1}{2}\alpha^3(\boldsymbol{\theta})+\frac{3}{2}\alpha(\boldsymbol{\theta})\beta(\boldsymbol{\theta})\right\}$, $\vartheta_2=\left\{\frac{6}{8}+\frac{1}{4}\alpha^4(\boldsymbol{\theta})+\frac{3}{8}\alpha^2(\boldsymbol{\theta})\beta(\boldsymbol{\theta})+\frac{3}{16}\beta^2(\boldsymbol{\theta})\right\}$, $\vartheta_3=\left\{-\frac{1}{6}\alpha^3(\boldsymbol{\theta})-\frac{1}{2}\alpha(\boldsymbol{\theta})\beta(\boldsymbol{\theta})\right\}$ and $\vartheta_4=\left\{-\frac{1}{8}-\frac{1}{24}\alpha^4(\boldsymbol{\theta})-\frac{1}{16}\alpha^2(\boldsymbol{\theta})\beta(\boldsymbol{\theta})-\frac{1}{32}\beta^2(\boldsymbol{\theta})\right\}$, $\alpha(\boldsymbol{\theta})=\frac{\pi_0(\boldsymbol{\theta})}{\pi_1(\boldsymbol{\theta})}$ and $\beta(\boldsymbol{\theta})=\frac{1}{\pi_1(\boldsymbol{\theta})}$.

The remainder of steps 5 – 7 encompasses the second retro-transformation from $\mathrm{P}_{\mathrm{Y}}^{4}(\Delta=1,\mathrm{y}|\mathrm{y}_0;\boldsymbol{\theta}))$ to $\mathrm{P}_{\mathrm{X}}^{4}(\Delta=1,\mathrm{x}|\mathrm{y}_0;\boldsymbol{\theta})$ to be followed by expressing $\mathrm{P}_{\mathrm{X}}^{4}(\Delta=1,\mathrm{x}|\mathrm{y}_0;\boldsymbol{\theta})$ in a conditional transition density format $\widetilde{\mathrm{P}}_{\mathrm{X}}^{\mathrm{J}}\big(\Delta=1,\mathrm{x}_{\mathrm{T}_i}\big|\mathrm{x}_{\mathrm{T}_{i-1}};\boldsymbol{\theta}\big)$, and changing the time-basis of the probability density function from continuous to discrete. The result of these steps obeys

$$\widetilde{P}_X^J(\Delta = 1, x_t|x_{t-1}; \boldsymbol{\theta}) = \frac{1}{\sqrt{|\sigma_2 x_{T_i}^2 + \sigma_1 x_{T_i} + \sigma_0|}} P_Y^J\left(\Delta = 1, F(x_{T_i})\middle|F(x_0); \boldsymbol{\theta}\right) =$$

$$\frac{1}{\sqrt{|\sigma_2 x_{T_i}^2 + \sigma_1 x_{T_i} + \sigma_0|}} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left\{DX_{T_i}\left(x_{T_i};\boldsymbol{\theta}\right) - DX_0(x_0;\boldsymbol{\theta})\right\}^2}{2\sigma_2}}$$

$$\Big[1 + \vartheta_0(\boldsymbol{\theta}) + \vartheta_1(\boldsymbol{\theta})\left\{DX_{T_i}\left(x_{T_i}; \boldsymbol{\theta}\right) - DX_0(x_0; \boldsymbol{\theta})\right\} - 2\vartheta_0(\boldsymbol{\theta})\left\{DX_{T_i}\left(x_{T_i}; \boldsymbol{\theta}\right) - DX_0(x_0; \boldsymbol{\theta})\right\}^2 - \frac{1}{3}\vartheta_1(\boldsymbol{\theta})\left\{DX_{T_i}\left(x_{T_i}; \boldsymbol{\theta}\right) - DX_0(x_0; \boldsymbol{\theta})\right\}^3 + \frac{1}{3}\vartheta_0(\boldsymbol{\theta})\left\{DX_{T_i}\left(x_{T_i}; \boldsymbol{\theta}\right) - DX_0(x_0; \boldsymbol{\theta})\right\}^4\Big] \tag{5.31}$$

where $DX_{T_i}(x_{T_i}; \boldsymbol{\theta}) = F\left(x_{T_i}; \boldsymbol{\theta}\right)$, $DX_0(x_0; \boldsymbol{\theta}) = DX_{T_0}\left(x_{T_0}; \boldsymbol{\theta}\right) = F(x_0; \boldsymbol{\theta})$, vector $\boldsymbol{\vartheta}(\boldsymbol{\theta})$: $\left[\left\{-\frac{3}{8} - \frac{1}{8}\alpha^4(\boldsymbol{\theta}) - \frac{3}{16}\alpha^2(\boldsymbol{\theta})\beta(\boldsymbol{\theta}) - \frac{3}{32}\beta^2(\boldsymbol{\theta})\right\}, \left\{\frac{1}{2}\alpha^3(\boldsymbol{\theta}) + \frac{3}{2}\alpha(\boldsymbol{\theta})\beta(\boldsymbol{\theta})\right\}\right]$, $\alpha(\boldsymbol{\theta}) = \frac{\pi_0(\boldsymbol{\theta})}{\pi_1(\boldsymbol{\theta})}$ and $\beta(\boldsymbol{\theta}) = \frac{1}{\pi_1(\boldsymbol{\theta})}$.

Note that the substitutions $\vartheta_2 = -2\vartheta_0$, $\vartheta_3 = -\frac{1}{3}\vartheta_1$ and $\vartheta_4 = \frac{1}{3}\vartheta_0$ have been made (see Appendix M4-B).

Function $F\left(x_{T_i}; \boldsymbol{\theta}\right)$ evaluates to:

(i) $\frac{1}{\sqrt{\sigma_2}}\sinh^{-1}\left[\frac{(\sigma_1 + 2\sigma_2 x_0)}{\sqrt{D}}\right]$ if $\sigma_2 > 0$ and the discriminant D of $\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0$ is greater than zero (*typical for operating cash flows*);

(ii) $\frac{-1}{\sqrt{-\sigma_2}}\sin^{-1}\left[\frac{(\sigma_1 + 2\sigma_2 x_0)}{\sqrt{-D}}\right]$ if $\sigma_2 < 0$ and the discriminant D of $\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0$ is smaller than zero (*typical for investing cash flows*).

From here on the arguments x_{T_i-1}, T_i and $\boldsymbol{\theta}$ are dropped from functions.

For the approximated conditional density function, parameters $\pi_0(\boldsymbol{\theta}; y_0)$ and $\pi_1(\boldsymbol{\theta}; y_0)$ will need to be expanded around x_0 instead of y_0[50]

$$\pi_0(\boldsymbol{\theta}; x_0) = \frac{(\mu_1 - \frac{1}{2}\sigma_2)x_0 + (\mu_0 - \frac{1}{4}\sigma_1)}{|\sigma_2 x_0^2 + \sigma_1 x_0 + \sigma_0|^{\frac{1}{2}}} \tag{5.32a}$$

[50] The function $\tilde{\mu}(X_t)$ is the conditional first central moment with x_0 as the transformed proxy for the average of random variable Y.

$$\pi_1(\boldsymbol{\theta}; x_0) = \frac{\left(\frac{1}{2}\mu_1\sigma_1 - \mu_0\sigma_2 - \frac{1}{4}\sigma_1\sigma_2 + \frac{1}{4}\sigma_2^2\right)x_0 - \frac{1}{2}\mu_0\sigma_1 + \mu_1\sigma_0 - \frac{1}{2}\sigma_0\sigma_2 + \frac{1}{8}\sigma_1\sigma_2}{|\sigma_2 x_0^2 + \sigma_1 x_0 + \sigma_0|^{\frac{3}{2}}} \tag{5.32b}$$

Variable x_0 is solved from

$$y_0 = F(x_0) = \frac{1}{\sqrt{\sigma_2}}\sinh^{-1}\left[\frac{(\sigma_1 + 2\sigma_2 x_0)}{\sqrt{D}}\right] \tag{5.33a}$$

for $D = \sigma_1^2 - 4\sigma_2\sigma_0 > 0$, or from

$$y_0 = F(x_0) = \frac{-1}{\sqrt{-\sigma_2}}\sin^{-1}\left[\frac{(\sigma_1 + 2\sigma_2 x_0)}{\sqrt{-D}}\right] \tag{5.33b}$$

for $D = \sigma_1^2 - 4\sigma_2\sigma_0 < 0$.

Solutions to Equation (5.33a) and (5.33b) are

$$x_0 = -\frac{\sigma_1}{2\sigma_2} + \frac{\sqrt{D}}{2\sigma_2}\sinh(\sqrt{\sigma_2}\, y_0) \tag{5.34a}$$

$$x_0 = -\frac{\sigma_1}{2\sigma_2} + \frac{\sqrt{-D}}{2\sigma_2}\sin(\sqrt{-\sigma_2}\, y_0) \tag{5.34b}$$

It is easy to see that if y_0 is set to zero, then both equations become

$$x_0 = -\frac{\sigma_1}{2\sigma_2} \tag{5.34c}$$

Substituting this result into Equations (5.32a) and (5.32b) gives

$$\pi_0(\boldsymbol{\theta}) = \frac{\mu_0 - \frac{\sigma_1}{2\sigma_2}\mu_1}{|\sigma_0 - \frac{\sigma_1^2}{4\sigma_2}|^{\frac{1}{2}}} = \frac{2\sqrt{\sigma_2}(\mu_0 - \frac{\sigma_1}{2\sigma_2}\mu_1)}{|-\frac{1}{2}D|^{\frac{1}{2}}} = \frac{2\sqrt{\sigma_2}\mu_0 - \frac{\sigma_1}{\sqrt{\sigma_2}}\mu_1}{|\frac{1}{2}D|^{\frac{1}{2}}} \tag{5.35a}$$

$$\pi_1(\boldsymbol{\theta}) = \frac{(4\sigma_2)^{\frac{3}{2}}(\mu_0\sigma_0 - \frac{\sigma_1^2}{4\sigma_2}\mu_1 + \frac{1}{8}D)}{|\frac{1}{2}D|^{\frac{3}{2}}} \tag{5.35b}$$

Conditional transitional density function

The conditional transitional density function governed by Equation (5.31), can be split in three different components (Chang and Chen (2011, pp. 2824-2825))

(i) Component $A_1 = \frac{1}{\sqrt{|\sigma_2 x_{T_i}^2 + \sigma_1 x_{T_i} + \sigma_0|}}$ which represents a non-linear scaling factor related to the Lamperti transform between x and $\check{y}$, vice versa;

(ii) Component $A_2 = \frac{1}{\sqrt{2\pi}} e^{-\frac{\{DX_{T_i} - DX_0\}^2}{2\sigma_2}} = \frac{1}{\sqrt{2\pi}} \frac{\sigma_2}{\sigma_2} e^{-\frac{\{DX_{T_i} - DX_0\}^2}{2\sigma_2}} = \sigma_2 \, \mathcal{N}(DX_0, \sigma_2)$, constitutes σ_2 times a normal distribution of DX_{T_i} with mean DX_0 and variance $\sigma_2 > 0$; and

(iii) Component $A_3 = 1 + \vartheta_0 + \vartheta_1(DX_{T_i} - DX_0) - 2\vartheta_0(DX_{T_i} - DX_0)^2 - \frac{1}{3}\vartheta_1 \, (DX_{T_i} - DX_0)^3 + \frac{1}{3}\vartheta_0(DX_{T_i} - DX_0)^4$, which adjusts the transition density for non-Gaussian characteristics of DX_{T_i} (that is, excess skewness and kurtosis if $J = 4$) relative to the same adjustment for DX_0.

Each part has its own significance in the complete transition density function and is discussed in more detail below.

Component $A_1 = \frac{1}{\sqrt{|\sigma_2 x_{T_i}^2 + \sigma_1 x_{T_i} + \sigma_0|}}$ has singularities if the roots of $\sigma_2 x_{T_i}^2 + \sigma_1 x_{T_i} + \sigma_0$ are real, common for operating cash flows (see Figure 3.5). Singularities may cause issues with maximum likelihood algorithms as they are recognised by statistical estimation routines as discontinuities. Around these singularities, function A_1 is highly nonlinear. If $x_{T_i} = x_{T_0} = x_0$ (that is, at time T_0 of the first observation), then A_1 evaluates to $|\frac{1}{2}D|^{-\frac{1}{2}}$ if $D > 0$, or to $|-\frac{1}{2}D|^{-\frac{1}{2}}$ if $D < 0$.

For operating cash flows, component $A_2 = \sigma_2 \, \mathcal{N}(DX_0, \sigma_2)$ yields

$$\sigma_2 \frac{1}{\sqrt{2\pi\sigma_2}} e^{-\frac{\left\{\sinh^{-1}\left[\frac{\sigma_1 + 2\sigma_2 x_{T_i}}{\sqrt{D}}\right] - \sinh^{-1}\left[\frac{\sigma_1 + 2\sigma_2 x_0}{\sqrt{D}}\right]\right\}^2}{2\sigma_2}} = \sigma_2 \frac{1}{\sqrt{2\pi\sigma_2}} e^{-\frac{\left\{\sinh^{-1}\left[\frac{\sigma_1 + 2\sigma_2 x_{T_i}}{\sqrt{D}}\right]\right\}^2}{2\sigma_2}} \quad (5.36)$$

since $\sinh^{-1}\left[\frac{\sigma_1 + 2\sigma_2 x_0}{\sqrt{D}}\right] = 0$. Observe that if $x_{T_i} = -\frac{\sigma_1}{2\sigma_2}$, then A_2 equals a constant $\frac{1}{\sqrt{2\pi}}$.

For investing cash flows, component A_2 produces

$$\sigma_2 \frac{1}{\sqrt{2\pi\sigma_2}} e^{-\frac{\left\{\sin^{-1}\left[\frac{\sigma_1 + 2\sigma_2 x_{T_i}}{\sqrt{-D}}\right] - \sin^{-1}\left[\frac{\sigma_1 + 2\sigma_2 x_0}{\sqrt{-D}}\right]\right\}^2}{2\sigma_2}} = \sigma_2 \frac{1}{\sqrt{2\pi\sigma_2}} e^{-\frac{\left\{\sin^{-1}\left[\frac{\sigma_1 + 2\sigma_2 x_{T_i}}{\sqrt{-D}}\right]\right\}^2}{2\sigma_2}} \quad (5.37)$$

Note that cash flows x_{T_i} are defined on a closed range $\varsigma_1 \leq x_{T_i} \leq \varsigma_2$, $\varsigma_{1,2} = \frac{-\sigma_1 \pm \sqrt{-D}}{2\sigma_2}$. If $x_{T_i} = -\frac{\sigma_1}{2\sigma_2}$ then $A_2 = \frac{1}{\sqrt{2\pi}}$.

For the value $DX_0 = 0$, component A_3 is simplified to

$$A_3 = 1 + \vartheta_0 + \vartheta_1 DX_{T_i} - 2\vartheta_0 DX_{T_i}^2 - \frac{1}{3}\vartheta_1\, DX_{T_i}^3 + \frac{1}{3}\vartheta_0 DX_{T_i}^4 \tag{5.38a}$$

where

(i) in case of operating cash flows

$$DX_{T_i} = F(x_{T_i}) = \frac{1}{\sqrt{\sigma_2}} \sinh^{-1}\left[\frac{\sigma_1 + 2\sigma_2 x_{T_i}}{\sqrt{D}}\right] \tag{5.38b}$$

or

(ii) in case of investing cash flows

$$DX_{T_i} = F(x_{T_i}) = \frac{-1}{\sqrt{-\sigma_2}} \sin^{-1}\left[\frac{\sigma_1 + 2\sigma_2 x_{T_i}}{\sqrt{-D}}\right] \tag{5.38c}$$

for cash flows x_t defined on a closed range $\varsigma_1 \le x_t \le \varsigma_2$, $\varsigma_{1,2} = \frac{-\sigma_1 \pm \sqrt{-D}}{2\sigma_2}$, and

$\vartheta_0 = \left\{-\frac{3}{8} - \frac{1}{8}\alpha^4(\boldsymbol{\theta}) - \frac{3}{16}\alpha^2(\boldsymbol{\theta})\beta(\boldsymbol{\theta}) - \frac{3}{32}\beta^2(\boldsymbol{\theta})\right\}$, $\vartheta_1 = \left\{\frac{1}{2}\alpha^3(\boldsymbol{\theta}) + \frac{3}{2}\alpha(\boldsymbol{\theta})\beta(\boldsymbol{\theta})\right\}$, $\alpha(\boldsymbol{\theta}) = \frac{\pi_0(\boldsymbol{\theta})}{\pi_1(\boldsymbol{\theta})}$, $\beta(\boldsymbol{\theta}) = \frac{1}{\pi_1(\boldsymbol{\theta})}$, $\pi_0(\boldsymbol{\theta}) = \frac{2\sqrt{\sigma_2}\mu_0 - \frac{\sigma_1}{\sqrt{\sigma_2}}\mu_1}{|\frac{1}{2}D|^{\frac{1}{2}}}$, $\pi_1(\boldsymbol{\theta}) = \frac{(4\sigma_2)^{\frac{3}{2}}(\mu_0\sigma_0 - \frac{\sigma_1^2}{4\sigma_2}\mu_1 + \frac{1}{8}D)}{|\frac{1}{2}D|^{\frac{3}{2}}}$.

Approximated likelihood function

The log-likelihood equality in Equation (5.31), is commonly solved by maximising the likelihood function $\widetilde{P}_X^J(\Delta = 1, x_{T_i}|x_{T_i-1}; \boldsymbol{\theta})$, employing specialised software that includes a suite of appropriate numerical optimisation algorithms, for example the ML-routine of Stata-15 (Gould et al. (2010)). The AMLE is

$$\ell_i(\boldsymbol{\theta}) = -\frac{1}{2}\ln\left[|\sigma_2 x_{T_i}^2 + \sigma_1 x_{T_i} + \sigma_0|\right] - \frac{1}{2}DX_{T_i}^2 + \ln\Big[|1 + \vartheta_0 + \vartheta_1 DX_{T_i} - 2\vartheta_0 DX_{T_i}^2 - \frac{1}{3}\vartheta_1\, DX_{T_i}^3 + \frac{1}{3}\vartheta_0 DX_{T_i}^4|\Big] \tag{5.39}$$

where $DX_{T_i} = \frac{1}{\sqrt{\sigma_2}}\sinh^{-1}\left[\frac{\sigma_1 + 2\sigma_2 x_{T_i}}{\sqrt{D}}\right]$ if $D > 0$ and $\sigma_2 > 0$, or $DX_{T_i} = \frac{-1}{\sqrt{-\sigma_2}}\sin^{-1}\left[\frac{\sigma_1 + 2\sigma_2 x_{T_i}}{\sqrt{-D}}\right]$ if $D < 0$ and $\sigma_2 < 0$.

Usually, maximum likelihood estimation works well for stochastic processes defined by a known density function with up to three estimation parameters. More complex, and therefore typically more generic stochastic processes, invariably include a larger number of estimation parameters. In the literature, these models are often called *high-parametrised* since their number of parameters is large relative to the number of observations. Additional complexity can arise from significant non-linearities in the underlying model. Usually, deriving an analytical parameter solution for a three-parameter-plus model is unrealistic, but nonetheless, even numerical algorithms have difficulties finding multiple global maximum parameter values simultaneously. To overcome this problem, different approaches have been tried and tested, for example, model augmentation (Hirose (2000)), decomposing the estimation problem into appropriate computationally tractable sub problems (Song et al. (2005), Hautsch et al. (2014)) and improved algorithms for approximated maximum likelihood estimators (Bertl et al. (2015)).

The linear-quadratic cash flow process is a good example of a high-parametrised model since it encompasses five parameters: $\boldsymbol{\theta}$: $\{\hat{\mu}_0, \hat{\mu}_1, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$ in an environment of low-frequency data sampling. In fact, the true model (i.e. the one to be estimated from uncoupled (raw) data) has no less than twelve parameters (two times five for the coupled operating and investing cash flows and two weight parameters): see Section 4-4. In such situations, a one-step optimization is (computationally or numerically) not possible and parameters have to be estimated in multiple steps (Hautsch et al. (2014, p. 2)).

Evidently, the estimation problem under consideration must be simplified. Three suggestions to reduce its complexity, are made below:

1. Perform the estimation on *uncoupled* data instead of coupled data. Uncoupled data facilitate a separate estimation of the operating cash flow and the investing cash flow equations. The original twelve parameters are now reduced to two (isolated) sets of five parameters $\boldsymbol{\theta}_{\mathrm{OCF}}$ and $\boldsymbol{\theta}_{\mathrm{ICF}}$. The remaining problem is the calculation of coupled data. Coupled data points are multiplied by the weight matrix as explained in Section 3-3: $\mathbf{v}_t = \begin{pmatrix} C'_t \\ I'_t \end{pmatrix} = \mathbf{Q^{-1}}.\mathbf{u}_t$ where $\mathbf{u}_t$ is the vector of coupled data $\begin{pmatrix} C_t \\ I_t \end{pmatrix}$, $\mathbf{Q^{-1}}$ is the inverse

matrix of the eigenvectors of $\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & -1 \end{pmatrix}$ and $\begin{pmatrix} C_t' \\ I_t' \end{pmatrix}$ is the vector of uncoupled data points. In Section 5-1 values of $\mathbf{Q^{-1}}$ are estimated as $\widehat{\mathbf{Q}}^{\mathbf{-1}} = \begin{pmatrix} 0.550 & 0.141 \\ -0.550 & 0.859 \end{pmatrix}$. An important follow-up question is whether these calculated weights can be applied indiscriminately to (groups of) firms alike, despite being derived from a very large sample of firms? The answer is allegedly affirmative since decoupling is a purely deterministic step prior to introducing stochastic variability into the model (Section 3-4). Sampling form a very large data-set, ideally the whole population, comes closest to the abstract of a deterministic variant of a process since all individual firm variability is 'averaged-out'.

2. For each of five-parameter sets $\boldsymbol{\theta}_{\mathrm{OCF}}$ and $\boldsymbol{\theta}_{\mathrm{ICF}}$, find the initial parameter estimates of the drift function, $\hat{\mu}_0^0$ and $\hat{\mu}_1^0$, independently from those of the diffusion function. This approach is considered feasible for two reasons:
   a. Doob-Meyer's decomposition theorem, Doob (1990), underpins the separation of the cash flow process into a deterministic (predictable) component (drift function) and a continuous random component (diffusion function).
   b. it is known that drift and diffusion functions are linked to each other via the stationary density function: $p_{st}(X_t) = K \exp\left[\int \frac{\alpha(\xi)}{\beta(\xi)} d\xi\right]$ where $\alpha(X_t)$ is a drift function, $\beta(X_t)$ is a diffusion function, K is a normalisation constant, X_t is some cash flow process described by $dX_t = \alpha(X_t)dt + \sqrt{\beta(X_t)}\, dW_t$ and $p_{st}(X_t)$ is the stationary density. In other words, the diffusion function can be derived from the drift function, or vice versa, if the stochastic properties of the (stationary) cash flow process are known. See also the comments made following Equation (5.10b) of this Chapter.

   The advocated approach estimates the complete parameter set in three steps:

   (i) first fit the drift parameters via a non-linear least-square estimation algorithm using the specification $X_t = -\frac{\mu_0}{\mu_1} + (X_0 + \frac{\mu_0}{\mu_1})e^{\mu_1 T}$, where $\{\mu_0, \mu_1\}$ are drift parameters to be estimated and X_0 is the first observation of the cash flow time series X_t. The result of this step is a set of initial values for the drift parameters $\hat{\mu}_0^0$ and $\hat{\mu}_1^0$ that can be used in the second step;

(ii) ML estimation of the overall parameter set $\boldsymbol{\theta}$: $\{\hat{\mu}_0^0, \hat{\mu}_1^0, \sigma_0, \sigma_1, \sigma_2\}$ where the initial values of $\boldsymbol{\theta}_\alpha^{\mathbf{0}}$: $\{\hat{\mu}_0^0, \hat{\mu}_1^0\}$ are already calculated;

(iii) once the parameter set $\boldsymbol{\theta}$: $\{\hat{\mu}_0^0, \hat{\mu}_1^0, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$ is obtained, the full parameter set is re-estimated, this time however with all five parameters optimised. The outcome is a final estimate of the full parameter set $\boldsymbol{\theta}$: $\{\hat{\mu}_0, \hat{\mu}_1, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$.

3. Even after implementing the simplifications suggested under 1 and 2, the likelihood Equation (5.39) remains difficult to estimate. Further moderation can be brought about by:

   a. applying the invariance principle: $\sup_{\boldsymbol{\eta}} \ell_i^*(\boldsymbol{\eta}) = \sup_{\boldsymbol{\eta}} \ell_\mathrm{i}(\mathrm{G}^{-1}(\boldsymbol{\eta})) = \sup_{\boldsymbol{\theta}} \ell_\mathrm{i}(\boldsymbol{\theta})$ with maximum estimates attained at $\boldsymbol{\eta} = \mathrm{G}(\boldsymbol{\theta}) = \mathrm{G}(\hat{\boldsymbol{\theta}})$ where commonly G is restricted to one-to-one functions. However, Casella and Berger (2002, p. 320) demonstrate that the invariance principle[51] effectively holds for all functions (regardless of whether they are one-to-one)[52]. The extension of the classical invariance principle is important and shows that from multiple maximum values $\mathrm{G}(\boldsymbol{\theta})$**,** MLE ensures that the supreme value is selected.

   b. maximizing the likelihood function in parts. This technique is described in Song et al. (2005) and considers a decomposition of the overall likelihood function in $\ell_\mathrm{i}(\boldsymbol{\theta}) = \ell_{\mathrm{i},1}(\boldsymbol{\theta}) + \ell_{\mathrm{i},2}(\boldsymbol{\theta})$. Likelihood function $\ell_{\mathrm{i},1}(\boldsymbol{\theta})$ is chosen such that the corresponding score function $\frac{\partial \ell_{\mathrm{i},1}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$ is easy to compute. From $\frac{\partial \ell_{\mathrm{i},1}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = 0$, the parameter set $\boldsymbol{\theta}^1$ is obtained. As expected, the efficiency of $\boldsymbol{\theta}^1$ is low because only part of the information contained in the full likelihood function is utilised. Therefore, In the next (iterative) step, the equality $\frac{\partial \ell_{\mathrm{i},1}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -\frac{\partial \ell_{\mathrm{i},2}(\boldsymbol{\theta}^1)}{\partial \boldsymbol{\theta}}$ is used to produce a set of more

[51] The authors call ℓ_i^* the induced likelihood function.

[52] The proof is the following. Observe that

(i) $\sup_{\boldsymbol{\eta}} \ell_i^*(\hat{\boldsymbol{\eta}}) = \sup_{\boldsymbol{\eta}} \sup_{\{\boldsymbol{\theta}:\, \mathrm{G}(\boldsymbol{\theta})=\boldsymbol{\eta}\}} \ell_\mathrm{i}(\boldsymbol{\eta}) = \sup_{\boldsymbol{\theta}} \ell_\mathrm{i}(\boldsymbol{\theta}) = \ell_\mathrm{i}(\hat{\boldsymbol{\theta}})$, and

(ii) (ii) $\ell_\mathrm{i}(\hat{\boldsymbol{\theta}}) = \sup_{\{\boldsymbol{\theta}:\, \mathrm{G}(\boldsymbol{\theta})=\mathrm{G}(\hat{\boldsymbol{\theta}})\}} \ell_\mathrm{i}(\boldsymbol{\theta}) = \ell_i^*\left(\mathrm{G}(\hat{\boldsymbol{\theta}})\right)$, so that the following equality holds:

(iii) $\ell_i^*(\hat{\boldsymbol{\eta}}) = \ell_i^*\left(\mathrm{G}(\hat{\boldsymbol{\theta}})\right)$.

to conclude: $\mathrm{G}(\hat{\boldsymbol{\theta}})$ is the MLE of $\mathrm{G}(\boldsymbol{\theta})$.

efficient estimates $\boldsymbol{\theta}^2$. This step is repeated until there is no marked efficiency improvement with $\boldsymbol{\theta}^{\mathrm{k}}$ (k = 2, 3, 4 …) denoting the final estimation value vector.

Employing some of the above described methods and techniques to the likelihood function under consideration, leads to the following proposal

1. The first term of Equation (5.39), $-\frac{1}{2}\ln\left[\left|\sigma_2 x_{T_i}^2+\sigma_1 x_{T_i}+\sigma_0\right|\right]$, is initially factorised into $(x_{T_i}-\lambda_1)(x_{T_i}-\lambda_2)$ where λ_1,λ_2 are its (real or complex) roots. Then, λ_1 is re-parametrised to $\theta_1-\theta_2$, and λ_2 to $\theta_1+\theta_2$, where $\theta_1=-\frac{\sigma_1}{2\sigma_2}$ and $\theta_2=\frac{\sqrt{|D|}}{2\sigma_2}$.

2. Define a third new parameter θ_3 as $\theta_3=\sqrt{|\sigma_2|}$. Notice that parameter mapping from $\{\sigma_0,\sigma_1,\sigma_2\}$ to $\{\theta_1,\theta_2,\theta_3\}$ covers the cases where (i) $\sigma_2>0$ and $D>0$ (operating cash flows) and (ii) $\sigma_2<0$ and $D<0$ (investing cash flows).

3. A transformation of the cash flow variable x_{T_i} is suggested such that $v_{T_i}=T(x_{T_i})=\frac{1}{\theta_3}\sinh^{-1}\left[\frac{x_{T_i}-\theta_1}{\theta_2}\right]$. This transformation has two advantages

   (i) the first term $-\frac{1}{2}\ln\left[\left|\sigma_2 x_{T_i}^2+\sigma_1 x_{T_i}+\sigma_0\right|\right]=-\frac{1}{2}\ln\left[(x_{T_i}-\lambda_1)(x_{T_i}-\lambda_2)\,|\right]$ is turned into this format: $-\frac{1}{2}\ln\left[|(x_{T_i}+\theta_1+\theta_2)|\right]-\frac{1}{2}\ln\left[(x_{T_i}+\theta_1-\theta_2)|\right]$;

   (ii) since $v_{T_i}=\frac{1}{\theta_3}\sinh^{-1}\left[\frac{x_{T_i}-\theta_1}{\theta_2}\right]=DX_{T_i}$, the second term $\frac{1}{2}DX_{T_i}^2$ vanishes from the likelihood function (being constant) and the third term becomes more manageable in terms of complexity.

4. Underlying transformation calculations are shown in Appendix M5. Solutions (if they exist) follow from $\frac{\partial\ell(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}=0$ where $\boldsymbol{\theta}$: $\{\hat{\mu}_0,\hat{\mu}_1,\sigma_0,\sigma_1,\sigma_2\}$ includes prior estimates of the drift function. The resulting system of derivatives must be numerically solved.

5. If calculating derivatives of the third term of Equation (5.39) (arguably the most complex term of the whole likelihood equation), turns-out to be too demanding, then the method described in Song et al. (2005) can be applied (see above).

6. Alternatively, if required the third term of Equation (5.39), after a variable transformation into a fourth-degree polynomial in v_{T_i}, could be factorised to

$$\ln\left[\left|1+\vartheta_0+\vartheta_1 v_{T_i}-2\vartheta_0 v_{T_i}^2-\frac{1}{3}\vartheta_1 v_{T_i}^3+\frac{1}{3}\vartheta_0 v_{T_i}^4\right|\right]=\ln\left[(v_{T_i}+\Lambda_1)(v_{T_i}+\Lambda_2)(v_{T_i}+\Lambda_3)(v_{T_i}+\Lambda_4)\right] \tag{5.40}$$

where $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ are the (real or complex) roots of $|1+\vartheta_0+\vartheta_1 v_{T_i}-2\vartheta_0 v_{T_i}^2-\frac{1}{3}\vartheta_1 v_{T_i}^3+\frac{1}{3}\vartheta_0 v_{T_i}^4|$. Then, in a following step, Equation (5.40) is approximated by a three-term Taylor expansion

$$\ln\left[(v_{T_i}+\Lambda_1)(v_{T_i}+\Lambda_2)(v_{T_i}+\Lambda_3)(v_{T_i}+\Lambda_4)\right]= \ln\left[\Lambda_1(1+\frac{v_{T_i}}{\Lambda_1})\right]+\ln\left[\Lambda_2(1+\frac{v_{T_i}}{\Lambda_2})\right]+\ln\left[\Lambda_3(1+\frac{v_{T_i}}{\Lambda_{31}})\right]+\ln\left[\Lambda_{41}(1+\frac{v_{T_i}}{\Lambda_4})\right] \approx \ln(\Lambda_1\Lambda_2\Lambda_3\Lambda_4)+\frac{(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4)}{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}v_{T_i}+\frac{(\Lambda_1^2+\Lambda_2^2+\Lambda_3^2+\Lambda_4^2)}{(\Lambda_1\Lambda_2\Lambda_3\Lambda_4)^2}v_{T_i}^2 \tag{5.41}$$

Notice that the suggested Taylor approximation is only accurate in a limited range $-0.5<\frac{v_{T_i}}{\Lambda_m}<1$ where $m=1..4$, and therefore the approximation should be avoided if possible.

Applying Vieta's formula[53], the approximation resulting from Equation (5.41) is re-written to

$$\ln(\Lambda_1\Lambda_2\Lambda_3\Lambda_4)+\frac{(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4)}{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}v_{T_i}+\frac{(\Lambda_1^2+\Lambda_2^2+\Lambda_3^2+\Lambda_4^2)}{(\Lambda_1\Lambda_2\Lambda_3\Lambda_4)^2}\ v_{T_i}^2=\ \ln\left(3\frac{\vartheta_0}{(1+\vartheta_0)}\right)+\frac{\vartheta_1}{3(1+\vartheta_0)}\theta_1^3\ \theta_2^3 v_{T_i}+\ \left[\frac{\vartheta_1^2}{\vartheta_0^2}+12\right]\frac{1}{\theta_1^2\theta_2^2}v_{T_i}^2 \tag{5.42}$$

Implementing the suggested procedure in a standard statistical package, like Stata, proved too arduous. Nevertheless, Stata's NL-routine (nonlinear estimation) was employed in a first step to estimate the initial parameters $\boldsymbol{\theta}_\alpha^{\mathbf{0}}$: $\{\hat{\mu}_0^0\ ,\hat{\mu}_1^0\}$ of the drift function. Tailor-made code had to be written to calculate first and second derivatives (of parameter-points) for the respective likelihood functions. In a second step, optimisation of the diffusion functions was achieved by setting first derivatives of the diffusion parameters $\boldsymbol{\theta}_\beta$: $\{\hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$ to zero under the appropriate condition of the Hessian matrix that ensure a maximum value is

[53] For a polynomial function $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \ldots a_0$, the sum of all roots is $-\frac{a_{n-1}}{a_n}$, the product of all roots is $(-1)^n\frac{a_0}{a_n}$ and the sum of all roots squared is $\left[\frac{a_{n-1}}{a_n}\right]^2-2\ \frac{a_{n-2}}{a_n}$.

obtained[54]. An evolutionary optimisation technique was utilised including a genetic algorithm. Fortunately, none of the simplifications described under 5. and 6. were required, albeit that the resulting derivatives, especially the second order ones, are, as expected, very complex formulas.

