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NUMERACY EQUIPMENT AND YEAR 3 CHILDREN: "BRIGHT, SHINY STUFF", OR SUPPORTING THE DEVELOPMENT OF PART-WHOLE THINKING?

LINDA BONNE

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Numeracy Equipment and Year 3 Children: "bright, shiny stuff", or supporting the development of part-whole thinking?

by
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Abstract

New Zealand teachers' use of equipment has increased as a result of their participation in the Numeracy Development Project. The purpose of this study was to discover how closely the children's reasons for their equipment choices matched their teachers' reasons for including the same pieces of equipment in their numeracy programmes. In the teachers' reasons for equipment choices, the surface features of equipment seemed equally important as the conceptual development the equipment can support. In contrast, the reasons given for equipment choices by the 34 Year 3 children who were interviewed were almost exclusively concerned with how the equipment might help them to solve the given problem. The children's success rates at solving the problem declined as the equipment became more structured; this paralleled the teachers' equipment choices. The equipment choices of the four teachers interviewed in this study were not strongly consistent with the equipment use recommended in the NDP materials.

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NUMERACY EQUIPMENT AND YEAR 3 CHILDREN: BRIGHT, SHINY STUFF, OR SUPPORTING THE DEVELOPMENT OF PART-WHOLE THINKING?

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List of Abbreviations

ANP	Advanced Numeracy Project
BSM	Beginning School Mathematics
CAN	Calculator Aware Number Project
ENL	Empty Number Line
ENP	Early Numeracy Project
NDP	Numeracy Development Project
NEMP	National Education Monitoring Project
RME	Realistic Mathematics Education
TIMSS	Third International Mathematics and Science Study

CHAPTER 1

Numeracy Teaching and Learning

1.1 Worldwide perspectives

Around the world, researchers and teachers are trying to discover ways to improve children's numeracy learning. Initiatives such as the National Numeracy Strategy in the UK, Realistic Mathematics Education (RME) in the Netherlands, Cognitively Guided Instruction in the US, and the Numeracy Development Project (NDP) here in New Zealand are all aimed at lifting the achievement of all children. Current theories about how children learn continue to be influenced by constructivism and the notion of child-centredness. The importance of helping children to construct a richly interconnected network of number understandings rather than a collection of isolated pieces of knowledge is central to recent initiatives worldwide. Inherent in this is a tension between using children's informal strategies as the starting point, while at the same time manipulating children's learning so that they each construct conventional mathematics knowledge.

While definitions of numeracy vary from country to country, the New Zealand version encapsulates the key common themes.

> To be numerate is to have the ability and the inclination to use mathematics effectively – in the home, at work and in the community (Ministry of Education, 2001, p.1)

An understanding of number underpins other areas of numeracy. The confident, flexible application of the key number concepts of place value and part-whole thinking are therefore central to being numerate (Ross, 1989; Baroody, 1990). In order to be able to help children become numerate, teachers also need to have a sound understanding of pedagogy and content (Fennema & Franke, 1992). This section will present an overview of developments and key issues in numeracy teaching and learning around the world, before shifting the focus to the New Zealand context.

1.1.1 Children's learning – the importance of building connections

There is some general agreement that constructivism provides the best explanation currently available about how children learn (Clements & Battista, 1990; Dengate, 1998; Jones, Thornton, Putt, Hill, Mogill, Rich & Van Zoest, 1996; Kamii & Lewis, 1990; Ritchie & Carr, 1997). Its central tenet is that children construct their own understanding by making connections between new experiences and their existing mental networks, or 'schemata'. Teaching numeracy is challenging because on the one hand the teacher takes children's informal strategies for problem-solving as a foundation on which to build, but on the other hand has to guide each child's construction of understanding so that they recreate pre-existing knowledge (Ball, 1993, 2001; Clements & Battista, 1990; Cobb, 1994). RME aims at some middle ground.

> RME involves an inherent tension between individual students' expressive creativity and their enculturation into established mathematical ways of knowing (Gravemeijer, Cobb, Bowers & Whitenack, 2000, p.227).

Rather than presenting children with pre-fabricated mathematical knowledge, the aim of RME is to have children re-invent it for themselves with the teacher's guidance. It is through this process of 'guided reinvention' that children engage in 'mathematizing' – essentially the organising and re-organising of mathematical information (Freudenthal, 1971, cited in Gravemeijer, 1994). RME favours beginning with the children's informal strategies, which can be thought of as a 'bottom-up' approach (Gravemeijer, 1997; Hiebert & Carpenter, 1992). In contrast, a 'top-down' approach involves the teacher presenting ideas to children by using equipment that is derived from abstract mathematical concepts in the hope that this will help them to internalise the concept embodied by the equipment. The 'top-down' approach is more in keeping with transmission and behaviourist views of teaching and learning.

Establishing connections with children's existing understandings is vital, but helping children to construct links between their different ideas is also crucial. The notion of a deep level of understanding of mathematics consisting of a richly interconnected web of concepts is not new to mathematics education, although terminology has changed. Skemp described two different types of understanding; one of these was 'relational understanding'.

> ...learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in

> principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point (Skemp, 1976, p.15).

In contrast, 'instrumental understanding' is consistent with understanding in terms of knowing set procedures to get from a starting point to a finishing point, and as such is consistent with having procedural knowledge without necessarily having a deep understanding of why a procedure works.

Helping children to make connections between their existing understandings and new learning is also a focus in the UK's National Numeracy Strategy, where children's mental strategies are also highlighted as the starting point for teaching and learning. Working in this setting, Askew, Brown, Rhodes, Wiliam and Johnson (1997) examined the link between teachers' practices, beliefs and knowledge and children's learning outcomes. They differentiated three different models of sets of beliefs: connectionist, transmission and discovery. Teachers with connectionist beliefs are thought to value children's methods and teaching strategies that emphasise establishing connections within mathematics, while those with a transmission orientation view mathematics as a body of knowledge to be passed on to children through their teaching. Teachers with discovery-based beliefs view mathematics as being discovered by children. The research findings indicated that teachers with a strong connectionist orientation were more likely to have children who made greater gains than teachers who had a strong transmission or discovery orientation (Askew et al., 1997).

The importance of establishing connections between concrete and symbolic representations, as well as between different pieces of equipment, is discussed in recent American research findings (Ball, 1992; Carpenter, Fennema, Franke, Levi & Empson, 1999). Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier and Human explained the significance of the relationship between different pieces of equipment.

> The different tools are different forms of representation, and each conveys a somewhat different message, and each emphasizes somewhat different features of the idea. Recognizing the similarities and differences between the forms is both an important and challenging task for students (Hiebert et al., 1997, p.58).

1.1.2 Models of development

Piaget's constructivist model of children's cognitive development from the concrete to the abstract is significantly modified in recent proposed models. In order to promote mathematics with understanding, it is thought that learning should begin with concrete experiences, then proceed to symbolic representations, with the means of bridging this gap often described in terms of children having mental images of the equipment (Baroody, 1989; Bobis, 1996; Fuson & Briars, 1990;). Other more detailed models of children's conceptual development have depicted children's mathematics learning as a recursive, multilevel process (Gravemeijer et al., 2000; Pirie & Kieren, 1989). A feature of the Pirie and Kieren model (shown in Figure 1) was the notion of 'folding back'. This referred to the situation of a child who is unable to solve a problem at their current level of understanding needing to move back to an earlier level. They maintained that it was through repeatedly reconstructing the child's earlier level knowledge that the child will further extend their knowledge at their current level. A simplified version of this model forms the basis of the teaching and learning model advocated in the NDP draft teachers' materials (see Figure 3, p.15).

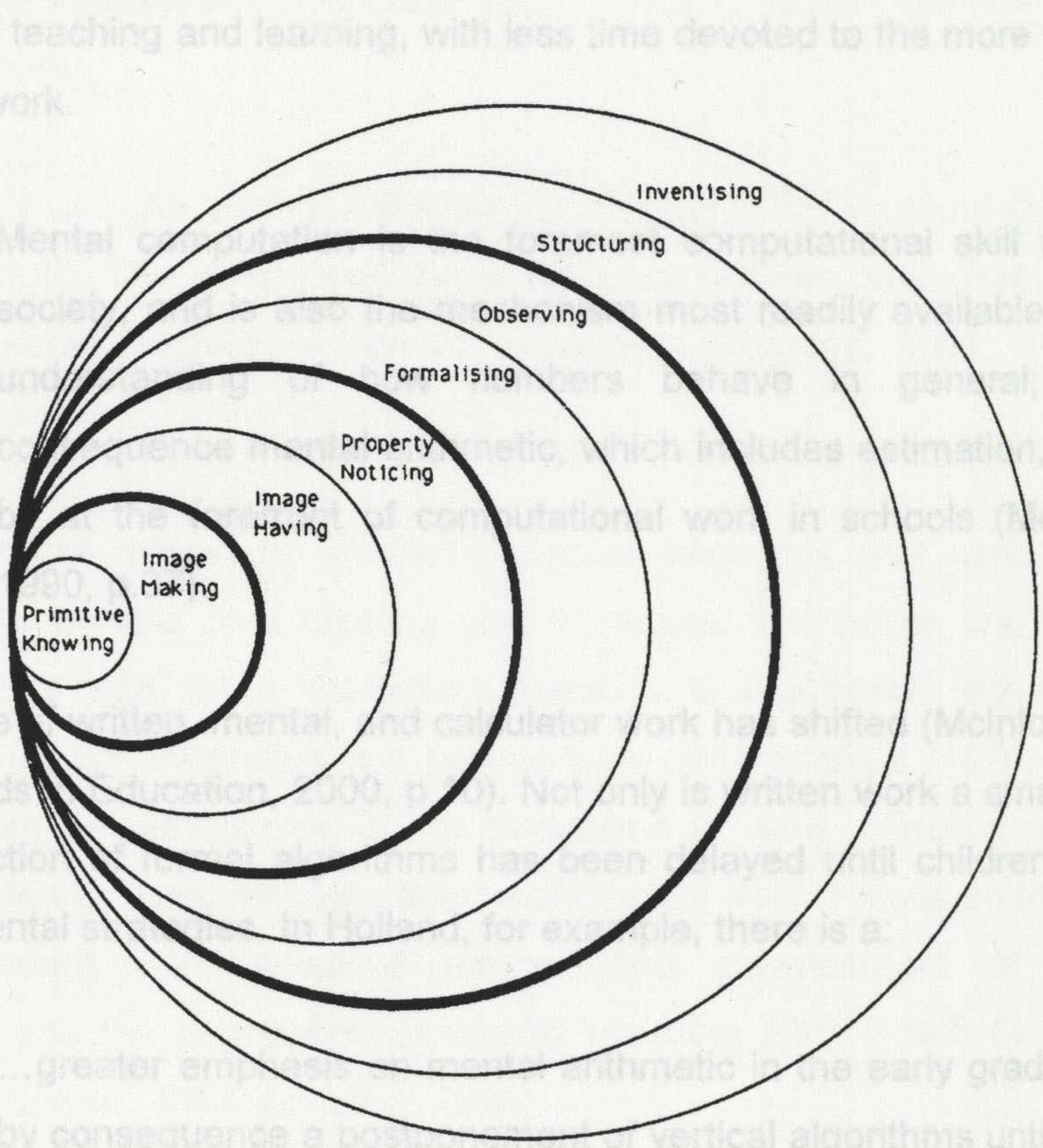

Figure 1:
Pirie and Kieren's (1989) diagrammatic representation of the model for growth of mathematical understanding

Various frameworks have been devised for describing and assessing children's mathematical understanding (Jones et al., 1996; Steffe, 1992; Wright, 1998; Young-

Loveridge, 1999a). Significant to an understanding of how children learn to count are descriptions of stages children typically go through and the principles they need to construct (Gelman & Gallistel, 1978).

There is much divergence in the details of the various models and frameworks. What they have in common, though, is that children's understanding of number can be roughly split into two areas – one based on a unitary concept of number, and the other, more advanced area is based on a multi-unit understanding of number. While children in the former group typically use counting-based strategies to solve problems, children in the latter range use strategies that involve the partitioning of numbers. Similarly, it is agreed that an understanding of place value plays a central role in children's understanding of number, and more particularly, their development of part-whole thinking (Ross, 1989; Jones et al., 1996; Wright, 1998; Young-Loveridge, 1999b).

1.1.3 Shifts in emphasis

In the various initiatives around the world, children's mental strategies are becoming the focus of teaching and learning, with less time devoted to the more traditional pencil and paper work.

> Mental computation is the foremost computational skill used in society, and is also the mechanism most readily available for the understanding of how numbers behave in general; as a consequence mental arithmetic, which includes estimation, should be at the forefront of computational work in schools (McIntosh, 1990, p.37).

The balance of written, mental, and calculator work has shifted (McIntosh, 1990; Office for Standards in Education, 2000, p.10). Not only is written work a smaller component, the introduction of formal algorithms has been delayed until children have a secure range of mental strategies. In Holland, for example, there is a:

> ...greater emphasis on mental arithmetic in the early grades, and by consequence a postponement of vertical algorithms until Grade 3 (Year 4) (Treffers & Beishuizen, 1999, p.34).

It is now thought that presenting children with standard algorithms too early can interfere with their continued development of mental strategies, which in turn also interferes with the development of children's part-whole thinking (Wright, Martland,

Stafford & Stanger, 2002; Young-Loveridge, 1999b). It has been found that children who have not been taught standard algorithms are likely to invent their own algorithms which are closely related to recognised mental strategies (Kamii, 1989; McClain, Cobb & Bowers, 1998; Thompson, 1999). Standard algorithms for multidigit computation:

> ...are the results of centuries of construction by adult mathematicians. Although it is not necessary for children to go through every historical step, it is unrealistic to expect them to skip the entire process of construction (Kamii & Dominick, 1998, p.131).

Children inventing their own algorithms that match their thinking is in line with the push towards teaching mathematics with understanding (Hiebert et al., 1997), as children use their own logic and reasoning to solve problems.

> The heart of all these self-devised algorithms is that the child tries to turn a difficult calculation into an easy one: 28 + 27 is hard, 30 + 25 is easy; 9 x 17 is hard, 10 x 17 – 17 is easier (McIntosh, 1998, p.46).

This contrasts with the mechanical following of steps in the standard algorithm (Kamii & Dominick, 1998). The current thinking is that once children have developed part-whole thinking, standard algorithms can be introduced (Hughes, 2001; Young-Loveridge, 1999b).

As the spotlight moves to children's mental strategies, the need for children to articulate and discuss their thinking also increases. The notion that such interaction with adults or with more advanced peers is a significant feature in children's construction of understanding is a central tenet of Vygotsky's (1978) social or socio-constructivism. In classrooms where discussion of, and reflection on, mathematics learning is valued, there has also been a shift away from a focus on the individual to a greater concern with collective mathematical development of the classroom community, and the negotiation of shared meaning (Ball, 1993; Gravemeijer et al., 2000; Hiebert et al., 1997; Lampert, 1989).

> New knowledge is constructed as a joint venture in the class rather than as a communication from teacher to students (Lampert, 1989, p.257).

The trends in mathematics curricula reform in Western countries during the 1990s saw greater emphasis placed on:

> ...solving problems in 'real-life' contexts, communication in mathematics, and the use of calculators and computers as learning tools (Garden, 1997, p.41).

Because children's understanding of number is central to their becoming numerate, teachers have made number a major focus in their numeracy programmes. In the National Numeracy Strategy, a general word of caution has been given about the balance of numeracy programmes in schools.

> ...teachers need to ensure that, in their enthusiasm for teaching number, the broader mathematics curriculum is not undermined (Office for Standards in Education, 2000, p.7).

1.1.4 Counting / part-whole thinking and place value

There is no question that a secure understanding of number concepts is pivotal in being numerate. A key understanding in being able to work flexibly with numbers to solve problems is being able to see a number not only as a complete unit in itself, but also as being made up of parts (Bobis, 1996; Ross, 1989). It is this ability to take numbers apart in order to combine them with others (or to subtract, multiply, divide) in ways that make the mental work easier and more efficient that is central to being numerate. In order to be able to choose the most efficient strategy with which to tackle a particular problem, children must have a repertoire of strategies from which to choose.

Understandings of place value and part-whole relationships are closely related.

> Interpreting number in terms of part-whole relationships makes it possible for children to think about number as compositions of other numbers (Bobis, 1996, p.20).

While counting has in the past been the focus of early number learning, it is now thought that early part-whole work with small numbers may make a later focus on place value a more straightforward progression for children (Wright et al., 2002). When dealing with small collections of items, young children are able to instantly recognise the number of items without counting. This is known as "subitising" (Labinowicz, 1985).

When items are randomly grouped, it is possible to subitise about five. When they are organised into patterns, such as finger patterns, our ability to recognise the number of items is increased. Calling on such patterns to "just know" the total number of dots on a 6-dot domino, for instance, is referred to as conceptual subitising. This is more complex than perceptual subitising that involves recognising the number in a group without using any other mathematical knowledge. Perceptual subitising is a more primitive form, closer to the original definition. Learning experiences with equipment such as the tens frame can contribute to children's development of conceptual subitising (see 2.3.2, p. 30).

Children's ability to subitise can help them develop mental images of five-based groupings for numbers up to ten. The progression to ten-based groupings is important in children's developing understanding of place value and also contributes to their part-whole thinking (Young-Loveridge, 1999a, 1999b).

1.1.5 Teachers' pedagogical content knowledge

Three domains of teachers' knowledge were differentiated in Shulman's (1986) framework. These were content knowledge, pedagogical content knowledge, and curricular knowledge. The content knowledge with which we are concerned here is the teacher's mathematical knowledge. Their pedagogical content knowledge includes such things as the clearest way to model an idea or the most powerful representation of a particular concept. Curricular knowledge involves teachers' knowledge about available instructional materials.

It is assumed that in order for teachers to help children become numerate, teachers must first have a sound mathematical knowledge themselves. Indeed, a teacher's conceptual understanding of mathematics has been shown to positively influence classroom instruction. Furthermore, teachers who themselves have a strong conceptual and connected understanding of mathematics have been found to be more conceptual in the way they teach (Fennema & Franke, 1992).

The mental organisation of the expert teacher's knowledge is likely to be similar to Skemp's relational understanding in that:

> Connections exist between ideas, the relationship between ideas can be specified, the links can differ among ideas, and the manner in which the knowledge is organized is relevant to understanding and application (Fennema & Franke, 1992, p.152).

A recent evaluation of the National Numeracy Strategy in the UK noted:

> ...weaknesses in teachers' subject knowledge, particularly the teaching of progression from mental to written methods of working; problem solving techniques; and fractions, decimals and percentages (Office for Standards in Education, 2000, p.6).

Two years later, although many teachers had improved their subject knowledge, weaknesses persisted (Office for Standards in Education, 2002). Among the areas of concern was:

> ...too little diagnosis and resolution of pupils' misconceptions and errors (Office for Standards in Education, 2002, p.10).

One of the National Numeracy Strategy's priorities has been to improve teachers' subject knowledge. In the latest evaluation report it was recommended that there should be ongoing monitoring of teaching:

> ... to identify weaknesses in teachers' subject knowledge and provide support and training where they are needed (Office for Standards in Education, 2002, p.4).

Research in Australia that examined teachers' conceptualisation of the key elements involved in teaching and learning about length, multiplication and decimal place value also raised concerns about the level of teachers' pedagogical content knowledge (Sullivan, Siemon, Virgona & Lasso, 2002).

If teachers do not themselves have a richly connected conceptual map of numeracy, then it is most unlikely that they will be able to guide children to construct deep, connected understandings.

1.2 The New Zealand context

Current numeracy initiatives in New Zealand draw on research from around the globe and are closely aligned with several other countries' strategies. Educators here face similar issues to their colleagues elsewhere; walking a tightrope as they balance essentially constructivist beliefs about teaching and learning on the one hand, and a largely behaviourist curriculum statement on the other (Begg, 1999; Neyland; 1995).

The same worldwide swing towards mental strategies and delaying of formal algorithms that was discussed in the previous section is happening in New Zealand.

1.2.1 Shifts in emphasis in New Zealand

Ritchie described the beginning of a shift in the way mathematics was being taught in New Zealand, which was in line with what has been happening around the world.

> From a culture of mathematics as reproduction (e.g. use of standard algorithms) to a culture of mathematics as creativity (e.g. construction of non-standard algorithms) (Ritchie, 1993, p.2).

Over the last two decades, the placement of algorithms in New Zealand curriculum statements for mathematics has varied. In *Mathematics: Junior Classes to Standard Four* it was indicated that children in Years 3 to 4 should be learning to:

> Solve addition or subtraction problems where renaming is required, to 999 (Department of Education, 1985, p.25).

In the current statement, *Mathematics in the New Zealand Curriculum* (Ministry of Education, 1992a), teaching of standard algorithms is not included until around Year 6. In the NDP teachers are urged to delay the introduction of standard algorithms until children have a repertoire of effective part-whole strategies for solving problems. This is likely to be around Years 4 to 6 for most children.

The role of communication in the mathematics setting has also gained greater prominence, in parallel with the shift towards the classroom becoming a community of mathematicians (Ball, 1993; Lampert, 1989). This is highlighted in the mathematical processes strand of the current curriculum statement. Research in New Zealand has examined what might constitute effective discussion among children in group situations (Thomas, 1994), as well as the role of the teacher in discussion (Higgins, 1991). Communication and discussion of mathematical ideas has necessarily been an area of focus in the NDP, where children's articulation of their mental strategies and teachers' questioning skills are central.

Of some concern is the recent finding that:

> Numeracy is generally interpreted as being the number strand of the mathematics curriculum rather than as embracing all the strands (Education Review Office, 2002, p.23).

This apparent confusion about how to interpret 'numeracy' is easy to understand. The current content of the NDP draft teachers' materials is exclusively related to number, while the definition of numeracy proliferated by the Ministry of Education (see p.1) implies a much broader meaning.

There is a danger that the other strands of the curriculum will be neglected if 'number' is treated as being synonymous with 'numeracy' (Education Review Office, 2002). This mirrors concerns about the balance of numeracy programmes in the UK (Office for Standards in Education, 2000).

1.2.2 Recent developments

The current curriculum statement for mathematics (Ministry of Education, 1992a) replaced the primary syllabus, *Mathematics: Junior Classes to Standard Four* (Department of Education, 1985). Shifts were in emphasis rather than any substantive changes to content. The use of calculators and computers was more strongly advocated, across all levels. Of greater significance was the emphasis of mathematics set in contexts that are meaningful to students. The strand of mathematical processes made explicit *how* children should learn mathematics, i.e. through problem solving, developing logic and reasoning, and communicating mathematical ideas. This is in keeping with a social constructivist approach, and is conducive to the idea of establishing communities of mathematicians in classrooms (Ball, 1993; Lampert, 1989). The current statement also stressed the importance of teachers encouraging greater participation from girls and Māori students, both groups that had been identified as underachieving.

The notion of a 'child-centred' approach to education, conventionally associated with the work of Piaget, continues to influence primary education in New Zealand (Higgins, 1998). *Mathematics in the New Zealand Curriculum* (Ministry of Education, 1992a):

> re-emphasises the student-centred approach to teaching, and beliefs about teaching and learning... (Garden, 1997, p.40).

Child-centred is generally understood to mean:

> ...that education should be oriented towards children's interests, needs and developmental growth and informed by an understanding of child development (Burman, 1994, p.164).

Key ideas of such an approach are readiness, choice, needs, play and discovery. A goal of the child-centered approach was to promote individual autonomy, but this focus on the individual also perpetuated social inequalities by masking differences that were based on gender, class or culture (Burman, 1994).

Despite international agreement that constructivism gives the best explanation of how children learn, the New Zealand curriculum statement for mathematics is in many ways more consistent with a behaviourist view, e.g. by giving lists of achievement objectives for each level. In the publication *In Time for the Future: A Comparative Study of Mathematics and Science Education* it was strongly suggested that:

> there should be a strong connection between what we know about learning and how we teach, but the connection is not necessarily a simple one (Education Review Office, 2000, p.58).

Teachers and researchers in New Zealand continue to grapple with this conundrum, as do their colleagues worldwide (Clements & Battista, 1990; Ball, 1993; Cobb, 1994; Gravemeijer et al., 2000).

The Third International Mathematics and Science Study (TIMSS) brought the achievement of New Zealand's Year 4 and 5 students under the spotlight, revealing that they:

> did poorly in number (place value, fractions, computation) and measurement, and algebra concepts (Ministry of Education, 1997, p.2).

In the same year, the Minster of Education set up the Mathematics and Science Taskforce. The purpose of the taskforce was to advise government on the nature of support needed by classroom teachers. It resolved that any interventions in mathematics teaching needed to raise the achievement of all students, that they should help teachers to translate the curriculum statement into practical 'hands-on' activities,

and that areas of identified weaknesses should be targeted, with an initial focus on teachers of Year 1 to 5 students.

Over the next few years, the Ministry of Education's policy development initiatives focussed on number learning in the junior primary school. Modifications were made to the National Administration Guidelines to emphasise the priority that needed to be given to developing children's literacy and numeracy in their first four years at school. This paved the way for a national pilot of Count Me In Too (New South Wales Department of Education and Training, 1999) in 2000.

Another Ministry of Education initiative has been the National Education Monitoring Project (NEMP), which began in 1993. The purpose of NEMP is to assess and report on New Zealand primary school children's achievement, in all curriculum areas. Children are assessed at Year 4 and at Year 8, with different curriculum areas assessed each year, over four-year cycles. In the mathematics results for 2001, it was reported that when:

> Asked to work on computations such as 36 + 29 or 9 x 98, few students at both levels chose the simplification of adjusting one of the numbers to a more easily handled adjacent number (making the 29 into 30, or the 98 into 100). Most relied instead on the standard algorithm for these tasks, indicating a lack of deep understanding of number operations (Crooks & Flockton, 2002, p.12).

At the next mathematics assessments in 2005, it must be hoped that a significant change will be evident in the strategies used by the Year 4 students in particular, as the part of the school population that has had the longest exposure to the NDP moves through the school system.

1.2.3 The Numeracy Development Project

The NDP began in New Zealand in 2000, and now encompasses three sequential projects: the Early Numeracy Project (ENP) for Years 1 to 3; the Advanced Numeracy Project (ANP) for Years 4 to 6; and an exploratory study in Years 7 to 10. The NDP aims to eventually raise the achievement of all students through increasing the capabilities of the teaching population via an intensive professional development programme. The particular programme for teachers' development includes several components that have been shown to make a positive impact on teachers; teachers participate in a series of after-school workshops – some with teachers from clusters of

local schools and others involving only their school's staff – and they also observe facilitators modelling lessons in their own classrooms with their own children before themselves being observed by the facilitator. Teachers are also encouraged to articulate what they are doing in their classrooms at parent education sessions. The in-depth professional development is the major focus for participating teachers for that school year.

From experience gained through the pilot of Count Me In Too as well as work being carried out around the world on early number learning, the New Zealand Number Framework (Ministry of Education, 2002a, see Appendix 1) was devised. Other frameworks (Jones et al., 1996; Young-Loveridge, 1999a) propose a similar progression from counting by ones strategies to being able to think about numbers in terms of their component parts in order to solve problems in many different ways. The New Zealand Number Framework provides a basis against which teachers can assess children's numeracy understanding, and as such, is a key feature in the NDP's professional development for teachers.

Given the trends around the world, it is not surprising that helping children to make connections is implicit in the NDP. The Number Framework presents teachers with the likely learning progressions in various domains for children at the various strategy stages. This allows teachers to make connections themselves between, for example, what is likely to be appropriate in the area of number sequence and order, and grouping/place value for the child who is at Stage 4: Advanced Counting. A recent Education Review Office (2002) report described the ENP having:

> …a *connectivist* approach whereby teachers of numeracy emphasise connections: within mathematics, between mathematics and the child's real-life experiences, and between what is being taught and the child's current thinking (p.52).

The general characteristics of effective teachers of numeracy that emerged from the most recent evaluation of the ENP included having a thorough knowledge of students' learning needs and a sound understanding of The Number Framework (Thomas & Ward, 2002). The Number Framework comprises two components: a framework for number strategies and a parallel framework for number knowledge. It should be stressed that this separation is an artificial one, designed to help teachers pinpoint likely learning progressions for children. The NDP draft teachers' materials stress the interrelationship between number knowledge and strategies, as shown in Figure 2.

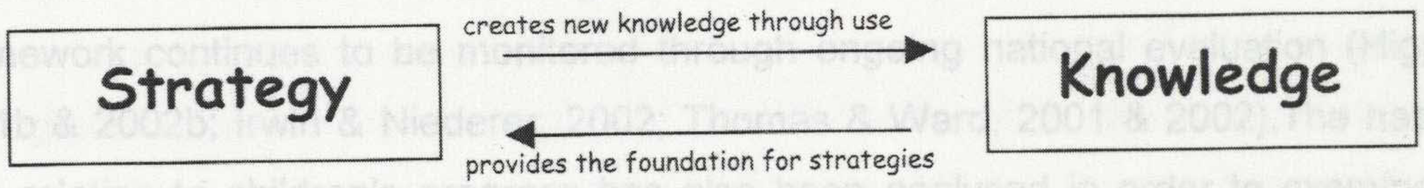

Figure 2
The interrelationship between number knowledge and strategies

The NDPs' teaching model draws from the work of Pirie and Kieren (1989) (see Figure 1, p. 4). Both the folding back idea and a simplified version of their model of learning, shown in Figure 3, are included in the teachers' materials. Throughout the various stages, new ideas are introduced through the use of equipment. The NDP resources advocate that equipment should be used by the teacher to scaffold the development of children's thinking, rather than simply as a means of representing their current thinking. It is thought that children then progress to working with mental images of number, although exactly what these images are remains uncertain. The Number Framework (Ministry of Education, 2002a, p.3) states that "Students at this stage are able to image visual patterns of the objects in their mind and count them." Bobis (1996) found that visualisation strategies helped young children to recognise part-whole relationships of numbers. Finally, children develop the ability to abstract number properties and use these to solve problems.

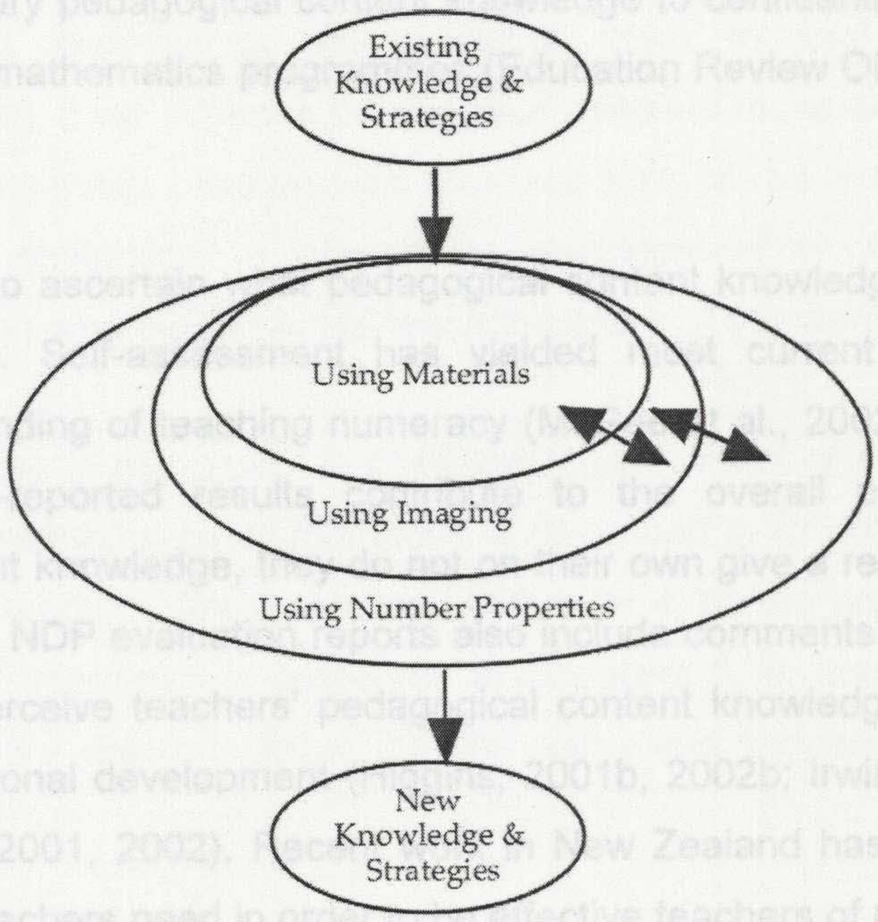

Figure 3:
The NDP teaching model

The effects on children's learning of teachers' implementation of The Number Framework continues to be monitored through ongoing national evaluation (Higgins, 2001b & 2002b; Irwin & Niederer, 2002; Thomas & Ward, 2001 & 2002).The national data relating to children's progress has also been analysed in order to examine the consistency of The Number Framework, particularly in the ANP (Years 4 to 6). This work has largely validated the learning progression as specified (Young-Loveridge & Wright, 2002).

1.2.4 New Zealand teachers' pedagogical content knowledge

A recent Education Review Office (2000) comparative study highlighted the need for:

> ...better information about the extent to which New Zealand primary school teachers have adequate and appropriate pedagogical content knowledge, and what is done to support and improve this (p.100).

And again two years later:

> It remains far from clear whether primary teachers have the necessary pedagogical content knowledge to confidently plan and deliver mathematics programmes (Education Review Office, 2002, p.22).

The issue of how to ascertain what pedagogical content knowledge teachers have is difficult to resolve. Self-assessment has yielded most current information about teachers' understanding of teaching numeracy (McGee et al., 2002; Thomas & Ward, 2002). While self-reported results contribute to the overall picture of teachers' pedagogical content knowledge, they do not on their own give a reliable picture. Partly for this reason, the NDP evaluation reports also include comments from the facilitators about how they perceive teachers' pedagogical content knowledge to have changed during the professional development (Higgins, 2001b, 2002b; Irwin & Niederer, 2002; Thomas & Ward, 2001, 2002). Recent work in New Zealand has begun to examine what knowledge teachers need in order to be effective teachers of numeracy, and what the characteristics of these teachers are likely to be (Higgins, 1998; Thomas & Ward, 2002).

According to a constructivist model, professional development facilitators should make no assumptions about teachers' current understandings. Rather, any development –

which is often new learning for teachers – should begin just as any good teaching practice begins, with some sort of diagnostic assessment which indicates appropriate starting points for learning. A strength of the NDP professional development model is the one-to-one in-class support component (Higgins, 2002a; Thomas & Ward, 2002); through working closely with each teacher in their classroom with their children the facilitator has the opportunity to determine the particular needs of the individual teacher. It is then possible for the needs of each teacher to be targeted by the facilitator modelling numeracy teaching, having focussed discussion with the teacher, and giving the teacher informed, critical feedback about their teaching practice.

Not all teachers are confident about the inclusion of equipment in their numeracy programmes. Teachers of Year 3 children are of particular interest in this regard, as it is often in children's third year at school (formerly Standard One) that there has traditionally been a shift to more formal mathematics teaching:

> ...which means fewer opportunities for the children to use equipment and more emphasis on written recording of the formal algorithm (Higgins, 2001a, p.17).

For some teachers of Year 3 children, the significant role of equipment in the NDP suggested learning experiences will challenge them to use different teaching strategies. For others, it will increase the range of concrete materials that are already a feature of their mathematics programmes. Most teachers will be challenged to consider how they will use equipment to support the development of children's part-whole thinking, rather than using equipment to teach the steps of standard algorithms.

In Chapter 2, the role of equipment in children's development of part-whole strategies is examined. The methodology of this study is described in Chapter 3. Findings from the interviews with teachers are presented in Chapter 4, before an analysis of findings from the children's interviews is presented in Chapters 5, 6, and 7. The concluding chapter – Chapter 8 – draws together the key points and suggests some possibilities for further research.

CHAPTER 2

Equipment Use

2.1 Worldwide perspectives

Constructivist theory tells us that individual children may each interpret a particular learning experience in profoundly different ways. Perhaps this is one reason that the results of studies on the effectiveness of using equipment[1] to develop children's understanding of numeracy are mixed, as reported by Raphael and Wahlstrom (1989), Hiebert and Carpenter (1992), and Thompson (1994). It is agreed, however, that simply using equipment is not enough to guarantee success; concrete materials do not automatically convey mathematical ideas to children (Ball, 1993; Baroody, 1989; Gravemeijer, 1994). As Thompson (1994) pointed out:

> The material may be concrete, but the idea that students are intended to see is not in the material. The idea is in the way the teacher understands the material and understands his or her actions with it (p.557).

Thompson also emphasised that the teacher needs to have a clear focus on what it is they want children to understand, rather than what they want children to do, when they use equipment. Furthermore, the conversations that teachers and children have about how they use a piece of equipment should provide opportunities to establish connections between concrete, mental, and written mathematics (Baroody, 1989; Hiebert & Carpenter, 1992; Miura, 2001; Thompson, 1994). As Hart (1989) suggested, it is bridging the gap between materials-based experiences and formal, symbolic mathematics that poses a challenge for teachers. Mental imagery is proposed as a bridge in a number of models (Baroody, 1989; Bobis, 1996; Pirie & Kieren, 1989).

The purpose of equipment use has shifted from it providing support for learning algorithms, to providing support for developing the repertoire of mental strategies that is the hallmark of part-whole thinking. Accordingly, the long-term goal of mathematics education is not for children to be able to use equipment proficiently; from equipment-based experiences we want children to construct a range of mental strategies for solving problems. A judiciously chosen piece of equipment can help to focus everyone's attention on the development of part-whole thinking. Concrete materials can

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[1] Equipment will also be referred to as 'concrete materials' and 'manipulatives' in this document.

be thought of as external representations that allow the communication of children's thinking – their internal representations (Hiebert & Carpenter, 1992). As such, teachers can learn about children's understandings as well as their misconceptions by observing children working with equipment (Labinowicz, 1985; Ross, 1989).

2.1.1 'Working/thinking' models

Gravemeijer (1994) described two different ways that equipment use could be thought about: as 'working' models and 'thinking' models. A 'working' model is conducive to procedural use in that the structure of the equipment might dictate particular steps for its use. When equipment such as place value blocks or an abacus is used to teach the steps of the standard algorithms for addition or subtraction, it can be considered to be a 'working' model. With the current shift towards the development of children's part-whole thinking, the way equipment is used also needs to shift in order to support the communication of their various strategies. When children use a 'thinking' model, they are representing the thinking they did when they solved the problem. A piece of equipment would therefore need to lend itself to flexible use in order to be considered a 'thinking' model. The empty number line[2] is a clear example of this type of model, as it can be used to illustrate a multitude of mental strategies.

Thompson (1994) urged teachers to help children to understand the various ways equipment use can be interpreted, rather than insisting that children use equipment in one correct way. This is consistent with the shift in emphasis from a procedural knowledge of mathematics to children having an interconnected understanding of mathematical concepts.

While some advocate presenting children with a variety of equipment (Young-Loveridge, 1999b), Boulton-Lewis (1996) suggests that children's development of place value understanding is likely to be enhanced if they use one particular representation of tens and ones regularly.

Effectiveness of equipment use may also be related to teaching experience (Raphael & Wahlstrom, 1989). The same writers cautioned against relying too heavily on the use of equipment as it may lead to poor pacing of content coverage.

The diversity of research findings previously reported (Hiebert & Carpenter, 1992; Raphael & Wahlstrom, 1989; Thompson, 1994) indicates that effective use of equipment to support children's learning is anything but straightforward. Many factors

[2] The empty number line is described in greater detail in section 2.3.1.

appear to play a role in determining whether or not the inclusion of concrete materials aids learning. These include the teacher's own pedagogical content knowledge; the learning needs of the children being matched with an appropriate model; and the reflective discussion that takes place about the equipment use. Also unclear is which particular equipment to use and for how long, in order to best serve the development of children's part-whole thinking.

2.1.2 How do teachers choose equipment?

Most teachers intuitively know that using equipment can help children's learning; 96% of teachers who responded to a survey conducted in New South Wales indicated that they believed manipulatives positively influence children's learning (Howard, Perry & Conroy, 1995 cited in Price, 1998). What exactly it is that causes equipment to positively effect children's cognitive development remains unclear.

Other research has shown that some teachers' belief that learning should be fun resulted in them including equipment-based learning experiences, aligning equipment with "fun maths" as opposed to "real maths" (Moyer, 2001). If using materials is trivialised, it seems possible that the use of materials for learning mathematics concepts may be devalued.

> No representations capture all aspects of an idea, nor are all equally useful for particular students (Ball, 1993, p.384).

In light of what Ball tells us, what *do* teachers consider when they are deciding which pieces of equipment to include in their numeracy programmes? It would be helpful if teachers had some clear guidelines available to consult as they contemplate their choice of manipulatives.

> But rarely are alternative manipulatives compared side by side. For example, in teaching place value, what are the relative merits of base-ten blocks and beansticks? Is money an equivalently workable model? How do bundled Popsicle sticks fit with other options available? (Ball, 1992, p.17).

A further consideration should be transparency of the equipment; are children able to see through it to the underlying concepts without being distracted by other features (Ball, 1992; Meira, 1998; Stacey, Helme, Archer & Condon, 2001)?

Others argue that the teacher needs to consider how using a particular piece of equipment might influence how children think.

> What seems to be important is not which tool a teacher chooses to introduce into the classroom, but rather that the teacher thinks carefully about the way in which students' thinking might be shaped by using particular tools (Hiebert et al., 1997, p.63).

It would still seem a helpful starting point for teachers to be provided with some comparative information about the conceptual understandings that various manipulatives might usefully support, as well as an indication of the difficulties that children typically encounter in their use.

2.1.3 Children's views

Hart and Sinkinson (1989) noted that children believed teachers' claims that concrete materials are helpful for understanding, but were often unable to successfully use equipment on which teaching had been based.

Children who are working symbolically can find it difficult to model problems using equipment.

> It is often thought that if a child is unable to work a particular question with symbols then a good teaching strategy is to provide the concrete materials again and this will provide 'success'. This is by no means obvious from the answers received on interview (Hart, Johnson, Brown, Dickson & Clarkson, 1989, p.45).

It appears that once children no longer use the support initially offered by the manipulation of concrete materials, it might become difficult for them to return to using it. If equipment is used to scaffold children while they develop new mental strategies, then by definition it is also intended to become obsolete as they move beyond the possibilities it offers them. Not all equipment can be readily manipulated in ways that might illustrate children's part-whole thinking. This has implications for any task involving the use of equipment that might be included in an assessment of children's number understandings.

Other research that explored children's views found that middle to upper primary school children in Western Australia generally preferred to use mental methods to

solve problems, rather than use written methods or the calculator (Swan & Bana, 1998). This is similar to McIntosh's (1990) findings about the methods most adults prefer to use. As the emphasis in numeracy teaching and learning around the world shifts to focus on developing children's mental strategies, with the aim of having all children become part-whole thinkers, the mathematics of the classroom is likely to become more closely aligned to the mathematics of everyday life.

2.2 The New Zealand context

> The use of apparatus to help children form mathematical concepts is an established feature of mathematics teaching in this country (Department of Education, 1985, p.16).