Estimation results

The two aggregated likelihood functions, derived in Appendix M5, that were used to obtain parameter estimates, are

Operating cash flows

$$\ell(\boldsymbol{\theta}; v_{T_i}) = -\frac{1}{2}\sum_{i=1}^{i=n} \ln[|2\theta_1 + \theta_2 \sinh[\theta_3 v_{T_i}] + \theta_2)|] - \frac{1}{2}\sum_{i=1}^{i=n} \ln[|2\theta_1 + \theta_2(\sinh[\theta_3 v_{T_i}] - \theta_2)|] + \sum_{i=1}^{i=n} \ln\left[|1 + \vartheta_0 + \vartheta_1\theta_1 v_{T_i} - 2\vartheta_0 v_{T_i}^2 - \frac{1}{3}\vartheta_1 v_{T_i}^3 + \frac{1}{3}\vartheta_0 v_{T_i}^4|\right] \tag{5.43a}$$

Investing cash flows

$$\ell(\boldsymbol{\theta}; v_{T_i}) = -\frac{1}{2}\sum_{i=1}^{i=n} \ln[|2\theta_1 + \theta_2 \sin[\theta_3 v_{T_i}] + \theta_2)|] - \frac{1}{2}\sum_{i=1}^{i=n} \ln[|2\theta_1 + \theta_2(\sin[\theta_3 v_{T_i}] - \theta_2)|] + \sum_{i=1}^{i=n} \ln\left[|1 + \vartheta_0 + \vartheta_1\theta_1 v_{T_i} - 2\vartheta_0 v_{T_i}^2 - \frac{1}{3}\vartheta_1 v_{T_i}^3 + \frac{1}{3}\vartheta_0 v_{T_i}^4|\right] \tag{5.43b}$$

where $DX_{T_i} = \frac{1}{\theta_3}\sinh^{-1}\left[\frac{x_{T_i} - \theta_1}{\theta_2}\right]$ if $D > 0$, or $DX_{T_i} = -\frac{1}{\theta_3}\sin^{-1}\left[\frac{x_{T_i} - \theta_1}{\theta_2}\right]$ if $D < 0$, $\pi_0(\boldsymbol{\theta}) = \frac{2\sqrt{2}\theta_3\hat{\mu}_0 - \sqrt{2}\,\theta_1\hat{\mu}_1}{2\theta_2\theta_3^2}$, $\pi_1(\boldsymbol{\theta}) = \frac{\sqrt{2}(\theta_1 - \theta_2^2)\hat{\mu}_0 - \sqrt{2}\theta_1^2\hat{\mu}_1}{\theta_2^3\theta_3^2} + \frac{\sqrt{2}}{2\theta_2}$, $\vartheta_0 = \left\{-\frac{3}{8} - \frac{1}{8}\alpha^4(\boldsymbol{\theta}) - \frac{3}{16}\alpha^2(\boldsymbol{\theta})\beta(\boldsymbol{\theta}) - \frac{3}{32}\beta^2(\boldsymbol{\theta})\right\}$, $\vartheta_1 = \left\{\frac{1}{2}\alpha^3(\boldsymbol{\theta}) + \frac{3}{2}\alpha(\boldsymbol{\theta})\beta(\boldsymbol{\theta})\right\}$, $\alpha(\boldsymbol{\theta}) = \frac{\pi_0(\boldsymbol{\theta})}{\pi_1(\boldsymbol{\theta})}$, $\beta(\boldsymbol{\theta}) = \frac{1}{\pi_1(\boldsymbol{\theta})}$.

First, estimates were attained for the whole population of observed firms (in total 340,159 data points from 5,202 different firms). The three-step estimation procedure, described above, was consistently applied and the results are reported in Tables 5-15 (operating cash flows) and 5-16 (investing cash flows).

[54] This condition is that the eigenvalues of the Hessian matrix are negative definite, or equivalently, that the leading principal minors of a matrix alternate in sign.

Secondly, the linear-quadratic cash flow model was tested at an industry level, more specifically on 73 different industries according to the Global Industry Classification Standard (GICS). The results are shown in Appendix S6.

A final test involves a set of cash flows of randomly selected individual firms. These results are detailed in Appendix S7.

To calculate t-statistics and confidence intervals for the parameter-set $\{\widehat{\sigma}_0, \widehat{\sigma}_1, \widehat{\sigma}_2\}$ from $\{\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3\}$-statistics, it was assumed that each of the parameters $\{\theta_1, \theta_2, \theta_3\}$ follows a normal distribution as per CLT and that these parameters are (almost) perfectly, positively correlated. The tests yielded consistent estimates with in most cases strong convergence to expected asymptotic values and in some cases only partial convergence regardless of multiple (repeated) optimisation runs.

Discussion of ML estimation results

<u>All observed firms</u>

From Table 5-15 it transpires that all parameter estimates are significant ($\widehat{\theta}_1, \widehat{\sigma}_1$) to highly significant ($\widehat{\theta}_2, \widehat{\theta}_3, \widehat{\sigma}_0, \widehat{\sigma}_2$). Obviously, parameter $\hat{\mu}_0$ is least significant. Recall that the drift function is described by the equation $X_t = -\frac{\mu_0}{\mu_1} + (X_0 + \frac{\mu_0}{\mu_1})e^{\mu_1 T}$, where in this instance X_t represents operating cash flow; $\beta_0 = \frac{\mu_0}{\mu_1}$ and μ_1 are parameters to be estimated. Since the data were examined for the whole population of firms, a possible explanation is that parameters μ_0 of individual firms largely cancel each other out at an aggregate level. The results are supportive of the linear-quadratic cash flow model.

When comparing parameters in Table 5-11 with those of Table 5-15, it should be noted that the two underlying estimation methods are dissimilar. The parameter estimates of Table 5-11 are based on *the expected values* of a solution to two equivalent SDEs, ignoring probability characteristics other than expected values. By contrast, the likelihood method takes a richer ambit of probability information into account: in this case the first four moments are included in the approximated likelihood function.

Comparing the set of estimated parameters $\widehat{\boldsymbol{\theta}}$: $\{\hat{\mu}_0, \hat{\mu}_1, \widehat{\sigma}_0, \widehat{\sigma}_1, \widehat{\sigma}_2\}$ in Table 5-15 with the ones in Table 5-11, it can be observed that the values are different, with some parameter pairs agreeing more than others. The largest variance is found for parameter $\widehat{\sigma}_0$. A relative elevated $\widehat{\sigma}_0$-value can be interpreted as follows: for smaller operating cash flows, constant $\widehat{\sigma}_0$ is the dominant term in the diffusion equation, and hence the diffusion process approximates an ABM-process with a Gaussian conditional density distribution if cash flows are small. Also, Table 5-11 shows a $\widehat{\sigma}_2$-value of less than 1/3 of the corresponding value in Table 5-15. A possible explanation is that $\widehat{\sigma}_2$, a parameter linked to the quadratic cash flow term, predominantly measures the 'jumpiness' of the cash flow diffusion process. For operating cash flows a relative low $\widehat{\sigma}_2$-value (as opposed to investing cash flows) is expected: empirical evidence as reported in Chapter 2 of this study, underpins this.

Table 5-15 Approximated Maximum Likelihood Estimation results – Operating Cash Flow- All observed firms

OPERATING CASH FLOWS	PARAMETER VALUES	T-STATISTIC	95% CI - LL	95% CI - UL	WALD STATISTIC
$\hat{\theta}_1$	437.8	3.1	158.4	717.1	9.4
$\hat{\theta}_2$	-2223.0	-108.0	-2263.3	-2182.6	11665.6
$\hat{\theta}_3$	-0.446	-8.2	-0.55	-0.34	67.4
$\hat{\mu}_0$	-0.406	-1.0	-0.4780	-0.3994	
$\hat{\mu}_1$	0.027	8.4	0.021	0.033	
$\hat{\sigma}_0$	-943389.6	-25.4	-1016161.8	-870617.5	
$\hat{\sigma}_1$	-173.9	-3.1	-284.9	-62.9	
$\hat{\sigma}_2$	0.199	84.2	0.19399	0.20323	
DISCRIMINANT	779715.7				
FIRST ROOT $\hat{\lambda}_1$	-1785.2				
SECOND ROOT $\hat{\lambda}_2$	2660.7				
$\hat{\pi}_0$	-0.018				
$\hat{\pi}_1$	-0.075				
$\hat{\alpha}$	0.244				
$\hat{\beta}$	-13.3				
$\hat{\vartheta}_0$	-16.7				
$\hat{\vartheta}_1$	-0.096				
HESSIAN	Maximum				
R^2 OF FIT $\hat{\mu}_0$, $\hat{\mu}_1$	0.72				

Based on significant t-statistics for all parameters except the σ_1-parameter, Table 5-16 below, the aggregate investing cash flow process is governed by a Modified Square Root process (a particular of the linear-quadratic model). As expected for investing cash flows, the discriminant displays a negative value. When comparing the results in Table 5-16 to those in Table 5-12, it should be noted that $\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0$ is equivalent to $-(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)$ where X_t is some cash flow process, refer to Section 3-4 for an explanation, and therefore there is consistency in parameter signs. In line with prior estimates, the $\hat{\sigma}_0$-value is very large relative to other parameters, and also greater than the $\hat{\sigma}_0$-value shown in Table 5-12, suggesting an approximate ABM process for smaller investing cash flows, more pronounced than that for operating cash flows. Following from the usually more jagged patterns of investing cash flows, unsurprisingly the $\hat{\sigma}_2$-value in Table 5-16 is more significant than the $\hat{\sigma}_2$-value in Table 5-15, reinforced by also a higher $\hat{\sigma}_1$-value.

Table 5-16 Approximated Maximum Likelihood Estimation results – Investing Cash Flow- All observed firms

INVESTING CASH FLOWS	PARAMETER VALUES	T-STATISTIC	95% CI - LL	95% CI - UL	WALD STATISTIC
$\hat{\theta}_1$	-27247.9	-0.9	-85463.2	30967.5	0.8
$\hat{\theta}_2$	-49238.9	-1647.7	-49297.5	-49180.3	2714849.5
$\hat{\theta}_3$	11.820	47.8	11.335	12.304	2285.1
$\hat{\mu}_0$	-160614.407	-246087.8	-160618.070	-160614.0737	
$\hat{\mu}_1$	-5.323	-4157.0	-5.326	-5.321	
$\hat{\sigma}_0$	-442424346456	2.4	-808776808443	-76071884469	
$\hat{\sigma}_1$	-7613164.8	-0.9	-23980830.6	8754500.9	
$\hat{\sigma}_2$	-139.702	-4.1	-206.676	-72.729	
DISCRIMINANT	-189270156008622				
FIRST ROOT $\hat{\lambda}_1$					
SECOND ROOT $\hat{\lambda}_2$					
$\hat{\pi}_0$	0.002				
$\hat{\pi}_1$	-0.445				
$\hat{\alpha}$	-0.004				
$\hat{\beta}$	-2.2				
$\hat{\vartheta}_0$	-0.8				
$\hat{\vartheta}_1$	0.00				
HESSIAN	Maximum				
R^2 OF FIT $\hat{\mu}_0$, $\hat{\mu}_1$	0.658				

From the foregoing results it follows that the general linear-quadratic model is supported as a suitable cash flow specification at an aggregated level, either in complete or in reduced parameter form. More precisely: the general linear-quadratic model is a more accurate description of an aggregate operating cash flow process than one of the particular benchmark specifications as outlined in Table 5-8, whilst investing cash flows follow a Modified Square Root process.

Industry-level

A total number of 73 different industries were examined. For each industry the same statistics as used in Tables 5-15 and 5-16, were calculated. Appendix S6 shows detailed estimates of the parameter sets $\{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3\}$ and $\{\hat{\mu}_0, \hat{\mu}_1, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2\}$ including their respective t-statistics, reported separately for operating and investing cash flows. Tables 5-17 (operating cash flow) and 5-18 (investing cash flow) summarise the results of the significance tests for parameters of the linear-quadratic model at an industry-level.

Parameter estimates, both for operating and investing cash flows, unveil a large variance between different parameters. Generally, parameters can be ranked from larger to smaller estimated values in the following sequence: $\hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2$. This is consistent with prior estimates at an aggregated level. Within each of the parameters separately, there is a marked variance too; however, with the restriction that outliers (extreme parameter values) are mostly not significant. A comparison of operating cash flow estimates to those of investing cash flow, leads to the conclusion that the diffusion parameters of investing cash flow are significantly greater, in particular $\hat{\sigma}_2$-values. A plausible explanation is the 'jumpiness' of cash flows. Commonly 'jumpiness' is more pronounced in investing cash flows than in operating cash flows. Interestingly, the estimated values of drift parameters show another contrast: for operating cash flows most industries (but certainly not all) display significant $\hat{\mu}_0$-parameters as opposed to $\hat{\mu}_1$-parameters, whilst the converse is generally true for investing cash flows. This suggests that investing cash flows are better fitted to an exponential trend and operating cash flows to a linear trend.

For operating cash flows (Table 5-17) a 97.3% convergence rate was achieved; only 2 industries exhibited partial convergence (which was likely to be caused by very slow convergence, given the fact that repeated convergence iterations only brought about marginally improved convergence). The test results in Table 5-17 reveal that a majority of 40 (56.3%) industries display significant t-statistics (at a 5%-level) for at least one diffusion parameter. This number seems low if compared to cases with no significant diffusion parameters at all: 31 (43.7%). The latter number is relatively elevated for a number of reasons: (i) it turns-out that all 31 cases have one or two significant drift parameters suggesting that in those industries the drift function takes precedence over the diffusion function; (ii) whilst parameters of all 31 cases almost fully converge, evidently diffusion parameter values were shown to be instable upon further convergence, i.e. marginally small additions to convergence often resulted in widely fluctuating parameter values; and (iii) the AMLE is derived under the condition that variable $\pi_1(\boldsymbol{\theta}) < 0$ (see Appendix M4-B, Equations (M4B.17a)- (M4B.17h), a limitation that potentially restricts the domain of the parameter optimisation routine.

A break-down of the 40 cases with significant diffusion parameters, yields 23 (32.4%) cases where all three diffusion parameters are significant, 7 (9.9%) cases with two significant diffusion parameters, and 10 (14.1%) cases with only one significant diffusion parameter. The 7 cases with two significant diffusion parameters are about evenly split between the parameter pair $\{\widehat{\sigma}_0, \widehat{\sigma}_2\}$, typical for the Modified Square Root (MSR) diffusion process, and the parameter pair $\{\widehat{\sigma}_1, \widehat{\sigma}_2\}$, not corresponding to any of the five benchmark diffusion specifications. The two-parameter diffusions found, and the complete, that is three-parameter, diffusion process, are related. Recall from Section 3-4 that the complete diffusion process can be written as $\sqrt{\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0} = \sqrt{(\sigma_2 X_t^2 + 2\rho\sqrt{\sigma_2\sigma_0}X_t + \sigma_0)}$ where X_t is some cash flow process, and ρ is shown to be the correlation coefficient between an ABM and a GBM. If $\rho = 0$ then the specification becomes a Modified Square Root (MSR) process: $\sqrt{\sigma_2 X_t^2 + \sigma_0}$.

The parameter pair $\{\widehat{\sigma}_1, \widehat{\sigma}_2\}$ leads to the following specification: $\sqrt{\sigma_2 X_t^2 + \sigma_1 X_t} = \sqrt{\upsilon_2 X_t}\ \sqrt{\eta_2 X_t + \eta_1}$, where X_t is some cash flow process, $\sigma_2 = \upsilon_2\eta_2$ and $\sigma_1 = \upsilon_2\eta_1$. It is not hard to see that the process is a multiplication of two CIR (square root) diffusion processes: $\sqrt{\upsilon_2 X_t}\ \sqrt{\upsilon_1 X_t'}$ where X_t' is a linear transformation of X_t: $X_t' = \eta_2 X_t + \eta_1$.

Of the 10 cases with one significant diffusion parameter, 2 cases have a significant $\widehat{\sigma}_0$-parameter corresponding to a ABM diffusion process, and 8 cases are defined by a significant $\widehat{\sigma}_2$-parameter suggesting an underlying GBM diffusion process.

Table 5-17 Operating Cash Flows - Significance of parameters - Industry level

Total number of industries	**73**	***100.0%***			
Partially converging	2	*2.7%*			
Fully converging	71	*97.3%*			
of which:					
Diffusion function			**71**	***100.0%***	
All 3 parameters significant			23	*32.4%*	
2 parameters significant			7	*9.9%*	
1 parameter significant			10	*14.1%*	
0 parameters significant			31	*43.7%*	
Drift function			**71**	***100.0%***	
All 2 parameters significant			47	*66.2%*	
1 parameter significant			21	*29.6%*	
0 parameters significant			3	*4.2%*	
Diffusion function					
$\hat{\sigma}_0$ significant			29	*40.8%*	*)
$\hat{\sigma}_1$ significant			26	*36.6%*	*)
$\hat{\sigma}_2$ significant			38	*53.5%*	*)
Drift function					
$\hat{\mu}_0$ significant			49	*69.0%*	*)
$\hat{\mu}_1$ significant			66	*93.0%*	*)

*) % of total fully converging cases

If the combined diffusion and drift functions are observed, a further break-down is provided in Table 5-18 of the 21 cases that have at least 4 significant parameters. The majority of those cases, 15 (65.2% of a total of 23 cases), must be described by the complete linear-quadratic model encompassing all five parameters.

Table 5-18 Operating Cash Flows - 4 and 5 parameters significant - Industry level

DIFFUSION AND DRIFT FUNCTION	**23**	
ALL 5 PARAMETERS SIGNIFICANT	15	*65.2%*
4 PARAMETERS SIGNIFICANT	8	*34.8%*

Only 3 industries, i.e. Speciality Retail, Real Estate Management & Development, and Wireless Telecommunication Services, have two drift parameters that are not significant. Here, the diffusion function appears to dominate the drift function since all 3 industries present highly significant drift parameters as opposed to the diffusion parameters.

Investing cash flows, Table 5-19, show a 100% convergence rate. The estimated parameters of just over half of all industries are all three significant. Of the industries with two significant parameters, all but one admit a Modified Square Root diffusion specification. In line with the observations under operating cash flows, of the 8 cases with only one significant diffusion parameter almost all pertain to the $\hat{\sigma}_2$-parameter, again, suggesting an underlying GBM diffusion process.

Table 5-19 Investing Cash Flows - Significance of parameters - Industry level

Total number of industries	**73**	***100.0%***			
Partially converging	0	*0.0%*			
Fully converging	73	*100.0%*			
of which:					
Diffusion function			**73**	***100.0%***	
All 3 parameters significant			40	*54.8%*	
2 parameters significant			10	*13.7%*	
1 parameter significant			8	*11.0%*	
0 parameters significant			15	*20.5%*	
Drift function			**73**	***100.0%***	
All 2 parameters significant			16	*21.9%*	
1 parameter significant			55	*75.3%*	
0 parameters significant			2	*2.7%*	
Diffusion function					
$\hat{\sigma}_0$ significant			46	*63.0%*	*)
$\hat{\sigma}_1$ significant			44	*60.3%*	*)
$\hat{\sigma}_2$ significant			58	*79.5%*	*)
Drift function					
$\hat{\mu}_0$ significant			19	*26.0%*	*)
$\hat{\mu}_1$ significant			68	*93.2%*	*)

*) % of total fully converging cases

For industries with all three diffusion parameters insignificant, the process is dominated by the exponential growth drift parameter $\hat{\mu}_1$. More generally, the parameter $\hat{\mu}_1$ plays an important role in investing cash flow processes, as can be observed in Table 5-19. In comparison to operating cash flows, investment processes overall appear to be more strongly conditioned by a (deterministic) drift component than a (random) diffusion component. This is not surprising, since investing processes are determined by more predictable management actions rather than by random outside events as is the case with operating cash flows.

Similar to operating cash flows, parameter $\hat{\sigma}_2$ is strongly represented in diffusion functions. More details are given in Appendix S6.

Table 5-20 Investing Cash Flows - 4 and 5 parameters significant - Industry level

DIFFUSION AND DRIFT FUNCTION	**30**	
ALL 5 PARAMETERS SIGNIFICANT	8	*26.7%*
4 PARAMETERS SIGNIFICANT	22	*73.3%*

Only 8 industries (26.7% of industries with at least 4 significant parameters) require that their investment processes are defined by all five model parameters (Table 5-20).

In summary, the results of the analyses corroborate the linear-quadratic cash flow model as a specification appropriate to describe operating and investing cash flow processes at an industry level. Around one-third of all industries have cash flow processes that involve at least four model parameters (three diffusion parameters, and one or two drift parameters).

Firm-level

Of all 73 randomly selected firms, all show (almost) full convergence of operating cash flow parameter estimates. *Grosso modo*, the same conclusions hold as reported for the analysis of industry level estimates.

In respect of operating cash flows (Table 5-21), just under 70% of all firms are characterised by at least one significant diffusion parameter. Of these 50 cases, 20 cases (27.4%) are exemplified by 3 significant diffusion parameters, 14 (19.2%) cases by 2 significant diffusion parameters, and 16 (21.9%) cases by only one significant diffusion parameter. The majority of two-parameter firms, 8 cases, are of the $\{\widehat{\sigma}_1, \widehat{\sigma}_2\}$-diffusion type (not one of the benchmark diffusion processes). The remaining 6 cases obey either a Modified Square Root (MSR) diffusion process (5 cases) or a Square Root (CIR) process (1 case). All 8 cases with one-parameter diffusion processes are GBM diffusions, underpinning again the relative importance of the $\widehat{\sigma}_2$ parameter in comparison to the two other parameters.

Table 5-21 Operating Cash Flows - Significance of parameters - Firm level

Total number of firms	**73**	***100.0%***			
Partially converging	0	*0.0%*			
Fully converging	73	*100.0%*			
of which:					
Diffusion function			**73**	***100.0%***	
All 3 parameters significant			20	*27.4%*	
2 parameters significant			14	*19.2%*	
1 parameter significant			16	*21.9%*	
0 parameters significant			23	*31.5%*	
Drift function			**73**	***100.0%***	
All 2 parameters significant			48	*65.8%*	
1 parameter significant			23	*31.5%*	
0 parameters significant			2	*2.7%*	
Diffusion function					
$\widehat{\sigma}_0$ significant			26	*35.6%*	*)
$\widehat{\sigma}_1$ significant			29	*39.7%*	*)
$\widehat{\sigma}_2$ significant			49	*67.1%*	*)
Drift function					
$\widehat{\mu}_0$ significant			51	*69.9%*	*)
$\widehat{\mu}_1$ significant			68	*93.2%*	*)

*) % of total fully converging cases

Just over 65% of cases, have a drift function that is defined by two parameters $\{\widehat{\mu}_0, \widehat{\mu}_1\}$. The relative importance of the $\widehat{\mu}_1$-parameter in drift functions of operating cash flows, is confirmed by the observation that the remaining cases almost exclusively have an exponential term.

Table 5-22 Operating Cash Flows - 4 and 5 parameters significant - Firm level

DIFFUSION AND DRIFT FUNCTION	22	
ALL 5 PARAMETERS SIGNIFICANT	12	*54.5%*
4 PARAMETERS SIGNIFICANT	10	*45.5%*

From Table 5-22 it is clear that of the 22 firms with at least four significant parameters, the operating cash flow process of 12 (54.5%) firms have to be described by all five parameters.

In Table 5-23 parameter estimates of investing cash flows at a firm-level are analysed. All but one case are fully converging. As expected, the diffusion parameters of the majority of firms are all three significant. Consistent with prior observations, cases with two significant diffusion parameters are about equally split between a $\{\hat{\sigma}_0, \hat{\sigma}_2\}$-diffusion process, and a Modified Square Root (MSR) diffusion process. Invariably, one-parameter processes are of the GBM-type. Again, the $\hat{\mu}_1$-parameter dominates the drift function whereas the $\hat{\mu}_0$-parameter is relatively unimportant.

Table 5-23 Investing Cash Flows - Significance of parameters - Firm level

Total number of firms	**73**	***100.0%***			
Partially converging	1	*1.4%*			
Fully converging	72	*98.6%*			
of which:					
Diffusion function			**72**	***100.0%***	
All 3 parameters significant			39	*54.2%*	
2 parameters significant			1	*1.4%*	
1 parameter significant			8	*11.1%*	
0 parameters significant			24	*33.3%*	
Drift function			**72**	***100.0%***	
All 2 parameters significant			16	*22.2%*	
1 parameter significant			55	*76.4%*	
0 parameters significant			1	*1.4%*	
Diffusion function					
$\hat{\sigma}_0$ significant			45	*62.5%*	*)
$\hat{\sigma}_1$ significant			43	*59.7%*	*)
$\hat{\sigma}_2$ significant			57	*79.2%*	*)
Drift function					
$\hat{\mu}_0$ significant			18	*25.0%*	*)
$\hat{\mu}_1$ significant			69	*95.8%*	*)

*) % of total fully converging cases

Table 5-24 Investing Cash Flows - 4 and 5 parameters significant - Firm level

DIFFUSION AND DRIFT FUNCTION	39	
ALL 5 PARAMETERS SIGNIFICANT	11	*28.2%*
4 PARAMETERS SIGNIFICANT	28	*71.8%*

Investment cash flow processes of 11 firms (28.2% of firms with at least four significant parameters) include all five model parameters (Table 5-24).

In accordance with the prior conclusion from industry-level analysis of cash flows, the linear-quadratic cash flow specification serves as a suitable model to describe operating and investing cash flow processes at a firm-level. Just over half of all firms require at least four parameters to model cash flow processes (three diffusion parameters, and one or two drift parameters).

5.3. Conclusions from Chapter 5

In this chapter the results of three different estimations are presented. Firstly, the results of estimates pertaining to the fundamental model (that theoretically justifies the linear-quadratic cash flow model), are discussed. It turns out that the parameters of the two linear difference equations that underpin the fundamental model, $\Delta \mathrm{C}_{t+1} = \alpha \mathrm{C}_t + \beta \mathrm{I}_t + \delta$ and $\Delta \mathrm{I}_{t+1} = \gamma \mathrm{C}_t - \mathrm{I}_t + \varepsilon$, are all statistically significant across the whole population of examined firms, with the exception of parameter ε. Overall, the model is sufficiently specified. The parameter values imply that the uncoupled operating and investing processes are converging to long-term stable values, not uncommon for a mature economy such as the USA where market dynamics see new firms entering at about the same rate as firms exit. Importantly, the estimation results support a recursive relationship between operating and investing cash flows, resulting in positive reinforcement effects between the two types of cash flows. These effects explain, for instance, the significant non-normal probability distribution of cash flows, as expected characterised by heavy tails and some skewness. The occurrence of non-normal densities plays an important role in selecting appropriate methods for statistical inference of the linear-quadratic cash flow model itself.

The centre piece of Chapter 5 is testing the linear-quadratic model. The test methodology must be equipped to deal with some significant statistical challenges. First and foremost, there is the challenge, thoroughly documented in the literature, of fitting discretely sampled data to a continuous-time stochastic process. Whilst high-frequency data can mitigate some reported difficulties, sampling low-frequent data (in this study, quarterly data) only tends to compound statistical challenges. Additionally, the already mentioned significant non-

Gaussian (conditional) densities pose a further challenge. Two main methods to derive parameter estimates are explored in this study: (i) directly estimating parameters from approximated solutions to the linear-quadratic SDE, and (ii) indirect estimations of parameters from an approximated probability density function and related approximated maximum likelihood estimator.

In Chapter 4 two approximated SDE solutions were derived that are found to be conducive to direct estimation of parameters. The estimation results, all valid for the whole population of examined firms, are encouraging. Estimated parameter values are plausible and in conformity with theoretical considerations and empirical observations described in this study. The results support, on an aggregated level, the superiority of the generic linear-quadratic cash flow model in complete specification.

Indirectly estimation of parameter values from an approximated density function, has proved to be particularly demanding. For the reasons set-out in the chapter, most of the usual stochastic inference techniques do not apply. After careful screening, the Ait-Sahalia-method is selected as the preferred technique to derive an approximated density function for the linear-quadratic model. The Ait-Sahalia- method is capable of dealing with a wide range of different stochastic processes characterised by non-normal densities and is suitable for problems including discrete low-frequency data sampling. In essence, the method transforms a (conditional) non-Gaussian density function into a Gaussian, after which the density function is expanded into a standard normal component and a polynomial term derived from a Hermite series expansion. The accuracy of the approximation, that is how close the approximation is to the true but unobservable density function, is determined by two important parameters: (i) parameter J indicating the number of terms included in the expansion, and (ii) parameter Δ, the time-interval over which discrete data are sampled. In this study, a five-term (J=4) approximation is used, in the expectation that the method's allegedly strong convergence to the true distribution sufficiently balances the very long sampling interval. Unlike Ait-Sahalia's original method, this study advocates calculating the Fourier coefficients (of the Hermitian expansion) from a (closed) system of moment ODEs. The final expression of the conditional density function, after retro-transformation, is appropriate for the proposed ML estimation. It consists of three components: (i) a nonlinear scaling factor related to the transformation from a non-normal to an approximated normal probability distribution, (ii) a standard normal distribution of the retro-transformed variable, and (iii) a term that adjusts the transition density for non-Gaussian characteristics (that is, excess skewness and kurtosis if $\mathrm{J} = 4$).

Converting the conditional transition density of the linear-quadratic model into a practically useable Approximated Maximum Likelihood Estimator (AMLE) has proved to be difficult. The convoluted specification of a mix of parameters and cash flow variables, requires a few considerable approximations, some compromising a desired high accuracy. Nevertheless, an attempt is made to apply the AMLE to three levels distinguished within the dataset of 5,202 North American firms examined (see Appendix S1): (i) an overall level including all firms, (ii) an industry-level, grouping firms into identical industries (by their GICS-codes), and (iii) a firm-level by selecting a random sample from all firms.

At an overall level, the ML-estimation results are not too dissimilar to those found for direct estimation; however, acknowledging that the ML-estimation technique encapsulates much richer probability information than the direct method. In addition, parameter estimates and related significance tests, were carried out at an industry level and on a random sample of individual firms. It turns out that the linear-quadratic cash flow model, in complete (five) parameter specification or in reduced parameter form, is appropriate to describe the vast majority of both operating cash flow and investing cash flow processes. In cases where less than three diffusion parameters are significant, four different reduced-form processes are observed. Two of these processes belong to the benchmark specifications: the Modified Square Root (MSR) diffusion process (two significant parameters) and a GBM-diffusion process (one significant $\widehat{\sigma}_2$-parameter). The second two-parameter diffusion process derived from the estimates, is of the $\{\widehat{\sigma}_1, \widehat{\sigma}_2\}$-type, a process not often found in the literature, but derivable from the complete quadratic diffusion processes as the multiplication of two different CIR (square root) processes. In only one case was a Square Root (CIR) process identified.

Drift functions pertaining to operating cash flows, are predominantly defined by their respective $\hat{\mu}_0$-parameters, suggesting a strong linear trend. In contrast, the exponential parameter $\hat{\mu}_1$ is in the vast majority of cases the only significant drift parameter relative to investing cash flows. Interestingly, for cases where the full specification appears to not be applicable, either the diffusion process (predominantly for operating cash flows) or the drift process (almost always for investing cash flows) qualifies as a dominating process.

6. Conclusions and Recommendations

In the epilogue, the most important findings and conclusions obtained from this study, are summarised. Furthermore, observations are made that serve to point-out areas where continuing development or refinement of the linear-quadratic model, as developed and analysed in this study, is seen as useful.

6.1. Conclusions and findings from this study

Cash flow is often recognised as the life-blood of an organisation. Models to predict, manage and control cash flows, are useful tools not only for timely identification of possible financial bottlenecks (and thus prevent cash crises), but also to support balanced business growth.

The focal point of this dissertation is stochastic continuous-time cash flow models. At present these models have been reported relatively sparsely in the literature, and hardly any practical applications are known. Yet, stochastic continuous-time cash flow models, as underpinned by the results of this study, prove to be very useful to describe the rich and diverse nature of trends and fluctuations in cash flow randomness. However, at a price of considerable mathematical and statistical complexity.

Before developing a generic stochastic continuous-time cash flow model from well-known stochastic principles, embodied in a general stochastic differential equation, the first chapter of this dissertation considers an important preliminary question: can cash flows be fully described in continuous time? Theoretical and empirical evidence (e.g. testing for jumps) shows that under some not too stringent regularities, operating cash flow processes can be well approximated by a diffusion equation, whilst investing processes -preferably-first need to be rescaled by a system-size variable to control for excess instantaneous change of variance.

Chapter 2 starts with investigation of the five stochastic continuous-time cash flow specifications that are frequently found in the literature. These are: the Geometric Brownian Motion, the Arithmetic Brownian Motion, the mean-reverting Vasicek and Cox, Ingersoll and Ross processes, and the Modified Square Root process. A pivotal question is whether these specifications are capable of sufficiently mimicking the behaviour of real-world cash flow processes. Consequently, the remainder of the chapter analyses the characteristics of cash flow processes from a theoretical and empirical perspective. Importantly, the mathematical form of the drift function and the diffusion function are examined in detail, as well as the evolution of the first four moments. The main conclusion is that an equation consisting of a linear drift function and a complete quadratic diffusion function (hereafter: “the linear-quadratic model”) is a specification that is preferred to the five processes commonly considered in the literature. This outcome is consistently supported by a multitude of theoretical considerations and a number of (preliminary) statistical tests. Additionally, but not less significantly, it turns out that the specifications from the literature can all be derived as particular cases of the advocated model. Hence, the linear-quadratic model is classed as a *hybrid* model since it is constructed from different basic stochastic models, in particular, the combination of a multiplicative GBM and an additive ABM. Hybrid models are powerful because their stochastic properties incorporate behaviour that is more complex and versatile than the sum of the properties of the component parts. Finally, as an introduction to Chapter three, a bi-causal relationship between operating and investing cash flows is examined, showing strong indications of support for the linear-quadratic model.

In Chapter 3, the presumed relationship between operating and investing cash flows is further developed into a two-variable, two equation (coupled) cash flow model, also called ‘the two fundamental relationships’. It is demonstrated that the model is rooted in well-studied and generally accepted business and financial knowledge. Furthermore, Chapter three lays the mathematical foundation for a *decoupled* (that is, a spectral decomposed) cash flow model, derived from the two fundamental relationships, that includes a hybrid linear-quadratic specification for each of the operating and investing cash flow variables in isolation. The decoupled system is described and analysed in both a deterministic and stochastic environment. To check its accuracy, simulation results for the model are compared to real-world cash flows. A large degree of similarity is found.

Chapter 4 deals with solutions to the hybrid decoupled cash flow model developed in this study. First it is shown that, despite meeting continuity and convergence conditions, there is no general closed-form solution to the hybrid linear-quadratic cash flow specification.

Nevertheless, three particular and two approximated exact solutions are derived under not too stringent parameter restrictions and cash flow domain limitations. In the absence of a strong general solution, Chapter 4 also explores weak solutions described by (forward or backward) Fokker-Planck- Kolmogorov equations. Supported by (preliminary) empirical evidence from Chapter 2, it is shown that since the process is *converging* in time, (uncoupled) investing cash flows can be described by a Pearson diffusion process with a stationary Pearson Type-IV probability density function, more appropriately a Student diffusion process. In contrast, (uncoupled) operating cash flow processes are *diverging* in time with no stable probability density function. Therefore, a dynamic analysis in a *bounded* cash flow domain is required. Chapter 4 sets-out why a bounded operating cash flow is not only mathematically appropriate but that it is also supported by financial and business considerations. Hence, the analysis is centred on solving the linear-quadratic non-stationary forward Kolgomorov (Fokker Planck) equation and backward Kolmogorov equation in a bounded domain.

It is well-known that the Fokker-Planck-Kolgomorov equations are notoriously difficult to solve analytically. Beginning with the Fokker-Planck equation, the method of separation of variables is proposed and defended (in comparison to other solution methods) to examine the intertemporal dynamics of the process. The suggested solution method comprises of several steps: first the general hypergeometric differential equation, pertaining to the cash flow component after separating the variables, is normalised on a bounded cash flow spectrum [0;1] and [-1;1] respectively. The [0;1]-normalisation leads to a composite Gaussian hypergeometric function that, depending on particular parameter values, includes a large number of special functions as specific or limiting cases. Furthermore, the [-1;1]-normalised ODE is transformed into a Sturm-Liouville specification, followed by three separate second transformations. These second transformations are the Jacobi, the Hermitian and the Schrödinger, each yielding a homonymous[55] equation. Only the Jacobi transformation provides an exact solution, albeit with a mix of combinatorial and polynomial terms that are difficult to interpret in a practical sense. The other two transformations require the inclusion of approximated terms to arrive at a closed-form general solution. Both transformations lead to a composite space-time density function of

[55] Having the same designation.

operating cash flow processes that can be constructed as the multiplication of two (independent) time-variant probability distributions: (i) the stationary, in the case of operating cash flows: the approximate long-time distribution, and (ii) the evolution of a standard normal distribution. It is shown that the long-time probability density function of operating cash flows is akin to the one that describes the (stationary) Pearson family of distributions with two real roots (Pearson's case 2). Additionally, it turns out that a *particular* solution to the prior mentioned Sturm-Liouville equation obeys a Pearson Type IV distribution similar to the one found for investing cash flows.