This quote, taken from the syllabus of the day, indicates that emphasis given to the importance of equipment use in *Suggestions for Teaching Mathematics in Infant Classes* (Lee, 1972) had by then become embedded in beliefs about the teaching and learning of mathematics in New Zealand. The 'Orange Handbook', as Lee's syllabus was also known, introduced heavily structured equipment such as Cuisenaire rods and adsum blocks. It advocated the use of quite a different collection of equipment to that which is the focus of this study, so it is difficult to draw comparisons.

Recent curriculum statements have not questioned the assumption that the use of equipment will help children at all levels of the primary school to develop abstract mathematical ideas. Higgins (2001a) pointed out that:

> While this use of equipment can be seen as part of a structuralist approach to teaching number, teachers have tended to justify their use of equipment in terms of providing children with "hands-on" experience as part of a child-centred approach (p.18).

2.2.1 Equipment references in key documents

Equipment currently plays a vital role in the NDP materials, providing the foundation experiences from which children move to using mental images and eventually number properties in order to solve number problems. There is an underlying assumption by all those who are engaged in the NDP that the use of equipment, particularly in the ENP, will help children learn. In the NDP materials, particular equipment is usually clearly specified. For example, in an activity called "The Missing Tens and Ones" (Ministry of Education, 2002d) the equipment suggested is:

> Ones and tens material – bundled and loose pop sticks, ten beans in film canisters and loose beans, unifix cubes, the Slavonic abacus, play money, and place value blocks or tens frames (p.23).

Whereas in the past it was expected that teachers would be able to choose an appropriate piece of equipment from a general category – and it was assumed they knew which equipment belonged in which category to begin with – they are now given clear guidance on which equipment might best suit the development of a particular numeracy concept at a particular stage in the development of children's part-whole thinking.

Another resource that gave teachers clear guidance about which equipment to use, was *Beginning School Mathematics* (BSM) (Department of Education, 1986, 1987, 1988, 1989; Ministry of Education, 1992b, 1993). A key mathematics resource in the junior primary school since the mid 1980s, the focus of this resource was children learning through their engagement with manipulatives and through talking about their experiences. Designed for use in the first three years of school, BSM is made up of twelve cycles, with the content of each cycle extending the content of the previous cycle. It reflected a Piagetian approach to learning where children's maturation and the provision of an appropriate learning environment were central considerations for the teacher (Higgins, 1991; Visser & Bennie, 1996; Young-Loveridge, 1987).

The inclusion and use of equipment mirrored the curriculum of the day. It could be said that one of the main goals of BSM was to have children learn to use the standard algorithms for addition and subtraction. Higgins (2001a) outlined the equipment included in BSM's teacher support material (see Appendix 2). In this resource, certain pieces of equipment that are used in the first stages of *The Number Framework* (Ministry of Education, 2002a) - for example the number line and the calculator - are not introduced until Cycle 9 (towards the end of most children's second year at school). The exception to this is play money, introduced in Cycle 5 in the context of ordering coins and notes according to their values. Dollar notes do not reappear until Cycle 9. Since the introduction of BSM, there have been several changes in this country's currency, making the money used in the many activities now obsolete. The activities were not updated at the time, because further currency changes were mooted. Calculators are used only in the final cycle (Year 3). This cycle was published along with Cycle 11 in 1993, so would have been in the writing stages before the publication of the current curriculum which advocates the use of calculators at all levels. Pattern

boards are used in almost every cycle from Cycles 3 to 11; tens frames were not included.

Recent syllabi and curriculum statements (which are synonymous terms) have often not specified particular pieces of equipment. Instead, they might be included by the use of such general phrases as "structured materials"[3] (Department of Education, 1985, p.24; Ministry of Education, 1992a, p.33) and "grouped discrete objects"[4] (Ministry of Education, 1992a, p. 34). References to equipment in these two curriculum documents are also included in Appendix 3.

In the current curriculum statement, calculators are used as soon as a child starts school in both the 1992 statement and the NDP materials. Previously, their inclusion was considered optional.

> Although systematic instruction in the use of calculators is not prescribed in this syllabus, teachers may wish to use them at times to help children's understanding of number relationships and patterns, and to enable them to perform calculations which arise from their investigations but which are temporarily beyond their capacity (Department of Education, 1985, p.17).

As teachers participate in the NDP professional development and implement the suggested learning experiences in the NDP materials, they will once again have a clear indication of specific equipment to use in their mathematics programmes. The significant difference between the recommendations for equipment use in the BSM resource and in the NDP materials, of course, is that equipment is used to support the development of children's part-whole thinking in the NDP materials, rather than the algorithmic use of equipment advocated in BSM.

In the evaluation of the ENP in 2001, changes in teachers' use of equipment were identified by NDP facilitators.

> Teachers demonstrated a greater awareness of the value of hands-on activities – identified by 33% of facilitators (Thomas & Ward, 2002, p.35).

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[3] "Structured materials" might include equipment such as tens frames and place value blocks, which were devised to illustrate the meaning of, and relationships between, numbers.

[4] "Grouped discrete objects" include bundled ice-block sticks – equipment where children can create the grouping themselves.

What exactly *is* "the value of hands-on activities" mentioned above? As has already been discussed, the mere inclusion of equipment-based experiences does not in itself aid children's learning. Any value attributed to children's use of concrete materials must lie in the teacher-child interactions around the equipment that children use to develop part-whole strategies. Similarly, "more effective use of equipment" would need to be considered in terms of its contribution to children's part-whole thinking for it to be of significance.

When, as part of an Education Review Office study, teachers were asked about strategies they used to teach a new mathematics concept, one of the strategies most often mentioned was:

> The use of concrete examples or equipment that students can see, touch, handle or manipulate in order to establish conceptual understanding (Education Review Office, 2002, p.30).

The notion of introducing a new concept through the use of manipulatives could be interpreted as being consistent with the teaching model in the NDP (see Figure 3, p. 15). It might also, however, show the continuing influence of a child-centred approach to teaching and learning, where the role of 'hands-on' experiences is central. From the findings of the NDP evaluation reports (Higgins, 2001b, 2002b; Irwin & Niederer, 2002; Thomas & Ward, 2001, 2002) and the Education Review Office (2000, 2002), it remains unclear how effectively teachers are using a variety of equipment in order to support children to develop part-whole thinking.

2.2.2 How do New Zealand teachers choose equipment?

Research in New Zealand adds to the international picture of how teachers might go about selecting equipment. Several writers have compiled helpful information regarding the conceptual understandings whose development is supported by particular manipulatives (Hughes, 2001; Ritchie, 1991; Young-Loveridge, 1999a). An indication of the difficulties that children typically encounter with various pieces of equipment has yet to be summarised as an accessible reference for teachers. A knowledge of how equipment can be used to support the development of children's part-whole thinking can be gained through teachers' engagement with the NDP professional development. It is perhaps largely through teaching experience that a thorough understanding of children's misconceptions as reflected in their equipment use, and how best to address these, is achieved.

While studies have commented on the presence of equipment in numeracy teaching (Higgins, 2001b; Thomas & Ward, 2002), questions have not yet been asked about whether or not teachers choose equipment according to the way in which it might shape children's part-whole thinking.

What is clear is that teachers must have the sort of expert knowledge described by Fennema and Franke (1992) if they are to select equipment that children can use as 'thinking' models.

> Without such pedagogical content knowledge teachers' instructional decisions in the choice and use of manipulatives and other classroom representations may be determined by the manipulative or techniques which happen to be in vogue at the time (Higgins, 2001a, p.27).

2.2.3 New Zealand children's views

Ritchie (1991) found that the children he interviewed chose to use equipment that was consistent with their current written strategies. He also noted that children did not always use the chosen equipment in the manner in which the teacher had presented it, e.g. children used the place value blocks as discrete objects for counting-on strategies. This study will build on Ritchie's work by examining why children make the equipment choices they do. Rather than comparing the ways children use equipment to how they tackle written number sentences, comparisons will be drawn between the children's mental strategies and their equipment-based strategies.

Using equipment did not prove helpful for all children interviewed in Ritchie's study. He found that for some children the equipment they used might have been in advance of their number understanding. For other children, whose mental strategies were more sophisticated, the use of equipment may have been relatively unfamiliar. He suggested that this might have contributed to these children seeming to be less successful at using equipment to solve problems. It may also have been that, as Hart et al. (1989) observed, children who are working symbolically tend to find it difficult to model problems with concrete materials. This redundancy of equipment was perhaps signalled in this statement from the 1985 syllabus.

> It should be remembered, however, that there is a point for a particular learner in a particular task at which the further use of

> apparatus becomes unnecessary because it no longer enhances the learning (Department of Education, 1985, p.16).

This does not sit well, however, with the suggestion in the same document that children's conceptual understandings be assessed through their modelling of problems using equipment.

Once children have moved from their reliance on physical models through to an imaging stage, and then are able to think about number properties in an abstract way, it may be inappropriate to expect them to return to using equipment to model problems (Hart et al., 1989). Just like training wheels being removed from the young rider's bike, if the equipment is brought back as they are gaining confidence and speed, it is more than likely that the intended supports will actually be a hindrance.

2.3 Specific equipment

As Ball (1993) said, no single piece of equipment represents every aspect of an idea. So six pieces of equipment that capture a range of mathematical ideas that contribute to children's part-whole thinking were chosen for this study. The six manipulatives are the number line, tens frames, bundled ice-block sticks, place value blocks, play money, and the calculator. This equipment was chosen because it is used across a wide range of stages for both strategy and number knowledge development (see Table 9 in Appendix 3). It was thought that most of the teachers and children would be familiar with using these pieces of equipment during the numeracy programme.

To have included more than six pieces of equipment was likely to have made the interviews unduly long, particularly for the children. Other equipment that could have been equally useful to explore includes the Slavonic abacus[5] and Unifix cubes. Equipment such as the hundreds board, on the other hand, is used for number knowledge in the NDP materials, so was omitted.

In this section, each of six pieces of equipment will be examined in light of the research to date. The six pieces of equipment are presented here roughly in the order in which children might encounter them in an NDP-based numeracy programme, with the notable exception of the calculator which is used at all stages. The particular merits of

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[5] See Young-Loveridge (1999a) for a comprehensive discussion of the merits of the Slavonic abacus as well as ten-structured bead-strings, both of which have some similar characteristics.

each manipulative in supporting the development of part-whole thinking will be presented and discussed, along with the weaknesses of each model.

2.3.1 The number line

The traditional, horizontal number line is a linear representation of number where units are shown by a set length. Because there is no visual separation between consecutive units, it is referred to as a continuous model.

Ernest (1985) suggests that the number line is best suited for the ordering of numbers, rather than for experiences with operations. While the number line is useful for supporting children's counting-based strategies (Gravemeijer, 1994), mental strategies that involve associativity are problematic to model.

The use of the number line for even the most basic addition and subtraction problems is made complex by the need for children to translate their thinking about these operations to a model that combines pictorial and symbolic information (Bright, Behr, Post, & Wachsmuth, 1988; Carr & Katterns, 1984; Ernest, 1985; Herbst, 1997). These writers questioned whether the demands on children to co-ordinate these two modes of information might result in the number line model being less clear than many teachers have believed it to be.

The clarity of structure of the number line has been questioned (English, 1993); the numerals represented do not correspond to the number of gradations, e.g. while four single jumps will take a child to the numeral 4, there are actually five gradations to this point. A common error is that children:

> …confuse the spaces between the numbers with the numbers themselves. In this case a student starting at '6' and adding on '4' might say, "six, seven, eight, nine – six and four is nine" (Carr, 1998, p.66).

Another misunderstanding that children can have is being what Carr and Katterns called "digit-bound", meaning that they focus on reading the numerals on the number line and overlook the size of the "jumps" between numbers (Carr & Katterns, 1984). Furthermore, the symbols may act as perceptual distracters, drawing children's focus away from their mental strategies.

Alternative models of the number line have been proposed. The empty number line (ENL) eliminates many of the issues related to the traditional number line. The ENL was devised during the revision of the Dutch primary mathematics curriculum in 1990. Children's informal strategies were encouraged in RME, and the ENL provided a model for illustrating these. Equally important, its use can also foster the development of more sophisticated strategies (Carr, 1998). The ENL is an example of a 'thinking' model that children and teachers can use to illustrate a variety of mental strategies.

> By using the empty number line, children can extend their counting strategies and raise the sophistication level of their strategies from counting by ones to counting by tens to counting by multiples of ten (Klein & Beishuizen, 1998, p.446).

These authors suggested, then, that the ENL can help children to develop part-whole thinking. As well as being useful in helping children to develop the flexibility of mental strategies once children are thinking about numbers in part-whole ways, the ENL is useful in providing diagnostic information about children's misconceptions about number operations (Beishuizen, 1999).

Another alternative, less often favoured, is the vertical number line. With the traditional horizontal number line, the numbers become larger as they progress from left to right. This is in direct conflict with the significance of directionality related to place value. When considering the place value of individual digits in a particular number, their value becomes greater as they occur further to the left. Bove (1995) instead suggested:

> Using a vertical number line with 0 at the bottom would give students an initial presentation that would not conflict with subsequent learning. On the vertical number line, the numbers *increase* in value as the student moves *up* the line; a left-to-right progression is not equated with an increase in numerical value as occurs with the traditional number line (p.544).

She argued that this arrangement of numbers makes it easier for children to see the repetitive nature of the decimal number system; the pattern of ten digits that is repeated, and the shift to the left as a second or third digit is needed.

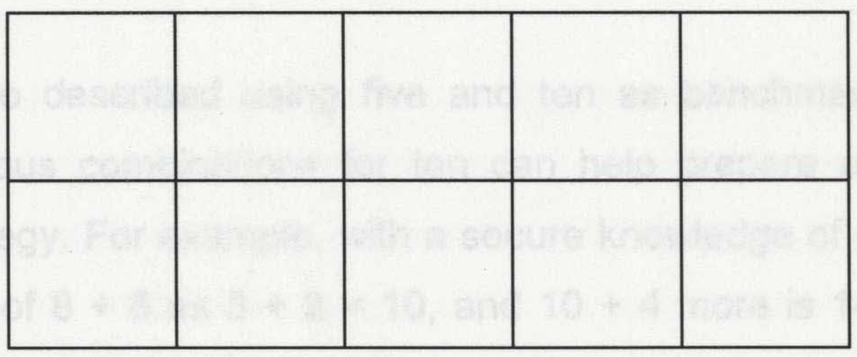
There is no single 'best' number line for teachers and children to use. And choosing the best number line to use in a particular situation is not straightforward. In the context of

fractions learning, Bright et al. (1988) recommended that children have experiences with multiple number line representations with different subdivisions marked in order to develop flexible thinking about number.

A variety of number lines is suggested in the NDP materials, some to develop children's number knowledge and some for mental strategies. As children's mental strategies advance, so the amount of labelling of number lines decreases. This is not only because children need less support as they begin to develop part-whole strategies, but also because too much support can cause children to revert to using counting-based strategies. Any number line model has a number of variables: the number of subdivisions and whether some (e.g. the decade numbers) will be more heavily marked than others; how many of these will be labelled; whether it ill be a vertical or horizontal number line; which section of numbers is to be represented. Or will an empty number line model be more appropriate? Teachers need to have an understanding of these complexities, and an ability to match a number line model to the intended learning outcomes for children, in order for them to select a number line that will help move children nearer to the goal of being part-whole thinkers.

2.3.2 Tens frames

The tens frame is a 2 X 5 grid of squares as shown here:



Those used in the NDP are large enough to fit a counter in each square (roughly 70 X 175cm). For some learning experiences, blank frames and counters are used. Others involve a set of frames with pre-printed dots to represent counters, i.e. one frame has one dot, another has two dots, and another has three, and so on up to 10.

Because it can be interpreted as two groups of five, it can often be connected to children's existing understanding by highlighting the similarities with their finger patterns for numbers to 10, i.e. five fingers on one hand and three on the other is a pattern of eight, just as five counters in one row and 3 in the other represents eight on the tens frame.

A tens frame provides a structure on which counters can be organised into patterns that lend themselves to conceptual subitising[6]. By developing conceptual subitising, children can be helped to establish a range of mental images of numbers. Such activities as briefly holding up a pre-printed tens frame and asking children "How many dots can you see?" are included in the NDP materials for this purpose.

Being able to visualise can also help children to develop counting-on strategies (Labinowicz, 1985). For example, a child thinking about seven and five might be able to count on from seven by using a visualised pattern of five as a mental tracking system.

Children's understanding of numbers both as a whole and as being made up of parts can be fostered through learning experiences that focus on the tens frame. For instance, when a child arranges five blue counters along one row, and three orange counters along the other row, the image can be interpreted in several ways; 5 and 3 make 8; 8 and 2 more make 10; 3 and 2 more make 5.

Count Me In Too introduces children to a fives frame before presenting them with a tens frame.

> ...using five as a base has the potential to greatly reduce reliance on counting by ones (Wright et al., 2002, p.16).

Bobis (1996) also described using five and ten as benchmark numbers. Teaching children the various combinations for ten can help prepare children to later use a bridging ten strategy. For example, with a secure knowledge of groupings within ten, a child might think of 8 + 6 as 8 + 2 = 10, and 10 + 4 more is 14. Seeing and thinking about ten as a group can also contribute to developing understanding of part-whole thinking and of place value. Because of the importance of knowing the groupings for ten, the NDP materials suggest learning experiences using pre-printed tens frames where children are asked not only how many dots they can see, but also how many more are needed to make ten. Learning experiences with tens frames are included from Stage 0: Emergent in the NDP materials, until Stage 5: Early Additive Part-Whole. Once children are confidently working with the different groupings within ten, the extension of this to numbers up to one hundred is supported by the use of the Slavonic abacus[7]

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6 See pp.7-8 for a detailed explanation of subitising

7 See Young-Loveridge (1999a) for a comprehensive discussion of the merits of the Slavonic abacus as well as ten-structured bead-strings, both of which have some similar characteristics.

The two parallel rows of five on a tens frame lend themselves to two quite different purposes – one based on the pairs of numbers, and the other on the groupings of five. Some writers (Bobis, 1996; Gervasoni, 1997) have used tens frames to highlight doubles and near-doubles, e.g. double four would be shown with two parallel columns of four counters. This is the same as the odd and even number patterns made explicit by the pattern boards that featured strongly in the BSM resource. Others recommend using tens frames to highlight groupings of '5 and how many more' for numbers up to ten (Clements, 1999; Ministry of Education, 2002c, 2002d; Thompson & Van de Walle, 1984).

Labinowicz (1985) described learning experiences that use both sorts of groupings on the tens frames, i.e. eight might be shown by double four, or by one row of five plus three more. This approach is more consistent with Thompson's (1994) recommendation that teachers should help children to understand the various ways in which a piece of equipment can be interpreted, rather than focussing on one, so-called 'right' way.

Thompson and Van de Walle (1984) suggested extending the tens frame model for use with larger numbers by adding a 10 X 10 cardboard frame of one hundred smaller squares containing dots. One thousand can then be shown by stacking and bundling ten one-hundred frames.

It is thought that through using various arrangements of counters on tens frames teachers can help children to develop powerful mental images of numbers to ten. Developing strong, visual images of numbers is believed to help children develop number sense (Bobis, 1996; Gervasoni, 1997). In turn, children who have more secure internal images are thought to produce better-organised external models, according to Thomas and Mulligan (1994).

2.3.3 Bundled ice-block sticks

Commonly known as bundled ice-block sticks in New Zealand classrooms, these manipulatives are also known as Popsticks, Popsicle sticks, ice cream sticks, and craft sticks. The ice-block sticks used in the interviews were brightly coloured; until recently, schools have more often had uncoloured sticks. Rubber-bands or pipe-cleaners are used to secure groups of ten sticks.

There are several advantages of using bundled ice-block sticks for place value and computation work. Initially the children themselves impose the groupings of ten on the

discrete materials, rather than being presented with pre-grouped materials such as place value blocks. Ten ones remain visible within a bundle of ten sticks, so the numbers still bear one-to-one correspondence to the materials (Ritchie, 1991). The making and breaking of bundles of ten during addition and subtraction is achieved by the removal of the rubber-band. Compared with play money or place value blocks where items must be removed and replaced in order to swap one ten for ten ones, renaming is relatively transparent with bundled ice-block sticks (Hughes, 2001).

The obvious disadvantage of bundled ice-block sticks is that the quantity of sticks becomes unwieldy when numbers beyond one hundred are involved.

Bundled ice-block sticks are often used as a lead-in to place value blocks, offering children more support as they explore place value (Liebeck, 1984). The recommended uses of bundled ice-block sticks in the NDP materials include composing two-digit numbers, and comparing this representation with the same number shown with, for example, Unifix cubes or play money. In the suggested learning experiences for addition, subtraction and place value, ice-block sticks are used for grouping in tens, and then adding tens and ones where no renaming is needed. Breaking a bundle of ten for subtraction is explored (Ministry of Education, 2002d, p.25), as is making another ten in addition (p.27).

2.3.4 Place value blocks

Place value blocks[8] are structured materials and are proportional in nature, i.e., the long block is a fixed grouping equivalent in length to ten unit blocks, and the flat block that represents one hundred is equivalent to ten longs laid side by side. The blocks were first introduced in the 1960s as an embodiment of place value.

> The structure of place-value blocks mirrors the structure of the base-ten numeration system analogically, with the size of each block being proportional to the value it represents, compared to the unit cube (Price, 1998, p.454).

A long can be renamed as ten units to allow multi-digit subtraction where renaming is necessary. The place value blocks can be used in such a way that the actions performed with the blocks match the steps of the written algorithm. This is how Cycle

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[8] Place value blocks are also referred to elsewhere as Dienes blocks, base-ten blocks, and multi-base arithmetic blocks (MAB). Dienes developed these for use in the instruction of standard algorithms.