Furthermore, Chapter 4 also demonstrates that the Schrödinger transform is the nexus between the backward linear-quadratic Kolgomorov equation and the corresponding Fokker-Planck equation. Solving the backward equation can be useful if the resulting ODEs are easier to solve. The last part of the chapter explains how to convert solutions to the decoupled system, back to solutions of the coupled system.

A final Chapter 5 estimates the parameters of the hybrid coupled model. The dataset used, comprises of quarterly cash flow data of 5,202 North-American firms over a period of (maximum) 120 consecutive quarters. The chapter reports the results of three different estimation methods. Firstly, the fundamental model, developed in Chapter 3 is estimated. Overall, the model is sufficiently specified. Parameters, with the exception of one, are statistically significant across the whole population of sample firms. Estimated parameter values support the conclusion that uncoupled (and therefore also coupled) operating and investing processes are converging to long-term stable values.

Secondly, the results of direct parameter estimation from approximated SDE solutions are plausible and in line with theoretical considerations and empirical observations discussed in this study. The results support, at least at an aggregated level, the superiority of the generic linear-quadratic cash flow model over any of the specific models derived from the complete specification.

Thirdly, overall parameter values are indirectly estimated from an approximated density function and its associated approximated maximum likelihood estimator. The so-called Ait-Sahalia- method is argued to be a superior technique to derive an approximated density function for the linear-quadratic model. In essence, the method transforms a (conditional) non-Gaussian density function into a Gaussian, after which the density function is expanded into a standard normal component and a polynomial term derived from a Hermite series expansion. Unlike Ait-Sahalia's original method, this study advocates calculating the Fourier coefficients (of the Hermite expansion) from a (closed) system of moment ODEs. Furthermore, converting the conditional transition density of the linear-quadratic model into a practically useable Approximated Maximum Likelihood Estimator (AMLE) requires a several transformations and approximations. The estimation procedure itself, is suitable for high-parametrised estimations and includes re-parametrisation (based on the extended invariance principle) and stepwise maximisation.

The most important conclusion derived from analysis of the reported AMLE results, is the corroboration of the hypothesised superiority of the linear-quadratic cash flow model, either in complete (five-parameter form) or in a reduced-parameter form, in comparison to the examined five benchmark processes. Reduced-parameter cases are found to be either (i) a Modified Square Root (MSR) diffusion process (two significant parameters), or (ii) a diffusion of the $\{\widehat{\sigma}_1, \widehat{\sigma}_2\}$-type (two significant diffusion parameters, each corresponding to two multiplied different CIR (square root) processes, or (iii) a GBM-diffusion process (one significant $\widehat{\sigma}_2$-parameter).

6.2. Recommendations for further research

The previous section showed that the results of parameter estimations (tentatively) support the hypothesis of the superiority of the linear-quadratic cash flow model over the common benchmark specifications. Nonetheless, this section includes three areas where further development and refinement of the linear-quadratic cash flow model, would be useful and beneficial.

Mixed diffusion-jump processes

This study examines cash flow processes as a pure diffusion process. Despite the results of the empirical tests being (under some restrictions and limitations) supportive of a linear-quadratic diffusion model, the question must be posed whether there are possibly better stochastic continuous-time models to describe cash flow processes. Section 1-4 mentions a more general class of Lévy processes represented by the stochastic differential equation

$$dC_t = \alpha(C_t, t)dt + \sqrt{\beta(C_t, t)}dW_t + \nu(C_t, t)dN_t \tag{6.1}$$

Whereas a Lévy process enables sample paths to include discontinuous random jumps, the associated probability distribution of cash flow C_t is nevertheless infinitely divisible. These processes are considered the simplest class of mathematically robust processes whose paths consist of continuous motion interspersed with jump discontinuities of random size appearing at random times (David Applebaum (2004)). Consequently, the mathematical treatment of Lévy processes is significantly more challenging than the pure diffusion process but, in most cases, still manageable. This study shows that for larger firms with high-frequent cash transactions, the diffusion model appropriately mimics operating cash flows as they are observed in practice. The diffusion model is, to a lesser degree (excluding one-off, relatively large investment outflows) and under restrictions (re-scaling cash flow variance), also adequately models investing cash flow processes. Indeed, by their nature, investing cash flows are more 'jumpy'. Consequently, it is expected that a Lévy process offers a more versatile specification and associated set of properties to further improve cash flow modelling. This is particularly true for smaller businesses with low frequency of cash transactions and more concentrated cash inflow and outflow time-patterns. In these cases, replacing a continuous-time diffusion model by a Lévy cash flow process, may prove to be beneficial. However, the suggestion is subject to supporting usefulness of empirical evidence.

Non-Markov processes

In financial asset markets the Markov assumption is assumed to hold reasonably well, at least for a short-time horizon (of up to several months). Regardless, there is some evidence against the Markov property: see B. Chen and Hong (2011) for a recent overview. Ait-Sahalia et al. (2010) describes an appropriate statistical test methodology to determine whether a process is Markovian or not. Market information is continuously updated, often described by the filtration: $I_t \subseteq I_{t-1} \subseteq I_{t-2}$ … where time $t > t-1 > t-2$. The most recent information set includes all (actual) information of prior sets. Additionally, the Efficient

Market Hypothesis (Fama (1970)) states that, at all times, current market prices incorporate and reflect all relevant information. In well-functioning competitive financial asset markets, there are a sufficiently large number of buyers and sellers who trade on the latest available (public) information. Trading happens because market parties will interpret the same information differently.

The question can be asked if similar conditions pertain to cash flow processes. Is information on cash flows continuously updated? Does information about the most recent reported cash flow include all (relevant and actual) prior cash flow information? Is the most recent cash flow the only input to future cash flow decisions? In Section 1-4, it is assumed that even if cash flow processes have memory time, they can be very well approximated by a continuous-time Markov process. But, do cash flow processes actually have only a small-time memory or, in contrast, perhaps a (much) longer memory? Obviously, firms are different from financial asset markets: instead of prices being the outcome of a set of heterogeneous market forces, financial decisions in firms are made by a few appointed insiders (managers). Therefore, firm decisions tend to be more homogeneous. Managers act on information that is largely privy to the firm, only to be publicly disseminated if required under the applicable legal framework. Although there is no imminent need for managers to instantaneously incorporate all available information into financial decisions, as opposed to the effect of trading in a competitive market, undoubtedly firms too benefit from the most current information. However, financial decisions firms make are vastly more complex than the usual simple 'buy-hold-sell' decisions of many traders. They include, for instance, the allocation of cash amounts over investment projects, in terms of amounts and timing, the assessment of how external factors will affect these projects, usually captured under the name of risk assessment, and a projection of possible outcomes. It is highly plausible that managers require more financial information than just the latest reported cash flows. They want to analyse financial trends including realised past cash flows. Hence, future cash flow states may not only depend on the last cash flow realisation but, more likely, on prior realised cash flow states as well. Fortunately, the Chapman-Kolmogorov equation is not restricted to non-Markov processes and a non-Markov process can still be described as an Itô process (Joseph L. McCauley (2010); J. L. McCauley (2012)). More realistic continuous-time cash flow models may have to be modelled as non-Markov stochastic processes, or, at least should include some statistics (such as trends) derived from multiple prior realisations.

Connection between microscopic to mesoscopic levels

Implicitly it is assumed in this study (Section 1-4) that firms behave like equivalent entities, similar to particles in physics. This assumption is critical to elevating the analysis from a microscopic (individual firm) level to a mesoscopic (multiple firms) level; without it every cash flow path generated by a SDE would be a unique realisation, specific to that firm only. By assuming equivalence, stochastic processes become *ergodic*, that is, repeatable at the same time, and stochastic properties are therefore derivable from a probability distribution. If this important assumption turns out to be flawed, the methodological consequences would be profound. Characteristics of non-ergodic processes can vary significantly between ensembles of firms at the same time and aggregated time-paths of individual firms (Peters and Klein (2013)). At an aggregated level the problem becomes irreducible (Bookstaber (2017)) resulting in, for example, a disconnection between the SDE paths of individual firms (microscopic level) and the probability density function that follows from solving the corresponding Fokker Planck equation (mesoscopic level). Then, as the only option left, one has to revert to modelling the cash flow process of each firm individually and their possible interrelationships with idiosyncratic variables and other specific and general economic variables. No doubt a mammoth task. However, if it is supposed that firm's cash flow processes share many common stochastic characteristics (a very reasonable abstraction from the position that firm's cash flow processes are truly unique), then a single cash flow model can be used, but with firm-specific parameter values. Despite varying individual parameter estimates, the statistical techniques underpinning the parameter estimates require the process to be ergodic, i.e. repeatable amongst firms in the ensemble. By contrast, if enough evidence is compiled that cash flow processes must be non-ergodic, then the approach outlined in this study will not satisfy and, consequentially, a fundamentally different research methodology is warranted. Irreducible, and likely much more complex, models will be required to accurately describe cash flow processes of individual firms[56].

[56] What this methodology could include is speculative but so-called agent-based models (abm) describing behavioural characteristics of specific actors, and, importantly, how they inter-act with each other and their environment, appear to be gaining popularity at the time of this writing.

In conclusion, development of the linear-quadratic cash flow model could benefit from (less than radical) improvements such as including a jump term or considering a (finite) non-Markovian underlying process. The value of such improvements is conditional on the assumption that cash flow processes are (close to) ergodic; if this premise was found to be wanting, the impact on the proposed linear-quadratic model would be major.

7. Mathematical Appendixes

Appendix M1 – Derivation of the general cash flow specification

The following derivation, adapted to cash flow processes, can be found in Gardiner (1985, chapter 3). Given its importance to the foundation of the general cash flow specification, the detailed derivation is included in this study.

The starting point is a stochastic continuous-time cash flow function $\mathrm{f(c,t)}$, twice differentiable with respect to c. The time-evolution of the expected value of $\mathrm{f(c,t)}$ can be expressed as $\frac{\partial}{\partial t}\int \mathrm{f(c,t)p(c_3,t_2 \mid c_1\,,t_1)dc} =$

$$\lim_{\Delta t\to 0}\frac{\int \mathrm{f(c,t)}[\mathrm{p}(c_3,t_2+\Delta t\,|c_1,t_1)-\mathrm{p}(c_3,t_2\mid c_1,t_1)]\mathrm{dc}}{\Delta t} \qquad \text{(M1.1)}$$

with $\mathrm{p}(c_3,t_2 \mid c_1,t_1)$ being the continuous and once-differentiable conditional transition probability between time $t_1$and t_2 ($t_2 > t_1$), c_1 are realised cash flows and c_3 are future cash flows. cash flow variable c is now defined as a continuous variable and the cash flow subscripts $\{c_1,c_2,c_3\}$ denote specific values (out of a number of infinite states) that the cash flow variable can take at times $\{t_1,t_2,t_3\}$

Since the process is Markovian, the Chapman-Kolmogorov equation can be used to expand $\mathrm{f(c,t)p}(c_3,t_2+\Delta t\,|c_1,t_1)$ to $\int \mathrm{f(c,t)p}(c_3,t_2+\Delta t\,|c_2,t_2)\,\mathrm{p}(c_2,t_2\,|c_1,t_1)\mathrm{dc}$ by introducing another future cash flow variable $[c_2,t_2]$ at t_2. The reason for this is to distinguish a continuous (smooth) process from a discontinuous process. For any $\varepsilon > 0$ a process is continuous if $\lim_{\Delta t\to 0}\int\frac{\mathrm{p}(c_3,t_2+\Delta t\,|c_1,t_1)\mathrm{dc}}{\Delta t} = 0$ where the integral is taken over $|c_3-c_1| > \varepsilon$. This means that in the limit the absolute difference between c_2 and c_3, $|c_3-c_2|$, goes faster to zero than Δt. If this condition is not met, the process will be exhibit jump-like behaviour.

Thus, Equation (M1.1) becomes

$$\lim_{\Delta t\to 0}\left\{\frac{\int\int \mathrm{f(c,t)p}(c_3,t_2+\Delta t\,|c_2,t_2)\,\mathrm{p}(c_2,t_2\,|c_1,t_1)\mathrm{d}c_2\,\mathrm{dc}}{\Delta t} - \frac{\int \mathrm{p}(c_3,t_2\mid c_1,t_1)\mathrm{dc}}{\Delta t}\right\} \qquad \text{(M1.2)}$$

Now, divide the integral over c in two regions: $|c_3-c_2| < \varepsilon$ and $|c_3-c_2| \geq \varepsilon$.

For $|c_3-c_2| < \varepsilon$ the twice differentiable function $\mathrm{f(c,t)}$ can be expanded around c_2 according to a (converging) Taylor series (with t kept constant):

$$\mathrm{f(c)} = \mathrm{f}(c_2) + \frac{\partial \mathrm{f}(c_2)}{\partial c}(c-c_2) + \frac{1}{2}\frac{\partial^2 \mathrm{f}(c_2)}{\partial c^2}(c-c_2)^2 + |\,c-c_2|^3\mathrm{R}(c,c_2) \qquad \text{(M1.3)}$$

where $c = c_3$ and $\mathrm{R}(c,c_2)$ is a rest term. Observe that $\mathrm{R}(c_3,c_2) \to 0$ as $|\,c_3-c_2| \to 0$.

In the following step, substitute Equation (M1.3) in the RHS of Equation (M1.2) and write the results as the sum of the following terms:

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\{\iint\left[\frac{\partial f(c_2)}{\partial c}(c-c_2)+\frac{1}{2}\frac{\partial^2 f(c_2)}{\partial c^2}(c-c_2)^2\right]dc_2dc\, p(c_3,t_2+\Delta t\,|c_2,t_2)p(c_2,t_2\,|c_1,t_1)+ \tag{M1.4a}$$

$$\iint|\,c-c_2|^3R(c,c_2)\,p(c_3,t_2+\Delta t\,|c_2,t_2)p(c_2,t_2\,|c_1,t_1)dc_2dc+ \tag{M1.4b}$$

$$\iint f(c,t)\;\, p(c_3,t_2+\Delta t\,|c_2,t_2)p(c_2,t_2\,|c_1,t_1)\;dc_2dc+ \tag{M1.4c}$$

$$\iint f(c_2)\;\, p(c_3,t_2+\Delta t\,|c_2,t_2)p(c_2,t_2\,|c_1,t_1)\;dc_3dc- \tag{M1.4d}$$

$$\iint f(c_2)\;\, p(c_3,t_2+\Delta t\,|c_2,t_2)p(c_2,t_2\,|c_1,t_1)\;dc_2dc\,\} \tag{M1.4e}$$

where the dc integral in (4a) and (4b) are over $|c_3-c_2|<\varepsilon$, the dc_2 integral in (4a) and (4b) are over $[-\infty;+\infty]$, both integrals in (4c) are over $|c_3-c_2|\geq\varepsilon$, both integrals in (4d) are over $|c_3-c_2|<\varepsilon$, and both integrals in (4e) are over $[-\infty;+\infty]$.

To evaluate (4a) define the following two continuous (in t, c and ε):

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\{\int(c-c_2)p(c_3,t_2+\Delta t\,|c_2,t_2)\,d=\alpha(c,t)+O(\varepsilon) \tag{M1.5a}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\{\int(c-c_2)^2p(c_3,t_2+\Delta t\,|c_2,t_2)\,d=\beta(c,t)+O(\varepsilon) \tag{M1.5a}$$

where both integrals are taken over $|c_3-c_2|<\varepsilon$, $\alpha(c,t)$ is a once-differentiable function and $\beta(c,t)$ is a twice-differentiable function.

Now, Equation (M1.4a) becomes

$$\int\left[\frac{\partial f(c_2)\alpha(c,t)}{\partial c}+\frac{1}{2}\frac{\partial^2 f(c_2)\beta(c,t)}{\partial c^2}\right]dc_2\;p(c_2,t_2\,|c_1,t_1)+O(\varepsilon) \tag{M1.6a}$$

In Equation (M1.4b) as $|\,c_3-c_2|\to 0$ also $|\,c_3-c_2|^3R(c_2,c_3)\to 0$ and consequently the whole term (M1.4b) approaches 0.

Equations (M1.4c), (M1.4d) and (M1.4e) can be grouped together to:

$$\iint f(c_2)\,[\,J(c_2|c_3,t_2)\;p(c_3,t_2\,|c_1,t_1)-\;J(c_3|c_2,t_2)\;\;p(c_2,t_2\,|c_1,t_1)]\;dc_2dc \tag{M1.6b}$$

where both integrals are over $|c_3-c_2|\geq\varepsilon$ and $J(c_3|c_2,t_1)=\lim\limits_{\Delta t\to 0}\frac{p(c_3,t_2+\Delta t\,|c_2,t_2)}{\Delta t}$.

Notice that $J(c_3|c_2,t_1)$ is a (discontinuous) jump function.

Taking the limit $\varepsilon\to 0$, the LHS of Equation (M1.1) can be expressed as

$$\frac{\partial}{\partial t}\int f(c_2,t)p(c_3,t_2 \mid c_1, t_1)dc = \int\left[\frac{\partial f(c_2)\alpha(c,t)}{\partial c} + \frac{1}{2}\frac{\partial^2 f(c_2)\beta(c,t)}{\partial c^2}\right] dc_2\, p(c_2,t_2 \mid c_1,t_1) + \iint f(c_2)\left[J(c_2|c_3,t_1)\, p(c_3,t_2 \mid c_1,t_1) - J(c_3|c_2,t_2)\; p(c_2,t_2 \mid c_1,t_1)\right] dc_2 dc \tag{M1.7}$$

Integrating Equation (M1.7) by parts, gives the following equation:

$$\int f(c_2)\frac{\partial}{\partial t}p(c_2,t_2 \mid c_1, t_1)dc_2 = \int f(c_2)\{\left[\frac{-\partial f(c_2)\alpha(c,t)}{\partial c}p(c_2,t_2 \mid c_1,t_1) + \frac{1}{2}\frac{\partial^2 f(c_2)\beta(c,t)}{\partial c^2}p(c_2,t_2 \mid c_1,t_1)\right] dc_2 + \int\left[J(c_2|c_3,t_2)\, p(c_3,t_2 \mid c_1,t_1) - J(c_3|c_2,t_2)\; p(c_2,t_2 \mid c_1,t_1)\right]\; dc_2\}\, c + S(c_{2,}\, c_3) \tag{M1.8}$$

where $S(c_{2,}\ c_3)$ are surface terms on the boundary of $[c_{2,}\ c_3]$ which are non-existent if functions $\{\alpha(c,t),\beta(c,t)\}$ are continuous. Then, Equation (M1.8) can be re-written in differential form as:

$$\frac{\partial}{\partial t}p(c_2,t_2 \mid c_1,t_1) = \frac{-\partial\alpha(c,t)p(c_2,t_2 \mid c_1,t_1)}{\partial c} + \frac{1}{2}\frac{\partial^2\ \beta(c,t)p(c_2,t_2 \mid c_1,t_1)}{\partial c^2} + \int\left[J(c_2|c_3,t_2)\, p(c_3,t_2 \mid c_1,t_1) - J(c_3|c_2,t_2)\; p(c_2,t_2 \mid c_1,t_1)\right] dc \tag{M1.9}$$

Equation (M1.9) can be evaluated in at least three different special forms:

a. as a *deterministic process* if $\alpha(c,t) \neq 0,\ \beta(c,t) = 0$ and $\{ J(c_2|c_3,t_2), J(c_3|c_2,t_2)\} = 0$. Then Equation (M1.9) reduces to $\frac{\partial}{\partial t}p(c_2,t_2 \mid c_1,t_1) = \frac{-\partial\alpha(c,t)p(c_2,t_2 \mid c_1,t_1)}{\partial c}$ (M1.10)

Substitute $p(c_2,t_2 \mid c_1,t_1) = \delta[c_2 - z(c_1)]$ with initial condition $p(c_2,t_2 \mid c_1,t_1) = \delta(c_2 - c_1)$ in the RHS of Equation (M1.10) to get

$$\frac{-\partial\alpha(c,t)\delta[c_2-c(c_1)]}{\partial c} = \frac{-\partial\alpha(z(c_1),t)\delta[c_2-c(c_1)]}{\partial c} = -\alpha(z(c_1),t)\frac{\partial\delta[c_2-z(c_1)]}{\partial c} \tag{M1.11}$$

where $\delta[\,]$ is Dirac's delta function.

The LHS of Equation (M1.10) can be evaluated to $\frac{\partial}{\partial t}\delta[c_2 - z(c_1)] = \frac{-\partial\delta[c_2-c(c_1)]}{\partial c}\frac{dz(c_1)}{dt}$ (M1.12)

Equating both the RHSs of Equations (M1.11) and (M1.12) leads to the ordinary differential equation

$$\frac{dz(c_1)}{dt} = \alpha(z(c_1),t) \tag{M1.13}$$

which describes a deterministic process.

b. as a *diffusion process* if $\alpha(c,t) \neq 0,\ \beta(c,t) \neq 0$ and $\{ J(c_2|c_3,t_2), J(c_3|c_2,t_2)\} = 0$. It is obvious that Equation (M1.9) now becomes the Fokker-Planck equation

$$\frac{\partial}{\partial t}p(c_2,t_2 \mid c_1,t_1) = \frac{-\partial\alpha(c,t)p(c_2,t_2 \mid c_1,t_1)}{\partial c} + \frac{1}{2}\frac{\partial^2\ \beta(c,t)p(c_2,t_2 \mid c_1,t_1)}{\partial c^2} \tag{M1.14}$$

c. as a *pure jump process* if $\alpha(\mathrm{c,t}) = 0$, $\beta(\mathrm{c,t}) = 0$ and $\{ \mathrm{J}(\mathrm{c}_2|\mathrm{c}_3, \mathrm{t}_2), \mathrm{J}(\mathrm{c}_3|\mathrm{c}_2, \mathrm{t}_2)\} \neq 0$. The Master Equation for the jump process is $\frac{\partial}{\partial \mathrm{t}} \mathrm{p}(\mathrm{c}_2, \mathrm{t}_2 \mid \mathrm{c}_1, \mathrm{t}_1) =$

$$\int [\, \mathrm{J}(\mathrm{c}_2|\mathrm{c}_3, \mathrm{t}_2)\, \mathrm{p}(\mathrm{c}_3, \mathrm{t}_2\, |\mathrm{c}_1, \mathrm{t}_1) - \mathrm{J}(\mathrm{c}_3|\mathrm{c}_2, \mathrm{t}_2)\, \mathrm{p}(\mathrm{c}_2, \mathrm{t}_2\, |\mathrm{c}_1, \mathrm{t}_1)]\, \mathrm{dc} \qquad (\mathrm{M1.15})$$

Equation (M1.15) can be approximated in a very small discrete time $\Delta \mathrm{t}$ by:

$$\mathrm{p}(\mathrm{c}_2, \mathrm{t} + \Delta \mathrm{t} \mid \mathrm{c}_1, \mathrm{t}) = \delta(\mathrm{c}_2 - \mathrm{c}_1)\, [1 - \Delta \mathrm{t} \int \mathrm{J}(\mathrm{c}_2|\mathrm{c}_1, \mathrm{t}_2)\, \mathrm{dc}\ ] - \Delta \mathrm{t}\, \mathrm{J}(\mathrm{c}_3|\mathrm{c}_1, \mathrm{t}_2) \qquad (\mathrm{M1.16})$$

with initial condition $\mathrm{p}(\mathrm{c}_2, \mathrm{t}_2\, |\mathrm{c}_1, \mathrm{t}_1) = \delta(\mathrm{c}_2 - \mathrm{c}_1)$.

where $\delta(\,)$ is Dirac’s delta function.

Appendix M2 – Expected value vector and co-variance matrix of the decoupled linear-quadratic cash flow model

Define the following matrices and vectors:

$\mathbf{P}$ the diagonal probability matrix corresponding to the independent states of the state-change matrix S $\{\Delta C_{\Delta t}, \Delta I_{\Delta t}\}$: $\mathbf{P} = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \Delta t$, the vector of cash flow variables $\mathbf{u} = \begin{pmatrix} C_t \\ I_t \end{pmatrix}$, the state-change variable parameter matrix $\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & -1 \end{pmatrix}$, the state-change constant parameter vector $\mathbf{b} = \begin{pmatrix} \delta \\ \varepsilon \end{pmatrix}$, vector $\boldsymbol{\varphi} = \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}$, matrix $\boldsymbol{\Phi} = \begin{pmatrix} \varphi^2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{P}' = \begin{pmatrix} 0 & p_3 \\ p_3 & 0 \end{pmatrix} \Delta t$ representing the dependency between the two parts of system S ,and the vector of decoupled cash flow variables $\mathbf{v} = \begin{pmatrix} C'_t \\ I'_t \end{pmatrix}$ belonging to the decoupled system $S'\{\Delta C'_{\Delta t}, \Delta I'_{\Delta t}\}$.

Deriving the expected value of the decoupled system

The expected value vector of S $\{\Delta C_{\Delta t}, \Delta I_{\Delta t}\}$ can be expressed as:

$$\mathbb{E}(\Delta \mathbf{u}_{\Delta t}) = \mathbf{P}.(\mathbf{A}.\mathbf{u}_{\Delta t} + \mathbf{b}) + \mathbf{P}'.\boldsymbol{\varphi} \quad \text{(M2.1)}$$

For notational convenience, the subscript Δt will be dropped from $\Delta \mathbf{u}_{\Delta t}$.

To derive the expected value of the decoupled system $S'\{\Delta C'_{\Delta t}, \Delta I'_{\Delta t}\}$, Equation (M2.1) will need to be diagonalised:

$$\mathbb{E}(\Delta \mathbf{u}) = (\mathbf{Q}.\mathbf{M}.\mathbf{Q}^{-1}.\mathbf{u} + \mathbf{P}.\mathbf{b}) + \mathbf{P}'.\boldsymbol{\varphi} \quad \text{(M2.2)}$$

where $\mathbf{M} = \begin{pmatrix} \mu_{C,1} & 0 \\ 0 & \mu_{I,1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(\alpha p_1 - p_2) + \frac{1}{2}\omega & 0 \\ 0 & -\frac{1}{2}(\alpha p_1 - p_2) - \frac{1}{2}\omega \end{pmatrix} \Delta t$ is the diagonal matrix of eigenvalues of $\mathbf{P}.\mathbf{A}$**,** where $\omega = \sqrt{(\alpha p_1 - p_2)^2 + 4(\beta\gamma + \alpha)p_1 p_2}$ and $\mathbf{Q} = \begin{pmatrix} \frac{\beta}{\Lambda_1 - \alpha} & \frac{\beta}{\Lambda_2 - \alpha} \\ 1 & 1 \end{pmatrix}$, the corresponding eigenvector matrix.

If the transformation $\mathbf{v} = \mathbf{Q}^{-1}.\mathbf{u}$ is defined then Equation (M2.2) can be re-written to:

$$\mathbf{Q}.\mathbb{E}(\Delta \mathbf{v}) = \mathbf{Q}(\mathbf{M}.\mathbf{v} + \mathbf{Q}^{-1}.\mathbf{P}.\mathbf{b}) + \mathbf{P}'.\boldsymbol{\varphi} \quad \text{(M2.3)}$$

Furthermore, define a new probability matrix $\mathbf{P}'' = \mathbf{Q}^{-1}.\mathbf{P}$ so that Equation (M2.3) becomes:

$$\mathbb{E}(\Delta \mathbf{v}) = \mathbf{M}.\mathbf{v} + \mathbf{P}''.\mathbf{b} + \mathbf{P}'.\boldsymbol{\varphi} \quad \text{(M2.4)}$$

In Equation (M2.5) replace $\mathbf{P}''.\mathbf{b} + \mathbf{P}'.\boldsymbol{\varphi}$ by the vector $\boldsymbol{\mu} = \begin{pmatrix} \mu_{C,2} \\ \mu_{I,2} \end{pmatrix}$ which leads to Equation (M2.5):

$$\mathbb{E}(\Delta \mathbf{v}) = \mathbf{M}.\mathbf{v} + \boldsymbol{\mu} \quad \text{(M2.5)}$$

which is linear *diagonal* in variable $\mathbf{v}$ with elements

$$\mathbb{E}\Delta C'_{\Delta t} = \mu_{C,1} C'_t \Delta t + \mu_{C,2}\, \Delta t \tag{M2.6a}$$

$$\mathbb{E}\Delta I'_{\Delta t} = \mu_{I,1} I'_t \Delta t + \mu_{I,2}\, \Delta t \tag{M2.6b}$$

Deriving the variance of the decoupled system

First derive an expression for the variance-covariance matrix of the coupled system S {$\Delta C_{\Delta t}$, $\Delta I_{\Delta t}$} by using the fact that $\mathbb{V}\Delta\mathbf{u}_{\Delta t} = \mathbb{E}[(\Delta\mathbf{u}_{\Delta t} - \mathbb{E}\Delta\mathbf{u}_{\Delta t}).(\Delta\mathbf{u}_{\Delta t} - \mathbb{E}\Delta\mathbf{u}_{\Delta t})^{T}] =$

$$\mathbb{E}\left[\Delta\mathbf{u}_{\Delta t}\Delta\mathbf{u}_{\Delta t}^{T}\right] - \mathbb{E}\Delta\mathbf{u}_{\Delta t}.\mathbb{E}\Delta\mathbf{u}_{\Delta t}^{T} - \mathbb{E}\Delta\mathbf{u}_{\Delta t}^{T}.\mathbb{E}\Delta\mathbf{u}_{\Delta t} + \mathbb{E}\Delta\mathbf{u}_{\Delta t}.\mathbb{E}\Delta\mathbf{u}_{\Delta t}^{T} \tag{M2.7}$$

It is easy to see that the last three terms of Equation (M2.9) are all of the order $\Delta^2 t$ and since $\Delta^2 t$ approaches 0 in a small-time Δt, the variance-covariance matrix in Equation (M2.7) can be approximated by

$$\mathbb{V}\Delta\mathbf{u}_{\Delta t} \approx \mathbb{E}\left[\Delta\mathbf{u}_{\Delta t}.\Delta\mathbf{u}_{\Delta t}^{T}\right] \tag{M2.8}$$

Consistent with the Markov State-Change matrix {ΔC_t, ΔI_t} defined in Section 2-4, the variance-covariance matrix of Equation (M2.8) can be expressed as follows:

$$\mathbb{V}\Delta\mathbf{u}_{\Delta t} = \mathbf{P}.(\mathbf{A}.\mathbf{u}_{\Delta t} + \mathbf{b})(\mathbf{A}.\mathbf{u}_{\Delta t} + \mathbf{b})^{\mathbf{T}} + \mathbf{\Phi}.\mathbf{P}' \tag{M2.9}$$

For notational convenience, the subscript Δt will be dropped from $\Delta\mathbf{u}_{\Delta t}$.

Now, diagonalise Equation (M2.9) by replacing $\mathbf{P}.\mathbf{A} = \mathbf{Q}.\mathbf{M}.\mathbf{Q}^{-1}$ as follows:

$$\mathbb{V}\Delta\mathbf{u} = (\mathbf{Q}.\mathbf{M}.\mathbf{Q}^{-1}.\mathbf{u} + \mathbf{P}.\mathbf{b})(\mathbf{P}^{-1}\mathbf{Q}.\mathbf{M}.\mathbf{Q}^{-1}.\mathbf{u} + \mathbf{b})^{\mathbf{T}} + \mathbf{\Phi}.\mathbf{P}' \tag{M2.10}$$

If the transformation $\mathbf{v} = \mathbf{Q}^{-1}.\mathbf{u}$ is defined then Equation (M2.10) can be re-written to:

$$\mathbb{V}\Delta\mathbf{u} = [\mathbf{Q}.(\mathbf{M}.\mathbf{v} + \mathbf{Q}^{-1}.\mathbf{P}.\mathbf{b})][\mathbf{P}^{-1}.\mathbf{Q}.(\mathbf{M}.\mathbf{v} + \mathbf{Q}^{-1}.\mathbf{b}]^{\mathbf{T}} + \mathbf{\Phi}.\mathbf{P}' \tag{M2.11}$$

Using the fact that $\mathbb{V}\Delta\mathbf{u} = \mathbf{Q}.\mathbb{V}\Delta\mathbf{v}.\mathbf{Q}^{\mathbf{T}}$, Equation (M2.11) becomes:

$$\mathbb{V}\Delta\mathbf{v} = [(\mathbf{M}.\mathbf{v} + \mathbf{Q}^{-1}.\mathbf{P}.\mathbf{b}][(\mathbf{M}.\mathbf{v} + \mathbf{Q}^{-1}.\mathbf{b}]^{\mathbf{T}}\mathbf{Q}^{-1}.\mathbf{P}^{\mathbf{T}^{-1}}.\mathbf{Q}^{\mathbf{T}^{-1}} + \mathbf{Q}^{-1}.\mathbf{\Phi}.\mathbf{P}'.\mathbf{Q}^{\mathbf{T}^{-1}} \tag{M2.12}$$

Replace (i) matrix $\mathbf{Q}^{-1}.\mathbf{P}^{\mathbf{T}^{-1}}.\mathbf{Q}^{\mathbf{T}^{-1}}$ by a transformed *diagonal* probability matrix $\mathbf{P}'''$, (ii) matrix $\mathbf{Q}^{-1}.\mathbf{\Phi}.\mathbf{P}'.\mathbf{Q}^{\mathbf{T}^{-1}}$ by a transformed dependency matrix $\mathbf{\Phi}'$ and (iii) define a new variable $\mathbf{x} = \mathbf{v} + \mathbf{M}^{-1}.\mathbf{Q}^{-1}.\mathbf{P}.\mathbf{b}$. Notice that since $\mathbf{M}^{-1}.\mathbf{Q}^{-1}.\mathbf{P}.\mathbf{b}$ is a constant, $\mathbb{V}\Delta\mathbf{x} = \mathbb{V}\Delta\mathbf{v}$.

After above substitutions, the transformed variance-covariance matrix becomes:

$$\mathbb{V}\Delta\mathbf{x} = \mathbf{M}.\mathbf{x}.(\mathbf{M}.\mathbf{x} + \mathbf{b}')^{\mathbf{T}}.\mathbf{P}''' + \mathbf{\Phi}' \tag{M2.13}$$

or

$$\mathbb{V}\Delta\mathbf{x} = \mathbf{M}.\mathbf{x}.\mathbf{x}^{\mathbf{T}}.\mathbf{M}^{\mathbf{T}}.\mathbf{P}''' + \mathbf{M}.\mathbf{x}.\mathbf{b}'^{\mathbf{T}}.\mathbf{P}''' + \mathbf{\Phi}' \tag{M2.14}$$

where $\mathbf{b}' = \mathbf{P^{-1}}.\mathbf{b}$. The elements of matrix $\mathbf{x}.\mathbf{x^T}$ are $\begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix}$. Recall that in Section 2-4 the dependency between simultaneous changes $\Delta C_{\Delta t}$ and $\Delta I_{\Delta t}$ in the coupled system was described as $\frac{\Delta C_{\Delta t}}{\Delta I_{\Delta t}} = \varphi$. This property will now be used to define $\Delta\mathbf{u}_{\Delta t} = \begin{pmatrix} \Delta C_{\Delta t} \\ \Delta I_{\Delta t} \end{pmatrix} = \begin{pmatrix} 1 \\ \varphi \end{pmatrix}$ which implies that $\Delta\mathbf{v}_{\Delta t} = \mathbf{Q}\begin{pmatrix} 1 \\ \varphi \end{pmatrix}$ and also $\Delta\mathbf{x}_{\Delta t} = \mathbf{Q}\begin{pmatrix} 1 \\ \varphi \end{pmatrix}$, again only for *simultaneous changes* of $\Delta C_{\Delta t}$ and $\Delta I_{\Delta t}$. Hence, the elements of the vector $\mathbf{x}_{\Delta t}$ are linked to each other by the vector $\mathbf{Q}\begin{pmatrix} 1 \\ \varphi \end{pmatrix} + \mathbf{k}$, where $\mathbf{k}$ is a vector of constants $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$. Replace $\mathbf{Q}\begin{pmatrix} 1 \\ \varphi \end{pmatrix} + \mathbf{k}$ by another vector of constants $\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$. Matrix $\mathbf{x}.\mathbf{x^T}$ can now be re-written to: $\begin{pmatrix} x_1^2 & \frac{m_2}{m_1}x_1^2 \\ \frac{m_1}{m_2}x_2^2 & x_2^2 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\begin{pmatrix} x_1 & \frac{m_2}{m_1}x_1 \\ \frac{m_1}{m_2}x_2 & x_2 \end{pmatrix}$.

In Equation (M2.14), $\mathbb{V}\Delta x$ can be replaced by $\mathbb{V}\Delta v$ and $\mathbf{x}$ by $\mathbf{v} + \mathbf{n}$ where $\mathbf{n}$ is a vector of constants $\mathbf{n} = \mathbf{M^{-1}}.\mathbf{Q^{-1}}.\mathbf{P}.\mathbf{b}$.

Since $\mathbf{M}$ and $\mathbf{P}'''$ are diagonal matrices, it is evident that the first RHS term of Equation (M2.14) admits a *quadratic diagonal* in variable $\mathbf{v} + \mathbf{n}$, and the second RHS term is *linear diagonal* in variable $\mathbf{v} + \mathbf{n}$. Consequently, $\mathbb{V}\Delta\mathbf{x}$ can be expressed as a matrix $\mathbf{\Lambda}$ with both first-row elements quadratic in C_t' and both second-row elements quadratic in I_t':

$$\mathbb{V}\Delta v = \mathbf{\Lambda} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \tag{M2.15}$$

where

$$\Lambda_{11} = \varsigma_{C,1}{C_t'}^2\Delta t + \varsigma_{C,2}C_t'\Delta t + \varsigma_{C,3}\Delta t \tag{M2.16a}$$

$$\Lambda_{12} = \varsigma_{C,4}{C_t'}^2\Delta t + \varsigma_{C,5}C_t'\Delta t + \varsigma_{C,6}\Delta t \tag{M2.16b}$$

$$\Lambda_{21} = \varsigma_{I,1}{I_t'}^2\Delta t + \varsigma_{I,2}I_t'\Delta t + \varsigma_{I,3}\,\Delta t \tag{M2.16c}$$

$$\Lambda_{22} = \varsigma_{I,4}{I_t'}^2\Delta t + \varsigma_{I,5}I_t'\Delta t + \varsigma_{I,6}\,\Delta t \tag{M2.16d}$$

Appendix M3 – Testing existence, continuity and convergence of the general cash flow process

Continuity and existence of solutions

For a general cash flow process $dX_t = \alpha(X_t, t)dt + \sqrt{\beta(X_t, t)}dW_t$, where $\alpha(X_t, t)$ and $\sqrt{\beta(X_t, t)}$ are adapted to the filtration generated by the Brownian motion W_t, solutions are unique and Lipschitz-continuous (Øksendal (2003, section 5.2.) if:

$$\left|\alpha\left(X_{t,2}, t\right) - \alpha\left(X_{t,1}, t\right) + \sqrt{\beta\left(X_{t,2}, t\right)} - \sqrt{\beta\left(X_{t,1}, t\right)}\right| \leq K_1|X_{t,2} - X_{t,1}| \tag{M3.1a}$$

$$\left|\alpha(X_t, t) + \sqrt{\beta(X_t, t)}\right| \leq K_2(1 + |X_t|) \tag{M3.1b}$$

where $X_{t,2}, X_{t,1} \in \mathbb{R}$ and K_1, K_2 are some constants.

For the functions $\alpha(X_t, t) = \mu_1 X_t + \mu_0$ and $\sqrt{\beta(X_t, t)} = \sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)}$ it can be shown that the process is indeed unique and continuous.
First, function $\sqrt{\beta(X_t, t)}$ can be expressed as $\sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)} = \sqrt{\sigma_2}X_t\sqrt{(1 + \frac{\sigma_1}{\sigma_2}X_t^{-1} + \frac{\sigma_0}{\sigma_2}X_t^{-2})}$. Provided that σ_2 is not very small relative to σ_1, the square root term can be approximated by a Taylor expansion[57]: $\sqrt{(1 + \frac{\sigma_1}{\sigma_2}X_t^{-1} + \frac{\sigma_0}{\sigma_2}X_t^{-2})} \approx 1 + \frac{\sigma_1}{2\sigma_2}X_t^{-1} + \mathcal{O}(X_t^{-2})$. Therefore, the following linear function serves as an approximation of the diffusion function: $\sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)} \approx \sqrt{\sigma_2}X_t + \frac{\sigma_1}{\sqrt{\sigma_2}}$.

Now, Equation (M3.1a) becomes

$$\mu_1|X_{t,2} - X_{t,1}| + \sqrt{\sigma_2}|X_{t,2} - X_{t,1}| \leq K_1|X_{t,2} - X_{t,1}| \tag{M3.1c}$$

and it is easy to set a value for the constant K_1 such that Equation (M3.1c) is true.
Similarly, Equation (M3.1b) can be written as

$$\mu_1|X_t| + \mu_0 + \sqrt{\sigma_2}|X_t| + \frac{\sigma_1}{\sqrt{\sigma_2}} \leq K_2(1 + |X_t|) \tag{M3.1d}$$

Since the LHS and RHS of Equation (M3.1d) are two straight lines, the following must hold: $K_2 > \mu_1 + \sqrt{\sigma_2}$ **and** $K_2 > \mu_0 + \frac{\sigma_1}{\sqrt{\sigma_2}}$. Again, it is not difficult to set a value for the constant K_2 that obeys Equation (M3.1d).

Admittedly, the above continuity test is only approximately valid. However, Ait-Sahalia (1996, p. 415) shows that strong solutions to the cash flow process studied exist under considerably less strict conditions then the Lipschitz conditions:

(i) The drift function $\alpha(X_t)$ and diffusion function $\sqrt{\beta(X_t)}$ are continuously differentiable in X_t and $\beta(X_t)$ is strictly positive on the whole state space;

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[57] $\sqrt{1+Z} \approx 1 + \frac{Z}{2} + \mathcal{O}(Z^2)$ where $Z = \frac{\sigma_1}{\sigma_2}X_t^{-1}$. The smaller Z is, the better the approximation. In this case Z is indeed small: cash flows X_t are normally sizeable amounts, and it was assumed that σ_2 is not very small relative to σ_1.

(ii) The integral of the scale density of the process, $\mathbb{s}(X_t) = \exp\left[-\int \frac{2\alpha(X_t)}{\beta(X_t)} dX_t\right]$, diverges at both boundaries of the diffusion state space;

(iii) The integral of the speed density of the process $\mathbb{m}(X_t) = \frac{2}{\beta(X_t)\exp\left[-\int \frac{2\alpha(X_t)}{\beta(X_t)} dX_t\right]} = \frac{2}{\beta(X_t)\,\mathbb{s}(X_t)}$, converges at both boundaries of the diffusion state space.

Condition (i) is easily met whilst a diverging scale density and a converging speed density implies that $\int \frac{2\alpha(X_t)}{\beta(X_t)} dX_t$ must be smaller than 0. The expression $\int \frac{2\alpha(X_t)}{\beta(X_t)} dX_t = \int \frac{2(\mu_1 X_t + \mu_0)}{\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0} dX_t$ in (ii) and (iii) is closely linked to the integral encapsulated by Equation (2.14) in Section 2-3 for which there are two solutions:

(i) The case where the discriminant of $\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0$ is less than 0 corresponding to a process with complex roots (empirically found for investing cash flow processes); and

(ii) The case where the discriminant of $\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0$ is greater than or equal to 0, a process with real roots (empirically found for the majority of operating cash flow processes).

In case (i) the integral $\int \frac{2\alpha(X_t)}{\beta(X_t)} dX_t$, after a similar transform as applied to Equation (2.15) in Section 2-3, becomes:

$$\int \frac{2(\mu_1 X_t + \mu_0)}{\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0} dX_t = K\left[(X_t + \frac{\sigma_1}{2\sigma_2})^2 + \lambda^2\right]^{\nu_1} \exp\left[\nu_2 \tan^{-1}\left[\frac{X_t + \frac{\sigma_1}{2\sigma_2}}{\lambda}\right]\right] \qquad \text{(M3.2a)}$$

where $\nu_1 = \frac{\mu_1}{\sigma_2 \lambda^2}$, $\nu_2 = \frac{\mu_0 - \frac{\sigma_1}{\sigma_2}}{\sigma_2 \lambda}$, $\lambda = \frac{\sqrt{4\sigma_0\sigma_2 - \sigma_1^2}}{2\sigma_2} > 0$ and K is an integrating constant.

For $K < 0$ Equation (M3.2a) is smaller than 0 and obeys the Aït-Sahalia conditions stated above.

In case (ii) the integral evaluates to:

$$\int \frac{2(\mu_1 X_t + \mu_0)}{\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0} dX_t = K\left[(X_t - \lambda_1)^{-(a\lambda_1 + b)\nu_3}(X_t - \lambda_2)^{(a\lambda_2 + b)\nu_3}\right] \qquad \text{(M3.2b)}$$

where $\lambda_{1,2} = \frac{-\sigma_1 \pm \sqrt{\sigma_1^2 - 4\sigma_0\sigma_2}}{2\sigma_2}$ are real roots of $\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0$, $a = -2(\mu_1 - \sigma_2)$, $b = 2(\mu_1 - \sigma_2)$, $\nu_3 = \frac{1}{\lambda_1 - \lambda_2}$ and K is an integrating constant. Again, if $K < 0$ with $\lambda_1 < X_t < \lambda_2$, Equation (M3.2b) is smaller than 0 and Aït-Sahalia's conditions are met.

Convergence and divergence of the process

Derivation of the hybrid and coupled model in Chapter 3, led to a system of continuous-time equations with a linear drift function and a quadratic diffusion function for operating and investing cash flow processes alike. The (uncoupled) general process equation is:

$$\mathrm{dX_t} = (\mu_1 \mathrm{X_t} + \mu_0)\mathrm{dt} + \sqrt{(\sigma_2 \mathrm{X_t^2} + \sigma_1 \mathrm{X_t} + \sigma_0)}\,\mathrm{dW_t} \tag{M3.3}$$

where $\mathrm{X_t}$ is some cash flow process.
One of the conditions under which a stochastic process converges, is the mean-square criterion (Pollard (1984)), i.e. $\lim_{t\to\infty} \mathbb{E}(\mathrm{X_t} - \mathrm{X_\infty})^2 = 0$ (M3.4)
where $-\infty < \mathrm{X_\infty} < \infty$ is a (finite) stable cash flow. This convergence criterion will be used to examine under which conditions the cash flow process governed by Equation (M3.3) is converging.

First, write Equation (M3.4) in alternative form: $\lim_{t\to\infty} \mathbb{E}(\mathrm{X_t^2}) = \lim_{t\to\infty}(\mathbb{E}^2\mathrm{X_\infty})$ (M3.5)
From the solution to the ODE governing the evolution of the first moment, it follows that for the expression $\lim_{t\to\infty}(\mathbb{E}^2\mathrm{X_\infty})$ to have a finite value, the drift function must be mean-reverting. In other words, $\mathbb{E}\mathrm{X_t} = \mathrm{K}e^{\mu_1 t} - \frac{\mu_0}{\mu_1}$, where K is an integration constant, and $\lim_{t\to\infty}(\mathbb{E}^2\mathrm{X_\infty})$ has a finite value $\frac{\mu_0{}^2}{\mu_1}$ only if $\mu_1 \leq 0$.
A further condition of Equation (M3.3) is that the diffusion must be also converging. To investigate this, Equation (M3.5) needs to be recast into a differential time format:

$$\lim_{t\to\infty} \frac{\mathrm{d}\mathbb{E}(\mathrm{X_t^2})}{\mathrm{dt}} = \lim_{t\to\infty} \frac{\mathbb{E}\mathrm{dX_t^2}}{\mathrm{dt}} = 0 \tag{M3.6}$$

Applying Itô's lemma, $\mathrm{dX_t^2}$ becomes:

$$\mathrm{dX_t^2} = [(2\mu_1 + \sigma_2)\mathrm{X_t^2} + (2\mu_0 + \sigma_1)\mathrm{X_t} + \sigma_0]\mathrm{dt} + 2\mathrm{X_t}\sqrt{(\sigma_2 \mathrm{X_t^2} + \sigma_1 \mathrm{X_t} + \sigma_0)}\,\mathrm{dW_t} \tag{M3.7}$$

Hence,

$$\frac{\mathbb{E}\mathrm{dX_t^2}}{\mathrm{dt}} = (2\mu_1 + \sigma_2)\mathrm{X_t^2} + (2\mu_0 + \sigma_1)\mathrm{X_t} + \sigma_0 \tag{M3.8}$$

and

$$\lim_{t\to\infty} \frac{\mathbb{E}\mathrm{dX_t^2}}{\mathrm{dt}} = (2\mu_1 + \sigma_2)\mathrm{X_\infty^2} + (2\mu_0 + \sigma_1)\mathrm{X_\infty} + \sigma_0 = 0 \tag{M3.9}$$

Equation (M3.9) is true for every $\mathrm{X_\infty}$ since the quadratic expression always has one or two real roots, or distinctive complex roots. Therefore, the diffusion function of Equation (M3.3) will converge in the mean-square sense. However, if $\mathrm{X_\infty}$ is infinitely large because of a diverging drift function as $t \to \infty$, the diffusion function will be overridden by the drift function and will also become infinitely large.

Appendix M4 – Approximate Maximum Likelihood Estimation method applied to the Linear-Quadratic Model

This appendix includes calculations that support derivation of the Approximate Maximum Likelihood Estimation Method (AMLE) in Section 5-2.

A. Calculation of Hermite Polynomials and corresponding Fourier coefficients

Hermite polynomials are calculated from $\mathcal{H}_j(z) = (-1)^n e^{\frac{z^2}{2}} \frac{d^n e^{-\frac{z^2}{2}}}{dz^n}$ for $n = 0..J$.

The expansion (Fourier) coefficients are conditional first moments of the Hermite polynomial terms:

$$\eta_Z^j(\Delta, z|y_0; \boldsymbol{\theta}) = \frac{1}{j!}\mathbb{E}_{\boldsymbol{\theta}}\left[\mathcal{H}_j\left(\frac{Y_t - Y_0}{\sqrt{\Delta}}\right)|Y_t = y_0\right] \qquad \text{(M4A. 1)}$$

where $z = \frac{y-y_0}{\sqrt{\Delta}}$ is close to a standard normal variable $\mathcal{N}(0,1)$. As $J \to \infty$, an exact approximation of the transformed transition density function $P_Z^J(\Delta, z|y_0; \boldsymbol{\theta})$ is obtained

$$P_Z(\Delta, z|y_0; \boldsymbol{\theta}) = \mathcal{N}(0,1)\sum_{j=0}^{\infty}\eta_Z^j(\Delta, z|y_0; \boldsymbol{\theta})\mathcal{H}_j(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\sum_{j=0}^{\infty}\left\{\frac{1}{j!}\mathbb{E}_{\boldsymbol{\theta}}\left[\mathcal{H}_j\left(\frac{Y_t-Y_0}{\sqrt{\Delta}}\right)|Y_t = y_0\right]\mathcal{H}_j(z)\right\} \qquad \text{(M4A. 2)}$$

For practical estimation purposes, Equation (M4A. 2) will need to be truncated. In this study, a five-term approximation, that is $J = 4$, is considered to be sufficiently accurate. Hence, the approximated density function becomes

$$P_Z^4(\Delta, z|y_0; \boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\sum_{j=0}^{4}\eta_Z^j(\Delta, z|y_0; \boldsymbol{\theta})\mathcal{H}_j(z) \qquad \text{(M4A. 3)}$$

The corresponding Hermite polynomials, expanded to the fifth term, yield

$$\mathcal{H}_0(z) = 1 \qquad \text{(M4A. 4a)}$$

$$\mathcal{H}_1(z) = z \qquad \text{(M4A. 4b)}$$

$$\mathcal{H}_2(z) = z^2 - 1 \qquad \text{(M4A. 4c)}$$

$$\mathcal{H}_3(z) = z^3 - 3z \qquad \text{(M4A. 4d)}$$

$$\mathcal{H}_4(z) = z^4 - 6z^2 + 3 \qquad \text{(M4A. 4e)}$$

Since variable z is close to a standard normal distribution, with mean zero and unit standard deviation, it follows that $\mu_z = \mathbb{E}(z) = 0$ and $\sigma_z = \mathbb{E}(z^2) - \mathbb{E}^2(z) = \mathbb{E}(z^2) = 1$. This result is useful when calculating the first conditional moments of Equations (M4A. 4a) – (M4A. 4e).

$$\mathbb{E}[\mathcal{H}_0(z)] = 1 \qquad \text{(M4A. 5a)}$$

$$\mathbb{E}[\mathcal{H}_1(z)] = \mathbb{E}(z\,) = 0 \qquad \text{(M4A. 5b)}$$

$$\mathbb{E}[\mathcal{H}_2(\mathrm{z})] = \mathbb{E}(\mathrm{z}^2) - 1 = 0 \qquad \text{(M4A. 5c)}$$

$$\mathbb{E}[\mathcal{H}_3(\mathrm{z})] = \mathbb{E}(\mathrm{z}^3) - 3\mathbb{E}(\mathrm{z}) = \mathbb{E}(\mathrm{z}^3) \qquad \text{(M4A. 5d)}$$

$$\mathbb{E}[\mathcal{H}_4(\mathrm{z})] = \mathbb{E}(\mathrm{z}^4) - 6\mathbb{E}(\mathrm{z}^2) + 3 = \mathbb{E}(\mathrm{z}^4) - 3 \qquad \text{(M4A. 5e)}$$

Equations (M4A. 5d) and (M4A. 5e) can be perceived as an adjustment to the leading Gaussian term for excess skewness and kurtosis. For a true Gaussian random variable $\mathrm{P}_\mathrm{Z}^4(\Delta, \mathrm{z}|\mathrm{y}_0, \boldsymbol{\theta})$, skewness ($\mathbb{E}(\mathrm{z}^3)$) and kurtosis ($\mathbb{E}(\mathrm{z}^4) - 3$) are zero and Equations (M4A. 5d) and (11e) effectively vanish from the Hermite expansion, reducing $\mathrm{P}_\mathrm{Z}^4(\Delta, \mathrm{z}|\mathrm{y}_0; \boldsymbol{\theta})$ to $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\mathrm{z}^2}{2}}$ (Singer (2006, p. 388).

After substituting Equations (M4A. 4a) – (M4A. 4e) and Equations (M4A. 5a) – (M4A. 5e) into Equation (M4A. 3), the approximated transitional density function $\mathrm{P}_\mathrm{Z}^\mathrm{J}$ for $\mathrm{J} = 4$, satisfies

$$\mathrm{P}_\mathrm{Z}^4(\Delta, \mathrm{z}|\mathrm{y}_0; \boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\mathrm{z}^2}{2}}\Big[1 - \frac{1}{8}(\mathrm{z}^4 - 6\mathrm{z}^2 + 3) + \frac{1}{6}\,(\mathrm{z}^3 - 3\mathrm{z}\,)\mathbb{E}(\mathrm{z}^3, \boldsymbol{\theta}) + \frac{1}{24}(\mathrm{z}^4 - 6\mathrm{z}^2 + 3)\mathbb{E}(\mathrm{z}^4, \boldsymbol{\theta})\,\Big] \qquad \text{(M4A. 6)}$$

B. Solving the system of central moment ODEs

Applying the first backward transformation in Equation (5.25), the transitional density function expressed in Y is calculated as

$$\mathrm{P}_\mathrm{Y}^4(\Delta, \mathrm{y}|\mathrm{y}_0; \boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\check{\mathrm{y}}^2\Delta}{2}}\Delta^{\frac{1}{2}}\Big[1 - \frac{1}{8}(\Delta^2\check{\mathrm{y}}^4 - 6\Delta\check{\mathrm{y}}^2 + 3) + \frac{1}{6}(\Delta^{\frac{3}{2}}\check{\mathrm{y}}^3 - 3\Delta^{\frac{1}{2}}\check{\mathrm{y}}\,)\Delta^{\frac{3}{2}}\mathbb{E}(\check{\mathrm{y}}^3; \boldsymbol{\theta}) + \frac{1}{24}(\Delta^2\check{\mathrm{y}}^4 - 6\Delta\check{\mathrm{y}}^2 + 3)\Delta^2\mathbb{E}(\check{\mathrm{y}}^4; \boldsymbol{\theta})\,\Big] \qquad \text{(M4B. 1)}$$

where $\check{\mathrm{y}} = \mathrm{y} - \mathrm{y}_0$.

To simplify calculations, from here on Δ will be set to 1. Accordingly, parameter values are calculated on a per quarter basis and, if required, can be rescaled to annual parameter values.

Recall that y_0 is considered a proxy for the average of random variable Y, and consequently the conditional moments $\mathbb{E}(\check{\mathrm{y}}^3; \boldsymbol{\theta})$ and $\mathbb{E}(\check{\mathrm{y}}^4; \boldsymbol{\theta})$ are to be interpreted as *central* moments. Furthermore, it is assumed that $\check{\mathrm{y}}_0 \approx \mathcal{N}(0,1)$.

Now, replacements for the two conditional central moments $\mathbb{E}(\check{\mathrm{y}}^3; \boldsymbol{\theta})$ and $\mathbb{E}(\check{\mathrm{y}}^4; \boldsymbol{\theta})$ have to be found such that the third and fourth moments are explicitly expressed in parameter vector $\boldsymbol{\theta}$. From Section 5-2, Equations (5.19) and (5.20), it becomes apparent that the applicable SDE equates to $\mathrm{dY_t} = \tilde{\mu}(\mathrm{Y_t})\mathrm{dt} + \mathrm{dW_t}$. Using the Fokker-Planck equation, Singer (2006, Appendix) proves that the following system of ODEs holds for associated conditional density functions:

$$\frac{d\mathbb{E}(\check{y};\boldsymbol{\theta})}{dt} = \mathbb{E}[\tilde{\mu}(Y_t)] \tag{M4B.2}$$

governing the first conditional moment, and

$$\frac{d\mathbb{E}(\check{y}^n;\boldsymbol{\theta})}{dt} = n\mathbb{E}\,\{\tilde{\mu}(Y_t)[\check{y}^{n-1} - \mathbb{E}(\check{y}^{n-1};\boldsymbol{\theta})]\} + \frac{1}{2}n(n-1)\,\mathbb{E}(\check{y}^{n-2};\boldsymbol{\theta}) \tag{M4B.3}$$

applicable to higher conditional moments ($n \geq 2$)

where $\tilde{\mu}(Y_t) = \frac{\left((\mu_1-\frac{1}{2}\sigma_2)F^{-1}(X_t)+(\mu_0-\frac{1}{4}\sigma_1)\right)}{\sqrt{\sigma_2[F^{-1}(X_t)]^2+\sigma_1 F^{-1}(X_t)+\sigma_0}}$ and $F(X_t) = \frac{1}{\sqrt{\sigma_2}}\ln\left|\sqrt{(\sigma_2 X_t^2 + \sigma_1 X_t + \sigma_0)} + \frac{2\sigma_2 X_t+\sigma_1}{2\sqrt{\sigma_2}}\right|$.

Obviously, function $\tilde{\mu}(Y_t)$ is a rather complex expression to employ in the calculations to follow. Therefore, $\tilde{\mu}(Y_t)$ will be approximated by a Taylor series expanded around $y = y_0$, up to a linear term[58].

$$\begin{aligned}\tilde{\mu}(Y_t) &= \pi_0(\boldsymbol{\theta};y_0) + \pi_1(\boldsymbol{\theta})(y-y_0) + \mathcal{O}(y^2)\\ &\approx \pi_0(\boldsymbol{\theta};y_0) + \pi_1(\boldsymbol{\theta};y_0)\check{y}\end{aligned} \tag{M4B.4}$$

where $\pi_0(\boldsymbol{\theta};y_0) = \frac{(\mu_1-\frac{1}{2}\sigma_2)y_0+(\mu_0-\frac{1}{4}\sigma_1)}{(\sigma_2 y_0^2+\sigma_1 y_0+\sigma_0)^{\frac{1}{2}}}$ and

$$\pi_1(\boldsymbol{\theta};y_0) = \frac{\left(\frac{1}{2}\mu_1\sigma_1-\mu_0\sigma_2-\frac{1}{4}\sigma_1\sigma_2+\frac{1}{4}\sigma_2^2\right)y_0-\frac{1}{2}\mu_0\sigma_1+\mu_1\sigma_0-\frac{1}{2}\sigma_0\sigma_2+\frac{1}{8}\sigma_1\sigma_2}{(\sigma_2 y_0^2+\sigma_1 y_0+\sigma_0)^{\frac{3}{2}}}.$$

The indicative example in Table 7-1 shows that a linear approximation of $\tilde{\mu}(Y_t)$ is reasonably accurate for most cash flow values[59]; clearly, for the presumed parameter values, the quality of the approximations of investing cash flows is higher than those of operating cash flows. Note that the parameter values are estimates reported in the first part of Section 5-2.

[58] Any higher-order approximation will make the system of moment ODEs non-recursive and thus unsolvable.

[59] Firms investigated in this study, show an average operating cash flow (uncoupled) of about 307 (STD 2816) and an average investing cash flow (uncoupled) of approximately 428 (STD 4751). All amounts are reported in millions of US$.

Parameters OCF		Operating Cash Flow						Investing Cash Flow					
m1	0.0552	x	y	Actual drift function	Approximated drift function	Variance	% variance	x	y	Actual drift function	Approximated drift function	Variance	% variance
m0	-141.69	-100000000	22.39338359	-1.821484296	-1.791888091 -	0.029596	1.62%	-100000000	22.26634301	-0.91213505	-0.91273718	0.00060213	-0.07%
s2	0.7018	-10000000	19.6448107	-1.736125707	-1.729920429 -	0.006205	0.36%	-10000000	19.53582934	-0.911767121	-0.912295873	0.000528752	-0.06%
s1	-145.85	-1000000	16.8963373	-1.657769624	-1.667955011	0.010185	-0.61%	-1000000	16.81083912	-0.911400065	-0.911855459	0.000455394	-0.05%
s0	6684.787946	-100000	14.14885824	-1.585559631	-1.60601201	0.020452	-1.29%	-100000	14.14084869	-0.911040542	-0.911423934	0.000383392	-0.04%
D	2506.685778	-10000	11.4112608	-1.518967081	-1.544291795	0.025325	-1.67%	-10000	11.9502754	-0.910745666	-0.911069893	0.000324227	-0.04%
Lambda1	68.24108497	-1000	8.766763926	-1.459145595	-1.484670568	0.025525	-1.75%	-1000	11.18285904	-0.910642383	-0.910945863	0.00030348	-0.03%
Lambda1	139.5816566	-100	6.727785712	-1.415744792	-1.438700999	0.022956	-1.62%	-100	11.08769084	-0.910629575	-0.910930481	0.000300906	-0.03%
y0	0	-10	5.984027933	-1.400461447	-1.421932686	0.021471	-1.53%	-10	11.07810181	-0.910628285	-0.910928932	0.000300647	-0.03%
pi0	-1.28702041	-1	5.87278306	-1.398199722	-1.419424626	0.021225	-1.52%	-1	11.07714258	-0.910628156	-0.910928777	0.000300621	-0.03%
pi1	-0.0225454	0	5.85967943	-1.397933721	-1.4191292	0.021195	-1.52%	0	11.077036	-0.910628141	-0.910928759	0.000300618	-0.03%
		1	5.846412652	-1.397664497	-1.418830095	0.021166	-1.51%	1	11.07692941	-0.910628127	-0.910928742	0.000300615	-0.03%
Parameters ICF		10	5.718861244	-1.395080588	-1.415954398	0.020874	-1.50%	10	11.07597012	-0.910627998	-0.910928587	0.000300589	-0.03%
m1	0	100	4.660410017	-1.373948737	-1.392091197	0.018142	-1.32%	100	11.06637496	-0.910626707	-0.910927036	0.00030033	-0.03%
m0	-8176.24	1000	8.519881474	-1.45376938	-1.479104506	0.025335	-1.74%	1000	10.97073688	-0.910613836	-0.910911579	0.000297743	-0.03%
s2	0.7108	10000	11.38669067	-1.518391702	-1.543737851	0.025346	-1.67%	10000	10.91850826	-0.910606807	-0.910903138	0.000296331	-0.03%
s1	-8176.37	100000	14.14640135	-1.585497568	-1.605956619	0.020459	-1.29%	100000	14.00470582	-0.911022213	-0.911401931	0.000379717	-0.04%
s0	45495150	1000000	16.89609161	-1.657762906	-1.667949472	0.010187	-0.61%	1000000	16.79719784	-0.911398228	-0.911853254	0.000455026	-0.05%
D	-62498784.1	10000000	19.64478613	-1.736124976	-1.729919875 -	0.006205	0.36%	10000000	19.53446518	-0.911766937	-0.912295653	0.000528715	-0.06%
y0	0	100000000	22.39338114	-1.821484216	-1.791888036 -	0.029596	1.62%	100000000	22.26620659	-0.912135032	-0.912737158	0.000602126	-0.07%
pi0	-0.90913848												
pi1	-0.00016162												

Table 7-1 Approximation of transformed drift function by a linear moment equation

After adopting the suggested linear approximation, the system of central moment ODEs is described by the following two equations

$$\frac{d\mathbb{E}(\check{y};\boldsymbol{\theta})}{dt} = \pi_0(\boldsymbol{\theta};\bar{y}_0) + \pi_1(\boldsymbol{\theta})\mathbb{E}(\check{y},\boldsymbol{\theta}) \tag{M4B.5a}$$

$$\begin{aligned}\frac{d\mathbb{E}(\check{y}^n;\boldsymbol{\theta})}{dt} &= n\mathbb{E}[(\pi_0(\boldsymbol{\theta};\bar{y}_0) + \pi_1(\boldsymbol{\theta};\bar{y}_0)\check{y})(\check{y}^{n-1} - \mathbb{E}(\check{y}^{n-1};\boldsymbol{\theta})] + \frac{1}{2}n(n-1)\mathbb{E}(\check{y}^{n-2};\boldsymbol{\theta}) \\ &= n\pi_1(\boldsymbol{\theta};\bar{y}_0)[\mathbb{E}(\check{y}^n;\boldsymbol{\theta}) - \mathbb{E}(\check{y},\boldsymbol{\theta})\mathbb{E}(\check{y}^{n-1};\boldsymbol{\theta})] \\ &+ \frac{1}{2}n(n-1)\mathbb{E}(\check{y}^{n-2};\boldsymbol{\theta})\end{aligned} \tag{M4B.5b}$$

Observe that y_0 is treated as a constant in the above equations, hence $\mathbb{E}[\pi_0(\boldsymbol{\theta};\bar{y}_0)] = \pi_0(\boldsymbol{\theta};\bar{y}_0)$ and $\mathbb{E}[\pi_1(\boldsymbol{\theta};\bar{y}_0)] = \pi_1(\boldsymbol{\theta};\bar{y}_0)\check{y}$. Moreover, $\bar{y}_0$ in the ODE calculations is considered the expected value of y_0 (being itself a stochastic variable), and hence $\mathbb{E}(y_0) = \bar{y}_0 = 0$ which is consistent with the prior assumption of $\check{y}_0 \approx \mathcal{N}(0,1)$ and $\mathbb{E}(\check{y}_0) \approx 0$.

The above system of ODEs is closed and thus solved recursively. The solution[60] to the first moment evolution equation satisfies

$$\mathbb{E}(\check{y},t;\boldsymbol{\theta}) = \frac{\pi_0(\boldsymbol{\theta})}{\pi_1(\boldsymbol{\theta})}\left[1 - e^{\pi_1(\boldsymbol{\theta})t}\right] \tag{M4B.6a}$$

Under the condition that $\pi_1(\boldsymbol{\theta}) < 0$, a condition seen for almost all investing cash flows and most operating cash flows[61], the stationary value (as $t \to \infty$) is

$$\mathbb{E}(\check{y},\infty;\boldsymbol{\theta}) = \frac{\pi_0(\boldsymbol{\theta})}{\pi_1(\boldsymbol{\theta})} \tag{M4B.6b}$$

Subsequently, using Equation (M4B.6b) a similar approximation will be applied to exponential terms of higher moments. Recognising that $\mathbb{E}(\check{y}^2,0,\boldsymbol{\theta}) = 1$, a solution to the ODE for $\mathbb{E}(\check{y}^2;\boldsymbol{\theta})$ is

$$\mathbb{E}(\check{y}^2,t;\boldsymbol{\theta}) = -\frac{\pi_0^2(\boldsymbol{\theta})}{\pi_1^2(\boldsymbol{\theta})} - \frac{1}{2\pi_1(\boldsymbol{\theta})} + e^{2\pi_1(\boldsymbol{\theta})t}\left[1 + \frac{\pi_0^2(\boldsymbol{\theta})}{\pi_1^2(\boldsymbol{\theta})} + \frac{1}{2\pi_1(\boldsymbol{\theta})}\right] \tag{M4B.6c}$$

with stationary value

$$\mathbb{E}(\check{y}^2,\infty;\boldsymbol{\theta}) = -\frac{\pi_0^2(\boldsymbol{\theta})}{\pi_1^2(\boldsymbol{\theta})} - \frac{1}{2\pi_1(\boldsymbol{\theta})} \tag{M4B.6d}$$

The solution to the ODE for the third central moment evolution equation is

[60] Notice that the solution to the system of ODEs is given in continuous-time despite the data being discretely sampled on an interval $\Delta = 1$. However, in the following derivations, the continuous-time representation will be replaced by discrete points in time.