12 of the BSM resource suggested teachers use the equipment (Ministry of Education, 1993). Much of the literature examines the use of place value blocks to teach place value in the context of formal algorithms (Fuson & Briars, 1990; Labinowicz, 1985; Liebeck, 1984; Resnick & Omanson, 1987; Thompson, 1992). Used this way, place value blocks can be said to provide a 'working' model of the algorithms (Higgins, 2001a).

A disadvantage of the place value blocks is that during the renaming process a long is replaced by ten different objects (Hughes, 2001). The same process with bundled ice-block sticks, for instance, simply involves the removal of the rubber-band to change the same sticks in one bundle of ten into ten loose sticks.

Place value blocks were heavily used in Cycles 9 to 12 of the BSM resource, with their suggested manipulation matching the steps of the standard algorithms. Their use in Cycle 9 was further structured by an accompanying 3 X 3 grid which:

> ...helps to emphasise the 3-place pattern of our number system, i.e., ones, tens, hundreds... Ministry of Education, 1992b, p.113).

The blocks were laid out on the 3 X 3 grid to represent the number problems in vertical, or working, form.

In the NDP materials, place value blocks are suggested as equipment appropriate for the development of an understanding of place value, and for two-digit addition and subtraction where no renaming is involved. In contrast to previous teachers' resources, place value blocks are not explicitly used alongside the written algorithms for addition and subtraction, which are introduced once children are part-whole thinkers, at Stage 6: Advanced Additive. Up to this point, the focus of the NDP booklet entitled *Teaching Addition, Subtraction, and Place Value, Draft Teachers' Materials* (Ministry of Education, 2002d) is to help children develop a raft of efficient mental strategies.

> Written forms are needed when the numbers get too large to handle mentally (Ministry of Education, 2002d, p.38).

As a support for the development of place value concepts in the NDP resources, the place value blocks are used in learning experiences such as "Tens and Ones" (Ministry of Education, 2002c, p.13). This involves children in composing numbers with place value blocks and then comparing these with other representations of the same

numbers, in order to help them construct connections between the various models (Ball, 1992; Carpenter et al., 1999; Hiebert et al., 1997). By connecting the different representations through their similarities and differences, children can construct a deep understanding of this aspect of number.

It is hoped that the inclusion of place value blocks in this study might identify a variety of ways that children use them beyond the narrow scope of being a model of algorithmic thinking. The way children use the place value blocks may also yield information about their place value understanding and their part-whole strategies.

2.3.5 Play money

The play money used in the interviews is the same as that used in the NDP materials and comprises notes of $1, $10, and $100 denominations. Each note has a picture of a different animal (moa, dolphin and seal respectively), the images of which are familiar to most New Zealand children. The notes are all the same size, and feature the appropriate number and words, e.g., the numeral 1 and the words "New Zealand One Dollar" appear on the $1 notes. The notes of each denomination are all the same colour, i.e., $1 notes might be yellow, $10 notes purple, and $100 blue. Although New Zealand currency no longer includes a $1 note, children seem willing to accept this in the context of play money. Coins are not used in this study.

Play money is considered non-proportional, pre-grouped material. Unlike bundled ice-block sticks and place value blocks, a $10 note does not look like ten $1 notes; the physical features of various notes do not suggest their values. Play money instead achieves value by arbitrary assignment.

> ...relationships between the denominations are not immediately discernible (English, 1993, p.21).

Play money provides a context for number work that is meaningful and interesting for most children (Hughes, 2001; Young-Loveridge, 1999a). For children who have had previous experiences with money – in games such as Monopoly played at home or school, playing shops, or with spending money of their own – using play money to develop their understanding of place value may be helpful. However, other writers caution that it is this very familiarity that can make money a difficult representation to use for place value understanding (English, 1993; Hiebert and Carpenter, 1992). Because money is used to buy things:

> ...it is unusual to let money stand for other objects, such as cows or candies. The consequence of its special role is that children may not recognize immediately the way in which money represents place value. Children's associations for money may remain tied to its use outside of class and fail to connect with place value ideas that the teacher is emphasizing in class (Hiebert & Carpenter, 1992, p.71).

As English (1993) summed it up:

> It is doubtful whether money would ever be used as an analog if it were not so pervasive in our society (p.22).

Apart from the contextual difficulties, there are some advantages to using play money. Large numbers are more easily represented using play money (Young-Loveridge, 1999a). The equipment is also very portable. One disadvantage, similar to that of place value blocks, is that in order to rename a $10 note as ten $1 notes, one item must be removed and then replaced by ten different items (Hughes, 2001).

In the NDP materials, play money is introduced for composing two-digit numbers which are also shown with arrow cards (Ministry of Education, 2002c, p.11). Most of the learning experiences for addition, subtraction, and place value in which play money is recommended, include problems that are directly related to money, such as:

> Mrs Jones takes her class to the circus. She has $237 to pay for the students to get in. Admission is $10 per person. She has 25 in her class. Does she have enough money? (Ministry of Education, 2002d, p.28).

In several instances, however, children are expected to decontextualise money and use it to represent numbers of sweets, or in this example, cans of drink:

> Marlene is planning a big party for 34 friends. Everyone needs a can of drink at the party, but she's got only 21 cans in her cupboard. How many cans of drink will she have to buy at the shop? (Ministry of Education, 2002d, p.23).

Play money comes complete with its own context – buying and selling using dollars and cents. To expect children to strip play money of its obvious context, and to use numbers of dollars to represent numbers of cans of drink may be a complex task for some children. As Hiebert and Carpenter (1992) suggested, the sort of context used in this last problem is likely to make children's use of play money less straightforward. Children in this study who choose to use play money to solve a problem about seagulls are likely to face the same difficulty.

2.3.6 The calculator

For various reasons, teachers and parents have in the past been less than enthusiastic about calculators being included in the numeracy programme. Despite official endorsement in curriculum statements for the use of calculators, this has not resulted in their widespread use in the junior primary school (Stacey & Groves, 1996). There have been concerns that children will become dependent on calculators and that their numeracy development will somehow be impeded (Duffin, 1997). Added to this is the fact that many of the current teaching population gained their mathematics education at a time when calculators were not yet readily available, so have not had the experience of learning with calculators.

> ...many teachers lacked confidence in using calculators as a teaching aid and in teaching pupils how to make the best use of them. Teachers were also unsure when pupils should use calculators and for what purposes (Office for Standards in Education, 2000, p.15).

Even when teachers have said they support calculator use, this has not translated into practice (Stacey & Groves, 1996). Teachers may be unaware of the calculator's potential for teaching and learning. Ingaram (2000) describes how children can discover generalisations about numbers, develop their prediction and estimation skills, increase their recall of basic facts, and learn about place value with regular calculator use. Obviously key to the successful inclusion of calculators in the mathematics programme is the purpose for which the teacher includes them. Ingaram suggests a rule of thumb for teachers to apply when considering calculator work:

> ...we have a principle that calculator work is acceptable **if it promotes mental calculation** and inappropriate if it inhibits it (Ingaram, 2000, p.13).

In keeping with this, Liebeck (1984) stressed the importance of children using calculators only when they have some idea of what the answer should be, in order that they can identify a reasonable answer.

There are many published resources that provide isolated activities – material for one-off calculator lessons – but this does not help teachers to integrate calculators into their numeracy programmes. McIntosh (1996) recommends the gathering up of proven calculator activities into some coherent form, which can make the meaningful, regular inclusion of calculators more accessible to teachers.

In the NDP resources, calculators are used at all stages to help develop children's number knowledge. Far less use is made of calculators in the area of strategies; they are included only for multiplication and division strategy work from Stage 5: Early Additive Part-whole onwards (See Table 9, Appendix 3).

Research has examined the effect on young children's learning of their free access to calculators. Both the *Calculator Aware Number Project* (CAN), begun in Britain in 1986 (Shuard, Walsh, Goodwin & Worcester, 1991), and the *Calculators in Primary Mathematics* (Stacey & Groves, 1996) project undertaken in Australia in the early 1990s, introduced calculators with the intention of providing children with a rich mathematical environment for them to explore, rather than giving them a computation tool on which they should become dependent. Both studies yielded positive results. The work in Australia found:

> ...some children were developing concepts related to large numbers, negatives and decimals at an earlier age than expected (Stacey & Groves, 1996, p.216).

This could, of course, create new challenges for teachers as they try to cater for an even wider range of learning needs.

Swan and Bana (1998) suggested that children who have ready access to calculators tend to choose to use a calculator when the numbers in a problem are large, difficult numbers (as opposed to, say, 5000 x 2000), decimals or fractions. Their research showed that children, on the whole, became discerning calculator users.

Calculator use is highly motivational for children (McIntosh, 1996). It would seem that teachers are not yet capitalising on children's enthusiasm for a technological tool that can be used to enhance the development of key number concepts.

2.4 Summary

As the emphasis in numeracy teaching and learning shifts from standard algorithms to children's part-whole thinking, so the use of equipment must change. 'Working' models were typically used to support children to learn the steps involved in formal algorithms. In order for children to communicate the variety of mental strategies which are now the focus of teaching, 'thinking' models are needed. The particular use of such equipment as place value blocks is broadening accordingly.

CHAPTER 3

Methodology

3.1 Rationale

Ongoing, national research (Higgins, 2001b, 2002b; Irwin & Niederer, 2002; Thomas & Ward, 2001, 2002) into the impact of the NDP has investigated the effects of the teacher professional development, as well as progress in children's achievement. While the viewpoints of teachers, principals and NDP facilitators have been sought, there has been very little included of the children's perspectives. Article 12 of the United Nations Convention on the Rights of the Child (New Zealand Office of the Commissioner for Children, 1990) asserts children's right to have their expressed views taken into consideration. In New Zealand, researchers, academics and policy makers are being urged to listen to the voices of children (Smith, Taylor and Gollop, 2000). To date, children's views have not been a focus of the research related to the NDP.

The purpose of this study, therefore, was to investigate the child's perspective regarding equipment use, including which materials the child finds the most helpful and why, and how this relates to the mental strategy stages on The Number Framework for strategies (Ministry of Education, 2002a, see Appendix 1). Furthermore, children's ideas about equipment were compared with those of teachers in order to identify likely implications for teaching and learning.

3.2 Research aims

The overarching research goal was to discover how closely the children's reasons for their equipment choices matched their teachers' reasons for including the same pieces of equipment in their numeracy programmes. The following questions were devised to underpin this goal:

1. Which manipulatives do children find the most helpful as supports for solving number problems?
2. What are the reasons that underlie their equipment choices?
3. What are the connections, if any, between children's strategy stages and their preferred equipment?
4. Which equipment are teachers choosing to include in their numeracy programmes?
5. What are the teachers' reasons for including or excluding particular equipment?
6. Is there a correlation between children's success with their use of structured equipment, and the teachers' equipment choices?

3.3 Research design

Face-to-face interviews were conducted with each of 34 Year 3 children from three central Wellington schools, referred to here as Faraway, Uphill, and Nearby Schools. The schools ranged in decile from eight to ten. The children's teachers were also interviewed. The focus of all the interviews was the use of equipment in numeracy teaching and learning. The teacher interviews were semi-structured, and were based around the questions included in Appendix 6. The structure of the children's interviews is detailed in the next section.

In two of the classrooms I spent one mathematics lesson roving among the children who were working independently, talking to them about their activities. I hoped that this would help them feel at ease when being interviewed over the following days. In the third classroom, the teacher felt this was probably unnecessary as the children were accustomed to having visitors in the classroom. There was no noticeable difference in the bearing of the children whom I had met before the interview took place, and those whom I had not.

A pilot study was conducted to verify the appropriateness of the data gathering methods. Five children from two strategy stages (Stage 4: Advanced Counting and Stage 5: Early Additive Part-Whole) were interviewed, as was their teacher. These interviews confirmed the usefulness of the chosen questions, and reinforced the importance of recording accurate field notes when the children were using equipment to solve a problem.

The Ethics Committee of Wellington College of Education gave its approval for this research project, which adhered to the ethics guidelines set by the New Zealand Association for Research in Education. Consent was sought from the teachers and the parents or guardians of the children involved in the study. Names of the children, teachers and their schools have been changed in this report in order to protect their anonymity.

3.4 Data gathering

The collection of data took place in the third term of the 2002 school year. There were two components to the data gathering: interviews with the children, and with their teachers. All the interviews were recorded on audiocassette for later transcription.

The children's interviews began with a series of questions designed to determine their strategy stages (see Appendix 7). These questions were modelled on those included in the NDP diagnostic interview (Ministry of Education, 2002b). Care was taken to modify the questions so that the children did not become too familiar with the original problems. These questions highlighted the strategies the children used to solve addition and subtraction problems. At the time of the interviews, I recorded children's responses on a recording sheet (see Appendix 8). As well as recording the children's verbal responses, note was also made of particular body language such as the use of fingers to support counting, or touching a finger to the lips when counting. Children's full verbal accounts of their thinking were sometimes completed after the interviews, using the tape of the interviews. Although video-taping the interviews would have created a visual record of the children's equipment use, I felt that this would have been an unnecessarily intrusive means of data collection.

Making a written record of those aspects of the interviews that were not captured by the audiotape was even more important during the second part of the interviews when the children were asked to choose and use equipment to solve a problem. The questions relating to equipment were developed from those used by Ritchie (1991). The pieces of equipment used in all the interviews were:

- a 1 to 100 number line
- tens frames and transparent counters
- bundled ice-block sticks
- place value blocks
- play money
- calculator

The choice of equipment is further explained in Section 2.3.

The problem that the children were asked to model and solve with their choice of equipment was the type of subtraction problem known as a separate problem (Carpenter & Moser, 1982). Separate problems involve a direct or implied action that takes place over time. Such problems, where the result is unknown, are the type of problems typically used to introduce subtraction to children; they are the least complex type of subtraction problem. Embedding the mathematics in a context that is familiar to children is thought to make the concept more accessible to them. Having considered these points, this was the problem with which the children were presented:

> There were 41 seagulls on the beach. 18 of the seagulls flew away. How many seagulls were left on the beach?

This particular problem gave the children an opportunity to show their understanding of place value; for example, those who chose to model the problem using the place value blocks might have traded one of the longs included in the model of 41 for ten units, in order to allow them to remove the eight ones in 18. Others might use a part-whole strategy like rounding the 18 up to 20, removing two longs and then replacing two unit blocks.

3.5 The teachers

The four teachers interviewed were drawn from three schools whose staff were participating in the ENP professional development during 2002. The teachers were selected first, and then children were drawn from their classes.

The teachers had been selected by asking NDP facilitators to suggest teachers who they considered provided a model of sound numeracy teaching. During the course of the development the facilitators worked alongside the teachers with their children. This included facilitators modelling for teachers as well as facilitators observing the teachers teaching, and giving them feedback.

Teachers from four schools were initially suggested. Three were chosen to approach because their schools were more centrally located. Principals were contacted first, and meetings with each principal and teacher were arranged to discuss the proposed research. Teachers signed consent forms (see Appendix 4) and arranged for the children's consent forms to be sent home to parents and caregivers.

Susan at Faraway School taught a composite Year 2-3 class. At Uphill School four classes of Year 3 children were cross-grouped for mathematics; in this case, children from Yvette's mathematics class were interviewed, rather than children from her home class. Two teachers at Nearby School each taught composite Year 3-4 classes and divided the children between them for mathematics instruction, according to children's abilities. Both teachers, Olivia and Tracy, were willing to participate in this research, so they were interviewed together. Interviewing Year 3 children from Olivia's home class therefore captured the experiences of children taught by both teachers (one of whom had been suggested by an NDP facilitator).

The interviews with teachers were arranged to accommodate their commitments. In two schools the teacher was interviewed after the interviews with their children were finished. At Nearby School, Olivia and Tracy were interviewed halfway through the interviews with their children.

3.6 The children

The children were drawn from the classes of the four teachers who participated in this study. All of the children for whom written consent had been gained, and who were present at school on the interview days, were included in the interviews. (A copy of the consent letter is included in Appendix 5). Of the 34 children, half were girls. The teachers were asked to supply the children's birth dates and their ethnicities. Ages of the children at the time of interview ranged from seven years and one month to eight years and two months. The average age of the children was seven years and eight months. It is interesting to note that the single child at Stage 3: Counting from One by Imaging and the child at Stage 6: Advanced Additive (Early Multiplicative) Part-Whole were both seven years and six months old and in the same class. This gives some indication of the range of strategy stages that might be represented by the children in one class.

The group comprised predominantly European children. Twenty-seven of the children were European, three were Asian, two came from Pacific Nations backgrounds, and one child was of each of two other ethnicities. Because numbers of children of ethnicities other than European were so small, it would not be valid to attribute any differences in the children's responses to their ethnicities.

3.6.1 The children's identified strategy stages

The limited number of strategy-related questions asked at the start of each interview was designed to give a snapshot of each child's strategies for solving addition and subtraction problems. Young-Loveridge (2002) suggested that:

> Making judgments about students' strategies can be problematic, and additional tasks may need to be used to provide a sharper picture of a student's thinking processes (p.31).

The information gained in the interviews provided an indication of the child's strategy stage.

The children's responses to the strategy questions indicated the range of Stages shown in Table 1. While I intended to interview ten children at each of three consecutive strategy stages, this proved unrealistic. The clear majority of the children appeared to be using strategies consistent with Stages 4 and 5 of the Number Framework. This is consistent with Thomas and Ward's (2002) findings that in higher

decile schools just 12% of children were below Stage 4: Advanced Counting at final testing.

Because only one child was assessed at each of Stages 3 and 6, it will be more helpful to talk about the children's strategy stages in terms of those who used counting-based strategies (Stages 3 and 4) and those who used part-whole strategies (Stages 5 and 6). Considering the results in these terms, the number of children who showed some part-whole thinking was almost twice the number of those who used counting strategies.

Table 1. The children by strategy stages

	Counting		Part-Whole	
Stage	3	4	5	6
Girls	1	7	9	-
Boys	-	4	12	1
Total	1	11	21	1
	12		22	

The teachers' comments about how equipment is used in their numeracy programmes are included in Chapter 4. Then the children's familiarity with the six pieces of equipment that are the focus of this study is discussed in Chapter 5, before considering the degree of flexibility with which the pieces of equipment were used by the children in Chapter 6. The reasons that underlie the children's choices of equipment are compared to those given by their teachers in Chapter 7.

CHAPTER 4

What the teachers said about equipment

4.1 An overview

Semi-structured interviews of around thirty minutes each were conducted with four teachers (see Appendix 6 for teacher interview questions). The teachers interviewed affirmed that equipment is an important feature of their numeracy programmes and that the use of equipment has increased as a result of their participation in the NDP professional development. The draft teachers' materials and some of the equipment were new to the teachers, so they were still developing their familiarity with, and confidence in using, these resources. The teachers talked about not feeling sure how to effectively use new equipment with their children. They were confused about the mathematical purposes of some equipment. Other available equipment was not used at all by some of the teachers. The reasons the teachers chose whether or not to include a piece of equipment, how they used it, and with which children, varied. Overall, the teachers' choices of equipment were not strongly consistent with the equipment use recommended in the NDP draft teachers' materials.

There appeared to be an assumption that by involving children in the manipulation of physical materials, particularly in the early stages of the number framework, they would have greater success with learning. Teachers knew that the visual features of equipment could influence children's engagement, and therefore their understanding, so teachers considered this when selecting equipment. These beliefs may be a legacy of the child-centred philosophy of teaching and learning that came to prominence in the 1960s (Burman, 1994; Higgins, 1998). The teachers' main focus seemed to be on involving the children in 'hands-on' experiences, rather than looking closely at how the manipulatives could be used by the children as 'thinking' models.

> Regarding manipulatives and other representations of mathematical ideas as 'thinking' models requires teachers to have a secure understanding themselves of the mathematical ideas being represented and a knowledge of where these ideas fit within the particular conceptual field and how they are related to other ideas within a conceptual field (Higgins, 2001a, p.27)

Teachers need to have a deep understanding about the various ways in which individual pieces of equipment might be used to develop concepts such as grouping

and place value at the various stages of the number framework; children will not develop a rich understanding of these ideas purely through the manipulation of materials (Baroody, 1989; Ball, 1993; Gravemeijer, 1994; Thompson, 1994).

4.2 A shift in emphasis

The fundamental shift in emphasis that underpins the NDP is one in favour of the development of children's mental strategies. Accompanying this is a de-emphasising of written work, although this is still an important component of numeracy programmes. Susan, a teacher from Faraway School, commented on a swing away from an emphasis on children's written work towards an emphasis on mental strategies in her teaching since she began the NDP professional development:

> I think what I've got from this numeracy project is, it's much more about what's going on in your head. And because it's new, I think I'm tending to go towards letting them work things out with not so many props. But I don't know if that's such a good way to go. Because for me it's a big move from writing things down, to just getting rid of all that – there's hardly a place for that.

In keeping with the emphasis on children developing part-whole thinking is the strong recommendation that formal algorithms should be introduced later than they have been formerly. One of the significant changes advocated in the NDP materials is for teachers to delay the introduction of standard algorithms for double-column addition and subtraction with renaming until children have reached Stage 5: Early Additive Part-Whole. Most teachers agreed with this comment from Olivia at Nearby School, that this was helpful to children's learning:

> ...I wouldn't teach it to them now. Not until they were ready for it, because a lot of them just aren't ready for it. They get mixed up. They get **so** mixed up, some of the kids, if they're trying to do working form in their heads.

Tracy continued to teach the algorithms for double-column addition and subtraction to all Year 3 children, regardless of their strategy stages. She described the difficulties involved in teaching her children (most of whom were at Stage 4: Advanced Counting) the standard algorithm for double-column subtraction with re-naming:

> Today we were doing subtraction and I said I want to know what the first – it was thirty-one minus eighteen or something – and I wanted them to tell me what to do first. And they couldn't **not** give me the answer. They couldn't do it, they didn't know how to do it. Because it was one take away eight, basically. So they reversed it to eight minus one, and they couldn't, they just couldn't give me the process. They wanted to... I said, just tell me, what to do first of all. And they're saying you just think. Well, what do you think about? What do you say?

This sort of top-down approach is not endorsed in the NDP materials which take children's informal strategies as the starting point for teaching and learning. One of the reasons for not imposing formal algorithms on children in this way is that they tend to apply the procedures without understanding what they are doing with the numbers. An example was the algorithmic method used by Tina (Nearby School) to mentally solve 54 – 27 = ? Her answer was 33. Tina explained that she had thought of it as "seven minus four equals three, then five take away two equals three".

Other comments from Tracy were perhaps more consistent with a traditional transmission philosophy of teaching:

> Whereas I think, another few years' time, even if you need equipment for that sort of basic thing, you probably won't use it. So it's a good time now to really get that basic stuff into them.

> ... we did fractions for a while, about a month ago, and I taught it and taught it...

She had been teaching for more than twenty years and seemed reluctant to change her teaching practice. The in-class modelling which is a feature of the NDP professional development:

> ...increases the likelihood of "teacher buy-in" through teachers being able to apply the new knowledge to their classrooms (Higgins, 2002a, p.44).

However, there are some teachers who are resistant to the fundamental changes advocated in the professional development.

4.3 Beliefs about the role of equipment in general

Many of the responses about the use of equipment made by the four teachers during their interviews were consistent with the overarching ideas inherent in the NDP. All four teachers asserted that equipment plays a very important role in their mathematics programmes. Some talked about the value of demonstrating with equipment when a new concept is being introduced to children at all strategy stages, and that equipment can provide support while children develop their understanding. Used in this way to scaffold children's learning, the teachers also commented on the importance of moving children on before they become reliant on any one piece of equipment. As Yvette from Uphill School explained:

> Use your ruler, use your fingers – it's there. I mean, they have to see it. It's a stage. As long as they don't get stuck on it.

The teachers described a progression from using materials, to imaging the materials, to later using number properties:

> I see it partly as a stage of learning a concept. (Olivia)

This teaching model, modified from Pirie and Kieren's (1989) work, is emphasised throughout each of the strategy booklets in the NDP draft teachers' materials (see Figure 3, p.15). Teachers also talked about dropping back to using materials when imaging became too difficult for children.

Still occasionally apparent is the long-standing belief among some teachers that as children's understanding grows, they need fewer hands-on experiences. Less equipment was used with children who were working at the higher strategy stages.

> With those top groups I don't use a lot of equipment other than, you know, number lines ... I suppose that's equipment though, isn't it ... number lines, hundreds board. (Susan)

The sort of equipment this teacher was using with her Stage 5 children, i.e. the number line and the hundreds board mentioned, are symbolic representations of the Arabic number system as opposed to equipment that represents actual amounts. Not only does the amount of equipment used by teachers seem to change as children's understanding develops, but the type of equipment chosen may also shift. And in a similar vein:

> I've got a group of kids who don't need the equipment so much now when they're practising. (Olivia)

This remark might reflect that the children in this group are able to use imaging or number properties to solve practice problems. Alternatively, it might again be a reflection of the teacher's belief that children need to use less and less equipment as their understandings develop. Table 9 (see Appendix 3) shows that a variety of equipment plays a significant role right through the stages of the number framework. The curriculum document also states clearly that equipment plays an important role at all levels of teaching and learning:

> Junior school teachers are used to choosing an appropriate range of apparatus to focus students' thinking on the concept to be developed and modifying the apparatus as the learner's understanding grows. Teachers know that students are capable of solving quite difficult problems when they are free to use concrete apparatus to help them think the problems through. Such an approach is equally valid with older students and should be used wherever possible (Ministry of Education, 1992a, p.13).

4.4 How did the teachers choose equipment?

During the interviews with the teachers, they were asked whether each piece of equipment was included in their numeracy programme. Table 2 gives an overview of how many teachers said they were using each manipulative. The teachers also talked about how they used the various manipulatives, and later explained their reasons for including or excluding particular pieces of equipment.

Table 2: The teachers' inclusion of equipment in their numeracy programmes

	Included	Not Included
Number line	4	0
Tens frames	1	3
Bundled ice-block sticks	3	1
Place value blocks	2	2
Play money	1	3
Calculator	2	2

The majority of children in these teachers' classes were at Stage 4 or beyond on The Number Framework. Table 9 (see Appendix 3) shows that in the NDP materials, each of these six pieces of equipment is included by the time children have reached Stage 4. This does not match what was happening in the classrooms included in this study. For example, in Stages 0 to 4 the tens frames are specifically mentioned on nine occasions in the addition, subtraction, and place value learning experiences. Yet only one of the four teachers interviewed had chosen to use tens frames with her children.

The reasons the teachers gave for their decisions about which equipment to include in their numeracy programmes ranged from the surface features of equipment – such qualities as colour, size and texture – through to the numeracy concepts that some equipment is thought to embody, to the influence of the curriculum. While the surface features of equipment were often mentioned as reasons for including a piece of equipment, the role of equipment in developing conceptual understanding was more typically talked about as a reason for excluding, as well as including, pieces of equipment from the numeracy programme.

4.4.1 Surface features of equipment (or what teachers think children like)

The teachers seemed to place more importance on the appearance of a piece of equipment than did the children who chose equipment on the basis of whether it would help them to solve the problem. The notion that learning should be fun was a theme that ran through many of the comments the teachers made about their reasons for including particular equipment. This notion of learning as fun or play harks back to the child-centred approach to education (Burman, 1994). It was epitomised by Yvette's remark when the equipment was brought out for discussion: "Toys!". The teachers made only positive remarks about the equipment's surface features. They commented that the visual characteristics of some equipment influenced their decision to include it in their numeracy programme. Tracy explained that:

> ...because the Unifix things are 3-dimensional, and brightly coloured and plastic, I actually prefer those... I prefer the 3-D, the bright, shiny stuff.

Yvette liked to use bundled ice-block sticks:

> Oh, I love those bundled ice-block sticks. Kids think they're fantastic. They love the colours.

There was a belief that if equipment is visually attractive then children will enjoy using it. Furthermore, there was an underlying assumption that if children engage with the materials (which is thought to be more likely if the materials are attractive) then they will learn. This comment from Yvette connects the two ideas:

> The [Slavonic] abacus I find has been really good, and that's quite a visual thing. The kids enjoy it, because they haven't really used an abacus before.

Not only has the Slavonic abacus been new to some teachers and children, it is a brightly coloured, well-finished piece of equipment.

Children's perceived preferences for equipment seemed to influence teachers' choices in some cases:

> My guys love, when we're adding big numbers like tens or nines or elevens, they like the hundreds board – that flip board. (Tracy)

However, in the case of calculators, their exclusion from numeracy programmes did not match teachers' comments about how much children enjoy using calculators:

> Which they all love. (Yvette)

4.4.2 Support for children's conceptual development

Most of the teachers used equipment to support their teaching of number knowledge. When the teachers explained why they chose to include particular pieces of equipment in their numeracy programme, they talked about the value of the bundled ice-block sticks for developing children's concepts of grouping in fives and tens. They also listed the teaching points for which they found the number line helpful. These included number sequence and order, counting forwards and backwards, and jumping in fives and tens.

The bundled ice-block sticks and the number line were the only two pieces of equipment included in the teachers' numeracy programmes that the teachers described in terms of helping children to develop part-whole strategies.

> I think the number line's good for adding and subtracting bigger numbers too. Like breaking it up to add fifteen to seventy-five, or ten and five and one for sixteen. (Tracy)

When the teachers were asked to talk about why they chose not to use a particular piece of equipment, their comments more often related to the effectiveness of equipment for teaching and learning rather than to the equipment's appearance. Yvette had used tens frames with her children to develop the concept of groupings of five, but did not seem satisfied that they were helpful:

> Not as good, not as clear. I don't know. Maybe I just haven't used them successfully, but I didn't like them as much. I mean, some of the other stuff is great. … Maybe it's just me. I didn't like it as much, or maybe I didn't feel it was as successful.

One teacher had difficulty justifying why she was not keen on using the place value blocks:

> Susan: Yes, there's something about those that I find a bit clumsy. Yes, I have used them with the class, but I don't tend to use them. I can't tell you **why** I don't use them. I think I choose to use other things to represent the same thing.
>
> Interviewer: So instead of that there would be things like …?
>
> Susan: Well, perhaps in the early stages I'd use the sticks [longs] and the ones, but not so much these (points to the flats that represent 100). One of those things in the back of the cupboard that I don't really warm to.

Teachers may have developed fluency in using a piece of equipment in their whole class maths warm-ups to develop number knowledge, but may not yet have included the equipment in their strategy teaching sessions with small groups. The teachers' use of the number line, for example, was largely restricted to number knowledge work with the whole class, as opposed to using it for exploring strategies with smaller groups of children. Yvette's comment about her use of the number line was fairly typical:

> Interviewer: So it's been used in the warm-ups with the whole class. Have you used them at all with group teaching?
>
> Yvette: Umm, I think we did earlier in the piece, but not recently, no. I've used more the one to twenty ones, the little ones. I can't remember what I did with that earlier, I'm sorry.

Number lines of various sorts are frequently used in the NDP draft teachers' materials for strategy development, as well as for developing children's number knowledge.

While the surface features of equipment were mentioned as reasons for including a piece of equipment, the role of equipment in developing conceptual understanding was talked about both as a reason for excluding a piece of equipment from the numeracy programme, and for including it.

4.4.3 Accessibility of equipment

For some teachers, the accessibility of equipment was an issue; with equipment kept in a centrally located resource room, it was not always on hand in the classroom at the moment it could have been the most helpful.

> That's so often the thing, that if it says something and you haven't got the resource right there, you'll actually just use the ice-block sticks, or use something else – use counters. … And a lot of the equipment is stored up in the Kowhai block, and unless you make the effort and go – really get yourself well organised and go and get it… And I've made some stuff myself so I actually have it in my room. I like to have it there, it's right where I have it. (Yvette)

For others, it seems to be more a question of being selective about which resources to introduce:

> …there's so much of it tucked in the cupboard that I haven't even dragged out… (Susan)

> …well, we haven't been given all the equipment, but we sort of selected a few. And it's kind of getting used to little bits, because otherwise you feel overwhelmed yourself, don't you? (Olivia)

It is interesting that the teachers did not mention the actual production of resources as an issue. The production of a large number of resources was identified as one of the least helpful elements of the ENP professional development programme (Thomas & Ward, 2002, p.iii). It should be noted, however, that the large number of black line masters with which teachers were provided in 2001 was significantly reduced the following year.

4.4.4 Teachers' familiarity with, and understanding of, NDP draft teachers' materials

Another theme in the teacher interviews was that some teachers have been working through the stages from the book, following the suggested learning experiences like a programme. As Yvette said:

> There is lots, that's right, I must admit, that I haven't looked at. And I haven't looked ahead either. I've looked a certain - but because it's - I've felt it's really been so different, and I'm going stage by stage. I'm just doing it.

This is not unique to the NDP resources: the introduction in the mid 1980s of the BSM resources met with a similar response from some teachers:

> Many teachers when familiarising themselves with the resource, or if lacking confidence in using it, tend to follow instructions accompanying the resource religiously. But, in time, with increasing familiarity and confidence, teachers should come to realise that the most effective way they can employ BSM is to use it in the manner most suited to the individual needs of their students. (Bennie, Henry & Ratcliff, 1990, p.163.)

Although some teachers may be working their way through the suggested learning experiences in the NDP draft teachers' materials, this is likely to be tempered by the teachers' access to the equipment demanded by some activities. The teachers' beliefs about particular equipment also impacted on the equipment with which they presented their children. For instance, all teachers are not using calculators. Only one of the four teachers described using calculators with her children for an activity other than checking an estimation, or computation work. Two of the teachers said they had not used them at all in their mathematics programme this year.

> I haven't used them this year. But last year I used to use them a lot for activities I could give the children. I think they're useful, I think they certainly have a place. Not for finding out things, not for the answer to such and such, but techniques and… (Susan)

> I actually think, for them, it's better to be using the materials and actually manipulating rather than going to something that's

> mechanical at this stage. I think they need to understand what they're doing. That's great for working stuff out, but anybody can do that. (Yvette)

In the NDP materials, calculators are used at all stages, mainly to support the development of number knowledge (see Table 9, Appendix 3). All the teachers interviewed said they had access to calculators to use with their children. Teachers could capitalise on the popularity of the calculator with children by including them in the numeracy programme for work that takes them beyond computation and checking answers. A multitude of commercially-produced teacher resources include such activities as skip counting and place value games. While there are learning experiences using calculators included in the NDP materials and in other Ministry of Education resources such as the *Figure It Out* series and the web-site www.nzmaths.co.nz, it may be timely for teachers to be provided with a compilation of calculator activities that would support the numeracy initiative.

Until they have actually used the learning experiences in the NDP materials, it may be difficult for teachers to judge which are absolutely vital to include and which may not be needed for particular children. Certainly, teachers need time to develop familiarity and confidence with selecting and implementing the learning experiences in the NDP materials.

It will also take time for teachers to develop their understanding of how some new pieces of equipment are most effectively used to help children build their understanding of number concepts. Until teachers have this understanding, some teachers will experiment with ways to use a piece of equipment:

> …we've got this bit of equipment, how can we teach that concept with this equipment? Instead of being so intimate with the equipment that it springs to mind instantly. (Tracy)

This could result in teachers using equipment in less effective manners. For instance, after it was explained to Tracy that tens frames are particularly helpful for fostering the strategy of making a ten, she was quick to think of another way to use them:

> Tracy: And I suppose for renaming when you're adding bigger numbers too, if you had more than one of them.

Interviewer: Yeah, um...

Tracy: Like, if you're adding forty-one you could have four, five of them and then adding ...

This was a good example of finding another way to use equipment, when the concept the teacher talked about – addition with renaming – would probably be better explored with bundled ice-block sticks or place value blocks, depending on what stage the children were at.

Some teachers may not have understood the intended purpose of some NDP equipment. The tens frames is a case in point. Two teachers were substituting pattern boards for tens frames, unaware that the two similar-looking frames support the construction of quite different mathematical concepts; while the pattern boards emphasise odd and even numbers, tens frames highlight groupings of five. In some cases this purpose may have been unclear to the teacher. Pattern boards were used a lot in the BSM resources; one of these two teachers said she had used a lot of BSM before her NDP professional development involvement. This may well have influenced her understanding of tens frames:

Susan: Oh, we've been using the plastic ones with the moulded...

Interviewer: Oh, the pattern boards. They're slightly different.

Susan: Yes, it is different, but you just flip them over and there's a flat board.

Similarly, Tracy immediately made a connection between the tens frame presented and pattern boards she had used previously:

Tracy: I've used them in the new entrants, with the old wooden things with holes in them.

Interviewer: The pattern boards?

Tracy: Is that what they're called? They look like that, but some of them have only got 6 holes and you bung the things in.

The Slavonic abacus has been new to some teachers. When asked whether they had used any pieces of numeracy equipment that had not been as helpful as they had hoped, Tracy talked about not knowing how to use the abacus effectively:

Tracy: For me it's the abacus, and that's because I don't know how to use it well. I use it very basically. And I know that it's got great potential, but I don't know how to use it properly.

Interviewer: So, how do you use it?

Tracy: I use it for breaking up tens – sixes and fours – and adding tens and things like that, but I'm sure there's zillions more things you can do.

4.4.5 The influence of *Mathematics in the New Zealand Curriculum*

The layout of the mathematics curriculum document, with its artificial compartmentalisation of content strands, may also have had a role in the possibilities the teachers saw for the ways equipment might be used. Money is dealt with in the measurement strand of the curriculum and it may not be easy for teachers to think about using it in the NDP for exploring place value, as the following comment illustrates:

Interviewer: What about play money? (Presents the play money.) This is the numeracy project play money.

Yvette: No, haven't done any money at all. That's most likely what we'll do in the fourth term for measurement, do some money.

The compartmentalisation of concepts may have made a strong impression with teachers; the curriculum's strong message about the use of calculators at all levels of the primary school seems to have had a much lesser impact on what actually happens in some classrooms. This is similar to the experience in Australia (Stacey & Groves, 1996).

4.5 Summary

This study included only four teachers, three of whom were recommended by NDP facilitators as providing models of strong numeracy teaching. Because of the size of the sample, it is not valid to generalise the findings to the wider teaching population. The teachers interviewed did not seem to be using the equipment presented with a clear focus on helping children's development of part-whole strategies; the use of equipment to support the teaching of number knowledge was more common. The equipment that represents some aspect of the decimal structure of our number system – the tens frames, place value blocks, and play money – did not feature strongly in these

teachers' numeracy programmes. Which of these pieces of equipment had been used in these particular classrooms during modelling by the NDP facilitators is unknown.

The teachers were still becoming familiar with the NDP draft teachers' materials, and during this phase some were using the collections of learning experiences almost as a programme. However, the equipment that was being used in these classrooms was not strongly consistent with the use of equipment advocated in the NDP draft teachers' materials.

Ideas of fun and engaging children in 'hands-on' experiences still appeared to pervade the teachers' beliefs about numeracy teaching and learning. There was little evidence in the interviews with the teachers that they had the strong conceptual and connected understandings of mathematics that Fennema and Franke (1992) found resulted in more conceptual and connected manners of teaching.

In the following three chapters, the children's perspectives will be presented. Similarities and differences between the teachers' and children's views will be highlighted and discussed.

CHAPTER 5

What the children knew about the equipment

5.1 An overview

Face-to-face interviews with each of the 34 children furnished information relating to their likely strategy stages, their familiarity with, and choices of, equipment, and exactly what happened when they used the equipment. The children's reasons for their equipment choices were also revealed before they talked about how they preferred to solve number problems. (See Appendix 5 for the children's interview script.)

The manner in which many of the children used the equipment showed that they had yet to develop the secure understanding of grouping and place value necessary for two-digit subtraction involving the renaming of a tens unit as ten ones units. As the equipment they chose to use became more structured, so their success rates in solving problems declined. This seemed to be connected to their teachers' choices of equipment often being influenced by factors other than a consideration of the children's concept development. A similar difficulty of modelling subtraction with counters was found in Anthony and Walshaw's (2002) NEMP Probe study. They suggested that children may not be experiencing sufficient modelling experiences before moving to imaging without materials.

5.2 The children's familiarity with the equipment

The children were asked if they had seen each piece of equipment. The results are shown in Table 3. Where they had seen a piece of equipment, they were asked about the sorts of things they had done with it. This provided an opportunity, for instance, to ascertain whether children who said they had seen the place value blocks had used them for number work, or only as wooden blocks. One girl, for whom English was a second language, described the context in which she had used place value blocks:

> Um, we were making towers and we got this bits of square to make it like twin tower. (Claudia)

Table 3: The children's familiarity with the equipment

	Familiar	Unfamiliar
Number line	31	3
Tens frames	16	18
Bundled ice-block sticks	25	9
Place value blocks	20	14
Play money	19	15
Calculator	34	-

Children who had used the place value blocks only for building were coded as being unfamiliar with this equipment in the number context of the interview questions. As the data in Table 3 shows, all of the children had seen a calculator and most recalled seeing a number line. The tens frames, place value blocks, and play money were unfamiliar to around half the children.

5.2.1 The number line

Just three children did not recall seeing the number line. The majority of those who had seen it described playing a game called "Squeeze" where one child selects a number on the number line and the rest of the class ask questions to find out what the secret number is. At the start of the game, two pegs are positioned at either end of the 1 to 100 number line, and in response to questions, the pegs are gradually drawn closer together, hence the name. "Squeeze – Guess My Number" is an activity from the NDP materials (Ministry of Education, 2002c, p.12) aimed at developing children's number knowledge.

5.2.2 Tens frames

Almost half the children said they had seen tens frames, yet only one teacher said she had used them. Some of Yvette's children were able to clearly explain how to arrange eight counters on a tens frame by first filling a row of five then putting three counters in the next row, starting at the end of the row. Two children, however, arranged the eight counters in four pairs.

The tens frames seem to have helped at least some of her children to think about numbers in terms of groupings of five. It is possible that some of the children mistook tens frames to be the same as pattern boards, as did some teachers. Furthermore, during the interviews some children confused tens frames with bingo cards, which can have a similar layout.

5.2.3 Bundled ice-block sticks

When the children were presented with the bundled ice-block sticks, they were asked if they had seen any bundled sticks; children from Susan's class said they had bundled nursery sticks in fives and tens. The day before the interviews began with her children, the NDP facilitator had visited Susan and modelled a strategy lesson with a group, using bundled ice-block sticks. Naturally, these children talked about their recent experience with bundled ice-block sticks. Children from another school described using loose sticks for fraction work e.g., to find half of a collection of 20 sticks, rather than to develop grouping concepts. So, although 25 children said they were familiar with this equipment, this was not necessarily as a result of having used bundled ice-block sticks for two-digit addition and subtraction.

One child described the use of bundled ice-block sticks being used to teach the algorithm for double-column addition:

Odette: We've had ten, and then Tracy, my maths teacher, tells people who are sitting nicely and quietly to come and hold their's up. Sometimes they're bundles of ten and sometimes they're bundles of one. And we did that yesterday. Yesterday we had one of those things that have, say, sixty-four plus twenty-eight. Then they have those two lines and you have put the answer in there.

Interviewer: Do you have to put it underneath those two numbers?

Odette: Yeah.

Interviewer: Oh, OK. Do you have to add up the columns?