[61] Nevertheless, the condition $\pi_1(\boldsymbol{\theta}) < \mathbf{0}$ will be imposed in the numerical optimisation routines applied to estimate the parameter vector $\boldsymbol{\theta}$.

$$\mathbb{E}(\check{y}^3,t;\boldsymbol{\theta}) = -\frac{\pi_0^3(\boldsymbol{\theta})}{\pi_1^3(\boldsymbol{\theta})} - \frac{3\pi_0(\boldsymbol{\theta})}{\pi_1^2(\boldsymbol{\theta})} + e^{3\pi_1(\boldsymbol{\theta})t}\left[\frac{\pi_0^3(\boldsymbol{\theta})}{\pi_1^3(\boldsymbol{\theta})} + \frac{3\pi_0(\boldsymbol{\theta})}{\pi_1^2(\boldsymbol{\theta})}\right] \tag{M4B.6e}$$

with stationary value

$$\mathbb{E}(\check{y}^3,\infty;\boldsymbol{\theta}) = -\frac{\pi_0^3(\boldsymbol{\theta})}{\pi_1^3(\boldsymbol{\theta})} - \frac{3\pi_0(\boldsymbol{\theta})}{\pi_1^2(\boldsymbol{\theta})} \tag{M4B.6f}$$

Finally, the fourth central moment evolution equation admits the following solution

$$\begin{aligned}\mathbb{E}(\check{y}^4,t;\boldsymbol{\theta}) = &-\frac{\pi_0^4(\boldsymbol{\theta})}{\pi_1^4(\boldsymbol{\theta})} - \frac{3\pi_0^2(\boldsymbol{\theta})}{2\pi_1^3(\boldsymbol{\theta})} - \frac{3}{4\pi_1^2(\boldsymbol{\theta})} \\ &+ e^{4\pi_1(\boldsymbol{\theta})t}\left[\frac{\pi_0^4(\boldsymbol{\theta})}{\pi_1^4(\boldsymbol{\theta})} + \frac{3\pi_0^2(\boldsymbol{\theta})}{2\pi_1^3(\boldsymbol{\theta})} + \frac{3}{4\pi_1^2(\boldsymbol{\theta})}\right]\end{aligned} \tag{M4B.6g}$$

again, with stationary value

$$\mathbb{E}(\check{y}^4,\infty;\boldsymbol{\theta}) = -\frac{\pi_0^4(\boldsymbol{\theta})}{\pi_1^4(\boldsymbol{\theta})} - \frac{3\pi_0^2(\boldsymbol{\theta})}{2\pi_1^3(\boldsymbol{\theta})} - \frac{3}{4\pi_1^2(\boldsymbol{\theta})} \tag{M4B.6h}$$

Plugging Equations (M4B. 6f)) and (M4B. 6h) into Equation (M4B. 1), leads to the following expression in which like terms are collected w.r.t. $\check{y}^n$, $n = 0\,..\,4$.

$$P_Y^4(\Delta = 1, y|y_0;\boldsymbol{\theta})) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\check{y}^2}{2}}[1 + \vartheta_0 + \vartheta_1\check{y} + \vartheta_2\,\check{y}^2 + \vartheta_3\,\check{y}^3 + \vartheta_4\,\check{y}^4] \tag{M4B.7}$$

where $\vartheta_0 = \left\{-\frac{3}{8} - \frac{1}{8}\alpha^4(\boldsymbol{\theta}) - \frac{3}{16}\alpha^2(\boldsymbol{\theta})\beta(\boldsymbol{\theta}) - \frac{3}{32}\beta^2(\boldsymbol{\theta})\right\}$, $\vartheta_1 = \left\{\frac{1}{2}\alpha^3(\boldsymbol{\theta}) + \frac{3}{2}\alpha(\boldsymbol{\theta})\beta(\boldsymbol{\theta})\right\}$, $\vartheta_2 = \left\{\frac{6}{8} + \frac{1}{4}\alpha^4(\boldsymbol{\theta}) + \frac{3}{8}\alpha^2(\boldsymbol{\theta})\beta(\boldsymbol{\theta}) + \frac{3}{16}\beta^2(\boldsymbol{\theta})\right\}$, $\vartheta_3 = \left\{-\frac{1}{6}\alpha^3(\boldsymbol{\theta}) - \frac{1}{2}\alpha(\boldsymbol{\theta})\beta(\boldsymbol{\theta})\right\}$ and $\vartheta_4 = \left\{-\frac{1}{8} - \frac{1}{24}\alpha^4(\boldsymbol{\theta}) - \frac{1}{16}\alpha^2(\boldsymbol{\theta})\beta(\boldsymbol{\theta}) - \frac{1}{32}\beta^2(\boldsymbol{\theta})\right\}$, $\alpha(\boldsymbol{\theta}) = \frac{\pi_0(\boldsymbol{\theta})}{\pi_1(\boldsymbol{\theta})}$ and $\beta(\boldsymbol{\theta}) = \frac{1}{\pi_1(\boldsymbol{\theta})}$.

Moreover, observe that $\vartheta_2 = -2\vartheta_0$, $\vartheta_3 = -\frac{1}{3}\vartheta_1$ and $\vartheta_4 = \frac{1}{3}\vartheta_0$ so that it is possible to express vector $\boldsymbol{\vartheta} = \begin{pmatrix}\vartheta_0\\ \vartheta_1\\ \vartheta_2\\ \vartheta_3\\ \vartheta_4\end{pmatrix}$ in a base of elements ϑ_0 and ϑ_1 only: $\begin{pmatrix}1 & 0\\ 0 & 1\\ -2 & 0\\ 0 & -\frac{1}{3}\\ \frac{1}{3} & 0\end{pmatrix}\begin{pmatrix}\vartheta_0\\ \vartheta_1\end{pmatrix}$. Also notice that functions $\alpha(\boldsymbol{\theta})$ and $\beta(\boldsymbol{\theta})$ are directly related to the first and second moments of the stationary probability function: $\alpha(\boldsymbol{\theta}) = \mathbb{E}(\check{y},\infty;\boldsymbol{\theta})$ and $\beta(\boldsymbol{\theta}) = -2[\mathbb{E}^2(\check{y},\infty;\boldsymbol{\theta}) - \mathbb{E}(\check{y}^2,\infty;\boldsymbol{\theta})]$.

Table 7-2 below provides information about the order of parameter values, given the parameter estimates found in the first part of Section 5-2.

Since the coefficient vector $\boldsymbol{\vartheta}$ is related to the probability density function transformed back from y back to x, the linear Taylor expansion of the Lamperti-transformed drift function should be expressed in x_0- values corresponding to $y_0 = 0$.

Table 7-2 Calculation of coefficients of polynomial function

Parameters OCF		**Operating Cash Flow**			
μ_1	0.0552	α	β	ϑ_0	ϑ_1
μ_0	-141.69	-68.00	-1381.06	-2190861.83	-16360.31
σ_2	0.7018				
σ_1	-145.85				
σ_0	6684.787946				
x_0	-430.2				
π_0	0.049239317				
π_1	-0.00072408				

Parameters ICF		**Investing Cash Flow**			
μ_1	0	α	β	ϑ_0	ϑ_1
μ_0	-8176.24	-1434.35	-0.87	-529091058712.03	-1475485576.23
σ_2	0.7108				
σ_1	-8176.37				
σ_0	45495150				
x_0	1655.41				
π_0	-1.15411952				
π_1	-0.000190277				

Appendix M5 – Numerical optimisation of the AMLE function.

The starting point is the re-parametrised maximum likelihood estimation function described by Equation (5.41) in Section 5-2. The mapping of the parameter spaces $\{\sigma_0, \sigma_1, \sigma_2\}$ to $\{\theta_1, \theta_2, \theta_3\}$, obeys $\theta_1 = -\frac{\sigma_1}{2\sigma_2}$, $\theta_2 = \frac{\sqrt{|D|}}{2\sigma_2}$ and $\theta_3 = \sqrt{|\sigma_2|}$. Note that each of the three transformations is monotonic. The re-parametrised maximum likelihood function is defined as

$$\ell_i(\boldsymbol{\theta}; x_{T_i}) = -\frac{1}{2}\ln[|(x_{T_i} + \theta_1 + \theta_2)|] - \frac{1}{2}\ln[(x_{T_i} + \theta_1 - \theta_2)|] - \frac{1}{2}DX_{T_i}^2 + \ln\left[|1 + \vartheta_0 + \vartheta_1 DX_{T_i} - 2\vartheta_0 DX_{T_i}^2 - \frac{1}{3}\vartheta_1 DX_{T_i}^3 + \frac{1}{3}\vartheta_0 DX_{T_i}^4|\right] \quad \text{(M5.1)}$$

where $DX_{T_i} = \frac{1}{\theta_3}\sinh^{-1}\left[\frac{x_{T_i}-\theta_1}{\theta_2}\right]$ if $D > 0$, or $DX_{T_i} = -\frac{1}{\theta_3}\sin^{-1}\left[\frac{x_{T_i}-\theta_1}{\theta_2}\right]$ if $D < 0$, $\pi_0(\boldsymbol{\theta}) = \frac{2\sqrt{2}\theta_3\hat{\mu}_0 - \sqrt{2}\,\theta_1\hat{\mu}_1}{2\theta_2\theta_3^2}$, $\pi_1(\boldsymbol{\theta}) = \frac{\sqrt{2}(\theta_1-\theta_2^2)\hat{\mu}_0 - \sqrt{2}\theta_1^2\hat{\mu}_1}{\theta_2^3\theta_3^2} + \frac{\sqrt{2}}{2\theta_2}$, $\vartheta_0 = \left\{-\frac{3}{8} - \frac{1}{8}\alpha^4(\boldsymbol{\theta}) - \frac{3}{16}\alpha^2(\boldsymbol{\theta})\beta(\boldsymbol{\theta}) - \frac{3}{32}\beta^2(\boldsymbol{\theta})\right\}$, $\vartheta_1 = \left\{\frac{1}{2}\alpha^3(\boldsymbol{\theta}) + \frac{3}{2}\alpha(\boldsymbol{\theta})\beta(\boldsymbol{\theta})\right\}$, $\alpha(\boldsymbol{\theta}) = \frac{\pi_0(\boldsymbol{\theta})}{\pi_1(\boldsymbol{\theta})}$, $\beta(\boldsymbol{\theta}) = \frac{1}{\pi_1(\boldsymbol{\theta})}$.

For operating cash flows (the case where $D > 0$) a transformation of the random variable x_{T_i} to $v_{T_i} = T(x_{T_i}) = \frac{1}{\theta_3}\sinh^{-1}\left[\frac{x_{T_i}-\theta_1}{\theta_2}\right]$ is considered. Likewise, a transformation of investing cash flows (the case where $D < 0$) obeys $v_{T_i} = T(x_{T_i}) = \frac{1}{\theta_3}\sin^{-1}\left[\frac{x_{T_i}-\theta_1}{\theta_2}\right]$ where $\theta_1 - \theta_2 \le x_{T_i} \le \theta_1 + \theta_2$.

Observe that $T(x_{T_i})$ is a bijective (one-to-one) and monotonic function, hence

$$\sup_{\boldsymbol{\eta}} \ell_i^*(\boldsymbol{\theta}, v_{T_i}) = \sup_{\boldsymbol{\eta}} \ell_i\left(T^{-1}(\boldsymbol{\theta}, vw_{T_i})\right) = \sup_{\boldsymbol{\theta}} \ell_i(\boldsymbol{\theta}, x_{T_i}) \quad \text{(M5.2)}$$

with maximum estimates attained at $\widehat{\boldsymbol{\theta}} = T(\widehat{\boldsymbol{\theta}}, v_{T_i}) = (\widehat{\boldsymbol{\theta}}, x_{T_i})$.

If the variable transform $v_{T_i} = \frac{1}{\theta_3}\sinh^{-1}\left[\frac{x_{T_i}-\theta_1}{\theta_2}\right]$ is applied, then Equation (M5.1) becomes

$$\ell_i(\boldsymbol{\theta}; v_{T_i}) = -\frac{1}{2}\ln[|2\theta_1 + \theta_2\sinh[\theta_3 v_{T_i}] + \theta_2)|] - \frac{1}{2}\ln[|2\theta_1 + \theta_2(\sinh[\theta_3 v_{T_i}] - \theta_2)|] - \frac{1}{2}v_{T_i}^2 + \ln\left[|1 + \vartheta_0 + \vartheta_1 v_{T_i} - 2\vartheta_0 v_{T_i}^2 - \frac{1}{3}\vartheta_1 v_{T_i}^3 + \frac{1}{3}\vartheta_0 v_{T_i}^4|\right] \quad \text{(M5.3a)}$$

or, neglecting the term $\frac{1}{2}v_{T_i}^2$ (which is constant w.r.t. $\boldsymbol{\theta}$),

$$\ell_i(\boldsymbol{\theta}; v_{T_i}) = -\frac{1}{2}\ln\left[|2\theta_1 + \theta_2 \sinh[\theta_3 v_{T_i}] + \theta_2)\,|\right] - \frac{1}{2}\ln\left[|2\theta_1 + \theta_2(\sinh[\theta_3 v_{T_i}] - \theta_2)|\right] + \ln\left[|1 + \vartheta_0 + \vartheta_1\theta_1 v_{T_i} - 2\vartheta_0 v_{T_i}^2 - \frac{1}{3}\vartheta_1 v_{T_i}^3 + \frac{1}{3}\vartheta_0 v_{T_i}^4|\right] \quad \text{(M5.3b)}$$

Similarly, the variable transform $v_{T_i} = \frac{1}{\theta_3}\sin^{-1}\left[\frac{x_{T_i} - \theta_1}{\theta_2}\right]$,

$$\ell_i(\boldsymbol{\theta}; v_{T_i}) = -\frac{1}{2}\ln\left[|2\theta_1 + \theta_2 \sin[\theta_3 v_{T_i}] + \theta_2)\,|\right] - \frac{1}{2}\ln\left[|2\theta_1 + \theta_2(\sin[\theta_3 v_{T_i}] - \theta_2)|\right] + \ln\left[|1 + \vartheta_0 + \vartheta_1\theta_1 v_{T_i} - 2\vartheta_0 v_{T_i}^2 - \frac{1}{3}\vartheta_1 v_{T_i}^3 + \frac{1}{3}\vartheta_0 v_{T_i}^4|\right] \quad \text{(M5.3c)}$$

where $\theta_1 - \theta_2 \le x_{T_i} \le \theta_1 + \theta_2$

The corresponding aggregated likelihood log-function $\ell(\boldsymbol{\theta}; v_{T_i})$ for operating cash flows yields

$$\ell(\boldsymbol{\theta}; v_{T_i}) = -\frac{1}{2}\sum_{i=1}^{i=n}\ln\left[|2\theta_1 + \theta_2 \sinh[\theta_3 v_{T_i}] + \theta_2)\,|\right] - \frac{1}{2}\sum_{i=1}^{i=n}\ln\left[|2\theta_1 + \theta_2(\sinh[\theta_3 v_{T_i}] - \theta_2)|\right] + \sum_{i=1}^{i=n}\ln\left[|1 + \vartheta_0 + \vartheta_1\theta_1 v_{T_i} - 2\vartheta_0 v_{T_i}^2 - \frac{1}{3}\vartheta_1 v_{T_i}^3 + \frac{1}{3}\vartheta_0 v_{T_i}^4|\right] \quad \text{(M5.4a)}$$

and the one for investing cash flows is

$$\ell(\boldsymbol{\theta}; v_{T_i}) = -\frac{1}{2}\sum_{i=1}^{i=n}\ln\left[|2\theta_1 + \theta_2 \sin[\theta_3 v_{T_i}] + \theta_2)\,|\right] - \frac{1}{2}\sum_{i=1}^{i=n}\ln\left[|2\theta_1 + \theta_2(\sin[\theta_3 v_{T_i}] - \theta_2)|\right] + \sum_{i=1}^{i=n}\ln\left[|1 + \vartheta_0 + \vartheta_1\theta_1 v_{T_i} - 2\vartheta_0 v_{T_i}^2 - \frac{1}{3}\vartheta_1 v_{T_i}^3 + \frac{1}{3}\vartheta_0 v_{T_i}^4|\right] \quad \text{(M5.4b)}$$

A numerical algorithm was developed that maximises Equations (M5.4a) and (M5.4b). The sum of derivatives w.r.t. each parameter $\boldsymbol{\theta}$: $\{\theta_1, \theta_2, \theta_3\}$ is numerically optimised to be as close as possible to zero under the condition that the eigenvalues of the Hessian matrix are negative-definitive. Fortunately, all derivatives, including the complicated ones pertaining to the term $\sum_{i=1}^{i=n}\ln\left[|1 + \vartheta_0 + \vartheta_1\theta_1 v_{T_i} - 2\vartheta_0 v_{T_i}^2 - \frac{1}{3}\vartheta_1 v_{T_i}^3 + \frac{1}{3}\vartheta_0 v_{T_i}^4|\right]$, were able to be calculated and the suggested simplifications (items 5. and 6.) in Section 5.2. under *Approximated likelihood function,* were not required.

8. Statistical Appendixes

Appendix S1 – Description of the dataset used for statistical testing

Introduction

The dataset was retrieved from Standard & Poor's Compustat North America database. The North America database was chosen because it comprises a relative homogeneous sample of firms that publish comparable cash flow and other financial data. These companies, for instance, report under the same accounting standards, operate in identical market conditions and in a similar legal environment. In total 5,202 companies were selected with at least 40 consecutive quarters of reported cash flow data. Of those 5,202 entities, 4,854 (93.3%) entities have their head quarter domiciled in the United States and 348 entities (6.7%) in Canada. Furthermore, the dataset contains both active (2,220 or 42.7% of total) and inactive (2,982 or 57.3% of total) companies. Companies can become inactive for various reasons, the most common are insolvency and acquisition by another legal entity. Inactive companies were included to avoid sample bias towards companies that are relatively successful by showing longevity. For groups the dataset encompasses consolidated financial data only since financial data of consolidated entities could be prone to inter-group transactions that may distort reporting true cash flows.

Cross-section and time-series characteristics

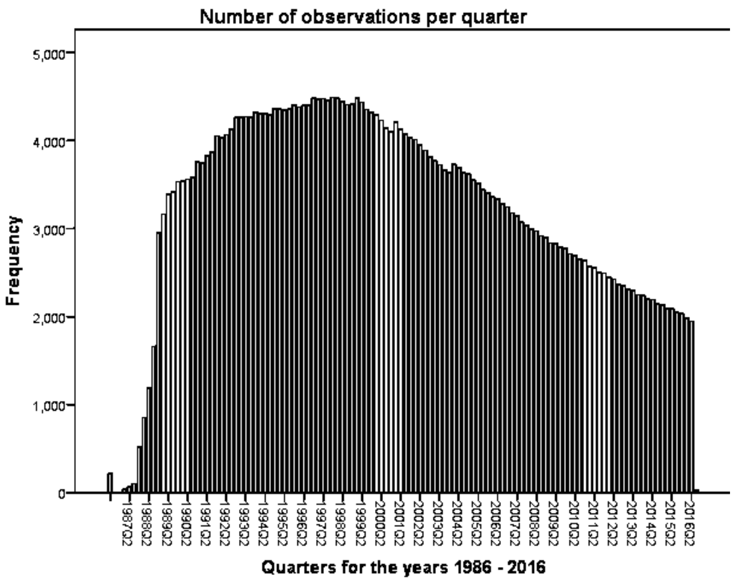
A total of 391,456 (excluding 215 observations with no reporting quarter allocated) cash flow data points were analysed. On average, there are 3,235 observations per quarter; for reasons of representativeness. In most analyses, quarters with less than 500 observations were excluded. Below are the descriptive statistics of the main financial variables analysed in this study.

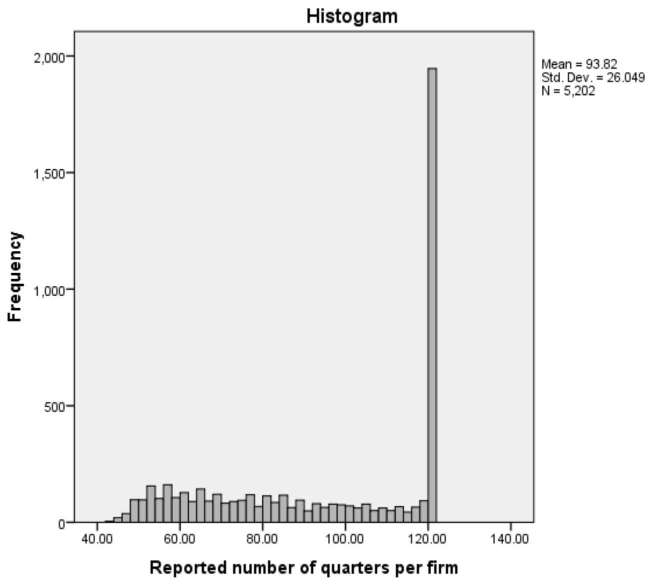
Descriptive Statistics Financial Variables

	N	Range	Minimum	Maximum	Mean		Std. Deviation	Variance	Skewness		Kurtosis	
	Statistic	Statistic	Statistic	Statistic	Statistic	Std. Error	Statistic	Statistic	Statistic	Std. Error	Statistic	Std. Error
Operating Activities - Net Cash Flow	5202	72605.00	-22907.00	49698.00	273.32	23.60	1702.18	2897404.71	14.34	0.03	319.57	0.07
Investing Activities - Net Cash Flow	5199	184367.00	-69504.00	114863.00	-199.16	42.48	3062.66	9379901.92	9.85	0.03	612.29	0.07
Revenue - Total	5046	120347.16	-28.16	120319.00	954.89	58.19	4133.75	17087910.27	11.66	0.03	208.67	0.07
Assets - Total	5201	3234893.00	0.00	3234893.00	12208.11	1348.92	97281.52	9463693678.91	19.94	0.03	486.94	0.07
Delta OA	5085	54379.00	-28215.00	26164.00	88.15	14.61	1041.49	1084703.99	2.32	0.03	278.69	0.07
Delta IA	4778	73370.00	-20323.00	53047.00	-46.45	21.14	1461.21	2135132.38	14.50	0.04	569.15	0.07
% Delta OC	5085	1802.00	-467.00	1335.00	1.73	0.37	26.43	6983548.95	28.82	0.03	1363.55	0.07
% Delta IA	4778	3952.26	-1055.26	2897.00	0.25	0.96	66.03	43601635.48	31.76	0.04	1460.95	0.07
Valid N (listwise)	4622											

Alongside operating cash flow and investing cash flow, their absolute changes (delta) and relative changes (% delta) were computed. Total revenue and total assets were used to de-scale cash flows by their approximate system size.

The reporting period extends from 1986Q2 – 2016Q3 including 121 quarters. The average number of quarters over which firms reported financial data was 75.3 quarters.



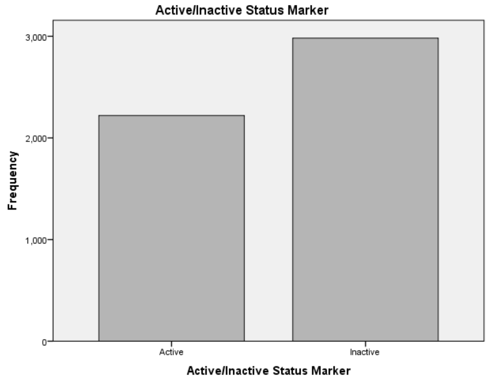


Representativeness

A methodological issue concerns how representative the selected firms are of the wider population of all firms. Since the selected firms are all public companies that are required to report their cash flow data regularly, the sample is likely to be biased in favour of larger, probably more professionally managed firms, excluding medium and small sized businesses which nevertheless constitute an important part of overall business activity. This may be a limitation on the application of the results of the study, but only if there are indications that cash flow processes of smaller, private businesses are fundamentally different from those of the sample. In this research, the number of observations in each quarter was considered sufficiently large to be a representative sample of the population of all (listed) companies. Therefore, analyses of time-series were based on heterogeneous samples in time as opposed to following the same firms over a time period (which would have drastically reduced the number of observations).

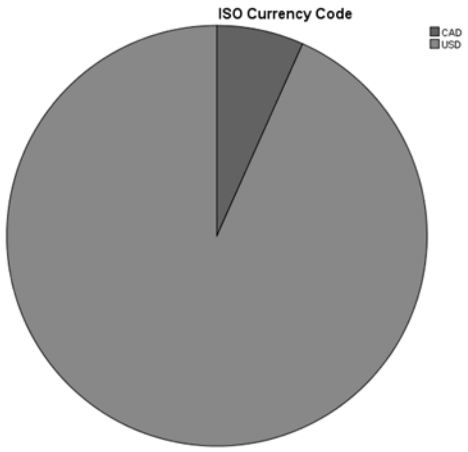
GIC Industries

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	101010	77	1.5	1.5	1.5
	101020	283	5.4	5.5	6.9
	151010	129	2.5	2.5	9.4
	151020	19	.4	.4	9.8
	151030	46	.9	.9	10.7
	151040	234	4.5	4.5	15.2
	151050	36	.7	.7	15.9
	201010	87	1.7	1.7	17.5
	201020	45	.9	.9	18.4
	201030	44	.8	.8	19.3
	201040	89	1.7	1.7	21.0
	201050	16	.3	.3	21.3
	201060	179	3.4	3.4	24.7
	201070	34	.7	.7	25.4
	202010	198	3.8	3.8	29.2
	202020	30	.6	.6	29.8
	203010	13	.2	.3	30.0
	203020	25	.5	.5	30.5
	203030	8	.2	.2	30.7
	203040	57	1.1	1.1	31.8
	203050	8	.2	.2	31.9
	251010	65	1.2	1.3	33.2
	251020	15	.3	.3	33.5
	252010	138	2.7	2.7	36.1
	252020	40	.8	.8	36.9
	252030	105	2.0	2.0	38.9
	253010	165	3.2	3.2	42.1
	253020	21	.4	.4	42.5
	254010	151	2.9	2.9	45.4
	255010	46	.9	.9	46.3
	255020	17	.3	.3	46.6
	255030	32	.6	.6	47.2
	255040	136	2.6	2.6	49.9
	301010	66	1.3	1.3	51.1
	302010	29	.6	.6	51.7
	302020	98	1.9	1.9	53.6
	302030	7	.1	.1	53.7
	303010	18	.3	.3	54.1
	303020	38	.7	.7	54.8
	351010	196	3.8	3.8	58.6
	351020	136	2.6	2.6	61.2
	351030	15	.3	.3	61.5
	352010	127	2.4	2.4	63.9
	352020	80	1.5	1.5	65.5
	352030	24	.5	.5	65.9
	401010	149	2.9	2.9	68.8
	401020	29	.6	.6	69.4
	402010	83	1.6	1.6	70.9
	402020	31	.6	.6	71.5
	402030	73	1.4	1.4	73.0
	402040	4	.1	.1	73.0
	403010	127	2.4	2.4	75.5
	404010	54	1.0	1.0	76.5
	404020	21	.4	.4	76.9
	404030	19	.4	.4	77.3
	451010	45	.9	.9	78.2
	451020	81	1.6	1.6	79.7
	451030	152	2.9	2.9	82.6
	452010	113	2.2	2.2	84.8
	452020	89	1.7	1.7	86.5
	452030	202	3.9	3.9	90.4
	452040	3	.1	.1	90.5
	452050	16	.3	.3	90.8
	453010	81	1.6	1.6	92.4
	501010	79	1.5	1.5	93.9
	501020	27	.5	.5	94.4
	551010	95	1.8	1.8	96.2
	551020	34	.7	.7	96.9
	551030	28	.5	.5	97.4
	551040	22	.4	.4	97.8
	551050	18	.3	.3	98.2
	601010	55	1.1	1.1	99.2
	601020	39	.7	.8	100.0
	Total	5191	99.8	100.0	
Missing	System	11	.2		
Total		5202	100.0		



Active/Inactive Status Marker

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Active	2220	42.7	42.7	42.7
	Inactive	2982	57.3	57.3	100.0
	Total	5202	100.0	100.0	



ISO Currency Code

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	CAD	348	6.7	6.7	6.7
	USD	4854	93.3	93.3	100.0
	Total	5202	100.0	100.0	

Number of Reported Quarters per Firm

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	43	4	.1	.1	.1
	44	1	.0	.0	.1
	45	19	.4	.4	.5
	46	13	.2	.2	.7
	47	24	.5	.5	1.2
	48	25	.5	.5	1.7
	49	72	1.4	1.4	3.0
	50	35	.7	.7	3.7
	51	61	1.2	1.2	4.9
	52	74	1.4	1.4	6.3
	53	82	1.6	1.6	7.9
	54	37	.7	.7	8.6
	55	65	1.2	1.2	9.8
	56	74	1.4	1.4	11.3
	57	87	1.7	1.7	12.9
	58	45	.9	.9	13.8
	59	61	1.2	1.2	15.0
	60	60	1.2	1.2	16.1
	61	67	1.3	1.3	17.4
	62	37	.7	.7	18.1
	63	52	1.0	1.0	19.1
	64	52	1.0	1.0	20.1
	65	91	1.7	1.7	21.9
	66	41	.8	.8	22.7
	67	50	1.0	1.0	23.6
	68	47	.9	.9	24.5
	69	73	1.4	1.4	25.9
	70	34	.7	.7	26.6
	71	48	.9	.9	27.5
	72	39	.7	.7	28.3
	73	50	1.0	1.0	29.2
	74	45	.9	.9	30.1
	75	49	.9	.9	31.0
	76	57	1.1	1.1	32.1
	77	61	1.2	1.2	33.3
	78	31	.6	.6	33.9
	79	37	.7	.7	34.6
	80	47	.9	.9	35.5
	81	66	1.3	1.3	36.8
	82	41	.8	.8	37.6
	83	44	.8	.8	38.4
	84	55	1.1	1.1	39.5
	85	61	1.2	1.2	40.6
	86	29	.6	.6	41.2
	87	34	.7	.7	41.8
	88	37	.7	.7	42.6
	89	58	1.1	1.1	43.7
	90	27	.5	.5	44.2
	91	22	.4	.4	44.6
	92	32	.6	.6	45.2
	93	48	.9	.9	46.2
	94	29	.6	.6	46.7
	95	35	.7	.7	47.4
	96	30	.6	.6	48.0
	97	48	.9	.9	48.9
	98	37	.7	.7	49.6
	99	38	.7	.7	50.3
	100	32	.6	.6	50.9
	101	38	.7	.7	51.7
	102	24	.5	.5	52.1
	103	37	.7	.7	52.8
	104	41	.8	.8	53.6
	105	37	.7	.7	54.3
	106	22	.4	.4	54.8
	107	28	.5	.5	55.3
	108	31	.6	.6	55.9
	109	30	.6	.6	56.5
	110	26	.5	.5	57.0
	111	24	.5	.5	57.4
	112	27	.5	.5	58.0
	113	40	.8	.8	58.7
	114	22	.4	.4	59.2
	115	22	.4	.4	59.6
	116	27	.5	.5	60.1
	117	38	.7	.7	60.8
	118	31	.6	.6	61.4
	119	61	1.2	1.2	62.6
	120	1917	36.9	36.9	99.4
	121	29	.6	.6	100.0
	Total	5202	100.0	100.0	

Number of Observations per Quarter

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		215	.1	.1	.1
	1986Q3	1	.0	.0	.1
	1986Q4	2	.0	.0	.1
	1987Q1	41	.0	.0	.1
	1987Q2	63	.0	.0	.1
	1987Q3	101	.0	.0	.1
	1987Q4	523	.1	.1	.2
	1988Q1	852	.2	.2	.5
	1988Q2	1191	.3	.3	.8
	1988Q3	1662	.4	.4	1.2
	1988Q4	2957	.8	.8	1.9
	1989Q1	3166	.8	.8	2.8
	1989Q2	3390	.9	.9	3.6
	1989Q3	3422	.9	.9	4.5
	1989Q4	3535	.9	.9	5.4
	1990Q1	3537	.9	.9	6.3
	1990Q2	3563	.9	.9	7.2
	1990Q3	3580	.9	.9	8.1
	1990Q4	3760	1.0	1.0	9.1
	1991Q1	3749	1.0	1.0	10.0
	1991Q2	3827	1.0	1.0	11.0
	1991Q3	3869	1.0	1.0	12.0
	1991Q4	4055	1.0	1.0	13.0
	1992Q1	4036	1.0	1.0	14.1
	1992Q2	4069	1.0	1.0	15.1
	1992Q3	4127	1.1	1.1	16.2
	1992Q4	4261	1.1	1.1	17.2
	1993Q1	4266	1.1	1.1	18.3
	1993Q2	4274	1.1	1.1	19.4
	1993Q3	4262	1.1	1.1	20.5
	1993Q4	4324	1.1	1.1	21.6
	1994Q1	4307	1.1	1.1	22.7
	1994Q2	4310	1.1	1.1	23.8
	1994Q3	4293	1.1	1.1	24.9
	1994Q4	4360	1.1	1.1	26.0
	1995Q1	4359	1.1	1.1	27.1
	1995Q2	4346	1.1	1.1	28.3
	1995Q3	4357	1.1	1.1	29.4
	1995Q4	4402	1.1	1.1	30.5
	1996Q1	4382	1.1	1.1	31.6
	1996Q2	4402	1.1	1.1	32.7
	1996Q3	4399	1.1	1.1	33.9
	1996Q4	4482	1.1	1.1	35.0
	1997Q1	4469	1.1	1.1	36.1
	1997Q2	4476	1.1	1.1	37.3
	1997Q3	4462	1.1	1.1	38.4
	1997Q4	4493	1.1	1.1	39.6
	1998Q1	4481	1.1	1.1	40.7
	1998Q2	4446	1.1	1.1	41.8
	1998Q3	4408	1.1	1.1	43.0
	1998Q4	4415	1.1	1.1	44.1
	1999Q1	4488	1.1	1.1	45.2
	1999Q2	4436	1.1	1.1	46.4
	1999Q3	4354	1.1	1.1	47.5
	1999Q4	4324	1.1	1.1	48.6
	2000Q1	4292	1.1	1.1	49.7
	2000Q2	4230	1.1	1.1	50.8
	2000Q3	4147	1.1	1.1	51.8
	2000Q4	4103	1.0	1.0	52.9
	2001Q1	4208	1.1	1.1	54.0
	2001Q2	4127	1.1	1.1	55.0
	2001Q3	4071	1.0	1.0	56.0
	2001Q4	4035	1.0	1.0	57.1
	2002Q1	4012	1.0	1.0	58.1
	2002Q2	3951	1.0	1.0	59.1
	2002Q3	3888	1.0	1.0	60.1
	2002Q4	3818	1.0	1.0	61.1
	2003Q1	3771	1.0	1.0	62.0
	2003Q2	3725	1.0	1.0	63.0
	2003Q3	3669	.9	.9	63.9
	2003Q4	3633	.9	.9	64.9
	2004Q1	3731	1.0	1.0	65.8
	2004Q2	3692	.9	.9	66.7
	2004Q3	3635	.9	.9	67.7
	2004Q4	3620	.9	.9	68.6
	2005Q1	3552	.9	.9	69.5
	2005Q2	3514	.9	.9	70.4
	2005Q3	3441	.9	.9	71.3
	2005Q4	3409	.9	.9	72.2
	2006Q1	3358	.9	.9	73.0
	2006Q2	3337	.9	.9	73.9
	2006Q3	3285	.8	.8	74.7
	2006Q4	3248	.8	.8	75.5
	2007Q1	3177	.8	.8	76.3
	2007Q2	3144	.8	.8	77.1
	2007Q3	3071	.8	.8	77.9
	2007Q4	3036	.8	.8	78.7
	2008Q1	2991	.8	.8	79.5
	2008Q2	2972	.8	.8	80.2
	2008Q3	2918	.7	.7	81.0
	2008Q4	2899	.7	.7	81.7
	2009Q1	2836	.7	.7	82.4
	2009Q2	2832	.7	.7	83.2
	2009Q3	2786	.7	.7	83.9
	2009Q4	2776	.7	.7	84.6
	2010Q1	2712	.7	.7	85.3
	2010Q2	2696	.7	.7	86.0
	2010Q3	2647	.7	.7	86.6
	2010Q4	2635	.7	.7	87.3
	2011Q1	2571	.7	.7	88.0
	2011Q2	2553	.7	.7	88.6
	2011Q3	2505	.6	.6	89.3
	2011Q4	2495	.6	.6	89.9
	2012Q1	2447	.6	.6	90.5
	2012Q2	2427	.6	.6	91.1
	2012Q3	2365	.6	.6	91.7
	2012Q4	2356	.6	.6	92.3
	2013Q1	2307	.6	.6	92.9
	2013Q2	2298	.6	.6	93.5
	2013Q3	2248	.6	.6	94.1
	2013Q4	2245	.6	.6	94.7
	2014Q1	2196	.6	.6	95.2
	2014Q2	2193	.6	.6	95.8
	2014Q3	2145	.5	.5	96.3
	2014Q4	2136	.5	.5	96.9
	2015Q1	2091	.5	.5	97.4
	2015Q2	2090	.5	.5	97.9
	2015Q3	2050	.5	.5	98.5
	2015Q4	2032	.5	.5	99.0
	2016Q1	1985	.5	.5	99.5
	2016Q2	1946	.5	.5	100.0
	2016Q3	29	.0	.0	100.0
	Total	391671	100.0	100.0	

Appendix S2 – Testing for jumps

The Barndorff-Nielsen Shephard jump test

The Barndorff-Nielsen Shephard test splits the total quadratic variation $[C]_t = \sum_{j=1}^{j=n}(C_{t_j} - C_{t_{j-1}})^2$ of a cash flow process C_t in two components: (i) a continuous part of the local martingale of C_t, $[C^c]_t$, and (ii) a pure discontinuous component $[C^d]_t$, such that $[C]_t = [C^c]_t + [C^d]_t$. In a high-frequency data environment each discrete time interval Δt is subdivided into j partitions. The variation split, and corresponding calculations, is made for each Δt separately, however, in this study having the restriction of having only low-frequency data available, a one-time interval (i.e. the full reporting length) is observed and partitioned into related quarters. Below the test is adapted to a low-frequency data environment.

Barndorff-Nielsen and Shephard (2006) show that the continuous component can be expressed as a realised 1,1 order Bi-power Variation Process (BPV): $[C^c] = \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{j=1}^{j=n} |C_{j-1}||C_j|$. Furthermore, they approximate the process variance by realised quad-power variation: $\tilde{q} = \sum_{j=1}^{j=n} |C_{j-3}||C_{j-2}||C_{j-1}||C_j|$. From the above information, they calculate the following linear ($\widehat{G}$) and ratio ($\widehat{H}$) test statistics: $\widehat{G} = \frac{\mu_1^{-2}\tilde{v} - \hat{v}}{\sqrt{\vartheta \mu_1^{-4} \tilde{q}}}$ and $\widehat{H} = \frac{\frac{\mu_1^{-2}\tilde{v}}{\hat{v}} - 1}{\sqrt{\vartheta \mu_1^{-2} \frac{\tilde{q}}{\tilde{v}^2}}}$ where $\tilde{v} = \sum_{j=1}^{j=n} |C_{j-1}||C_j|$, $\hat{v} = \sum_{j=1}^{j=n}(C_j - C_{j-1})^2$ and $\mu_1 = \frac{\sqrt{2}}{\sqrt{\pi}} \approx 0.7979$, $\vartheta \approx 0.6090$ are normalisation factors. Both test statistics converge asymptotically to $\mathcal{N}(0,1)$ as $n \to \infty$. Significant left-side (negative) statistic values indicate the occurrence of one or more jumps in the time-series under investigation.

A number of 5,103 firms (operating cash flow) and 5,176 firms (investing cash flow) were included in the tests, after removing (significant) outliers from the available total of 5,202 firms. The results of the analysis are reported in Section 1.6. For smaller values both test statistics agreed but differences were identified for larger negative values (indicating jumps). Therefore, the reported test statistic is an (unweighted) average of the linear and ratio test statistic: $\hat{J} = \frac{\widehat{G} + \widehat{H}}{2}$.

The modified Barndorff-Nielsen Shephard jump test

To accommodate the objections to the Barndorff-Nielsen Shephard test raised in the more recent literature, as suggested in Buckle et al. (2016), the original definitions of test statistics were modified. The test statistic $\tilde{\mathrm{v}} = \sum_{\mathrm{j=1}}^{\mathrm{j=n}} |\, \mathrm{C_{j-1}}||\mathrm{C_j}|$ was replaced by a median-based statistic defined in the following equation: $\tilde{\mathrm{v}} = \sum_{\mathrm{j=1}}^{\mathrm{j=n}} (\mathrm{med}\left|\mathrm{C_{j-4}}\right|, \left|\mathrm{C_{j-3}}\right|, \left|\mathrm{C_{j-2}}\right|, |\,\mathrm{C_{j-1}}|, |\mathrm{C_j}|)^2$. The authors claim that this estimator is more robust to detect multiple closely located jumps which would remain undetected with the original Barndorff-Nielsen Shephard estimator. The results of the modified jump test are also reported in Section 1.6.

Detailed results (per firm) and associated calculations of all of the above tests are available on request from the author.

Appendix S3 – Test of common drift functions

The quarterly cash flow data of 5,202 firms were examined for a fit with (i) a linear growth trend and (ii) an exponential growth trend. A linear growth trend is defined as $C_T = \alpha T + \beta$ and an exponential growth trend as $C_T = \beta e^{\alpha T}$ or alternatively as $\ln C_T = \alpha T + \ln \beta$, where $T: [1, 2,3, \dots . n]$ stands for a number representing sequential quarters.

For all individual firms, regressions were performed with respect to both equations. Generally, the coefficients of determination R^2 are, as expected, low which can be explained by significant stochastic variability around the growth trend. Hence the main test statistic is the F-statistic which was evaluated at different significance levels.

Admittedly, the test for the exponential growth trend is a little artificial since firms with consistent positive cash flows during their life time are very rare. Among the firms selected for this test, less than 10% have reported only positive cash flows in the observed period (and thus no negative cash flows).

Detailed results (per firm) and associated calculations of both tests are available on request from the author.

Appendix S4 – Tests of Pearson distribution function

To approximate a stationary probability density function, operating and investing cash flow data were normalised between quarters by applying the following transform: $c_n' = \frac{c_n-\mu_n}{\sigma_n}$ where n is the number in the sequence of the quarters, μ_n is the average of cash flows of quarter n, and σ_n is the standard deviation of cash flows of quarter n. Quarters with less than 468 observations were excluded from the analysis. For each month, the mean (m), variance (v), skewness (s) and kurtosis (k) of the transformed cash flows c_n' were calculated. Using the Method of Moments, the set of parameters $\{\sigma_0, \sigma_1, \sigma_2\}$ of the Pearson differential equation $\frac{p_{st}'}{p_{st}} = \frac{2(\mu_1-\sigma_2)c_n'+(2\mu_0-\sigma_1)}{\sigma_2 {c_n'}^2+\sigma_1 c_n'+\sigma_0}$ are estimated as follows: $\widehat{\sigma}_{0,n} = \frac{-v_n(4k_n-3s_n^2)}{10k_n-12s_n^2-18}$, $\widehat{\sigma}_{1,n} = \frac{-\sqrt{v_n}\, s_n(k_n+3)}{0k_n-12s_n^2-18}$, $\widehat{\sigma}_{2,n} = \frac{2k_n-3s_n^2-6)}{10k_n-12s_n^2-18}$.

From the quarterly estimates $\{\widehat{\sigma}_0, \widehat{\sigma}_{1,} \widehat{\sigma}_2\}$ the discriminants of the transformed quadratic diffusion function $\sigma_2 {c_n'}^2 + \sigma_1 c_n' + \sigma_0$ were computed: $D_n = \sqrt{4\sigma_0\sigma_2 - \sigma_1^2}$ or alternatively expressed as $k_n = \frac{\sigma_1^2}{4\sigma_0\sigma_2}$ where $D_n = \sigma_1^2(1-\frac{1}{k_n})$. Values of $0 < k_n < 1$ correspond to complex roots ($D_n > 0$), values of $k_n \geq 1$ or values of $k_n < 0$ amount to real roots ($D_n \leq 0$) . If real roots were identified, these were calculated by the usual formula: $r_{1,2} = \frac{-\sigma_1 \pm \sqrt{\sigma_1^2-4\sigma_0\sigma_2}}{2\sigma_2}$.

Averaged over all quarters the following parameter estimates were found:

OPERATING CASH FLOWS	$\widehat{\sigma}_{0,av-n}$	$\widehat{\sigma}_{1,av-n}$	$\widehat{\sigma}_{2,av-n}$	$\hat{k}_{av-n}$	$\widehat{D}_{av-n}$
PARAMETER ESTIMATES	**1.630**	**0.976**	**0.356**	**5.644**	**0.274**
STD	9.446	0.435	0.203	34.885	0.278
5%-UL	3.364	1.056	0.393	12.048	0.325
5%-LL	-0.104	0.896	0.319	-0.760	0.222

INVESTING CASH FLOWS	$\widehat{\sigma}_{0,av-n}$	$\widehat{\sigma}_{1,av-n}$	$\widehat{\sigma}_{2,av-n}$	$\hat{k}_{av-n}$	$\widehat{D}_{av-n}$
PARAMETER ESTIMATES	**-0.081**	**0.095**	**-0.303**	**0.044**	**0.014**
STD	0.048	0.025	0.014	0.078	0.010
5%-UL	-0.072	0.100	-0.301	0.058	0.016
5%-LL	-0.089	0.091	-0.306	0.030	0.012

Detailed results (per quarter) and associated calculations of both tests are available on request from the author.

Appendix S5 – Tests of relationship between operating and investing cash flow

The relationship between operating and investing cash flow at a macroscopic level
Full period 1988 Q1 – 2016 Q2

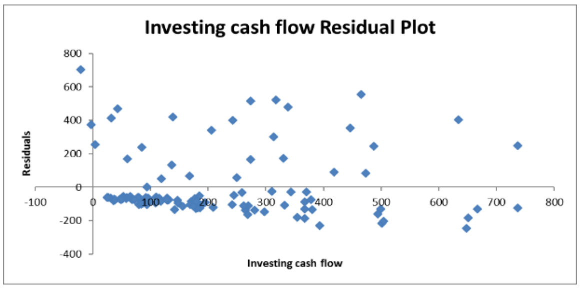
SUMMARY OUTPUT

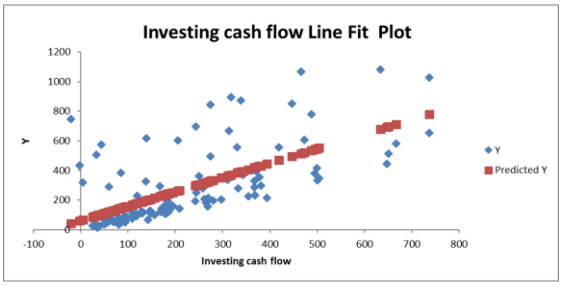
Regression Statistics	
Multiple R	0.655134
R Square	0.429201
Adjusted R Square	0.424105
Standard Error	195.2177
Observations	114

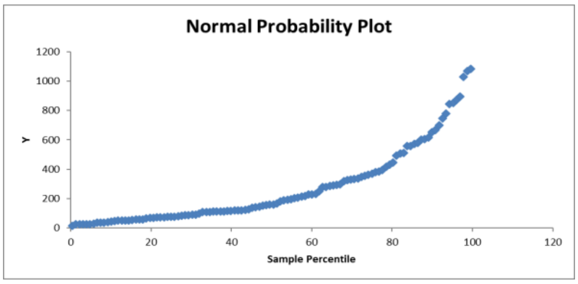
ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	3209475	3209475	84.21618	2.62E-15
Residual	112	4268315	38109.96		
Total	113	7477790			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	62.03494	28.94325	2.14333	0.03425	4.687592	119.3823	4.687592	119.3823
Investing cash flow	0.9714	0.105852	9.176938	2.62E-15	0.761667	1.181132	0.761667	1.181132







The relationship between operating and investing cash flow at a macroscopic level
Pre-GFC period 1988 Q1 – 2008 Q2

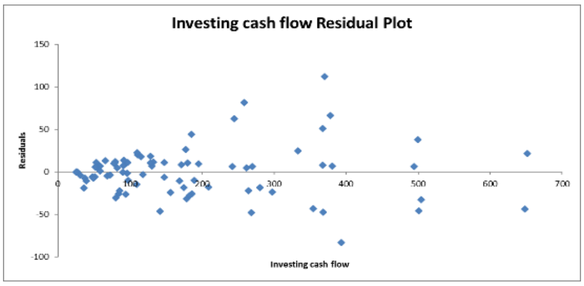
SUMMARY OUTPUT

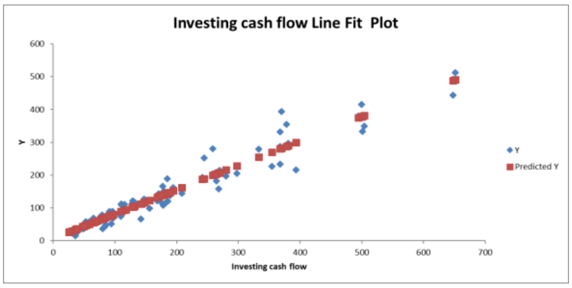
Regression Statistics	
Multiple R	0.965629
R Square	0.932439
Adjusted R Square	0.931595
Standard Error	29.32077
Observations	82

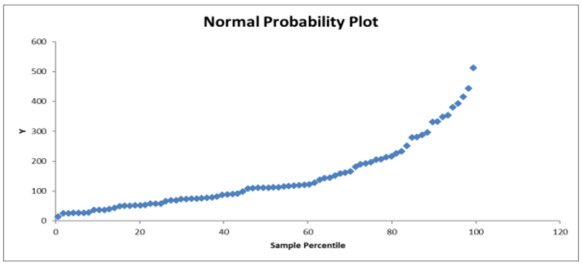
ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	949220.4	949220.4	1104.12	1.42E-48
Residual	80	68776.6	859.7075		
Total	81	1017997			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	7.480162	5.219669	1.433072	0.155734	-2.90731	17.86763	-2.90731	17.86763
Investing cash flow	0.741806	0.022325	33.2283	1.42E-48	0.697379	0.786234	0.697379	0.786234







The relationship between operating and investing cash flow at a macroscopic level Post-GFC period 2008 Q1 – 2016 Q2

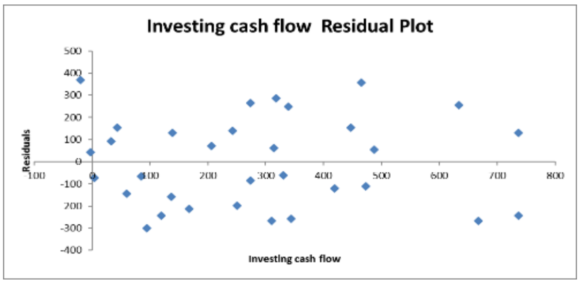
SUMMARY OUTPUT

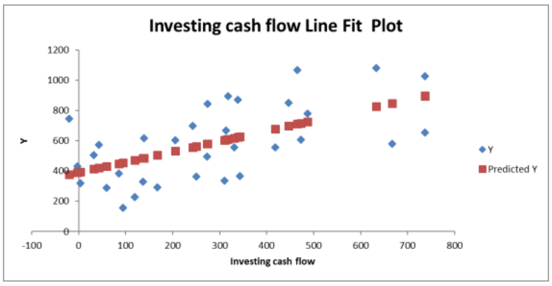
Regression Statistics	
Multiple R	0.590541
R Square	0.348739
Adjusted R Square	0.32703
Standard Error	205.0899
Observations	32

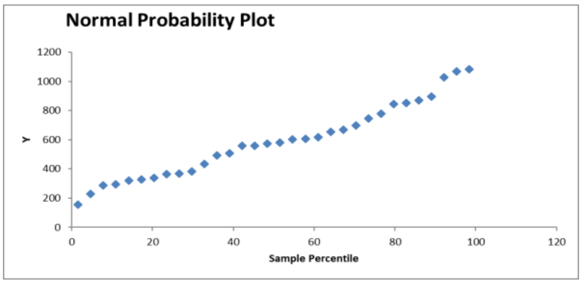
ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	675702	675702	16.06447	0.000373
Residual	30	1261856	42061.88		
Total	31	1937559			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	391.242	60.77961	6.437061	4.14E-07	267.1135	515.3706	267.1135	515.3706
Investing cash flow	0.685566	0.171047	4.008051	0.000373	0.336241	1.034892	0.336241	1.034892







Appendix S6 – Results of the approximated likelihood estimations – industry level

Operating Cash Flows

GISC	Industry	theta1	t	theta2	t	theta3	t	mu0	t	mu1	t	sigma0	t	sigma1	t	sigma2	t	Comment
101010	Energy Equipment & Services	1,420.1	0.0	(87,781.7)	(0.0)	(0.7)	(0.0)	(121,363.9)	(13,507.2)	17.63	4,325.6	(3,702,559,796.3)	0.0	(1,365.1)	(0.0)	0.48	0.0	
101020	Oil, Gas & Consumable Fuels	437.3	5.0	(2,223.1)	(106.8)	0.5	29.7	(0.6)	(12.7)	0.02	3.9	(1,166,066.8)	65.1	(214.7)	(5.0)	0.25	895.5	
151010	Chemicals	(26.3)	(3.1)	89.6	104.9	0.2	39.1	0.3	15.3	0.01	2.7	(358.4)	15.0	2.6	3.1	0.05	7,838.6	
151020	Construction Materials	3,748.3	0.9	14,024.6	472.3	5.2	14.0	(40,198.0)	(0.0)	7.29	842.9	(4,900,444,387.6)	4.6	(201,144.9)	(0.9)	26.83	1.8	
151030	Containers & Packaging	4,362.3	0.1	34,890.7	809.0	8.9	19.2	(323,635.6)	(28,779.8)	(8.92)	(787.9)	(95,730,831,278.7)	0.1	(696,980.6)	(0.1)	79.89	1.1	
151040	Metals & Mining	(5.9)	(5.5)	(2.7)	(61.7)	1.0	153.1	(1.1)	(128.9)	0.03	3.1	30.3	(2.2)	13.0	5.5	1.09	5,360.0	
151050	Paper & Forest Products	1,098.4	0.0	(80,812.8)	(26.2)	(0.5)	(0.0)	(63,011.0)	(0.0)	(16.55)	(319.4)	(1,662,953,559.4)	0.0	(559.5)	(0.0)	0.25	0.0	
201010	Aerospace & Defense	4.9	0.4	1.0	1.7	1.1	0.6	(1.4)	(238.6)	0.03	3.8	26.5	(0.0)	(11.3)	(0.1)	1.14	0.0	
201020	Building Products	94,046.5	0.0	34,416.4	0.0	(11.8)	(0.0)	(2,812.5)	(0.0)	(7.35)	(748.6)	1,069,570,147,607.2	(0.0)	(26,262,655.2)	(0.0)	139.63	0.0	
201030	Construction & Engineering	9.9	57.8	1.4	157.5	1.8	125.6	(1,325.1)	(0.5)	(0.44)	(84.4)	326.6	(27.6)	(67.2)	(57.8)	3.39	1,163.2	
201040	Electrical Equipment	30.0	31.8	1.1	98.1	2.0	119.6	(408,352.3)	(91,996.2)	20.00	5,514.4	3,546.3	(15.6)	(236.5)	(31.8)	3.94	908.6	
201050	Industrial Conglomerates	11.2	17.9	2.8	46.7	2.0	56.0	(776,447.9)	(0.0)	(11.65)	(1,905.8)	483.5	(8.0)	(91.8)	(17.8)	4.09	191.4	
201060	Machinery	1,095.7	0.0	(54,320.1)	(0.0)	(0.7)	(0.0)	(65,564.4)	(995.3)	(19.73)	(6,166.5)	(1,323,923,928.2)	0.0	(983.6)	(0.0)	0.45	0.0	
201070	Trading Companies & Distributors	16.7	11.4	(1.3)	(50.3)	1.1	15.1	(0.7)	(101.6)	0.02	2.5	317.9	(5.1)	(38.3)	(11.3)	1.15	49.8	
202010	Commercial Services & Supplies	1,498.3	1.2	3,089.5	26.5	1.1	2.9	(3.4)	(3,947.3)	0.10	10.6	(8,312,843.6)	2.9	(3,412.3)	(1.1)	1.14	1.7	
202020	Professional Services	6,801.9	1.4	96,776.8	11.3	0.4	2.8	(4.1)	(384.0)	0.17	28.6	(1,735,079,422.1)	11.8	(2,532.7)	(1.4)	0.19	10.1	
203010	Air Freight & Logistics	20.8	0.5	14.9	43.2	1.3	35.8	(10,399.6)	(84.2)	(0.18)	(84.2)	369.5	(0.6)	(73.0)	(0.5)	1.75	182.6	
203020	Airlines	(15,283.7)	(1.0)	76,262.6	3.8	0.7	2.0	(855.4)	(9,532.4)	18.62	2,100.3	(2,504,191,035.8)	3.8	13,712.1	0.9	0.45	2.1	
203030	Marine	9,144.5	0.3	65,667.1	914.2	5.7	6.4	(221,809.2)	(4,040.4)	(9.72)	(2,028.1)	(135,208,904,188.3)	0.5	(584,792.8)	(0.3)	31.98	0.3	
203040	Road & Rail	111.7	0.0	(2.6)	(0.0)	5.8	0.0	(83.4)	(10.4)	(9.69)	(2,656.7)	417,876.2	(0.0)	(7,488.7)	(0.0)	33.53	0.0	
203050	Transportation Infrastructure	5.2	21.7	0.9	62.1	1.3	464.1	(3,791.3)	(0.0)	1.40	86.7	44.5	(10.5)	(17.5)	(21.7)	1.67	32,259.4	
251010	Auto Components	1,655.8	0.0	(81,751.9)	(42.1)	(0.8)	(0.0)	(104,423.1)	(8,228.4)	15.48	3,174.8	(4,310,193,048.5)	0.0	(2,136.5)	(0.0)	0.65	0.0	
251020	Automobiles	25,296.5	0.5	52,706.2	421.3	8.5	5.2	(77,440.4)	(0.0)	10.53	1,965.4	(154,436,652,860.6)	0.1	(3,654,482.4)	(0.2)	72.23	0.1	
252010	Household Durables	(37,216.5)	(5.1)	(97,715.4)	(3.5)	0.6	10.5	(0.4)	(755.4)	0.02	2.2	(2,970,188,357.8)	11.7	27,082.4	5.1	0.36	74.8	
252020	Leisure Products	3.0	0.2	1.3	1.2	(0.7)	(23.4)	(400,853.1)	(0.0)	6.11	888.6	3.0	(0.7)	(2.6)	(0.2)	0.43	313.8	
252030	Textiles, Apparel & Luxury Goods	11,392.1	0.1	95,835.8	1,001.7	6.0	8.9	(416,337.9)	(43,022.9)	(6.73)	(1,424.5)	(329,344,661,526.3)	0.2	(828,724.9)	(0.1)	36.37	0.5	
253010	Hotels, Restaurants & Leisure	(19,846.4)	(3.1)	63,455.6	5.3	0.6	6.4	(0.3)	(160.3)	0.21	5.2	(1,227,990,430.8)	19.0	13,417.6	3.1	0.34	29.7	
253020	Diversified Consumer Services	5.8	28.7	1.4	91.5	1.0	121.9	(23.7)	(0.1)	(0.00)	(0.2)	34.3	(13.4)	(12.6)	(28.7)	1.09	3,411.2	
254010	Media	21.8	0.0	(1.1)	(0.0)	1.9	0.0	(0.5)	(44.7)	0.02	6.5	1,686.6	(0.0)	(154.9)	(0.0)	3.55	0.0	
255010	Distributors	13,238.2	0.1	53,671.1	1,358.6	8.4	16.5	(241,560.3)	(25,005.1)	(4.44)	(816.1)	(189,186,570,211.2)	(0.3)	(1,851,513.7)	(0.1)	69.93	1.0	
255020	Internet & Direct Marketing Retail	(10,673.6)	(0.5)	(48,906.0)	(456.5)	4.8	4.3	(47,889.5)	(0.0)	(7.32)	(136.0)	(52,802,100,875.4)	0.3	494,837.3	0.3	23.18	0.2	
255030	Multiline Retail	8,693.3	0.1	75,939.7	850.0	5.0	1.8	(237,963.2)	(0.0)	(11.80)	(1,324.4)	(145,081,530,657.4)	0.0	(443,219.9)	(0.1)	25.49	0.0	
255040	Specialty Retail	39.5	1.6	0.9	2.8	(1.6)	(43.6)	(0.1)	(0.1)	(0.00)	(0.0)	4,229.1	(1.0)	(214.1)	(1.6)	2.71	175.7	
301010	Food & Staples Retailing	7.5	4.1	(4.8)	(46.6)	(1.3)	(69.4)	(1,255,937.4)	(0.1)	15.23	3,314.3	53.7	(1.3)	(24.5)	(4.1)	1.64	732.8	
302010	Beverages	28,536.6	2.3	62,209.2	2,128.8	13.1	45.6	(264,579.0)	(347.6)	(10.14)	(1,522.2)	(525,705,012,712.9)	66.2	(9,819,062.3)	(2.2)	172.04	3.0	

Operating Cash Flows

GISC	Industry	theta1	t	theta2	t	theta3	t	mu0	t	mu1	t	sigma0	t	sigma1	t	sigma2	t	Comment
302020	Food Products	10,341.3	2.4	24,880.6	918.3	10.3	63.5	(195,936.8)	(68,707.2)	16.25	1,981.4	(54,641,478,580.5)	11.7	(2,206,822.7)	(2.4)	106.70	9.4	
302030	Tobacco	664.1	0.0	(54,098.2)	(0.1)	(0.3)	(0.0)	(10,972.4)	(0.0)	(9.64)	(965.7)	(219,865,700.1)	0.0	(99.8)	(0.0)	0.08	0.0	
303010	Household Products	(2,872.5)	(0.1)	18,737.2	555.5	6.0	34.3	(290,081.8)	(2,071.6)	(9.44)	(4,307.2)	(12,297,605,436.0)	(0.4)	206,081.3	0.1	35.87	8.2	
303020	Personal Products	0.1	2.3	1.1	37.9	(0.4)	(45.5)	978,152.5	0.2	19.52	4,241.0	(0.2)	17.9	(0.0)	(2.3)	0.16	3,270.3	Partial convergence
351010	Health Care Equipment & Supplies	12.3	2.3	(5.7)	(27.3)	1.3	82.7	(71.9)	(92.1)	0.01	3.0	207.9	(1.1)	(43.2)	(2.3)	1.76	973.0	
351020	Health Care Providers & Services	24.5	2.9	(1.4)	(10.6)	1.6	5.6	(0.7)	(19,899.8)	0.03	8.4	1,561.9	(1.0)	(127.7)	(2.6)	2.60	3.0	
351030	Health Care Technology	(25,632.7)	(0.9)	(50,579.8)	(1,660.7)	9.5	22.1	(78,089.4)	(23,312.6)	(3.86)	(804.6)	(170,849,149,893.1)	1.0	4,606,713.4	0.9	89.86	1.4	
352010	Biotechnology	3,784.0	0.0	46,294.3	842.4	4.4	0.1	(201,261.1)	(0.0)	2.44	277.8	(40,606,501,181.5)	0.0	(144,355.2)	(0.0)	19.07	0.0	
352020	Pharmaceuticals	7.0	0.0	13.2	0.0	(0.5)	(0.0)	(11.3)	(244.6)	0.05	6.4	(28.7)	0.0	(3.2)	(0.0)	0.23	0.0	
352030	Life Sciences Tools & Services	(2,801.6)	(0.1)	31,404.3	944.1	4.6	21.9	(271,068.5)	(24,000.5)	(5.09)	(782.4)	(20,880,811,813.2)	(0.0)	119,583.5	0.1	21.34	5.6	
401010	Banks	11,580.1	1.1	49,490.9	1,603.2	7.5	8.8	(165,349.8)	(0.0)	(14.27)	(1,157.9)	(130,146,840,056.9)	0.6	(1,301,905.5)	(0.6)	56.21	0.3	
401020	Thrifts & Mortgage Finance	6,774.7	0.2	50,234.2	1,146.1	5.6	12.1	(195,934.8)	(7,080.7)	(4.56)	(223.6)	(76,479,952,603.2)	1.1	(418,253.3)	(0.2)	30.87	1.2	
402010	Diversified Financial Services	5.0	0.0	2.2	0.0	1.1	0.0	(4.1)	(9,176.7)	0.03	5.7	26.2	(0.0)	(12.9)	(0.0)	1.29	0.0	
402020	Consumer Finance	3,658.9	0.1	36,773.1	347.9	3.2	1.9	(90,124.4)	(0.0)	7.62	1,060.0	(13,673,950,899.2)	0.1	(74,737.8)	(0.1)	10.21	0.1	
402030	Capital Markets	9.9	23.1	0.9	63.7	1.3	161.6	(0.4)	(275.9)	0.28	8.0	170.2	(11.4)	(34.5)	(23.1)	1.74	3,759.5	
402040	Mortgage Real Estate Investment Trusts (REITs)	3,763.6	0.0	62,671.9	0.0	(2.2)	(0.0)	(203,361.2)	(34,553.5)	(19.97)	(1,039.6)	(18,198,278,886.5)	0.0	(35,001.7)	(0.0)	4.65	0.0	
403010	Insurance	15.2	29.7	2.4	87.9	2.0	110.2	(33,152.8)	(285.6)	0.61	127.5	922.1	(14.2)	(124.5)	(29.7)	4.10	740.4	
404010	Internet Software & Services (old)	3.8	11.6	(0.1)	(35.6)	1.9	21.8	(2.5)	(374.3)	0.68	49.2	49.5	(4.9)	(26.3)	(11.5)	3.50	34.0	
404020	IT Services (old)	49,246.1	9.7	62,341.2	2,143.2	13.9	74.9	(139,788.4)	(4,582.4)	14.10	2,849.4	(282,065,633,322.9)	3.3	(19,011,987.8)	(8.1)	193.03	7.3	
404030	Real Estate Management & Development	(2.8)	(4.1)	(5.7)	(99.5)	(0.4)	(123.6)	(2.6)	(0.5)	(0.00)	(0.1)	(3.7)	6.2	0.8	4.1	0.14	26,437.2	
451010	Internet Software & Services	8.0	15.4	1.4	38.7	(1.4)	(121.5)	(2,502.0)	(1.0)	(10.17)	(1,231.7)	126.9	(7.4)	(32.8)	(15.4)	2.05	1,796.4	
451020	IT Services	30,800.2	201.4	38,486.7	509.0	1.0	403.1	(0.7)	(95.8)	0.02	5.5	(537,382,793.4)	56.5	(62,157.1)	(201.4)	1.01	40,257.7	
451030	Software	2,301.5	0.0	(62,265.6)	(0.0)	(1.3)	(0.0)	(139,095.8)	(956,748.2)	16.68	3,961.0	(7,031,930,422.4)	0.0	(8,360.1)	(0.0)	1.82	0.0	
452010	Communications Equipment	33.6	0.0	(0.9)	(0.0)	4.9	0.0	(107,598.7)	(11,954.5)	(13.70)	(4,173.2)	27,262.5	(0.0)	(1,621.5)	(0.0)	24.09	0.0	
452020	Technology Hardware, Storage & Peripherals	0.2	0.2	(17.4)	(125.5)	(2.8)	(619.6)	6.8	361.2	(0.09)	(24.0)	(2,449.8)	194.0	(3.5)	(0.2)	8.07	11,886.2	
452030	Electronic Equipment, Instruments & Components	4,925.0	0.1	76,354.7	200.8	2.0	0.3	(61,709.7)	(65,551.9)	14.72	1,693.4	(23,874,086,057.7)	0.0	(40,504.4)	(0.0)	4.11	0.0	
452040	Semiconductors & Semiconductor Equipment (old)	(3,265.5)	(0.0)	31,561.7	32.4	5.1	0.0	(333,638.5)	(0.1)	(7.71)	(250.5)	(25,450,214,442.3)	0.0	168,662.5	0.0	25.83	0.0	
452050	Semiconductor Equipment & Products	(3,862.6)	(0.1)	26,462.6	635.4	6.3	26.4	(298,507.2)	(1.9)	9.27	420.3	(27,206,361,541.3)	0.0	306,672.4	0.1	39.70	4.4	
453010	Semiconductors & Semiconductor Equipment	(37,229.9)	(0.7)	(71,799.6)	(1,335.7)	13.6	43.6	(337,225.3)	(0.1)	(8.99)	(2,699.0)	(695,404,053,036.2)	0.1	13,737,902.5	0.7	184.50	2.6	
501010	Diversified Telecommunication Services	9.0	17.1	(6.6)	(82.1)	(1.5)	(189.1)	(16,211.9)	(218.8)	(17.19)	(6,063.4)	90.2	(4.0)	(42.6)	(17.1)	2.36	3,790.8	
501020	Wireless Telecommunication Services	0.0	0.0	(94.2)	(1,018.0)	1.8	3,192.0	4.8	13.1	0.00	1.0	(29,867.2)	100.5	(0.3)	(0.0)	3.37	756,310.9	
551010	Electric Utilities	1,908.8	0.0	(58,788.8)	(3.4)	(1.3)	(0.1)	(127,300.5)	(5,127.2)	11.79	2,952.7	(5,872,321,542.1)	0.0	(6,493.5)	(0.0)	1.70	0.0	
551020	Gas Utilities	2.0	0.0	(2.1)	(0.0)	(0.6)	(0.0)	(0.9)	(60.9)	(0.01)	(1.6)	(0.1)	0.0	(1.2)	(0.0)	0.31	0.0	Partial convergence
551030	Multi-Utilities	4,302.2	0.0	56,073.9	0.0	(2.6)	(0.0)	(165,712.2)	(673.0)	(14.73)	(4,464.5)	(20,875,192,491.0)	0.0	(57,464.0)	(0.0)	6.68	0.0	
551040	Water Utilities	13,720.9	0.4	65,993.7	1,094.8	14.9	82.0	(1,354,796.3)	(41,707.2)	19.67	4,210.4	(925,530,644,375.2)	1.8	(6,095,226.6)	(0.4)	222.11	7.6	
551050	Independent Power and Renewable Electricity Producers	5.1	11.9	(3.3)	(38.7)	1.1	54.7	(7.9)	(113.5)	0.03	3.2	19.7	(3.4)	(13.2)	(11.9)	1.29	581.0	
601010	Equity Real Estate Investment Trusts (REITs)	14.0	27.1	(1.9)	(87.0)	1.9	127.9	(109.3)	(49.3)	0.01	1.6	657.3	(13.1)	(96.0)	(27.1)	3.44	1,189.6	
601020	Real Estate Management & Development	3,943.9	0.0	50,371.4	0.0	(2.7)	(0.0)	(147,993.6)	(53,550.8)	(11.53)	(1,728.4)	(17,796,149,395.7)	0.0	(55,665.0)	(0.0)	7.06	0.0	