Odette: Yeah. And she asked some people to hold up – six people to hold up a ten bundle, and some people to hold up… and they're single people, which was me – I just held up one. Then she got two people holding up groups of ten, and then… eight people holding up just one Popsicle stick.

Interviewer: Yeah. Then what happened?

Odette: And then we had to figure out… all of us had to figure out how much it was altogether, and we guessed it. And we went outside to do it, and umm… we had to do – we had to take away one, coz I think it was 13…

Interviewer: Yeah.

Odette: And then so she gave ten Popsicle sticks to me, and all the other people that were holding one, they went on saying "Oh!" because they weren't holding up one. And so I joined the other people and we went off together and counted it. And we knew it was ninety-something.

Interviewer: So you swapped over your… when you got ten ones you swapped them over for a bundle, did you?

Odette: Yeah, yeah.

It is of concern that this procedural use of the equipment is still featuring when teachers are nearing the end of their NDP professional development, which strongly advocates that the introduction of algorithms should be delayed[9]. In this instance, the bundled ice-block sticks were used as a 'working' model, matching the steps of the formal algorithm. In Section 6.1.3 children's use of the same equipment as 'thinking' models will be outlined.

5.2.4 Place value blocks

Of the twenty children who may have used the place value blocks in a number context, five children described composing two-digit numbers with them. A further eight children said they had used them for addition and subtraction, counting in tens, or counting on. Seven could not recall exactly how they had used them.

Ten children explained that they had used place value blocks as building blocks for making towers and castles.

5.2.5 Play money

Children talked about using play money in games such as Monopoly, and when playing shops – both of these happened at home and at school. However, 15 children said they had not seen any play money, or that they had used only coins.

5.2.6 The calculator

The calculator was the single piece of equipment that all 34 children said they had seen. Experience with calculators ranged from the children who had used them during

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[9] The role of standard algorithms within current worldwide initiatives is examined more closely in Section 1.1.3.

mathematics time the same day as they were interviewed, to the children who had not used them at school since the previous year, to those who talked animatedly about how they used a calculator at home to explore negative numbers.

5.3 The children's choices of equipment

During the interviews, children were asked to choose a piece of equipment to use in order to solve this problem:

> There were 41 seagulls on the beach. 18 of the seagulls flew away. How many seagulls were left on the beach?

When they had used one piece of equipment, they were invited to work it out with a second, and then a third choice of equipment. A total of ninety-nine choices of equipment were made by the children. Their choices are shown in Table 4.

Table 4. The children's choices of equipment

	1st choice	2nd choice	3rd choice	4th choice	Total
Number line	1	8	9	1	19
Tens frames	1	-	4	-	5
Bundled ice-block sticks	14	10	5	1	30
Place value blocks	1	5	4	-	10
Play money	1	1	3	1	6
Calculator	16	9	3	1	29

One child chose to use only the calculator, while four children wanted to use four choices of equipment to model and solve the problem. The majority of children chose to use three pieces of equipment.

Easily the most popular choices were the bundled ice-block sticks and the calculator, with the latter proving to be the most popular first choice. Just four children made a first choice of equipment other than the calculator or bundled ice-block sticks. Fifteen children chose to use some combination of the bundled ice-block sticks, the calculator and the number line. The tens frames were the least popular choice, with just five children choosing them; for four of these children it was their third choice of equipment.

When the children's equipment choices are compared with the teachers' inclusion of equipment in their numeracy programmes (see Table 2, p. 50), there is some correlation. The two pieces of equipment that were least used by the teachers – tens frames and play money – were also the two pieces of equipment chosen least often by the children. Nineteen children chose the number line, which was being used by all the teachers. It seems likely from the teachers' comments, however, that the number line is more commonly used for teaching number knowledge than strategies, so some children might not normally choose to use it to solve a subtraction problem such as the one they were given. The calculator proved very popular with the children, but noticeably less so with their teachers. The children's choice of bundled ice-block sticks was higher than might have been expected, considering that three of the four teachers were using them in various ways in their classrooms.

5.3.1 Gender

Table 5: The children's choices of equipment by gender

	1st choice		2nd choice		3rd choice		4th choice	
	Boys	Girls	Boy	Girls	Boys	Girls	Boys	Girls
Number line	-	1	4	4	5	4	1	-
Tens frames	-	1	-	-	2	2	-	-
Bundled ice-block sticks	5	9	7	3	2	3	1	-
Place value blocks	-	1	3	2	3	1	-	-
Play money	1	-	-	1	1	2	1	-
Calculator	11	5	3	6	1	2	-	1

The calculator was the first choice of eleven boys and five girls for solving the problem. The same number of girls and boys used the bundled ice-block sticks as their first or second choices of equipment. Otherwise, there were only slight variations in the numbers of girls and boys who chose to use each piece of equipment, as is shown in Table 5.

In contrast to Ritchie's (1991) work, there was little difference related to girls' and boys' choice of place value blocks. Similar to Ritchie's (1991) findings is the:

> ...evenness of the samples preferences across all the types of equipment presented (p.89).

5.3.2 Strategy stage

There are some clear distinctions that can be made on the basis of children's strategy stages and their choices of equipment (see Table 6). The part-whole children seemed to have a stronger preference for the more structured equipment than did the counting children. Only Stage 5 children chose to use the play money to solve the problem. More part-whole children than counting children chose to use the place value blocks. A disproportionate number of children who showed part-whole thinking chose the bundled ice-block sticks as their first choice, even when it is taken into account that there were almost twice as many part-whole children in the sample as counting children. These findings are similar to those of Ritchie (1991), although the particular pieces of equipment presented to the children differed.

The ways in which the equipment was actually used will be examined in Chapter 6.

Table 6: The children's choices of equipment by strategy stage

	Counting				Part-Whole			
Choices	1st	2nd	3rd	4th	1st	2nd	3rd	4th
Number line	1	-	5	1	-	8	4	-
Tens frames	1	-	1	-	-	-	3	-
Bundled ice-block sticks	3	6	1	-	11	4	4	1
Place value blocks	1	2	-	-	-	3	4	-
Play money	-	-	-	-	1	1	3	1
Calculator	6	3	1	-	10	6	2	1

On 13 occasions children elected to use a piece of equipment which they said they had not seen. Seven of these instances occurred with the bundled ice-block sticks. The degree of success that the children had using equipment with which they were unfamiliar is discussed later (see 6.2, p. 78).

5.4 Summary

The part-whole children tended to choose the pieces of equipment which are structured in groups of ten – bundled ice-block sticks, place value blocks and play money. However, a significant number of children indicated that they had not had experiences that used some of the more structured equipment included here – tens frames, place value blocks and play money – which could be used to help develop their concepts of grouping and place value. The impact this had on the children's attempts to use this equipment to solve a problem is presented in section 6.2 (see p. 78).

The children's comments relating to their familiarity with the equipment were not always consistent with the teachers' comments about which equipment they had chosen to include in their numeracy programmes. Some children may have confused tens frames with pattern boards or even bingo cards. Some of the teachers were also uncertain about the differences. The tens frames were likely to have been a new piece of equipment for most teachers and it seems likely that their potential for developing children's understanding of five- and ten-based groupings within numbers may not yet have been understood by some teachers.

More than half of the children had had experiences with play money, often in the context of games played at home or at school, but three of the teachers did not show an awareness of how money could be used to help children develop place value understanding. The calculator was familiar to all the children, but only two teachers actually included calculators in their numeracy programmes – one of these used a problem-solving approach while the other used calculators in a procedural manner. Although the curriculum statement for mathematics includes statements about the inclusion of calculators for exploring the number system and number patterns, as well as being a tool for problem solving, three of the teachers described calculators only as computational tools. Two of these teachers maintained that children should not need to use calculators.

The reasons behind the children's equipment choices will be presented after consideration is given to what actually happened when the children used the equipment to solve the given problem.

CHAPTER 6

How the children used the equipment

6.1 What the children did with the equipment

There was a wide range of thinking reflected in the children's equipment use. This did not always match the type of thinking that would have been consistent with their strategy stages as they had been gauged at the beginning of the interviews.

Some children were quite accepting of different solutions yielded by using different equipment to solve the same problem. Other children seemed aware that their answer should be the same each time they solved the same problem, regardless of the equipment they chose to use. The concern that this second group of children had with consistency appeared to influence their manipulation of the equipment.

6.1.1 The number line

Of the 19 children who chose to use the number line, 18 used a count-by-ones strategy to solve the problem, regardless of their strategy stages. This supports Gravemeijer's (1994) observation that the number line supports the use of counting-based strategies.

Seven counting children successfully used the number line to solve a problem involving two-digit numbers when they had earlier been unwilling to attempt similar problems when asked to solve them without the support of equipment. Each of these children counted back in ones from 41, with 40 being one, 39 being two, 38 being three, and so on until they arrived at 23 being the 18 count. The process of children's tracking of their counting is described in more detail by Wright et al. (2002, p.173-174). Earlier in the interviews, several of these children had stumbled when they counted back to solve 36 – 8 = ? For example, one child who had counted back in her head to get an answer of 27, verbalised her counting: "35, 34, 33, 32, 31, 29, 28, 27."

Nine part-whole children also used the number line to count back 18 in ones from 41. These children had earlier attempted to use standard partitioning to solve problems mentally, and had been reasonably successful with addition problems. For instance, some children had successfully used partitioning to solve the problem 43 + 28 = ? They had described their thinking as four (tens) and two (tens) is six (tens), and three and eight is 11, so it's 71 altogether. When some of these children tried to apply similar thinking to the subtraction problem 54 – 27 = ? all but one arrived at an incorrect

answer. While dealing with 50 take away 20 was straightforward, they came unstuck when dealing with the ones. One or two used a difference approach[10], taking seven minus four and getting three, giving a final answer of 23. Another child arrived at the same answer by another means; they started with 50 take away 20, giving 30. They then subtracted seven from 30, resulting in 23. Even with prompting, this child was sure they needed to do nothing with the four from 54.

With the support of the number line these children geared down to a less sophisticated strategy, consistent with counting-based thinking. Using a count back in ones strategy, they all got the correct answer.

One boy had used a mental strategy that could readily have been illustrated on the number line. Alan solved the problem 54 – 27 = ? by thinking of it as 54 – 4 = 50, 50 – 3 = 47, 47 – 20 = 27. When faced with the problem 41 – 18 = ? he did not, however, use the number line to model 41 – 1 = 40, 40 – 7 = 33, 33 – 10 = 23, which would have been consistent with his mental strategy. Instead, he counted back in ones.

In his discussion of children's development of increasing levels of abstraction, Baroody (1990) suggested that children need to make connections between various representations of the same concept. It seems likely that these children may not yet have connected the way they can mentally manipulate numbers to the number line model, ie they did not translate their part-whole thinking to the number line.

The particular number line used in the interviews was similar to those used in the children's classrooms, and showed every number from zero to one hundred, with the decade numbers in a larger font. The manner in which the children used the number line supports Gravemeijer's (1994) observation that children are more likely to revert to counting strategies when they are presented with a number line such as this one. Had the number line shown only the decade numbers, the children might have used other strategies. Another issue was that the partitioning strategies used elsewhere by some part-whole children cannot easily be shown on the number line, so their choice of the number line may have in turn restricted their choices of strategy.

Three part-whole children from Faraway School used different strategies to solve the problem with the number line. Eliza counted the numbers from 41 back to 18. Another girl, Carly, counted the numbers from 18 to 41 to solve the problem. Only one child,

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[10] Using a "difference approach" refers to the common error of subtracting the top number from the bottom number when the top number is smaller. It is also referred to as the "smaller from larger" bug, identified by Brown and Van Lehn (1982).

Pearl, counted back by ten to get to 31, then counted back eight more by ones to get to an answer of 23.

6.1.2 Tens frames

None of the five children who chose to use the tens frames demonstrated any part-whole thinking when they used them, although three children had used part-whole strategies when working mentally. While some of them used the empty tens frames as groups of ten in order to quickly assemble a model of 41, they lost their way during the subtraction process.

Two children who used counting strategies elsewhere chose the bag of tens frames and counters to solve the problem. However, one of those children used only the counters; she did not use the actual tens frames. The other child used five frames, filling four with counters, then putting one counter in the corner square of the fifth frame. Starting with this last counter, she counted off 18 counters, before glancing at the arrangement of counters left on three frames and announcing the total.

The other three children – all part-whole thinkers – who chose to use the tens frames lost track of where they were up to with subtracting 18, and gave up. For two of these children, solving the problem was made more difficult by their using the tens frames without counters. Nathan made 41 with four empty tens frames and one counter on the table. Wendy spread out five frames, with no counters, and said “And we’ll use that one there”, pointing to a corner square on the fifth frame. When she stopped part way through subtracting by counting back in ones, looking unsure of herself, she was neither able to go back and work out where she might have gone wrong, nor move forward to finish the subtraction. Wendy was unwilling to start again, and packed the equipment away.

The tens frames appeared to be problematic for all the children who used them, regardless of their strategy stages. Because such a small number of children chose to use the tens frames, it is not possible to note any other trends in their use.

6.1.3 Bundled ice-block sticks

Thirty children used the bundled ice-block sticks, making it the most popular choice of equipment. The children used the bundled ice-block sticks in a greater variety of ways than they used any other piece of equipment, showing a range of thinking. Twenty-one of the thirty children who chose to use the bundled ice-block sticks began by making 41

with four bundles of ten sticks and one loose stick. Before looking at how these children proceeded, the other children's methods will be outlined.

The other nine children used count-by-ones strategies when they manipulated the ice-block sticks. Four children undid bundles, one by one, and counted the sticks by ones until they reached 41 sticks. One child left the sticks in their bundles, but counted them by ones.

Vinnie did not make 41; he began by getting 18 sticks.

Interviewer:	And what were you doing with the eighteen to work out the answer, Vinnie?
Vinnie:	Minusing them.
Interviewer:	So you pointed to each one and did you go forty, thirty-nine, thirty-eight… were you counting backwards on them, or what were you doing?
Vinnie:	No, I was counting backwards and these ones went first (pointed to the sticks on his left), these ones went last (pointed to the sticks on his right).
Interviewer:	OK, so I started at the wrong side – you started over here.

Although several children had demonstrated this method in the pilot study, Vinnie was the only child to use this method during the study itself.

Another three children, neither of whom actually solved the problem, began by making 41 (or, in one case, 40) and 18. Each of these part-whole children needed much prompting to begin to see that they needed to take the 18 from the collection of 41, rather than introduce additional ice-block sticks. These children had not shown this difficulty when working mentally to solve subtraction problems at the start of the interview.

Of the twenty-one children who made 41 with four bundles of ten sticks and one loose stick, there was much variation in the ways the children employed the bundled ice-block sticks from this point. The most commonly used method (13 children) was to remove one bundle of ten sticks, then the single stick, then seven sticks from a bundle that they had broken into ten ones. The children then counted the remaining collection of sticks, usually counting "ten, twenty, twenty-one, twenty-two, twenty-three" or "ten twenty, twenty-three", to find the solution. One child actually swapped a bundle of ten

sticks for ten loose sticks from the container to allow the subtraction, rather than removing the rubber band from the bundle.

Three of the children that chose to use the bundled sticks this way were unsuccessful in their attempts. With one child, this was due to miscounting the sticks they took away; the other two children were unable to ungroup a bundle of ten sticks into ten loose sticks in order to allow the subtraction.

The second most common method was for children to remove the rubber bands from all four bundles, removing 18 sticks from the total, then counting how many sticks remained. Five children tackled the problem like this, four of them successfully. Undoing the bundles like this gave the children ungrouped materials to work with.

Two children got to the point of having removed 18 from the original 41 sticks, then used these 18 sticks to keep track of counting back 18 from 41 (rather than count the remaining sticks).

Probably the most elegant solution, used by two part-whole thinkers, was to remove two intact bundles of ten and then add to the remaining 21 sticks, two loose sticks from the container. Nick confidently used this strategy of rounding and compensating with the play money, the bundled ice-block sticks, then the place value blocks. This was inconsistent with the mental strategies he described at the start of the interview. He had moved fluently up and down through tens to add and subtract small numbers, e.g., 36 – 8 was dealt with as 36 – 6 = 30 and 30 – 2 = 28. However, when presented with two two-digit numbers to add – in this instance 43 and 28 – Nick had used neither the rounding and compensating strategy that he demonstrated with equipment, nor the standard partitioning strategy that 16 of the 21 part-whole thinkers used. Instead, he had given a description of algorithmic thinking:

> Eight plus three equals eleven, took one over to the two, and then I had one left over. And then I added the four and the three together, which is seventy, and here I put that one.

While he had applied the procedure correctly in this case, he was not successful with the subtraction problem 54 – 27. Nick was unsure how he had worked out his answer of 23, but it seems likely that he misapplied the algorithm and took a difference approach (see footnote p.69) when thinking about the ones column. So in Nick's case,

using equipment seemed to support the use of a more sophisticated and reliable strategy.

Largely due to the semi-structured nature of the bundled ice-block sticks – where children could take advantage of the grouping in ten or could instead break the groups in order to work with a collection of discrete objects – the way the children manipulated the sticks could be said to provide a model of how they were thinking about the problem. The children used the bundled ice-block sticks as 'thinking' models. That there was such a divergence in the way the children used the bundled ice-block sticks might indicate that their classroom use had been fairly open-ended. It might also be due to their having had limited experience with this equipment, from the teachers' comments.

6.1.4 Place value blocks

Ten children used the place value blocks – three counting children and seven part-whole children. The place value blocks yielded the lowest rate of success of the six pieces of equipment. None of the children seemed familiar with the idea that a long could be traded for ten units in order for them to remove part of a group of ten.

Just one successful strategy was modelled with the place value blocks. Two part-whole thinkers made 41 with four longs and one unit block, then used a rounding and compensating strategy, removing two longs and returning two unit blocks from the bag. (They both did the same when they used the bundled ice-block sticks, and the play money.)

Of the other eight children, six made 41 with four longs and one unit block, but reached an impasse when they tried to remove 18 from this combination. Harry's response was typical:

> Um… (moved one long and the unit block to one side, then stopped.) …Oh, this is a hard one.

Ivan had counted in tens as he had unpacked some of the longs from the bag. He was perplexed by the subtraction situation:

> You don't count eighteen when you go in tens… hmm…

These attempts were abandoned. Ronnie tried to overcome the trading challenge:

> Let's see... so I'll take... (Removes one of the longs from 41). So, there's a whole block... (Then uses the unit block he has just picked up to count along another long to eight. Looks unhappy with this.)

He did eventually swap a long for ones, but ended up at 22 as his answer.

Uma counted out 41 unit blocks to begin solving the problem. She reached the same wrong answer as Ronnie, but in her case this was because her pronunciation of number words was not always synchronised with her rhythmic moving of the blocks as she counted.

The two children who used the place value blocks successfully did not use them in the manner that parallels the steps of the written algorithm. Instead, the way they used the equipment probably matches their mental strategy, i.e., they made the problem easier by rounding 18 up to 20, and later adjusting by two. This was the only strategy that reflects part-whole thinking. The other children did not yet seem to have a key place value understanding – that one ten is the same as ten ones.

6.1.5 Play money

None of the children who used counting strategies chose the play money. Of the six part-whole children who chose to use the play money, just two children who said they had not seen play money before succeeded in solving the problem. They began by making 41 with four $10 notes and one $1 note. They each proceeded differently from this point.

Amber took away a $10 note, then swapped another $10 for ten $1 notes to allow the subtraction of $18.

Nick removed two $10 notes then returned two $1 notes – the same strategy that he used with the bundled ice-block sticks and the place value blocks.

Four children failed to solve the problem by actually using the play money to work through the subtraction process. Two children who had $23 when they had finished, essentially changed the original $41 into $23 without actually going through the subtraction process. These children had already solved the problem using at least two

other pieces of equipment and probably understood that the solutions gained with various pieces of equipment should be consistent.

Lisa also made $41 initially, but then she made $18 as well, just as she did with the bundled ice-block sticks. As with the ice-block sticks, this attempt was abandoned.

Ronnie's approach was different again. The play money was his third choice of equipment and one that he had not used before. Having just tried unsuccessfully to use the place value blocks and getting muddled during the trading process, he began with the play money by counting out all the $1 notes, probably to try and avoid repeating his recent experience. (There had deliberately been fewer than 18 $1 notes included, so that the children might be forced to use non-counting strategies.)

Interviewer:	How many one dollars are there?
Ronnie:	Sixteen. But I can still... but I have to use my brain there. I may as well just have to use two of these, OK? (Picked up two unit blocks from the bag of place value blocks, and added these to the 16 $1 notes.)
Interviewer:	OK. Now what are you going to do? You've got eighteen there, what are you going to do?
Ronnie:	(Gets three $10 notes and puts these in a separate pile on the table.)
Interviewer:	What have you got here, Ronnie? What's this pile of $10 notes?
Ronnie:	Yeah.
Interviewer:	So there's thirty here. So how much have you got there? (Indicated the pile of 18 items.)
Ronnie:	Eighteen.
Interviewer:	You've got eighteen over there and thirty here, so how many have you got altogether?
Ronnie:	(Pauses while he thinks.) Forty-eight.
Interviewer:	Is that helpful?
Ronnie:	No, I don't think so.

Although Ronnie tried to recover the situation, he eventually said that he did not know what to do, and the play money was put away.

Three part-whole children were unable to solve the problem using play money. The effect of using money to stand for other objects – in this case, seagulls – may have contributed to the difficulty of the task (Hiebert & Carpenter, 1992).

6.1.6 The calculator

Twenty-nine children chose to use the calculator. For 16 children it was their first choice of equipment. The children needed to follow a procedure in order to solve the problem with the calculator. For the children who knew how to use the correct procedure, it could be said there was no thinking work needed. Twenty of the 29 children who chose to use the calculator correctly entered:

On/c, 4, 1, –, 1, 8, =.