Investing Cash Flows

GISC	Industry	theta1	t	theta2	t	theta3	t	mu0	t	mu1	t	sigma0	t	sigma1	t	sigma2	t	Comment
101010	Energy Equipment & Services	5,774.9	1.4	19,945.6	504.7	6.51	21.8	(84,584.5)	(13,486.0)	12.75	3,127.3	1.83E+10	7.8	488,985.7	18.9	(42.34)	4,745.1	
101020	Oil, Gas & Consumable Fuels	(5,186.0)	(0.4)	36,403.5	1,438.0	7.23	65.7	(456,259.7)	(0.0)	7.95	1,102.2	7.07E+10	(56.9)	(542,651.6)	(5.0)	(52.32)	895.5	
151010	Chemicals	5,003.9	0.0	(88,285.3)	(0.0)	(1.51)	(0.0)	(81,460.0)	(0.0)	(18.51)	(383.3)	1.78E+10	(15.1)	22,735.7	3.1	(2.27)	7,838.6	
151020	Construction Materials	8,283.9	2.1	29,072.6	3,797.5	9.34	153.0	(182,419.4)	(0.0)	(4.70)	(86.0)	7.98E+10	7.4	1,446,132.4	(15.1)	(87.29)	708.8	
151030	Containers & Packaging	(8,175.2)	(0.0)	49,240.4	1,219.8	8.87	49.9	(816,210.0)	(0.0)	8.16	1,071.6	1.96E+11	0.0	(1,286,786.0)	(0.0)	(78.70)	0.0	
151040	Metals & Mining	(6,066.7)	(0.0)	(50,006.9)	(0.0)	2.93	0.0	(85,713.9)	(447.3)	13.30	1,672.5	2.19E+10	2.2	(104,491.4)	5.5	(8.61)	5,360.0	
151050	Paper & Forest Products	(20,173.2)	(1.7)	79,472.3	3.8	0.59	3.6	(161.9)	(1.4)	8.12	178.8	2.31E+09	0.5	(13,893.1)	(1.2)	(0.34)	1,791,386.6	
201010	Aerospace & Defense	34,402.0	0.1	96,551.0	5.4	0.72	1.7	(0.7)	(141.9)	0.01	1.0	5.47E+09	0.0	35,813.5	(0.1)	(0.52)	0.0	
201020	Building Products	22,501.0	4.1	(48,907.6)	(6.3)	0.69	8.5	(26.4)	(907.0)	3.44	29.5	1.37E+09	6.2	21,299.5	(12.7)	(0.47)	65,172.1	
201030	Construction & Engineering	(6,370.8)	(0.0)	28,042.9	801.8	9.31	7.3	(533,367.0)	(0.3)	9.70	1,083.4	7.17E+10	30.9	(1,104,719.9)	(61.7)	(86.70)	1,273.0	
201040	Electrical Equipment	(46,101.0)	(0.5)	(96,828.4)	(802.8)	11.17	6.1	(112,445.0)	(0.0)	(8.38)	(127.7)	1.43E+12	0.5	(11,499,325.6)	(1.2)	(124.72)	82.9	
201050	Industrial Conglomerates	(34,121.3)	(0.0)	(76,938.9)	(0.0)	9.94	0.0	(131,738.4)	(7,790.2)	(4.82)	(251.6)	7.00E+11	8.4	(6,744,508.6)	(17.2)	(98.83)	181.5	
201060	Machinery	9,566.4	0.4	44,358.9	1,194.7	11.15	55.6	(441,006.4)	(0.0)	(9.34)	(1,170.0)	2.56E+11	14.6	2,377,265.2	(29.8)	(124.25)	886.6	
201070	Trading Companies & Distributors	1,446.7	0.0	(73,206.6)	(0.4)	(0.87)	(0.1)	(111,845.9)	(0.0)	11.39	1,624.9	4.06E+09	0.0	2,190.4	(0.0)	(0.76)	0.0	
202010	Commercial Services & Supplies	19,174.1	1.2	66,452.4	1,742.9	8.93	12.2	(209,994.9)	(0.0)	(11.64)	(2,680.8)	3.81E+11	(0.7)	3,055,659.3	(1.1)	(79.68)	1.7	
202020	Professional Services	(10,626.5)	(0.1)	(42,480.4)	(390.5)	5.54	1.5	(70,742.4)	(0.0)	(6.88)	(175.6)	5.88E+10	(8.8)	(651,879.6)	(1.4)	(30.67)	10.1	
203010	Air Freight & Logistics	5,907.9	4.6	(13,964.5)	(6.8)	0.67	9.4	(0.3)	3.0	0.20	3.0	1.04E+08	0.1	5,363.6	(1.0)	(0.45)	448.2	
203020	Airlines	1,150.2	0.0	(44,494.3)	(0.0)	(0.89)	(0.0)	(78,483.7)	(0.0)	13.88	973.4	1.56E+09	(0.0)	1,809.6	0.0	(0.79)	0.0	
203030	Marine	(5,921.6)	(0.0)	(62,483.5)	(0.0)	2.57	0.0	(72,914.9)	(0.0)	8.33	899.9	2.60E+10	4.8	(78,187.0)	(8.4)	(6.60)	33.4	
203040	Road & Rail	(21,407.4)	(0.1)	(28,205.0)	(3,521.9)	12.01	4.2	(81,481.0)	(1.1)	2.51	361.6	1.81E+11	0.0	(6,173,278.9)	(0.0)	(144.19)	0.0	
203050	Transportation Infrastructure	9,902.8	0.9	38,312.2	1,497.8	10.69	65.8	(309,789.1)	(0.0)	(8.60)	(122.2)	1.79E+11	5.5	2,262,316.4	(11.4)	(114.23)	42,038.6	
251010	Auto Components	(6,843.3)	(0.2)	26,456.5	766.8	9.98	67.1	(559,701.5)	(12,376.0)	12.63	1,553.7	7.44E+10	(0.0)	(1,363,083.8)	0.0	(99.59)	0.0	
251020	Automobiles	2,793.1	0.0	(78,542.3)	(0.0)	(1.40)	(0.0)	(162,741.9)	(0.0)	15.94	1,124.0	1.21E+10	(0.1)	10,977.8	(0.1)	(1.97)	9,925.9	
252010	Household Durables	2,639.7	0.0	(88,521.1)	(6.0)	(1.24)	(0.5)	(148,129.8)	(3,545.8)	(14.19)	(2,905.4)	1.20E+10	(8.8)	8,106.2	5.1	(1.54)	74.8	
252020	Leisure Products	(21,404.5)	(1.2)	(39,315.6)	(1,258.4)	13.95	86.6	(342,844.4)	(0.0)	(11.34)	(2,261.7)	3.90E+11	0.0	(8,333,932.5)	(0.3)	(194.68)	325.4	
252030	Textiles, Apparel & Luxury Goods	6,728.7	0.4	23,741.6	1,299.4	10.83	94.7	(266,768.3)	(0.0)	5.82	115.7	7.15E+10	(0.2)	1,579,241.9	(0.1)	(117.35)	0.5	
253010	Hotels, Restaurants & Leisure	(5,617.8)	(0.1)	77,098.8	528.7	4.64	12.3	(820,430.9)	(0.0)	(14.44)	(376.2)	1.29E+11	(8.3)	(241,608.8)	3.1	(21.50)	29.7	
253020	Diversified Consumer Services	4,573.2	1.0	16,872.5	375.1	5.02	9.9	(38,113.8)	(0.0)	9.05	426.0	7.70E+09	13.5	230,343.2	(28.7)	(25.18)	3,411.2	
254010	Media	1,871.7	0.1	(92,377.0)	(0.4)	(0.07)	(0.2)	(254.7)	(2,276.0)	(9.54)	(378.4)	3.86E+07	0.0	16.9	(0.0)	(0.00)	0.0	
255010	Distributors	(43,039.0)	(5.0)	(99,671.0)	(7,756.5)	0.60	1,846.7	(0.1)	(19.5)	(0.00)	(0.0)	4.31E+09	(13.1)	(31,447.1)	0.0	(0.37)	328.6	
255020	Internet & Direct Marketing Retail	(4,732.6)	(0.0)	(78,044.4)	(0.0)	1.86	0.0	(50,259.6)	(0.0)	(8.81)	(378.2)	2.11E+10	5.1	(32,695.6)	(10.2)	(3.45)	93,762,735.2	
255030	Multiline Retail	35,736.5	2.5	77,800.4	3,370.9	18.62	231.1	(1,121,474.2)	(0.0)	13.77	287.9	2.54E+12	(5.2)	24,774,463.9	1.3	(346.63)	176,024.8	
255040	Specialty Retail	(10,846.0)	(0.4)	47,315.8	1,110.8	12.17	111.8	(1,480,888.5)	(0.0)	(18.65)	(1,647.2)	3.49E+11	16.1	(3,210,244.1)	(31.2)	(147.99)	476.2	
301010	Food & Staples Retailing	4,727.2	0.3	31,946.7	905.6	5.32	20.3	(143,963.5)	(0.0)	5.16	175.1	2.96E+10	1.2	267,991.6	(4.1)	(28.35)	733.3	
302010	Beverages	(38,865.9)	(0.0)	(94,104.9)	(512.4)	9.44	0.3	(68,405.0)	(0.0)	(8.45)	(1,101.2)	9.23E+11	(1.6)	(6,923,930.8)	(2.2)	(89.07)	3.0	

Investing Cash Flows

GISC	Industry	theta1	t	theta2	t	theta3	t	mu0	t	mu1	t	sigma0	t	sigma1	t	sigma2	t	Comment
302020	Food Products	16,141.3	*2.2*	67,531.2	*2,250.2*	11.34	*69.9*	(490,007.1)	*(0.0)*	(17.80)	*(516.6)*	6.20E+11	*14.8*	4,148,465.3	*(18.7)*	(128.50)	*13.1*	
302030	Tobacco	4,497.6	*0.0*	(82,913.7)	*(0.0)*	(1.90)	*(0.0)*	(158,001.1)	*(0.0)*	19.55	*752.1*	2.50E+10	*(0.0)*	32,574.1	*(0.0)*	(3.62)	*0.0*	
303010	Household Products	4,705.5	*0.4*	39,753.9	*892.6*	4.87	*17.2*	(175,090.9)	*(0.0)*	6.90	*842.4*	3.80E+10	*2.4*	223,264.1	*(5.1)*	(23.72)	*3,551.9*	
303020	Personal Products	(5,674.7)	*(0.0)*	(92,069.5)	*(0.2)*	2.04	*0.0*	(83,181.1)	*(0.0)*	6.86	*574.0*	3.54E+10	*(0.0)*	(47,170.4)	*1.2*	(4.16)	*34.6*	
351010	Health Care Equipment & Supplies	6,729.5	*0.4*	32,240.5	*668.8*	10.97	*31.3*	(312,056.3)	*(0.1)*	(13.68)	*(722.9)*	1.31E+11	*2.5*	1,619,595.0	*(6.5)*	(120.34)	*1,215.2*	
351020	Health Care Providers & Services	(11,274.0)	*(0.0)*	(76,582.1)	*(0.0)*	3.92	*0.0*	(126,391.6)	*(0.2)*	12.96	*1,037.0*	9.19E+10	*2.2*	(345,955.4)	*(2.6)*	(15.34)	*3.0*	
351030	Health Care Technology	3,963.7	*1.5*	54,164.6	*82.4*	0.03	*4.8*	(5.8)	*(1,059.1)*	0.84	*35.9*	2.14E+06	*7.1*	5.8	*(13.8)*	(0.00)	*123.6*	
352010	Biotechnology	5,927.9	*3.2*	(8,482.3)	*(5.1)*	0.72	*10.1*	0.0	*2.9*	0.04	*0.8*	5.51E+07	*20.1*	6,103.4	*(39.4)*	(0.51)	*727.3*	
352020	Pharmaceuticals	7,085.6	*0.3*	21,566.4	*165.5*	10.82	*27.6*	(160,818.3)	*(0.0)*	(8.54)	*(525.4)*	6.04E+10	*(2.7)*	1,660,427.3	*(1.0)*	(117.17)	*5,416.5*	
352030	Life Sciences Tools & Services	68,124.2	*3.6*	(97,879.1)	*(6.3)*	0.65	*10.3*	(0.2)	*(73.0)*	0.07	*2.0*	6.02E+09	*5.2*	57,638.3	*(10.7)*	(0.42)	*43,638.3*	
401010	Banks	(4,632.6)	*(0.0)*	57,938.2	*726.1*	5.08	*7.9*	(690,669.8)	*(0.0)*	(9.65)	*(673.7)*	8.73E+10	*(0.2)*	(239,403.0)	*(0.6)*	(25.84)	*0.3*	
401020	Thrifts & Mortgage Finance	15,138.3	*0.6*	69,276.4	*1,739.5*	7.64	*6.8*	(206,037.5)	*(20,366.8)*	(11.00)	*(728.7)*	2.94E+11	*5.0*	1,767,425.0	*(10.8)*	(58.38)	*748.2*	
402010	Diversified Financial Services	(79,505.8)	*(3.6)*	(93,318.0)	*(5,820.6)*	19.96	*169.6*	(493,721.2)	*(0.0)*	10.54	*574.1*	5.98E+12	*0.1*	(63,321,259.0)	*(0.6)*	(398.22)	*7.8*	
402020	Consumer Finance	(3,896.3)	*(0.5)*	25,774.6	*1,002.4*	6.56	*47.0*	(307,118.1)	*(0.0)*	10.87	*229.8*	2.92E+10	*(0.0)*	(335,190.6)	*(0.1)*	(43.01)	*0.1*	
402030	Capital Markets	3,481.2	*0.4*	20,255.5	*467.3*	4.45	*10.0*	(66,748.0)	*(0.0)*	6.93	*532.6*	8.35E+09	*12.5*	137,641.6	*(25.2)*	(19.77)	*5,760.2*	
402040	Mortgage Real Estate Investment Trusts (REITs)	13,271.2	*1.6*	57,093.5	*2,419.2*	11.01	*90.9*	(469,195.6)	*(0.0)*	(11.73)	*(222.0)*	4.17E+11	*2.2*	3,219,685.7	*(8.2)*	121.30	*1,368.3*	
403010	Insurance	3,680.1	*0.0*	42,908.5	*768.7*	5.96	*0.0*	(252,446.9)	*(0.0)*	(9.14)	*(555.0)*	6.60E+10	*11.6*	261,771.7	*(24.1)*	35.57	*842.6*	
404010	Internet Software & Services (old)	1,218.7	*0.0*	(99,775.6)	*(0.0)*	(0.51)	*(0.0)*	(70,461.0)	*(0.3)*	19.92	*1,121.2*	2.62E+09	*6.9*	642.1	*(11.5)*	(0.26)	*34.0*	
404020	IT Services (old)	2,953.3	*0.0*	(74,375.6)	*(1.2)*	(1.74)	*(0.0)*	(181,226.9)	*(0.0)*	10.83	*210.2*	1.68E+10	*(1.8)*	17,948.1	*(8.1)*	(3.04)	*7.3*	
404030	Real Estate Management & Development	3,407.2	*0.0*	(57,527.4)	*(68.8)*	(2.35)	*(1.0)*	(201,853.4)	*(0.0)*	12.94	*1,851.1*	1.84E+10	*(6.3)*	37,773.1	*4.1*	(5.54)	*26,437.2*	
451010	Internet Software & Services	(24,224.8)	*(9.0)*	(36,403.8)	*(6,336.5)*	15.64	*462.0*	(284,768.6)	*(0.0)*	(7.87)	*(57.6)*	4.68E+11	*8.1*	(11,851,651.9)	*(15.6)*	(244.62)	*187.3*	
451020	IT Services	8,468.8	*1.1*	45,434.8	*2,020.7*	8.14	*80.0*	(363,680.6)	*(0.1)*	8.69	*617.8*	1.42E+11	*(56.4)*	1,122,673.6	*(201.4)*	(66.28)	*40,257.7*	
451030	Software	51,645.4	*2.1*	(99,573.3)	*(9,364.4)*	(0.74)	*(4,438.7)*	(0.1)	*(0.0)*	0.00	*0.0*	6.80E+09	*(6.5)*	55,844.8	*(134.6)*	(0.54)	*397,917.7*	
452010	Communications Equipment	10,355.7	*0.2*	98,482.3	*4.3*	(0.07)	*(0.6)*	(3.4)	*(252.0)*	(0.96)	*(8.2)*	5.50E+07	*0.0*	116.1	*(0.0)*	(0.01)	*0.0*	
452020	Technology Hardware, Storage & Peripherals	(4,609.4)	*(0.1)*	29,166.9	*1,196.5*	7.26	*61.2*	(389,202.6)	*(0.0)*	5.80	*338.3*	4.60E+10	*(188.5)*	(486,539.5)	*(0.2)*	(52.78)	*11,886.2*	
452030	Electronic Equipment, Instruments & Components	(3,913.9)	*(3.4)*	(77,813.2)	*(5.4)*	0.27	*6.8*	(1.6)	*(120.4)*	0.18	*3.9*	4.46E+08	*3.2*	(574.8)	*(6.5)*	(0.07)	*333.8*	
452040	Semiconductors & Semiconductor Equipment (old)	(37,284.6)	*(108.8)*	59,362.0	*142.3*	0.76	*217.6*	(0.5)	*0.4*	0.17	*1.5*	2.86E+09	*7.0*	(43,371.9)	*(14.3)*	(0.58)	*187,063.9*	
452050	Semiconductor Equipment & Products	26,363.4	*0.0*	68,357.5	*334.5*	7.98	*0.4*	(95,897.4)	*(18.2)*	11.24	*334.2*	3.42E+11	*(65.7)*	3,360,845.9	*(262.1)*	(63.74)	*2,277.6*	
453010	Semiconductors & Semiconductor Equipment	(3,491.3)	*(0.0)*	29,211.0	*935.7*	5.59	*10.2*	(292,948.6)	*(0.0)*	5.26	*132.8*	2.70E+10	*24.2*	(218,141.3)	*(47.6)*	(31.24)	*1,206.3*	
501010	Diversified Telecommunication Services	(5,533.8)	*(0.0)*	(93,213.5)	*(17.1)*	1.87	*0.0*	(136,756.3)	*(394.9)*	20.00	*4,419.6*	3.06E+10	*0.6*	(38,825.3)	*(1.5)*	(3.51)	*9.8*	
501020	Wireless Telecommunication Services	(9,715.8)	*(3.9)*	(25,321.9)	*(25.3)*	0.16	*7.4*	(3.9)	*(0.2)*	1.57	*102.9*	1.82E+07	*(101.2)*	(481.1)	*(0.0)*	(0.02)	*756,310.9*	
551010	Electric Utilities	2,469.9	*0.0*	(79,856.0)	*(0.0)*	(1.43)	*(0.0)*	(252,258.0)	*(52.9)*	20.00	*5,010.1*	1.30E+10	*(5.5)*	10,075.5	*(29.9)*	(2.04)	*245,977.6*	
551020	Gas Utilities	3,291.4	*2.9*	11,082.4	*1,348.8*	6.57	*75.2*	(57,781.7)	*(0.0)*	7.64	*803.2*	5.77E+09	*(0.0)*	284,145.5	*(0.0)*	(43.17)	*0.0*	
551030	Multi-Utilities	1,372.5	*0.0*	(36,781.5)	*(0.0)*	(1.29)	*(0.0)*	(120,324.9)	*(0.0)*	18.10	*1,020.1*	2.27E+09	*0.0*	4,595.8	*(0.2)*	(1.67)	*0.1*	
551040	Water Utilities	(2,338.5)	*(0.1)*	(43,674.4)	*(127.7)*	1.50	*0.5*	(46,625.8)	*0.0*	(11.04)	*(1,223.2)*	4.28E+09	*1.8*	(10,463.7)	*(3.5)*	(2.24)	*33.7*	
551050	Independent Power and Renewable Electricity Producers	19,011.1	*0.5*	87,523.1	*283.5*	5.44	*3.1*	(113,606.5)	*(0.0)*	14.54	*1,819.3*	2.37E+11	*3.4*	1,124,135.0	*(11.9)*	(29.57)	*581.0*	
601010	Equity Real Estate Investment Trusts (REITs)	(5,253.6)	*(0.1)*	(66,801.6)	*(227.1)*	2.25	*0.5*	(50,470.1)	*0.0*	(9.84)	*(751.2)*	2.27E+10	*13.4*	(53,155.9)	*(27.1)*	(5.06)	*1,189.6*	
601020	Real Estate Management & Development	(39,191.2)	*(0.2)*	(52,135.4)	*(0.6)*	(0.58)	*(2.5)*	(0.0)	*(1.7)*	0.05	*5.2*	1.45E+09	*6.3*	(26,669.9)	*(27.8)*	(0.34)	*684.2*	

Appendix S7 – Results of the approximated likelihood estimations – firm level

Operating Cash Flows

GCC	Ticker Symbol	Company Name	theta1	t	theta2	t	theta3	t	mu0	t	mu1	t	sigma0	t	sigma1	t	sigma2	t	Comment
1045	AAL	AMERICAN AIRLINES GROUP INC	(3,630.6)	(0.1)	34,469.7	483.9	5.01	18.5	(426,394.6)	(200,113.8)	(12.19)	(2,990.1)	(29,506,824,735.1)	(0.4)	182,349.8	0.1	25.11	3.4	
1121	AE	ADAMS RESOURCES & ENERGY INC	437.3	5.0	(2,223.1)	(106.8)	0.50	29.7	(0.6)	(12.7)	0.02	3.9	(1,166,066.8)	(56.9)	(214.7)	(5.0)	0.25	895.5	
1186	AEM	AGNICO EAGLE MINES LTD	51.8	4.6	1.9	3.6	(1.00)	(11.0)	(1.1)	(1.5)	(0.70)	(125.8)	2,692.7	2.4	(104.0)	(4.6)	1.00	30.1	
1234	ATRI	ATRION CORP	29,203.2	0.0	(88,412.2)	(0.0)	(8.28)	(0.0)	(180,537.1)	(0.0)	12.77	1,476.5	(477,635,071,771.7)	(0.0)	(4,005,947.8)	(0.0)	68.59	0.0	
1356	AA	ALCOA INC	32.8	2.4	7.7	146.8	1.24	150.1	(1,062.9)	(39.4)	0.01	1.3	1,566.8	1.1	(101.0)	(2.4)	1.54	3,660.1	
1388	AMR1	AMERICAN AIRLINES INC	23,671.8	0.0	(88,368.7)	(0.0)	(5.35)	(0.0)	(111,340.0)	(241,733.7)	(19.37)	(2,130.1)	(207,583,100,467.8)	(0.0)	(1,355,799.7)	(0.0)	28.64	0.0	
1449	AFL	AFLAC INC	9.1	34.3	1.6	146.4	0.89	200.3	(6,643.6)	(0.0)	(1.25)	(24.1)	63.0	16.6	(14.3)	(34.3)	0.79	12,733.0	
1487	AIG	AMERICAN INTERNATIONAL GROUP	(8,041.8)	(0.3)	17,969.6	821.5	13.84	132.4	(571,057.2)	(118,541.6)	16.89	2,212.7	(49,430,126,242.0)	(0.2)	3,078,623.5	0.3	191.41	22.9	
1690	AAPL	APPLE INC	7,339.5	0.4	70,287.9	1,628.4	8.98	44.3	(669,873.8)	(0.3)	(10.60)	(1,080.6)	(393,749,898,223.2)	(3.2)	(1,182,817.1)	(0.4)	80.58	6.1	
1712	TREC	TRECORA RESOURCES	10.8	1.1	4.2	10.6	1.31	35.0	(31,602.6)	(11.4)	(0.49)	(93.6)	169.4	0.4	(37.0)	(1.1)	1.72	177.6	
1794	ASH	ASHLAND GLOBAL HOLDINGS INC	28.6	0.4	4.5	12.7	1.45	18.5	(1,210,123.6)	(534,929.0)	17.65	4,865.8	1,676.4	0.1	(120.1)	(0.4)	2.10	40.6	
1860	ATW	ATWOOD OCEANICS	21.1	0.7	71.6	505.2	1.57	436.4	(896,469.9)	(0.0)	(10.32)	(1,688.4)	(11,597.3)	(2.4)	(104.3)	(0.7)	2.48	19,227.8	
1864	REX	REX AMERICAN RESOURCES CORP	10.2	1.8	10.9	97.9	1.00	91.6	(23,030.1)	(346.7)	(19.83)	(6,196.5)	(14.8)	(0.1)	(20.4)	(1.8)	1.00	2,096.0	
1920	AVP	AVON PRODUCTS	5.0	14.3	(4.9)	(80.2)	(1.04)	(135.3)	(210,659.9)	(159,333.2)	(13.71)	(2,160.3)	0.7	0.2	(10.7)	(14.3)	1.08	4,248.6	
1968	BMI	BADGER METER INC	16,896.7	1.5	62,567.4	6.6	1.00	3.3	(7.2)	(591.4)	0.20	21.4	(3,651,235,124.4)	(1.8)	(33,998.8)	(1.5)	1.01	2.6	
2052	BRN	BARNWELL INDUSTRIES	6,801.9	1.4	96,776.8	11.3	0.43	2.8	(4.1)	(384.0)	0.17	28.6	(1,735,079,422.1)	(8.8)	(2,532.7)	(1.4)	0.19	10.1	
2337	RFP	RESOLUTE FOREST PRODUCTS INC	17.4	0.8	32.1	221.4	1.13	160.6	(2,353.4)	1,368.5	2.94	1,368.5	(925.6)	(0.7)	(44.4)	(0.8)	1.27	5,059.3	
2444	BC	BRUNSWICK CORP	(16,627.4)	(0.0)	(99,703.5)	(0.0)	0.72	0.0	(951.1)	(9,908.5)	19.85	2,239.3	(5,025,016,851.9)	(0.0)	17,291.1	0.0	0.52	0.0	
2556	CSS	CSS INDUSTRIES INC	25.6	0.0	1.3	0.0	4.44	0.0	(442.1)	(23.8)	(3.08)	(643.0)	12,859.7	0.0	(1,008.5)	(0.0)	19.72	0.0	
2787	CRS	CARPENTER TECHNOLOGY CORP	111.7	0.0	(2.6)	(0.0)	5.79	0.0	(83.4)	(10.5)	(9.69)	(2,656.7)	417,911.3	0.0	(7,489.0)	(0.0)	33.53	0.0	
3093	CLC	CLARCOR INC	5.2	8.0	0.9	26.6	1.28	462.0	(86,845.0)	(0.1)	0.88	54.6	43.6	3.9	(17.1)	(8.0)	1.63	32,631.0	
3105	IHRT	IHEARTMEDIA INC	28.8	0.5	6.2	27.6	1.30	31.8	(135,712.2)	(11,270.6)	15.15	3,108.2	1,350.3	0.1	(98.2)	(0.5)	1.70	148.5	
3138	COKE	COCA-COLA BTLNG CONS	4,894.8	0.1	36,971.1	847.5	5.21	10.1	(181,345.6)	(0.0)	(4.91)	(916.1)	(36,474,403,286.6)	(0.2)	(265,896.9)	(0.1)	27.16	0.9	
3226	CMCSA	COMCAST CORP	(37,216.5)	(5.1)	(97,715.4)	(3.5)	0.60	10.5	(0.4)	(755.4)	0.02	2.2	(2,970,188,357.8)	(8.8)	27,082.4	5.1	0.36	74.8	
3429	CTO	CONSOLIDATED TOMOKA LAND CO	3.5	8.6	2.4	38.3	(0.71)	(60.5)	(226,219.5)	(0.0)	5.41	787.5	3.5	2.3	(3.6)	(8.6)	0.51	1,797.4	
3622	CRWS	CROWN CRAFTS INC	11,392.1	0.1	95,835.8	1,001.7	6.03	8.9	(416,337.9)	(43,022.9)	(6.73)	(1,424.5)	(329,344,661,526.3)	(0.2)	(828,724.9)	(0.1)	36.37	0.5	
3813	TGT	TARGET CORP	(19,846.4)	(3.1)	63,455.6	5.3	0.58	6.4	(0.3)	(160.3)	0.21	5.2	(1,227,990,430.8)	(8.3)	13,417.6	3.1	0.34	29.7	
4145	PKI	PERKINELMER INC	5.8	28.7	1.4	91.5	1.04	121.9	(23.7)	(0.1)	(0.00)	(0.2)	34.3	13.5	(12.6)	(28.7)	1.09	3,411.2	
4201	EV	EATON VANCE CORP	21.8	0.0	(1.1)	(0.0)	1.88	0.0	(0.5)	(44.7)	0.02	6.5	1,686.6	0.0	(154.9)	(0.0)	3.55	0.0	
4275	ELSE	ELECTRO-SENSORS INC	11.0	7.9	(45.2)	(54.3)	(0.18)	(17.7)	(2.0)	(70.7)	(0.41)	(74.7)	(63.3)	(35.5)	(0.7)	(7.9)	0.03	2,379.1	
4485	KINS	KINGSTONE COS INC	(4,096.5)	(0.0)	30,669.0	734.3	6.00	20.7	(347,158.4)	(0.0)	7.60	141.2	(33,245,167,812.3)	(0.1)	294,845.4	0.0	35.99	3.0	
4600	FDML	FEDERAL-MOGUL HOLDINGS CORP	4,874.2	0.0	(62,820.2)	(3.0)	(1.67)	(0.0)	(50,968.6)	(0.0)	(16.21)	(1,819.3)	(10,993,994,642.4)	(0.0)	(27,322.0)	(0.0)	2.80	0.0	
4605	FRT	FEDERAL REALTY INVESTMENT TR	(77,458.6)	(18.0)	47,721.9	221.4	(11.98)	(30.5)	(13,840.8)	(0.0)	(14.69)	(3,503.6)	534,140,603,403.8	2.0	22,229,301.4	3.2	143.49	1.6	
4885	BEN	FRANKLIN RESOURCES INC	7.4	4.7	(4.8)	(50.6)	(1.28)	(73.5)	(879,118.2)	(0.1)	10.58	2,302.7	53.2	1.4	(24.4)	(4.7)	1.64	821.9	
4926	FUL	FULLER (H. B.) CO	28,536.6	2.3	62,209.2	2,128.8	13.12	45.6	(264,579.0)	(347.6)	(10.14)	(1,522.2)	(525,705,012,712.9)	(1.6)	(9,819,062.3)	(2.2)	172.04	3.0	

Operating Cash Flows

GCC	Ticker Symbol	Company Name	theta1	t	theta2	t	theta3	t	mu0	t	mu1	t	sigma0	t	sigma1	t	sigma2	t	Comment
4988	TGNA	TEGNA INC	21.2	1.8	7.9	108.4	0.95	77.2	(123,460.0)	(4,863.4)	1.31	159.5	353.2	0.7	(38.6)	(1.8)	0.91	1,640.6	
5046	GD	GENERAL DYNAMICS CORP	1,341.1	0.0	(51,289.2)	(0.0)	(0.76)	(0.0)	(47,643.7)	(0.0)	(12.98)	(1,300.1)	(1,537,857,539.7)	(0.0)	(1,569.1)	(0.0)	0.59	0.0	
5087	SPXC	SPX CORP	7,531.3	0.0	69,274.5	0.0	(4.86)	(0.0)	(572,223.5)	(1,913.4)	19.99	9,116.8	(112,157,525,701.2)	(0.0)	(356,243.8)	(0.0)	23.65	0.0	
5125	GPC	GENUINE PARTS CO	(3.9)	(12.8)	5.6	264.1	(0.60)	(288.2)	(18,355.5)	(0.0)	4.59	996.9	(5.9)	(6.7)	2.8	12.8	0.36	57,701.7	
5237	GRC	GORMAN-RUPP CO	12.2	0.0	(5.7)	(0.0)	1.33	0.0	(68.7)	(102.4)	0.01	3.3	205.8	0.0	(43.1)	(0.0)	1.76	0.0	
5256	GWW	GRAINGER (W W) INC	24.5	2.9	(1.4)	(10.6)	1.61	5.6	(0.7)	(19,899.8)	0.03	8.4	1,561.9	2.2	(127.7)	(2.6)	2.60	3.0	
5639	HRC	HILL-ROM HOLDINGS INC	5.7	67.4	1.1	201.1	0.88	135.4	(13,426.6)	(21,768.8)	3.57	743.2	24.3	32.4	(8.9)	(67.4)	0.78	5,894.3	
5680	HD	HOME DEPOT INC	4,779.2	0.5	27,678.7	864.9	8.95	31.1	(225,657.1)	(0.0)	(11.33)	(1,287.7)	(59,542,484,913.3)	(1.8)	(765,715.5)	(0.5)	80.11	3.0	
5690	HNI	HNI CORP	7.3	2.2	(5.5)	(29.4)	1.00	35.7	(15.4)	(564.6)	0.08	10.8	22.7	0.4	(14.7)	(2.2)	1.01	316.3	
5783	JBHT	HUNT (JB) TRANSPRT SVCS INC	10.7	2.8	5.1	53.4	1.01	55.1	(1,681.5)	(5,366.1)	0.96	147.2	89.1	1.1	(21.7)	(2.8)	1.02	743.5	
5860	ITT	ITT INC	7.4	2.6	3.3	9.5	(1.73)	(22.5)	(260,040.9)	(0.0)	(12.67)	(1,027.9)	131.8	1.0	(44.3)	(2.6)	2.98	42.4	
5862	RYN	RAYONIER INC	(15.7)	(1.4)	10.4	34.2	1.65	42.2	(96,651.5)	(11,983.7)	(6.69)	(327.8)	373.5	0.4	84.9	1.4	2.71	164.5	
6013	INS	INTELLIGENT SYSTEM CORP	2,906.8	1.4	48,364.3	90.1	0.89	3.0	(1,663.6)	(408.9)	5.58	1,008.9	(1,865,371,046.4)	(2.3)	(4,653.0)	(1.3)	0.80	2.6	
6104	IP	INTL PAPER CO	3,632.9	0.1	36,188.6	344.8	3.19	1.7	(87,882.6)	(0.0)	7.50	1,043.5	(13,196,418,450.6)	(0.0)	(73,960.3)	(0.1)	10.18	0.1	
6266	JNJ	JOHNSON & JOHNSON	(10.4)	(8.4)	(3.2)	(17.7)	1.00	114.3	(0.3)	(309.6)	0.28	8.1	98.7	3.8	20.8	8.4	1.00	3,271.6	
6307	SHLD	SEARS HOLDINGS CORP	8,548.0	0.0	48,923.7	0.0	(5.54)	(0.0)	(289,659.8)	(29,727.1)	(17.87)	(930.2)	(71,326,551,364.1)	(0.0)	(525,497.6)	(0.0)	30.74	0.0	
6349	KATY	KATY INDUSTRIES INC	10.6	6.8	2.4	20.4	1.58	22.9	(113,802.0)	(338.4)	1.69	354.8	270.2	3.4	(53.4)	(6.8)	2.51	52.1	
6379	KELYA	KELLY SERVICES INC -CL A	3.8	11.6	(0.1)	(35.6)	1.87	21.8	(2.5)	(374.3)	0.68	49.2	49.5	6.9	(26.3)	(11.5)	3.50	34.0	
6543	LZB	LA-Z-BOY INC	49,246.1	9.7	62,341.2	2,143.2	13.89	74.9	(139,788.4)	(4,582.4)	14.10	2,849.4	(282,065,633,322.9)	(1.8)	(19,011,987.8)	(8.1)	193.03	7.3	
6669	LEN	LENNAR CORP	(25,188.0)	(1.0)	(54,839.7)	(1,793.1)	8.38	14.2	(57,588.8)	(1.2)	(3.86)	(509.2)	(166,813,771,677.5)	(0.3)	3,541,333.1	0.8	70.30	0.7	
6730	LLY	LILLY (ELI) & CO	6,040.5	0.4	29,797.1	682.4	5.98	17.3	(124,507.5)	(82.4)	6.22	752.8	(30,445,649,031.3)	(0.9)	(432,023.4)	(0.4)	35.76	2.1	
6742	LNC	LINCOLN NATIONAL CORP	30,800.2	201.4	38,486.7	509.0	1.00	403.1	(0.7)	(95.8)	0.02	5.5	(537,382,793.4)	(56.4)	(62,157.1)	(201.4)	1.01	40,257.7	
6791	FAC	FIRST ACCEPTANCE CORP	48.2	22.9	(2.6)	(93.1)	2.52	240.1	(202,399.3)	(42,546.5)	5.46	1,296.2	14,702.2	11.5	(611.4)	(22.9)	6.34	2,273.9	
6821	LPX	LOUISIANA-PACIFIC CORP	32,202.3	1.6	56,930.0	1,607.4	12.39	43.1	(185,598.2)	(73,925.3)	7.30	2,224.8	(338,455,206,263.3)	(1.0)	(9,890,079.4)	(1.6)	153.56	3.0	
7117	MLP	MAUI LAND & PINEAPPLE CO	2,368.0	0.2	43,732.2	120.7	1.25	0.5	(16,041.2)	(16,080.1)	11.40	3,034.4	(2,975,133,931.6)	(0.0)	(7,389.0)	(0.0)	1.56	0.0	
7138	MAYS	MAYS (J.W.) INC	20,687.8	0.5	64,365.7	1,782.9	10.10	31.9	(272,065.7)	(17,554.0)	(4.98)	(573.2)	(378,900,080,073.9)	(0.8)	(4,220,030.5)	(0.5)	101.99	2.5	
7146	MKC	MCCORMICK & CO INC	5,548.1	0.0	(72,616.0)	(0.0)	(2.73)	(0.0)	(212,746.4)	(0.0)	19.83	644.3	(38,927,656,296.3)	(0.0)	(82,396.2)	(0.0)	7.43	0.0	
7241	CVS	CVS HEALTH CORP	5,330.7	0.0	75,751.5	295.4	3.12	0.2	(183,421.0)	(1.0)	10.94	496.3	(55,488,359,567.3)	(0.0)	(103,606.2)	(0.0)	9.72	0.0	
7316	MIK	MICHAELS COS INC	15.4	47.6	(0.5)	(146.2)	(1.44)	(100.0)	0.6	0.1	(0.00)	(0.0)	489.3	24.2	(63.7)	(47.6)	2.07	1,206.3	
7343	MU	MICRON TECHNOLOGY INC	28.1	0.9	11.3	36.9	1.64	38.7	(1,424.2)	(324.0)	(1.05)	(370.0)	1,783.6	0.3	(151.2)	(0.9)	2.69	139.2	
7481	MOCO	MOCON INC	0.0	0.0	(94.2)	(1,018.0)	1.84	3,192.0	4.8	13.1	0.00	1.0	(29,867.2)	(101.2)	(0.3)	(0.0)	3.37	756,310.9	
7906	NKE	NIKE INC	17,796.0	0.0	68,465.6	1,342.6	10.48	2.7	(537,571.0)	(234,532.0)	6.30	1,579.0	(480,432,547,101.5)	(0.0)	(3,912,185.3)	(0.0)	109.92	0.0	
7985	NOC	NORTHROP GRUMMAN CORP	6,148.6	0.0	84,018.2	797.1	4.42	0.0	(297,636.1)	(3,419.5)	(15.12)	(4,128.2)	(137,109,911,748.3)	(0.0)	(240,137.3)	(0.0)	19.53	0.0	
7991	TEX	TEREX CORP	9,921.3	0.0	(46,328.1)	(0.0)	(3.40)	(0.0)	(36,104.4)	(174.0)	(12.36)	(3,748.0)	(23,660,416,202.4)	(0.0)	(229,257.2)	(0.0)	11.55	0.0	
8123	OLN	OLIN CORP	(3,644.4)	(0.0)	42,809.7	457.1	4.37	7.1	(442,757.1)	(23,226.3)	(11.42)	(2,445.4)	(34,669,154,383.3)	(0.0)	138,891.8	0.0	19.06	0.6	
8148	OLP	ONE LIBERTY PROPERTIES INC	5.1	11.9	(3.3)	(38.7)	1.13	54.7	(7.9)	(113.5)	0.03	3.2	19.7	3.4	(13.2)	(11.9)	1.29	581.0	
8214	OC	OWENS CORNING	14.0	27.1	(1.9)	(87.0)	1.85	127.9	(109.3)	(49.3)	0.01	1.6	657.3	13.4	(96.0)	(27.1)	3.44	1,189.6	
8253	PCAR	PACCAR INC	3,437.3	0.0	(77,047.4)	(0.0)	(1.64)	(0.0)	(175,226.9)	(98,045.5)	19.98	2,994.4	(16,008,058,759.6)	(0.0)	(18,575.0)	(0.0)	2.70	0.0	