These children quickly arrived at 23 and were confident that it was right. A further six children did this after making incorrect entries, then pressing 'on/c' to start again. These children realised they had pressed an incorrect key by watching the display and noticing an error, either during the computation process or when they reached an answer with which they were unhappy. These children self-corrected their errors with a minimum of prompting.

Three children did not arrive at 23 as their answer to 41 – 18 =?
Liam was one:

Liam:	(Enters 4, 1, M-, 1, 8, =. Display shows 18 and M in the top left corner.)
Interviewer:	Did that work?
Liam:	It just says eighteen!

He recognised that his answer was incorrect and appealed for help. Liam was able to solve the problem only when he was shown the location of the appropriate minus key. His difficulty, then, seemed to be due to being unfamiliar with the calculator presented.

Olive was unsure about the answer she got.

Olive:	(Enters 4, 1, – . Then touches the 1 key too lightly for it to register. Carries on without realising this, pressing 8, =.) Thirty-three.
Interviewer:	Does that sound right to you?

Olive: I don't know.

She was beginning to put the calculator away at this point, and needed prompting to try it again, eventually arriving at 23.

Emma was also quite uncertain about what might have been a sensible answer:

Emma: Fourteen and –
Interviewer: Forty-one seagulls and eighteen flew away.
Emma: So forty-one take away eighteen.
Interviewer: OK.
Emma: (Enters 4, 1, –, 1, =.) Forty.
Interviewer: You think that's right?
Emma: (Looks unsure.)

And when prompted to check her answer:

Emma: (Enters on/c, 4, 1.)
Interviewer: That's better… forty-one…
Emma: Take away… (Enters +, 1, 8, =.) Fifty-nine.
Interviewer: Fifty-nine. Is that right? Forty-one take away eighteen – it could be fifty-nine, could it? It could be bigger, could it?
Emma: No…

Olive and Emma both seemed unable to estimate the solution, so could not be confident that the number in the calculator's display was the correct answer. Liebeck (1984) stressed that children need to be able to estimate in order to identify a reasonable answer on a calculator. Liam, on the other hand, would probably learn very quickly to use the calculator effectively for subtraction; in other parts of the interview, he had demonstrated a greater facility with numbers than had the other two children.

A few children talked about negative, decimal, or very large numbers while they were using the calculator, which supports Stacey and Groves' (1996) findings that calculators can help children to develop some number concepts at an earlier age than usual. This interchange with Eddie illustrates the point:

Eddie: And, and I always, like, do big numbers then the number after it, and then I know it's negative something.

Interviewer: Wow! So you know about negative numbers as well?

Eddie: And it goes, if I go, like, five hundred minus two thousand, it's like negative two thousand or negative one thousand or something.

Interviewer: So, what does negative mean, do you think?

Eddie: Negative is lower than zero, I know.

The high number of children who chose to use the calculator, and also their positive attitudes about its use, clearly show the calculator was popular with the children, as was also noted by McIntosh (1996).

6.2 The children's success with the chosen equipment

Of a total of ninety-nine choices of equipment, just over two-thirds (67) resulted in children successfully solving the problem. The overall success rate generally declined as children chose more structured equipment. Almost one-third of the children's attempts – with the equipment of their choosing – failed.

Relative success rates overall with each piece of equipment are shown in Figure 4. Children met with the lowest success rate when they chose the place value blocks; just two of the ten attempts to use the place value blocks to solve the problem were successful. The highest success rate, perhaps not surprisingly, was with calculator use (26 of 29 attempts).

Ten of the thirty-two unsuccessful attempts were made with children's first choices of equipment. In the thirteen cases where children chose to use equipment they had earlier said they had not seen, six of their attempts were successful.

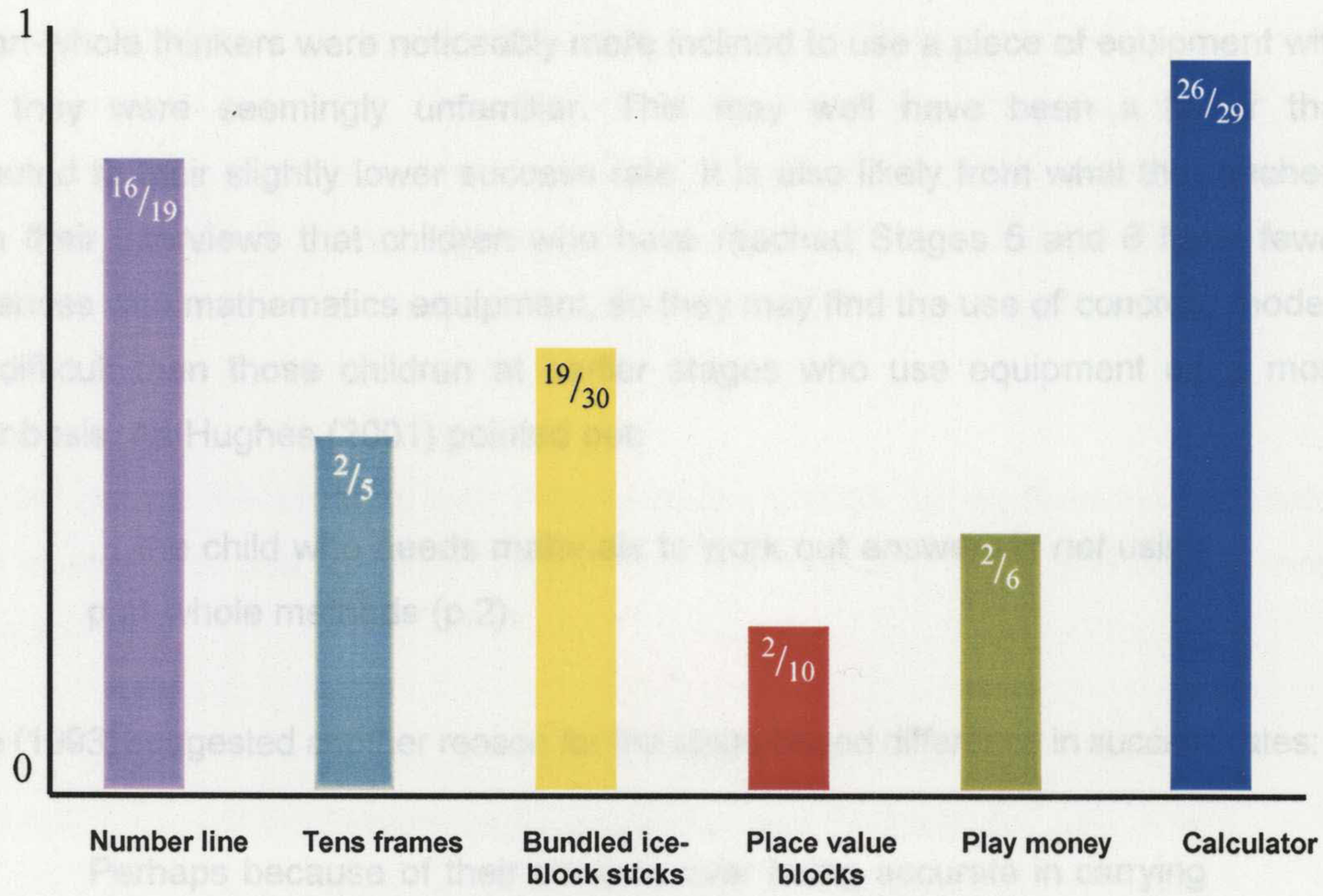


Figure 4
The children's success rates with equipment

NB. Each fraction represents the number of successful attempts out of the total number of attempts with that equipment.

6.2.1 Gender

The boys' overall success rate was very slightly lower than that of the girls; the boys solved the problem in 34 of 51 attempts and the girls succeeded in 34 of 48 attempts.

The boys made more choices of unseen equipment than the girls in this study – eight and five respectively. Both genders made three errors with unseen equipment. The boys succeeded on five occasions using unseen equipment, compared to twice for the girls.

A larger sample would be needed in order for any significance to be attributed to this information.

6.2.2 Strategy stage

The children who used counting strategies made 24 successful attempts out of a total of 32. Of these attempts, only one involved the choice of unseen equipment.

The children who used part-whole strategies were successful with 45 of their 67 attempts. Unseen equipment was chosen on twelve of these 67 occasions.

The part-whole thinkers were noticeably more inclined to use a piece of equipment with which they were seemingly unfamiliar. This may well have been a factor that contributed to their slightly lower success rate. It is also likely from what the teachers said in their interviews that children who have reached Stages 5 and 6 have fewer experiences with mathematics equipment, so they may find the use of concrete models more difficult than those children at earlier stages who use equipment on a more regular basis. As Hughes (2001) pointed out:

> ... the child who needs materials to work out answers is *not* using part-whole methods (p.2).

Ritchie (1993) suggested another reason for the stage-based difference in success rates:

> Perhaps because of their concern over being accurate in carrying out a procedure, the children doing poorly at mathematics were actually better at using mathematics equipment than those doing well at mathematics (p.2).

6.2.3 Consistency of answers

The children were given opportunities to solve the same problem with several different manipulatives. Some children seemed to treat each attempt as a new task, and were not concerned if they arrived at different answers with different pieces of equipment. For other children, the need for consistency of answers, regardless of which pieces of equipment were used, seemed to distract them from what might otherwise have been the effective use of the equipment. Many of these children made 41 to begin with, then essentially swapped the equipment that showed 41 for equipment that showed 23, which they knew to be the answer. In these cases, the children did not actually use the equipment to help them to solve the problem. For example, Louie used the calculator to quickly solve the problem, then chose to use the bundled ice-block sticks.

> Louie: (Gets four bundles of ten and one loose stick.)
> Interviewer: And very speedily there you've got forty-one – four bundles and one loose one. Now, what happens if you're taking away eighteen?
> Louie: Um, then you, like, do ... um, wait. We've got ... I forgot what the answer was.

Interviewer: Ah, but I want you to work it out – show me how you can work it out with your ice-block sticks.

Louie: OK. Twenty-two. (Having said that, he takes two bundles of ten and the loose stick from 41, and gets another loose stick out of the box to make 22, which he seems to think is the answer he got on the calculator.)

Abby had solved the problem with the bundled ice-block sticks, then the calculator, before choosing to use the play money. She has four $10 notes and one $1 note, making $41.

Abby: Wait. So, let's see. One, two, three, four, five, six, seven, eight (picks up one $1 from the $41, then gets another seven $1 notes from the pile to make eight ones. Takes a $10 note from the $41 and puts it with the $8 to make $18. At this stage, she has three $10 notes left of the original $41. One of these she puts back into the pile, swapping it for three $1 notes. This gives $23.). Hey now, let's … twenty… three (with nervous giggling).

Interviewer: Now what happened there? You counted out twenty-three because you know twenty-three's the answer. How much did you actually take away?

Abby: Ah… must've taken away eighteen dollars.

Adam had worked out the answer with the calculator, and used this to diagnose, and later self-correct, his mistake with the bundled ice-block sticks.

> Twenty-four. I think I might've put one more in than I was supposed to, or I mis-replaced one, or I did one twice.

Ritchie (1991) found that:

> The tendency towards being consistent was more evident in children doing well in mathematics (p.95).

The seven children in this study who had an obvious concern that their answers should be consistent were a mix of gender (five boys and two girls) and strategy stage (four counting, three part-whole). That is not to say that other children were not using previous answers to cross-check the solutions they reached with various pieces of equipment. For the other children, however, the need to make consistent answers did not override their using the equipment to show their thinking through the subtraction process.

6.3 Summary

Overall, the children's choices of equipment were not strongly consistent with the equipment they used in their classrooms, chosen for inclusion by the teachers. However, their low level of use of play money and tens frames matched the level of inclusion by the teachers. If the children's equipment choices did not seem to be closely linked to the equipment presented to them by the teachers, then it might have been expected that their choices would parallel the equipment with which the children had said they were familiar. Yet only in the case of the calculator and the bundled ice-block sticks did the children's selections resemble their stated familiarity with the equipment.

Almost all the children who used the number line used a count-back-in-ones method to solve the problem, regardless of their strategy stages. Being presented with a number line that showed all the numbers seemed to provide the children with a cue for which method they should use. For those children who had described more advanced thinking than they demonstrated with this number line, this particular piece of equipment is unlikely to be useful in extending their number understandings, so is largely redundant.

Children used the bundled ice-block sticks in a multitude of ways. While it might be argued that such diversity of use is a strength of this equipment, it is perhaps indicative that children need some guidance from teachers about how to use bundled ice-block sticks in ways that support their progression towards part-whole thinking.

The higher error rate of the part-whole children may also be due to the challenge posed by the manipulation of equipment. As children's thinking becomes more abstract, it seems to also become more difficult for them to return to using equipment in ways that mirror their mental strategies (Hart et al., 1989).

An implication for teachers is that they need to consider the effect on children's learning of the prolonged use of a particular piece of equipment for whole-class activities. To do so may help the teacher to develop confidence in how to use what may be a new piece of equipment, but this may be at the expense of some children's learning. While this number line may be of value for some children, it may be necessary to also include work with, say, a number line with only the decade numbers shown or an empty number line, in order to cater for the needs of children with more advanced thinking. Teachers also need to use group teaching situations to introduce children at different strategy stages with equipment that specifically targets their learning needs. This might mean that during one lesson, the teacher might work with various groups using number lines that range from the type used in the interviews, to a number line with decade numbers marked, to an empty number line model where it is the children's role to mark the numbers they need.

CHAPTER 7

Why the Children Chose the Equipment

7.1 The children's reasons for their choices of equipment

The children justified their equipment choices by describing how various pieces of equipment were reliable, fast, easy (or hard) to use, provided visual support, were grouped in a helpful way, or supported a counting strategy. Only one child mentioned the surface features of equipment - such as colour, size and texture - as a factor that influenced his choice.

The range of comments from children at the different stages was similar; for example, children across the stages included references to the number line as being easy for them to use, providing visual support, and lending itself to counting forwards and backwards. There were no noticeable differences between the responses given by girls and those given by boys.

7.1.1 Surface features

Nick, the single child who included the physical attributes of equipment, explained why he chose to use the bundled ice-block sticks:

> Um, it's just all the colours that I liked, so I chose them.

He talked about why he used the place value blocks:

> Nick: Because they're sort of, like, fun. Like ... I don't really know, but they're just fun to use.
>
> Interviewer: Yeah? Because when I brought them out, you sort of went "Yay!" I could hear you thinking, "Oh yeah! Place value blocks!"
>
> Nick: (Laughs.)
>
> Interviewer: Is that because you like doing maths with them, or building with them, or what?
>
> Nick: Yeah, I like building castles with them.

This child's preference for the place value blocks was influenced by his enjoyment when he had previously used them in a context other than number. The element of fun

also influenced the teachers' decisions about equipment, perhaps a legacy of a child-centred approach to education (Higgins, 1998).

7.1.2 Fast and/or reliable

Seven children talked about the calculator being a fast way to solve problems. Thirteen explained that it is reliable, with statements such as:

> Because it will probably have the right answer. (Amelia)
>
> Um, because you don't have to work it out and it gives you the right answer. (Teresa)
>
> Well, calculators are for finding out answers, aren't they? (Abby)

Most of the children were aware that for the calculator to display the correct answer, they had to have keyed in the problem correctly.

It could be argued that the reasons the children gave for choosing to use the calculator might also be the reasons for two of the teachers choosing not to include it: because it quickly gave children a reliable answer as the result of following a procedure.

7.1.3 Easy

Half of the children said that the calculator was an easy means of finding the solution. The children's comments were supported by their results when they used the calculator. The children's success rate with the calculator was the highest for any piece of equipment, with 26 of 29 attempts proving successful. Three children found the number line an easy piece of equipment to use. Again, the children's claims are matched by a high degree of success when they chose to use the number line; 16 out of 19 attempts met with success. The calculator and the number line had the two highest success rates of all the equipment used, so the children were justified in stating that they were (relatively) easy to use.

Two children said that they had expected the place value blocks to make answering the problem easy; both of these children, however, had been unable to solve the problem with the place value blocks. Wendy's response to a question about why she thought the place value blocks would be good to use was:

> Because I didn't think of taking away eighteen, but I thought it'd be easier.

7.1.4 Hard

While many children opted for equipment they perceived to be easy to use, a few chose to challenge themselves. The bundled ice-block sticks were deliberately selected by three children because they thought they would be hard to use:

> They're too compul … cated. (Paul)

Like one other child who said he had chosen them because they were hard, earlier in the interview Paul had said he was unfamiliar with the bundled ice-block sticks and had not managed to solve the problem with them.

7.1.5 Visual support

Six children said that they chose to use the number line specifically because they could see the numbers:

> Because it's a line and you can actually see the numbers. (Marty)

> Because they have pictures of the numbers, it's got pictures of the numbers, it's easy for me. (Alan)

Being able to see each number would have helped children to solve the problem by counting in ones, so there was naturally some overlap between the idea of the number line offering visual support, and it supporting the strategy of counting in ones.

7.1.6 Supported counting in ones

Again, the number line was the most commonly-mentioned piece of equipment when children were talking about using counting in ones to solve the problem. Twelve children made comments similar to Ian's:

> Coz I could just count backwards on it.

One child differentiated between counting collections of items and counting back on the number line:

> Because you can count backwards instead of, um, like, using stuff to count. (Claudia)

Some children chose to use the bundled ice-block sticks because they found these helpful materials to use for counting in ones. Four children undid bundles of ten ice-block sticks to allow them to count each loose stick. Alan removed 18 sticks from his collection of 41 sticks, then used the 18 sticks to support counting back from 41 to find the answer. Alan had chosen to use the bundled ice-block sticks:

> Because they're easy to count backwards with.

7.1.7 Grouped in tens and ones

The bundles of ice-block sticks were the equipment that children referred to most often as being chosen because of the way in which they were grouped. Thirteen of the thirty children who chose to use this equipment found the grouping helpful.

Four children made similar comments in relation to the place value blocks, although only two children actually used these successfully. Two children thought the tens frames would be good to use because they represented groups of ten, and two children used the play money because of its grouping.

7.1.8 Other reasons

Other reasons given by the children for their choices included using the calculator because:

> I know how to use them. (Odette)

Reasons for selecting the play money included:

> Umm, I just like money. (Nick)
>
> Because you can divide things up. (Ronnie)

Odette justified her choice of the bundled ice-block sticks:

> Because we've done these on the board without the Popsicle sticks, and so I thought these would help me with take-aways as well. But they didn't.

Tina made this comment about why she chose to use the place value blocks:

> Because I could have used... I was going to use the... take away eighteen, but then I realised that you couldn't take away ten and then I only take away seven. But I thought that it would, like, work in a different way.

7.2 Additional equipment that the children liked

At the end of the interview the children were asked what other equipment they liked to use. Fingers and rulers – neither of which are normally very colourful or shiny – were the most frequently-mentioned alternatives, with nine children saying they liked to use each of these. Counters and rods were both named by four children as other equipment they liked to use to help them solve maths problems.

Both fingers and rulers, of course, have limited usefulness; Ivan showed an awareness of this when he remarked that he used his ruler:

> Sometimes, if it's not over thirty.

Each of the following pieces of equipment was named by one child as something else they liked to use: bottle tops, mosaic shapes, and a times table chart. Two children said that the hundreds charts in the backs of their maths books were helpful. One child, Tina, liked to get a piece of paper and draw the problem.

Children's perceived preferences for equipment seemed to influence teachers' choices in some cases:

> My guys love, when we're adding big numbers like tens or nines or elevens, they like the hundreds board – that flip board. (Tracy)

7.3 Summary

The Year 3 children and teachers interviewed in this study used different criteria when deciding which equipment to use; the children valued equipment that would help them to solve a problem, while the teachers felt it was equally important to present children with equipment that was attractive and enjoyable to use, as equipment that had significance for children's concept development. This disparity of children's and teachers' perceptions may have contributed to the children's problem-solving success rate decreasing as they chose to use more structured equipment to solve the given problem.

As children develop part-whole thinking, they appear to be less inclined to use equipment to solve problems, and to have less success when they do. This is in line with Hart et al.'s (1989) observations that once children are able to work abstractly, it is very difficult for them to model problem solving with concrete materials. Children who have part-whole strategies for solving problems mentally are likely to prefer mental methods more than their counting peers (Swan & Bana, 1998), who still need the scaffolding that equipment provides. It seems that as children's strategies advance, and they are thinking more flexibly about making and breaking numbers, the transferral of this thinking to concrete materials becomes a more complex task. Hughes (2001) comments on the role of equipment for part-whole thinkers:

> What characterises part/whole thinking is the ability of a child to regard numbers as abstractions and to use number properties to compute answers. That is to say the child who needs materials to work out answers is *not* using part/whole methods (p.6).

It must be remembered that the role of equipment is a temporary one; it is not the goal of numeracy teaching that all children should become proficient users of equipment. Rather, the equipment should be employed to introduce children to new concepts on their journey to part-whole thinking. Teachers need to make informed professional judgements about when the equipment should be withdrawn, and when children should be expected to use either imaging of the equipment or generalised number properties to apply a particular strategy.

CHAPTER 8

Conclusions

8.1 Using equipment to support the development of part-whole thinking

Around the world there is a shift in emphasis in numeracy teaching and learning from developing children's procedural knowledge of standard algorithms, to their developing part-whole thinking. In accordance with this, there needs to be an accompanying shift in the way teachers use equipment, from using it to support the teaching of algorithms to using equipment to support the development of children's part-whole thinking, or more specifically, a repertoire of mental strategies for solving number problems. It is for this reason that the use of equipment, once focussed on a collection of 'working' models, should now move to including more 'thinking' models. This does not necessarily demand the invention of new representations; it is through the ways in which equipment is used by teachers to support children's thinking that this can be achieved.