Investing Cash Flows

GCC	Ticker Symbol	Company Name	theta1	t	theta2	t	theta3	t	mu0	t	mu1	t	sigma0	t	sigma1	t	sigma2	t	Comment
1045	AAL	AMERICAN AIRLINES GROUP INC	5,718.4	1.0	19,814.1	396.1	6.31	16.9	(83,893.9)	(12,585.5)	12.63	3,098.9	-1.41E+10	7.8	455,006.7	18.9	(39.78)	4,745.1	
1121	AE	ADAMS RESOURCES & ENERGY INC	(5,186.0)	(0.1)	34,050.8	877.7	7.11	37.2	(454,956.3)	(0.0)	7.89	1,093.8	-1.97E+11	(56.9)	(524,198.0)	(5.0)	(50.54)	895.5	
1186	AEM	AGNICO EAGLE MINES LTD	1,754.0	0.0	(81,555.1)	(0.0)	(0.87)	(0.0)	(123,574.7)	(0.0)	(20.00)	(414.1)	-1.89E+63	(15.1)	2,661.7	3.1	(0.76)	7,838.6	
1234	ATRI	ATRION CORP	7,948.2	1.1	28,622.5	2,032.1	9.66	88.7	(199,010.9)	(0.0)	(5.37)	(98.2)	-6.54E+10	7.4	1,482,975.6	(15.1)	(93.29)	708.8	
1356	AA	ALCOA INC	(8,174.8)	(0.0)	47,988.3	63.3	9.05	0.1	(812,071.5)	(0.0)	8.12	1,066.7	3.27E+22	0.0	(1,339,850.8)	(0.0)	(81.95)	0.0	
1388	AMR1	AMERICAN AIRLINES INC	(3,800.1)	(0.0)	(53,501.3)	(0.0)	1.84	0.0	(78,091.1)	(267.0)	18.67	2,347.0	-4.89E+70	2.2	(25,869.9)	5.5	(3.40)	5,360.0	
1449	AFL	AFLAC INC	(20,420.1)	(1.4)	78,622.1	3.6	0.55	3.0	(160.8)	(1.4)	8.07	177.5	1.76E+09	0.5	(12,563.5)	(1.2)	(0.31)	1,791,386.6	
1487	AIG	AMERICAN INTERNATIONAL GROUP	34,548.8	4.4	96,589.2	29.8	0.73	8.4	(0.7)	(93.4)	0.01	1.0	4.28E+09	0.0	36,569.9	(0.1)	(0.53)	0.0	
1690	AAPL	APPLE INC	22,495.6	1.2	(49,083.1)	(3.3)	0.52	3.8	(26.0)	(840.8)	3.40	29.1	4.39E+08	6.2	12,165.2	(12.7)	(0.27)	65,172.1	
1712	TREC	TRECORA RESOURCES	(6,364.3)	(0.0)	32,423.6	583.1	8.58	2.2	(549,316.8)	(0.3)	10.03	1,120.3	-8.35E+14	30.9	(937,981.9)	(61.7)	(73.69)	1,273.0	
1794	ASH	ASHLAND GLOBAL HOLDINGS INC	(46,072.8)	(0.0)	(96,741.6)	(0.0)	11.17	0.0	(111,968.1)	(0.0)	(8.37)	(127.6)	4.08E+65	0.5	(11,492,443.2)	(1.2)	(124.72)	82.9	
1860	ATW	ATWOOD OCEANICS	(34,556.7)	(0.6)	(76,617.1)	(1,385.4)	10.32	17.9	(145,878.1)	(8,200.6)	(5.20)	(271.4)	3.51E+11	8.4	(7,356,319.0)	(17.2)	(106.44)	181.5	
1864	REX	REX AMERICAN RESOURCES CORP	9,729.8	0.4	44,508.8	1,121.9	11.21	52.1	(441,878.9)	(0.0)	(9.39)	(1,175.8)	1.51E+11	14.6	2,447,030.1	(29.8)	(125.75)	886.6	
1920	AVP	AVON PRODUCTS	(7,349.4)	(0.0)	(76,053.0)	(0.0)	2.08	0.0	(41,364.6)	(0.0)	14.01	1,999.1	1.15E+69	0.0	(63,358.1)	(0.0)	(4.31)	0.0	
1968	BMI	BADGER METER INC	19,174.1	1.1	66,452.4	1,689.5	8.93	11.8	(209,994.9)	(0.0)	(11.64)	(2,680.8)	7.31E+11	(0.7)	3,055,659.3	(1.1)	(79.68)	1.7	
2052	BRN	BARNWELL INDUSTRIES	(10,626.5)	(0.1)	(42,480.4)	(384.0)	5.54	2.2	(70,742.4)	(0.0)	(6.88)	(175.6)	1.02E+13	(8.8)	(651,879.6)	(1.4)	(30.67)	10.1	
2337	RFP	RESOLUTE FOREST PRODUCTS INC	9,071.7	0.7	(15,738.8)	(3.9)	0.35	3.9	(1.7)	15.6	1.01	15.6	-5.13E+05	0.1	2,215.3	(1.0)	(0.12)	448.2	
2444	BC	BRUNSWICK CORP	1,130.1	0.0	(46,988.9)	(0.0)	(0.85)	(0.0)	(84,387.1)	(0.0)	15.97	1,119.4	4.29E+68	(0.0)	1,639.9	0.0	(0.73)	0.0	
2556	CSS	CSS INDUSTRIES INC	(7,264.6)	(0.0)	(98,995.8)	(0.0)	2.29	0.0	(151,952.5)	(0.0)	19.67	2,126.5	8.07E+71	4.8	(76,018.8)	(8.4)	(5.23)	33.4	
2787	CRS	CARPENTER TECHNOLOGY CORP	(20,800.4)	(1.9)	(27,419.7)	(5,456.7)	11.76	116.6	(74,558.6)	(1.0)	2.28	328.6	2.81E+10	0.0	(5,755,593.5)	(0.0)	(138.35)	0.0	
3093	CLC	CLARCOR INC	11,179.0	1.4	36,540.3	2,103.3	10.73	87.5	(253,908.9)	(0.0)	(7.13)	(101.4)	1.32E+11	5.5	2,574,687.4	(11.4)	(115.16)	42,038.6	
3105	IHRT	IHEARTMEDIA INC	7,321.4	1.1	25,042.1	1,347.8	11.45	111.7	(322,011.7)	(7,476.2)	10.44	1,284.3	6.93E+10	(0.0)	1,918,325.7	0.0	(131.01)	0.0	
3138	COKE	COCA-COLA BTLNG CONS	1,426.0	0.0	(70,892.9)	(0.0)	(0.89)	(0.0)	(118,258.8)	(0.0)	12.36	871.7	8.52E+58	(0.1)	2,244.9	(0.1)	(0.79)	9,925.9	
3226	CMCSA	COMCAST CORP	15,857.2	0.0	(70,247.3)	(32.9)	(4.10)	(0.0)	(72,550.5)	(971.0)	(20.00)	(4,095.4)	3.29E+30	(8.8)	534,231.6	5.1	(16.85)	74.8	
3429	CTO	CONSOLIDATED TOMOKA LAND CO	(21,404.5)	(1.0)	(39,315.6)	(1,191.1)	13.95	81.0	(342,844.4)	(0.0)	(11.34)	(2,261.7)	1.20E+11	0.0	(8,333,932.5)	(0.3)	(194.68)	325.4	
3622	CRWS	CROWN CRAFTS INC	6,723.8	0.1	22,164.5	1,157.8	11.02	45.6	(250,184.0)	(0.0)	5.66	112.6	-1.71E+12	(0.2)	1,633,825.3	(0.1)	(121.50)	0.5	
3813	TGT	TARGET CORP	(5,617.8)	(0.0)	77,098.8	499.2	4.64	9.7	(820,430.9)	(0.0)	(14.44)	(376.2)	-6.08E+11	(8.3)	(241,608.8)	3.1	(21.50)	29.7	
4145	PKI	PERKINELMER INC	4,575.8	0.7	16,972.1	356.9	4.76	6.3	(30,743.2)	(0.0)	7.43	349.9	1.33E+10	13.5	207,419.3	(28.7)	(22.66)	3,411.2	
4201	EV	EATON VANCE CORP	1,871.7	0.1	(92,377.0)	(0.2)	(0.07)	(0.1)	(254.7)	(2,010.4)	(9.54)	(378.4)	3.15E+10	0.0	16.9	(0.0)	(0.00)	0.0	
4275	ELSE	ELECTRO-SENSORS INC	15.1	0.0	(80,710.0)	(0.0)	(0.01)	(0.0)	(27.6)	(0.0)	(0.02)	(0.4)	7.49E+63	(13.1)	0.0	0.0	(0.00)	328.6	
4485	KINS	KINGSTONE COS INC	(4,729.4)	(0.1)	(94,139.8)	(171.8)	1.82	0.9	(90,593.5)	(0.0)	(15.66)	(672.0)	4.38E+12	5.1	(31,170.6)	(10.2)	(3.30)	93,762,735.2	
4600	FDML	FEDERAL-MOGUL HOLDINGS CORP	35,749.9	2.5	76,809.9	3,314.6	18.60	226.2	(1,085,273.6)	(0.0)	13.51	282.4	1.53E+12	(5.2)	24,724,836.8	1.3	(345.80)	176,024.8	
4605	FRT	FEDERAL REALTY INVESTMENT TR	(10,768.3)	(0.2)	48,251.4	898.5	11.70	86.5	(1,477,713.6)	(0.0)	(18.60)	(1,643.3)	-1.00E+11	16.1	(2,948,402.2)	(31.2)	(136.90)	476.2	
4885	BEN	FRANKLIN RESOURCES INC	4,727.2	0.3	31,946.7	840.3	5.32	18.5	(143,963.5)	(0.0)	5.16	175.1	2.07E+10	1.2	267,991.6	(4.1)	(28.35)	733.3	
4926	FUL	FULLER (H. B.) CO	(38,431.3)	(0.0)	(93,840.6)	(85.0)	9.31	0.0	(66,443.4)	(0.0)	(8.35)	(1,087.5)	1.42E+29	(1.6)	(6,669,031.1)	(2.2)	(86.77)	3.0	

Investing Cash Flows

GCC	Ticker Symbol	Company Name	theta1	t	theta2	t	theta3	t	mu0	t	mu1	t	sigma0	t	sigma1	t	sigma2	t	Comment
4988	TGNA	TEGNA INC	16,726.4	1.5	63,445.2	1,529.4	12.15	53.7	(490,578.4)	(0.0)	(16.83)	(488.4)	5.39E+11	14.8	4,937,209.9	(18.7)	(147.59)	13.1	
5046	GD	GENERAL DYNAMICS CORP	550.6	0.0	(75,964.3)	(0.0)	(0.30)	(0.0)	(49,148.5)	(0.0)	17.45	671.3	5.79E+71	(0.0)	100.1	(0.0)	(0.09)	0.0	
5087	SPXC	SPX CORP	4,330.4	0.2	32,967.9	692.4	4.58	9.0	(118,583.2)	(0.0)	4.77	582.9	1.19E+10	2.4	181,851.5	(5.1)	(21.00)	3,551.9	
5125	GPC	GENUINE PARTS CO	(1,822.6)	(0.0)	(93,863.6)	(0.0)	0.71	0.0	(70,197.1)	(0.0)	12.54	1,048.9	1.55E+72	(0.0)	(1,862.3)	1.2	(0.51)	34.6	
5237	GRC	GORMAN-RUPP CO	6,712.9	0.4	31,873.9	661.4	10.82	30.4	(300,357.0)	(0.1)	(13.48)	(712.0)	8.41E+10	2.5	1,570,555.6	(6.5)	(116.98)	1,215.2	
5256	GWW	GRAINGER (W W) INC	(3,848.2)	(0.0)	(97,550.8)	(6.1)	1.34	0.0	(124,677.9)	(0.1)	19.86	1,589.4	7.16E+38	2.2	(13,735.5)	(2.6)	(1.78)	3.0	
5639	HRC	HILL-ROM HOLDINGS INC	(1,349.9)	(0.0)	(93,693.5)	(0.9)	0.02	0.1	(17.2)	(3,368.2)	2.48	106.6	1.05E+10	7.1	(1.4)	(13.8)	(0.00)	123.6	
5680	HD	HOME DEPOT INC	5,925.8	2.4	(9,314.6)	(5.8)	0.72	7.3	(0.4)	(30.2)	0.04	0.9	2.40E+07	20.1	6,121.5	(39.4)	(0.52)	727.3	
5690	HNI	HNI CORP	7,515.3	0.5	21,590.7	744.8	10.64	32.9	(144,215.5)	(0.0)	(8.60)	(529.0)	2.67E+10	(2.7)	1,702,374.2	(1.0)	(113.26)	5,416.5	
5783	JBHT	HUNT (JB) TRANSPRT SVCS INC	68,172.8	3.1	(98,139.8)	(5.6)	0.65	9.8	(0.2)	(64.3)	0.07	2.0	1.94E+09	5.2	57,658.6	(10.7)	(0.42)	43,638.3	
5860	ITT	ITT INC	(4,651.2)	(0.3)	98,667.9	2,174.3	4.39	24.8	(683,475.1)	(0.0)	(9.54)	(666.4)	1.82E+11	(0.2)	(179,009.3)	(0.6)	(19.24)	0.3	
5862	RYN	RAYONIER INC	15,834.9	0.2	67,031.2	1,005.7	7.65	2.8	(202,890.3)	(20,753.8)	(10.34)	(685.3)	5.94E+13	5.0	1,851,312.5	(10.8)	(58.46)	748.2	
6013	INS	INTELLIGENT SYSTEM CORP	(78,888.1)	(4.6)	(92,325.8)	(5,993.2)	19.96	187.7	(490,851.5)	(0.0)	10.76	586.4	8.01E+11	0.1	(62,881,364.6)	(0.6)	(398.55)	7.8	
6104	IP	INTL PAPER CO	(4,265.3)	(0.0)	25,908.0	506.0	6.20	13.4	(305,883.9)	(0.0)	10.87	229.7	-4.05E+11	(0.0)	(327,638.8)	(0.1)	(38.41)	0.1	
6266	JNJ	JOHNSON & JOHNSON	3,428.3	0.1	19,221.9	385.4	4.16	1.4	(48,513.6)	(0.0)	5.31	408.6	2.53E+12	12.5	118,521.1	(25.2)	(17.29)	5,760.2	
6307	SHLD	SEARS HOLDINGS CORP	13,011.0	1.4	48,951.4	1,678.3	11.20	63.8	(369,161.7)	(0.0)	(12.06)	(228.2)	-2.69E+11	2.2	(3,267,118.8)	(8.2)	125.55	1,368.3	
6349	KATY	KATY INDUSTRIES INC	8,117.1	0.3	48,010.4	973.9	5.87	4.9	(152,521.6)	(0.0)	(9.28)	(563.7)	-7.26E+11	11.6	(559,079.5)	(24.1)	34.44	842.6	
6379	KELYA	KELLY SERVICES INC -CL A	8,117.1	0.3	48,010.4	973.9	5.87	4.9	(152,521.6)	(0.0)	(9.28)	(563.7)	-7.26E+11	6.9	(559,079.5)	(11.5)	34.44	34.0	
6543	LZB	LA-Z-BOY INC	(8,041.2)	(0.0)	(64,179.5)	(0.0)	3.10	0.0	(114,200.6)	(0.0)	20.00	388.1	4.00E+70	(1.8)	(154,982.3)	(8.1)	(9.64)	7.3	
6669	LEN	LENNAR CORP	3,467.8	0.0	(62,687.2)	(0.0)	(2.21)	(0.0)	(196,806.4)	(0.0)	12.99	1,859.0	1.72E+62	(6.3)	33,805.5	4.1	(4.87)	26,437.2	
6730	LLY	LILLY (ELI) & CO	(24,815.3)	(2.2)	(37,021.4)	(2,423.7)	15.81	168.5	(277,291.7)	(0.0)	(7.73)	(56.5)	1.57E+11	8.1	(12,409,990.7)	(15.6)	(250.05)	187.3	
6742	LNC	LINCOLN NATIONAL CORP	8,966.5	0.6	46,021.2	1,202.0	7.78	42.9	(318,488.0)	(0.1)	8.56	608.3	1.12E+11	(56.4)	1,086,404.5	(201.4)	(60.58)	40,257.7	
6791	FAC	FIRST ACCEPTANCE CORP	50,387.5	2.7	90,830.9	1,672.9	12.79	34.5	(215,808.5)	(0.0)	10.31	158.9	9.49E+11	(6.5)	16,474,945.4	(134.6)	(163.48)	397,917.7	
6821	LPX	LOUISIANA-PACIFIC CORP	10,692.3	0.2	98,652.5	3.8	(0.07)	(0.5)	(3.4)	(215.7)	(0.97)	(8.3)	3.71E+07	0.0	120.1	(0.0)	(0.01)	0.0	
7117	MLP	MAUI LAND & PINEAPPLE CO	(4,609.4)	(0.0)	28,713.9	208.4	7.26	0.4	(386,702.8)	(0.0)	5.78	337.4	7.32E+17	(188.5)	(486,494.5)	(0.2)	(52.77)	11,886.2	
7138	MAYS	MAYS (J.W.) INC	(3,915.8)	(3.1)	(77,815.0)	(4.9)	0.27	6.2	(1.6)	(100.0)	0.18	3.9	4.48E+08	3.2	(575.3)	(6.5)	(0.07)	333.8	
7146	MKC	MCCORMICK & CO INC	(39,037.0)	(0.3)	54,983.4	0.8	0.59	5.9	(4.1)	3.4	1.47	13.0	-5.03E+09	7.0	(27,056.0)	(14.3)	(0.35)	187,063.9	
7241	CVS	CVS HEALTH CORP	26,674.3	0.9	68,384.8	436.0	7.76	6.0	(82,553.4)	(15.4)	11.15	331.5	2.96E+12	(65.7)	3,212,657.7	(262.1)	(60.22)	2,277.6	
7316	MIK	MICHAELS COS INC	(3,491.8)	(0.0)	28,888.6	946.6	5.68	5.1	(298,705.7)	(0.0)	5.24	132.2	-6.27E+12	24.2	(225,076.0)	(47.6)	(32.23)	1,206.3	
7343	MU	MICRON TECHNOLOGY INC	(5,692.8)	(0.0)	(96,739.8)	(0.0)	1.87	0.0	(135,817.1)	(402.4)	19.67	4,347.2	1.27E+72	0.6	(39,766.1)	(1.5)	(3.49)	9.8	
7481	MOCO	MOCON INC	(9,438.5)	(2.2)	25,422.2	15.2	0.16	4.3	(3.3)	(0.2)	1.47	96.0	1.34E+07	(101.2)	(468.4)	(0.0)	(0.02)	756,310.9	
7906	NKE	NIKE INC	22,385.0	0.0	(60,633.1)	(0.0)	(8.21)	(0.0)	(133,380.8)	(40.5)	13.88	3,477.3	2.42E+65	(5.5)	3,017,227.4	(29.9)	(67.39)	245,977.6	Partial convergence
7985	NOC	NORTHROP GRUMMAN CORP	3,299.1	2.0	11,559.1	1,000.1	6.27	50.2	(55,265.4)	(0.0)	7.36	773.9	4.73E+09	(0.0)	259,678.6	(0.0)	(39.36)	0.0	
7991	TEX	TEREX CORP	55,738.4	2.0	99,699.9	963.3	13.57	25.4	(294,525.7)	(0.0)	16.23	914.6	1.52E+12	0.0	20,513,365.4	(0.2)	(184.01)	0.1	
8123	OLN	OLIN CORP	(2,338.5)	(0.0)	(43,674.4)	(119.4)	1.50	0.3	(46,625.8)	0.0	(11.04)	(1,223.2)	2.50E+14	1.8	(10,463.7)	(3.5)	(2.24)	33.7	
8148	OLP	ONE LIBERTY PROPERTIES INC	19,011.1	0.4	87,523.1	267.8	5.44	2.9	(113,606.5)	(0.0)	14.54	1,819.3	1.30E+13	3.4	1,124,135.0	(11.9)	(29.57)	581.0	
8214	OC	OWENS CORNING	(5,253.6)	(0.0)	(66,801.6)	(213.4)	2.25	0.0	(50,470.1)	0.0	(9.84)	(751.2)	2.22E+22	13.4	(53,155.9)	(27.1)	(5.06)	1,189.6	
8253	PCAR	PACCAR INC	(39,197.6)	(2.9)	(54,498.9)	(6.4)	(0.57)	(10.1)	(0.0)	(1.2)	0.05	5.2	4.18E+08	6.3	(25,764.4)	(27.8)	(0.33)	684.2	

9. Other Appendixes

Appendix O1 – Stochastic characteristics of well-known cash flow specifications

	Geometric Brownian Motion (GBM)	Arithmetic Brownian Motion (ABM)	Vasicek process	Cox, Ingersoll and Ross (CIR) process	Modified Square Root (MSR) process
SDE specification	$dC_t = \mu C_t dt + \sigma C_t dW_t$	$dC_t = \mu dt + \sigma dW_t$	$dC_t = \alpha(m - C_t)dt + \sigma dW_t$	$dC_t = \alpha(m - C_t)dt + \sigma\sqrt{C_t}dW_t$	$dC_t = \mu C_t dt + \sqrt{k_1^2 + k_2^2 C_t^2 dW_t}$
Random process	Wiener	Wiener	Wiener	Wiener	Wiener
Parameters	(constant)drift rate μ, (constant) σ (> 0) volatility rate	(constant)drift rate μ, (constant) σ (> 0) volatility rate	m is the long term average, α (> 0) is a parameter that measures how rapidly this convergence in time to m occurs and σ (> 0) is a (constant) volatility parameter	m is the long term average, α (> 0) is a parameter that measures how rapidly this convergence in time to m occurs and σ (> 0) is a (constant) volatility parameter	μ is the (constant) drift rate, k_1^2 and k_2^2 are constant parameters defining the (constant) volatility rate
SDE has closed solution	Yes	Yes	Yes	Yes	Only in special case if $k_2^2 = \pm 2\mu$
Domain of cash flow	C_t [0 , ∞]	C_t $[-\infty, +\infty]$	C_t $[-\infty, +\infty]$	C_t [0, ∞]	C_t $[-\infty, +\infty]$
Transitional probability density function	Log-normal	Normal	Normal	Noncentral chi-square	Difference of two log-normal distributions
Stationary probability density function	No	No	Yes	Yes	No
Drift function	Exponential growth trend	Linear growth trend	Mean-reverting	Mean-reverting	Exponential growth trend

Appendix O1 - Stochastic characteristics of well-known cash flow specifications (cont'ed)

	Geometric Brownian Motion (GBM)	Arithmetic Brownian Motion (ABM)	Vasicek process	Cox, Ingersoll and Ross (CIR) process	Modified Square Root (MSR) process
Diffusion function	Multiplicative	Additive	Additive	Multiplicative	Multiplicative
Evolution of first moment	Diverging to +∞ or converging to 0, depending on choice of μ and σ; exponential growth trend.	Diverging to +∞ for μ > 0; Growth trend linear in t.	Exponentially fast converging to m.	Exponentially fast converging to m.	Diverging to +∞ or converging to 0, depending on choice of μ and σ; exponential growth trend.
Evolution of second moment	Diverging to +∞ or converging to 0, depending on choice of μ and σ; exponential growth trend.	Diverging to +∞; quadratic function of t.	Exponentially fast converging to $m^2 + \frac{\sigma^2}{4\alpha}$.	Exponentially fast converging to $\frac{\alpha m^2 \sigma^2}{\alpha}$, the higher the moment the faster the convergence.	Exponentially fast converging to stationary value, the higher the moment the faster the convergence.
Evolution of third moment	Diverging to +∞ or converging to 0, depending on choice of μ and σ; exponential growth trend.	0	0	Exponentially fast converging to stationary value, the higher the moment the faster the convergence	Exponentially fast converging to stationary value, the higher the moment the faster the convergence
Evolution of fourth moment	Diverging to +∞ or converging to 0, depending on choice of μ and σ; exponential growth trend.	Diverging to +∞; quadratic function of t.	Can be expressed as function of the evolution of the second moment $M_2(t)$.	Exponentially fast converging to stationary value, the higher the moment the faster the convergence.	Exponentially fast converging to stationary value, the higher the moment the faster the convergence.

10. Definitions of terms used

Akaike Information Criterion (AIC): is a measure of the relative quality of statistical models for a given set of data. Since AIC estimates the quality of each model, relative to the other models, the criterion is AIC is used to select the best model, especially if models have a different number of parameters. The model with the lowest AIC is preferred.

Approximated Maximum Likelihood Estimation (AMLE): particular form of MLE (see Maximum Likelihood Estimation) where complexity is reduced by deriving informative but low dimensional summaries from the data sets. Often this information is incorporated in a stochastic gradient algorithm that approximates the maximum likelihood estimate.

Bayesian Information Criterion (BIC): see Akaike Information Criterion. The formula to calculate BIC is slightly different to that of AIC.

Birth and Death process: a continuous-time Markov process where the state transitions are of only two types: "births", which increase the state variable by one and "deaths", which decrease the state by one.

Boundary conditions: a set of additional constraints to differential equations that require solutions to be found in a bounded region. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.

Cash flow model: an abstract (usually mathematical) representation of real-world cash flows with the aim of describing and explaining how cash flows can be predicted, managed, and controlled

Cash flow process: sequence of interdependent changes to cash that comes in and goes out of a company which are described and linked in time.

Cash flow specification: is a formal expression defining the mathematical behaviour of a cash flow process.

Conditional (or transitional) probability density function: for two jointly distributed random variables X and Y, the conditional probability distribution function of Y given X is the probability distribution of Y when X is known to be a particular value.

Continuous process: a type of well-behaved stochastic process with time as a continuous variable.

Continuous-time: variables measured in continuous time have a particular value for potentially only an infinitesimally short amount of time. Between any two points in time there are an infinite number of other points in time.

Converging process: a stochastic process that in the limit of $t \to \infty$ exhibits unchanged randomness which can be described by a time-invariant (stationary) probability distribution.

Coupled linear-quadratic cash flow model: the cash-flow model developed in this study, consisting of a coupled operating cash flow and an investing cash flow process, where both equations are defined by a linear drift function and a quadratic diffusion function.

Coupled process: a multi-variable stochastic process where the process dynamics are described by equations of which at least one equation includes more than one variable. To find solutions to a coupled process, the process has to be decoupled first by eigen-decomposition.

Decoupled process: a multi-variable stochastic process where the process dynamics are expressed in a solvable problem, described by equations that each contain exactly one variable.

Deterministic: not random, having only one, pre-determined outcome.

Diffusion function: the mathematical representation of the random component of a continuous-time Markov process with almost surely continuous sample paths.

Discrete-time: variables measured in discrete time have values at distinct, separate "points in time", or equivalently as being unchanged throughout each non-zero region of time ("time period").

Diverging process: a stochastic process with no stationary probability distribution and characterised by random values becoming more dispersed as $t \to \infty$.

Drift function: the mathematical representation of the deterministic component of a continuous-time Markov process with almost surely continuous sample paths.

Eigen-decomposition: is the factorisation of a diagonalisable matrix into a canonical form, whereby the matrix is represented in terms of its eigenvalues and eigenvectors (or spectral decomposition).

Financing cash flow: cash flow that results from external activities that allow a firm to raise capital. In addition to raising capital, financing activities also include repaying investors, adding or changing loans, or issuing more shares.

Fokker-Planck equation: is a partial differential equation that describes the time evolution of the probability density function of an underlying Brownian motion. The equation is also known as the Kolmogorov forward equation.

Hybrid stochastic differential equation: a stochastic differential equation that is the combination of two or more simpler stochastic differential equations. The properties of a hybrid SDE are significantly more complex than the sum of the properties of the underlying SDEs.

Hypergeometric ordinary differential equation: a differential equation of the functional form $x(1-x)\frac{\mathrm{d}^2\mathrm{y}(\mathrm{x})}{\mathrm{dx}^2} + [\gamma - (\alpha + \beta + 1)]\frac{\mathrm{dy}(\mathrm{x})}{\mathrm{dx}} - \alpha\beta\mathrm{x} = 0$ which has three regular singular points: 0, 1 and ∞. Any second order differential equation with three regular singular points can be converted to the hypergeometric differential equation by a change of variables.

Investing cash flow: under US GAAP the main components of investing cash flow are: acquisition of debt instruments of other entities, sale of debt instruments of other entities, acquisition of equity instruments of other entities, sale of equity instruments of other entities, acquisition of property, plant and equipment, sale of property, plant and equipment, capital expenditures, and payment for purchase of another entity.

Jump process: type of stochastic process that has discrete movements, called jumps, with random arrival times, rather than continuous movement, typically modelled as a simple or compound Poisson process.

Kernel density estimation: is a non-parametric way to estimate the probability density function of a random variable. Kernel density estimation is a fundamental data smoothing problem where inferences about the population are made, based on a finite data sample.

Kolmogorov equations: parabolic partial differential equations that describe how the probability density function of a stochastic process changes between states over time. There are two equivalent (leading to the same solutions) specifications: the forward Kolmogorov equation (or the Fokker-Planck equation) and the backward Kolmogorov equation.

Lamperti transform: transforms a state-dependent stochastic process into a state-independent process with unit instantaneous variance.

Leptokurtic probability distribution: a distribution where the points along the X-axis are clustered, resulting in a higher peak, or higher kurtosis, than the curvature found in a normal distribution. This high peak and corresponding fat tails mean the distribution is more clustered around the mean than in a mesokurtic or platykurtic distribution and has a relatively smaller standard deviation.

Levy process: is a stochastic process with independent, stationary increments representing the motion of a point whose successive displacements are random and independent, and statistically identical over different time intervals of the same length. A Lévy process may thus be viewed as the continuous-time analog of a random walk.

Lipschitz conditions: conditions that test how fast a function can change to remain continuous: there exists a definite real number such that, for every pair of points on the graph of this function, the absolute value of the slope of the line connecting them is not greater than this real number.

Macroscopic: this approach considers that the system is made up of a very large number of individual entities, for instance firms. Each firm has different stochastic characteristics; however, the analysis is focused on the “average” behaviour of the system and only few properties are required to fully describe the system.

Markov process: a stochastic process where the behavior of the process in the future is stochastically independent of its behavior in the past, given the current state of the process. The sequence of steps over time is called Markov chain.

Master Equation: describes the time evolution of a system that can be modelled as being in a probabilistic combination of states at any given time and the switching between states is determined by a transition rate matrix. The equations are a set of differential equations over time of the probabilities that the system occupies each of the different states.

Maximum Likelihood Estimation (MLE): is a method of estimating the parameters of a statistical model given observations, by finding the parameter values that maximize the likelihood of making the observations given the parameters.

Mean-square convergence: if the distance (square of differences) of a random variable to some defined value, measured over a long time, on average is very small, the variable is said to be converging in the mean-square.

Mesoscopic: an approach that features between the microscopic and the macroscopic approach. It describes both aggregated system behaviour and higher-level randomness as a common denominator for all entities, for instance firms.

Method of Moments: to estimate population parameters one starts with deriving equations that relate the population moments (i.e., the expected values of powers of the random variable under consideration) to the parameters of interest. A sample is drawn and the population moments are estimated from the sample. The equations are then solved for the parameters of interest, using the sample moments in place of the (unknown) population moments. This results in estimates of those parameters.

Microscopic: this approach examines the detailed behaviour of individual entities, for instance firms. Usually a larger number of variables and/or complex relationships are needed to describe such a system and to model specific randomness.

Particular solution: a solution that is only valid for specific parameter values as opposed to a general solution that is valid for all possible parameter values.

Pearson diffusion: a flexible class of diffusions defined by linear mean-reverting drift function and a quadratic diffusion function.

Perturbation theory: a set of approximation schemes directly related to mathematical perturbation for describing a complicated system in terms of a simpler one. From known solution to a simple system, the more complicated system can be described as weak disturbances, being small compared to the size of the simple system. Disturbances can be calculated using approximate methods such as asymptotic series. The complicated system can therefore be studied based on knowledge of the simpler one.

Principle of superposition: states that, for all linear systems, the net response at a given place and time caused by two or more stimuli is the sum of the responses that would have been caused by each stimulus individually.

Probability current: is a mathematical quantity describing the flow of probability in terms of probability per unit time per unit area (or probability flux).

Pseudolikelihood estimation: estimation method where the probability density distribution is replaced by a collection of random variables that can provide an approximation to the likelihood function of a set of observed data which may either provide a computationally simpler problem for estimation, or may provide a way of obtaining explicit estimates of model parameters.

Operating cash flow: under US GAAP the main components of operating cash flow are: cash received from sale of goods or services, cash paid to suppliers and employees, receipt of dividends, receipt of interest, payment of interest, receipt of insurance proceeds, and income taxed paid.

Q-Q plot: is a probability plot, which is a graphical method for comparing two probability distributions by plotting their quantiles against each other. Also called quantile-quantile plot.

Space-time density function: a continuous three-dimensional function, the first dimension consisting of the variable considered, the second dimension the probability density function of the variable at a specific moment in time, and as a third dimension, time to show the evolution of the probability density.

Stationary probability distribution: is a probability distribution that remains unchanged in the Markov chain as time progresses.

Stochastic: random in time.

Stochastic differential equation: is a differential equation, describing a stochastic process, that contains a variable which represents calculated random white noise (Wiener process).

Student diffusion: a diffusion process that is governed by a Student probability density function.

Sturm-Liouville problem: a problem defined by a second order differential equation that can be expressed in a specific form, usually derived from separation of variables of a partial differential equation. The form includes a separation parameter that can be determined, if it exists, by satisfying given boundary conditions. Solutions consist of eigenvalues and corresponding eigenfunctions.

Unconditional (or marginal) probability density function: for two jointly distributed random variables X and Y, the unconditional or marginal probability distribution function of Y is the probability distribution of Y without reference to values of X.

11. Glossary of abbreviations used

ABM: Arithmetic Brownian Motion.

AIC: Akaike Information Criterion

AMLE: Approximated Maximum Likelihood Estimator

ARIMA model: Autoregressive Integrated Moving Average model.

BIC: Bayesian Information Criterion

CIR process: Cox, Ingersoll and Ross process (or Square Root process).

CLT: Central Limit Theorem.

GBM: Geometric BrownianM.

GFC: Global Financial Crisis

GMM: Generalised Method of Moments

KDE: Kernel Density Estimation

MLE: Maximum Likelihood Estimator

MSR process: Modified Square Root process.

ODE: Ordinary Differential Equation.

PDE: Partial Differential Equation.

PDF: Probability Density Function.

SDE: Stochastic Differential Equation.

SPDE: Stochastic Partial Differential Equation.

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