It is against this backdrop that we now turn to a consideration of the findings of this research in relation to the original aims.

8.2 Which manipulatives did the children find the most helpful as supports for solving number problems?

The two pieces of equipment most favoured by the children were the bundled ice-block sticks and the calculator. There was much diversity in the ways the children used the bundled ice-block sticks to solve the given problem. Because the children could take advantage of the grouping in tens, or could instead break the groups in order to work with a collection of discrete objects, this equipment provided a 'thinking' model. The only other equipment that two part-whole children used as a 'thinking' model was the place value blocks.

Many of the children knew that the calculator would quickly give them an accurate answer to the problem if they keyed it in correctly. While they seemed motivated to use the calculator, it was often not included in their classroom numeracy programme. Overall, the children's equipment choices were not strongly consistent with the equipment that was used in their classrooms for strategy development, as reported by the teachers.

8.3 What were the connections, if any, between the children's strategy stages and their preferred equipment?

A wide range of thinking was reflected in the ways the children used the equipment. This did not always reflect the type of thinking that would have been consistent with their strategy stages as assessed at the beginning of the interviews. Only a small portion of the children's number understanding was gauged at the start of the interview – their mental strategies for addition and subtraction. Their grasp of place value concepts was not assessed in this interview. However, the way some children used the bundled ice-block sticks, the place value blocks and the play money suggests that their understanding of grouping and place value may be out of kilter with their mental strategies. At Stage 4 on The Number Framework for strategies, it states that:

> The student sees 10 as a completed count composed of 10 ones (Ministry of Education, 2002a).

While there is no specific mention of decomposing numbers on the framework, it would be consistent for children operating at Stage 4 to be exploring this idea in the context of subtraction, initially with the support of materials. Certainly by the time children reach Stage 5, this concept needs to be secure.

Overall, when the children independently solved the problem with equipment, they used methods consistent with, or less sophisticated than, their mental strategies. The children who relied on counting-based strategies to mentally solve problems, consistently used equipment to count on or back, although with the support of concrete materials like the number line they were able to work successfully with two-digit numbers. The difference was that they could keep a visual track of where they were up to when they used the number line. This highlights the crucial role of the teacher in guiding children to use equipment in ways that support their development of part-whole thinking. From the evidence presented here, it seems highly unlikely that children's strategies can develop through their working independently with equipment.

The children who used part-whole mental strategies often used a less sophisticated strategy when working with the equipment. The most obvious examples of this were those who used the number line to count back in ones. Whether these children might have used a part-whole strategy if they had been presented with a number line with only the decade numbers marked, or an empty number line, is a question for further investigation. On other occasions, part-whole children used the bundled ice-block

sticks to count back to reach a solution. Their use of equipment may have been related to one teacher's comment that as children's thinking becomes more sophisticated, their need to use equipment diminishes. This might mean that the part-whole thinkers were less likely to be regular users of equipment than those children who used counting strategies.

8.4 Which equipment did the teachers choose to include in their numeracy programmes?

The teachers affirmed that their use of equipment had increased as a result of their engagement with the NDP professional development. There is a close parallel in the equipment that the teachers chose to include in their programmes and children's success rates in using the various manipulatives; the less structured materials seemed to be used more by the teachers, and it was with the less structured equipment that the children enjoyed the greatest success in solving a problem. The bundled ice-block sticks and the number line were the only two pieces of equipment that the teachers included in their programmes that they talked about in terms relating to the development of children's part-whole thinking.

The equipment being used by the teachers did not consistently match the use of equipment such as tens frames and play money that is advocated in the NDP draft teachers' materials. Equipment was more typically being used to support the teaching of number knowledge than strategies. While only a very small number of teachers were interviewed for this study, the findings suggest that some teachers do not yet understand the importance of including the more structured pieces of equipment in their strategy teaching. The consequences of excluding such manipulatives may be to delay children's development of part-whole thinking.

8.5 What were the teachers' reasons for including or excluding particular equipment? What were the reasons underlying the children's equipment choices?

Almost all of the reasons children gave for their choices of equipment related to the ways that the equipment helped them to solve problems. The children explained that particular pieces of equipment gave them visual support for counting, were fast and reliable, or were easy to use because they were grouped in tens and ones. A single child was influenced by the colour of equipment and by the possibility of having fun with a piece of equipment.

This trend does not match the point of view presented by the teachers for whom the surface features of equipment seemed equally important when selecting equipment as the consideration of the conceptual understandings they wanted children to develop. Other influences on the teachers' decisions included the accessibility of equipment, the ongoing impact of the curriculum document, and the teachers' familiarity with, and understanding of, the NDP draft teachers' materials.

If these children's teachers did not select equipment to include in their numeracy programme on the basis of the contribution it could make to children's part-whole thinking, then it is not surprising that some of the children had yet to develop this. It is possible that some teachers were not yet sure about how to use the more structured pieces of equipment in ways that were consistent with the development of children's part-whole thinking. This may develop with their continued implementation of the NDP, as they become less concerned with getting to know the new resources and equipment and so are better able to focus on children's learning progressions.

However, it seems likely that some teachers who have participated in the NDP need to further develop their pedagogical content knowledge. It is unlikely that such teachers can be identified from the current available information which is largely based on teachers' self-assessment. How these teachers might be identified and their needs addressed is clearly problematic, but needs to be considered in order for there to be a long-term, nationwide strength in the teaching of numeracy.

8.6 Was there a correlation between the children's success with their use of structured equipment and the teachers' equipment choices?

From the data presented here, there is an apparent relationship between the materials becoming more structured and a decline in the children's success in solving the problem. A critical understanding for the children who chose to use either the bundled ice-block sticks, the place value blocks, or the play money was that one ten unit was interchangeable with ten one units. From the way the children used these particular pieces of equipment, it seemed that only a small number had this understanding. Generally, when the children chose to use the more structured equipment, they were able to make the starting number, 41, using tens units and ones units. With a few exceptions, the children's understanding of combining tens and ones to compose a number was secure. It was when the problem could be solved by their decomposing a more structured unit of ten into ten ones that a number of the children's methods unravelled.

The children's declining success rates with the more structured equipment match a declining rate of this equipment's use in their teachers' numeracy programmes. This seems to be connected to the teachers' choices of equipment being influenced by factors other than a consideration of children's concept development.

The part-whole children had a higher error rate than the counting children had. This may be due to it being difficult for them to return to using manipulatives when they are developing confidence in working abstractly. It may also be the result of being provided with fewer equipment-based experiences than the counting children.

8.7 Support for teachers

It may be helpful for teachers who participate in the NDP professional development programme to be provided with a clear indication of the particular strengths and weaknesses of some key pieces of numeracy equipment, to help them to more readily choose equipment on the basis of the concepts it embodies or can help to develop. An overview of which equipment can be effectively used at which stage could also be helpful. Supposedly this would align with Table 9 (see Appendix 3) which summarises equipment use in the NDP draft teachers' materials. Even more useful might be a description of the difficulties children typically encounter with the key pieces of equipment, and some of the misconceptions they are known to construct. Once able to identify children's misconceptions, teachers would obviously also need to know how to resolve these in ways that are consistent with developing children's strategies towards the goal of part-whole thinking.

8.8 Limitations of the research

The scope of this research was small. It examined the use of a particular collection of numeracy equipment in three Year 3 classes. Four teachers were interviewed, with the main interest lying in the perspectives of the 34 children interviewed. All but two of the children were assessed at Stages 4 and 5 of the Number Framework. Most of the children were identified by their teachers as being of European ethnicity. All of the schools involved were high decile schools. Because of these limitations, it is not valid to generalise any findings beyond the scope of this research.

The number of teachers involved was too small for any generalisations to be made about the wider teaching population. Some of the findings from the teachers' interviews, however, might provide some useful starting points for further research.

8.9 Further research

Further research is needed, in the context of the NDP, to compare the effects on children's learning of using structured equipment, with the effects of not using it. The effects on children's part-whole thinking of the use of a variety of number line models could provide teachers with clear guidelines for their use.

The issue of how to identify teachers whose pedagogical content knowledge needs developing is yet to be tackled. Teachers' self-assessments are unlikely to reveal any gaps in their conceptual maps of numeracy. Research into teachers' effective use of equipment in helping children to construct part-whole strategies is also needed.

8.10 Concluding comments

In order for teachers in New Zealand to be able to confidently use equipment to help children in their development of part-whole thinking, they themselves need to have an understanding of which pieces of equipment best support the development of particular aspects of number knowledge and strategies. This also necessitates teachers having an understanding of the difficulties children are likely to encounter when using different manipulatives, and how best to resolve these.

The ultimate goal for teacher educators must be for all teachers to have a richly-connected conceptual map of numeracy in order for teachers to be able to effectively use equipment in ways that help children to construct their own meaningful connections as they learn about number. Rather than talking about equipment as "bright, shiny stuff", teachers' must have a clear focus on the role equipment can play in the development of children's part-whole thinking.

Appendix 1

The Number Framework – Strategies

Global Stage		Operational Domains: Addition and Subtraction	Multiplication and Division	Proportions and Ratios
Counting	Zero: Emergent	**Emergent** The student is unable to count a given set or form a set of up to ten objects.		
	One: One-to-one Counting	**One-to-one Counting** The student is able to count a set of objects but is unable to form sets of objects to solve simple addition and subtraction problems.	**One-to-one counting** The student is able to count a set of objects but is unable to form sets of objects to solve simple multiplication and division problems.	**Unequal sharing** The student is unable to share a region or set into two or more equal parts.
	Two: Counting from One on Materials	**Counting – from One** The student solves simple addition and subtraction problems by counting all the objects, e.g., 5 + 4 as 1, 2, 3, 4, 5, 6, 7, 8, 9. The student needs supporting **materials**, like fingers.	**Counting – from One** The student solves multiplication and division problems by counting one to one with the aid of **materials.**	**Equal Sharing** The student is able to share a region or set into two or more equal parts using materials.
	Three: Counting from One by Imaging	**Counting – from One** The student **images** all of the objects and counts them. The student does not see ten as a unit of any kind and solves multi-digit addition and subtraction problems by counting all the objects.	**Counting – from One** The student **images** the objects in simple multiplication and division problems, e.g., 4 × 2 as **1**, **2**, 3, **4**, 5, **6**, 7, **8**.	**Equal Sharing** The student is able to share a region or set into two or more equal parts by using materials or by **imaging** the materials for simple problems, e.g., $\frac{1}{2}$ of 8.
	Four: Advanced Counting	**Counting On** The student uses counting on or counting back to solve simple addition or subtraction tasks, e.g., 8 + 5 by 8, 9, 10, 11, 12, 13 or 52 – 4 as 52, 51, 50, 49, 48. Initially, the student needs supporting **materials** but later **images** the objects and counts them. The student sees 10 as a completed count composed of 10 ones. The student solves addition and subtraction tasks by incrementing in ones (38, 39, 40, ...), tens counts (13, 23, 33, ...), and/or a combination of tens and ones counts (27, 37, 47, 48, 49, 50, 51).	**Skip-counting** On multiplication tasks, the student uses skip-counting (often in conjunction with one-to-one counting), e.g., 4 × 5 as 5, 10, 15, 20.	

Global Stage	Operational Domains: Addition and Subtraction	Multiplication and Division	Proportions and Ratios
Part-Whole	**Five: Early Additive Part-Whole** **Early Addition and Subtraction** The student uses a limited range of mental strategies to estimate answers and solve addition or subtraction problems. These strategies involve deriving the answer from known basic facts, e.g., 8 + 7 is 8 + 8 – 1 (doubles) or 5 + 3 + 5 + 2 (fives) or 10 + 5 (making tens). Their strategies with multi-digit numbers involve using tens and hundreds as abstract units that can be partitioned, e.g., 43 + 25 = (40 + 20) + (3 + 5) = 60 + 8 = 68 (standard partitioning) or 39 + 26 = 40 + 25 = 65 (compensation).	**Multiplication by Repeated Addition** On multiplication tasks, the student uses a combination of known multiplication facts and derivation from addition fact knowledge, e.g., 4 × 6 as (6 + 6) + (6 + 6) = 12 + 12 = 24.	**Fraction of a Number by Addition** The student finds a fraction of a number mentally using addition fact knowledge, e.g., $^1/_4$ of 12 is 3 because 3 + 3 + 3 + 3 = 12. The student estimates answers and solves proportions and ratios problems by duplicating the proportion or ratio repeatedly with the support of materials.
	Six: Advanced Additive (Early Multiplicative) Part-Whole **Advanced Addition and Subtraction** The student can estimate answers and solve addition and subtraction tasks involving whole numbers mentally by choosing appropriately from a broad range of advanced mental strategies, e.g., 63 – 39 = 63 – 40 + 1 = 24 (rounding and compensating) or 39 + 20 + 4 = 63 so 63 – 39 = 24 (reversibility).	**Derived Multiplication** The student uses a combination of known facts and a limited range of mental strategies to derive answers to multiplication and division problems, e.g., 4 × 8 is double 2 × 8 = 16, 2 × 16 = 32 (doubling) or 9 × 6 is (10 × 6) – 6 = 54 (rounding) or 63 ÷ 7 = 9 because 9 × 7 = 63 (reversibility).	**Fraction of a Number by Multiplication** The student derives from known multiplication and division facts to estimate answers and solve fractions and proportions problems, e.g., $^1/_3$ of 36, 3 × 10 = 30, 36 – 30 = 6, 6 ÷ 3 = 2, 10 + 2 = 12 (compensating from a known fact). In the absence of a known related fact, the student will use strategies based on adding or skip-counting.

	Global Stage	Operational Domains: Addition and Subtraction	Operational Domains: Multiplication and Division	Operational Domains: Proportions and Ratios
Part-Whole	Seven: Advanced Multiplicative (Early Proportional) Part-Whole	**Decimal Addition and Subtraction** The student can estimate answers and solve addition and subtraction tasks involving decimal numbers mentally by choosing appropriately from a broad range of advanced mental strategies, e.g., 3.2 + 1.95 = 3 + 2 + 0.2 − 0.05 = 5.2 − 0.05 = 5.15 (compensation); 8.65 − 4.2 = (8 − 4) + (0.6 − 0.2) + 0.05, 8.65 − 4 = 4.65, then 4.65 − 0.2 = 4.45 (place value); 6.03 − 5.8 = □ as 5.8 + □ = 6.03 (reversibility); □ + 3.98 = 7.04 as 3.98 + □ = 7.04 (commutativity).	**Advanced Multiplication and Division** The student is able to choose appropriately from a broad range of mental strategies to estimate answers and solve multiplication and division problems. These strategies involve partitioning one or more of the factors, e.g., 24 × 6 = (20 × 6) + (4 × 6) (place value partitioning); 25 × 6 − 6 (rounding and compensating); 3 × 27 = 9 × 9 = 81 (trebling and dividing by three); 96 ÷ 4 = 25 − 1, since 25 × 4 = 100 (reversibility).	**Early Fractions, Ratios, and Proportions** The student uses a range of mental strategies based on multiplication and division to estimate answers and solve problems with fractions, proportions, and ratios. These strategies involve finding equivalent fractions and using unit fractions, e.g., $\frac{3}{4}$ of 24 = (24 ÷ 4) × 3 = 6 × 3 = 18; 3:5 as □:40, 8 × 5 = 40, 8 × 3 = 24 so □ = 24. The student renames fractions as decimals and percentages using multiplication and division, e.g., 3 out of 4 is equivalent to 75 out of 100 (multiplying by 25).
	Eight: Advanced Proportional Part-Whole		**Multiplication and Division of Fractions and Decimals** The student can estimate answers and solve problems involving the multiplication and division of fractions and decimals using mental strategies. These strategies involve recognising the effect of number size on the answer and converting decimals to fractions where appropriate, e.g., 3.6 × 0.75 = $\frac{3}{4}$ × 3.6 = ~~0.9~~ 2.7 (conversion and commutativity); 7.2 ÷ 0.4 as 7.2 ÷ 0.8 = 9 so 7.2 ÷ 0.4 = 18 (doubling and halving with place value).	**Advanced Fractions, Ratios, and Proportions** The student chooses appropriately from a broad range of mental strategies to estimate answers and solve problems involving fractions, proportions, and ratios. These strategies involve finding relationships between units of different quantities and converting between fractions, decimals, and percentages, e.g., 6:9 as □:24, 6 × $1\frac{1}{2}$ = 9, □ × $1\frac{1}{2}$ = 24, □ = 16 (between unit multiplying); 65% of 24: 50% of 24 is 12, 10% of 24 is 2.4 so 5% is 1.2, 12 + 2.4 + 1.2 = 15.6 (partitioning percentages).

The Number Framework - Knowledge

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Written Recording
Stage Zero: Emergent Stages One, Two, Three: Counting from One	The student identifies: • all of the numbers in the range 0–10.	The student says: • the number word sequences, forwards and backwards, in the range 0–10 at least; • the number before and after a given number in the range 0–10. The student orders: • numbers in the range 0–10.	The student instantly recognises: • patterns to 5, including finger patterns.		
	The student identifies: • all of the numbers in the range 0–20.	The student says: • the number word sequences, forwards and backwards, in the range 0–20; • the number before and after a given number in the range 0–20; • the skip-counting sequences, forwards and backwards, in the range 0–20 for twos and fives. The student orders: • numbers in the range 0–20.	The student knows: • groupings within 5, e.g., 2 and 3, 4 and 1; • groupings with 5, e.g., 5 and 1, 5 and 2 , …; • groupings within 10, e.g., 5 and 5, 4, and 6, … The student instantly recognises: • patterns to 10 (doubles and 5-based), including finger patterns.	The student recalls: • addition and subtraction facts to five, e.g., 2 + 1, 3 + 2, 4 − 2, … etc; • doubles to 10, e.g., 3 + 3, 4 + 4, … etc.	The student records: • the results of counting and operations using symbols, pictures, and diagrams.

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Written Recording
Stage Four: Advanced Counting / Stage Five: Early Additive	The student identifies: • all of the numbers in the range 0–1000; • symbols for halves, quarters, thirds, and fifths.	The student says: • the number word sequences, forwards and backwards, in the range 0–100; • the number before and after a given number in the range 0–100; • the skip-counting sequences, forwards and backwards, in the range 0–100 for twos, fives, and tens. The student orders: • numbers in the range 0–100.	The student knows: • groupings with 10, e.g., 10 and 2, 10 and 3, … and the pattern of "-teens"; • groupings within 20, e.g., 12 and 8, 6 and 14; • the number of tens in decades, e.g., tens in 40, in 60.	The student recalls: • addition facts to 10, e.g., 4 + 3, 6 + 2, …; • doubles to 20 and corresponding halves, e.g., 6 + 6, 7 + 7, $^1/_2$ of 14; • multiples of 10 that add to 100, e.g., 30 + 70, 40 + 60.	The student records: • the results of mental addition and subtraction using equations, e.g., 4 + 5 = 9, 12 – 3 = 9.
	The student identifies: • all of the numbers in the range 0–1000; • symbols for halves, quarters, thirds, fifths, and tenths, including improper fractions, e.g., $\frac{5}{4}$.	The student says: • the number word sequences, forwards and backwards, by ones, tens, and hundreds in the range 0–1000; • the number 1, 10, 100 before and after a given number in the range 0–1000; • the skip-counting sequences, forwards and backwards, in the range 0–100 for twos, threes, fives, and tens. The student orders: • numbers in the range 0–1000; • fractions with like denominators, e.g., $^1/_4$, $^2/_4$, $^3/_4$, … etc.	The student knows: • groupings within 100, particularly multiples of 5, e.g., 25 and 75, 49 and 51; • groupings of two that are in numbers to 20; • groupings of five in numbers to 50; • groupings of ten that can be made from a three-digit number, e.g., tens in 763; • the number of hundreds in centuries and thousands, e.g., hundreds in 800, in 4000.	The student recalls: • addition facts to 20, e.g., 7 + 5, 8 + 7; • multiplication facts for the 2, 5, and 10 times tables and the corresponding division facts; • multiples of 100 that add to 1000, e.g., 400 and 600, 300 and 700.	The student records: • the results of addition, subtraction, and multiplication calculations using equations, e.g., 35 + 24 = 59, 4 × 5 = 20, and diagrams, e.g., an empty number line.

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Vertical Written Forms
Stage Six: Advanced Additive	The student identifies: • all of the numbers in the range 0–1 000 000; • decimals to three places; • symbols for any fraction including tenths, hundredths, thousandths, and improper fractions.	The student says: • the whole number word sequences, forwards and backwards, by ones, tens, hundreds, and thousands in the range 0–1 000 000; • the number 1, 10, 100, 1000 before and after a given whole number in the range 0–1 000 000; • forward and backward word sequences for halves, quarters, thirds, fifths, and tenths, e.g., $\frac{1}{3}$, $\frac{2}{3}$, 1, $\frac{4}{3}$, $\frac{5}{3}$, etc. The student orders: • whole numbers in the range 0–1 000 000; • unit fractions for halves, thirds, quarters, fifths, and tenths.	The student knows: • groupings within 1000, e.g., 240 and 760, 498, and 502, …; • groupings of two, three, five, and ten that are in numbers to 100 and the resulting remainders, e.g., threes in 17, fives in 48. • groupings of 10 and 100 that can be made from a four-digit number, e.g., tens in 4562, hundreds in 7894.	The student recalls: • addition and subtraction facts to 20, e.g., 9 + 5, 13 – 7; • multiplication facts for the two, three, five, and ten times tables and the corresponding division facts; • multiplication facts for squares to 100, e.g., 4 × 4, 6 × 6, …	The student records: • the results of calculations using addition, subtraction, multiplication, and division equations, e.g., 349 + 452 = 350 + 451 = 801, 45 ÷ 9 = 5, and diagrams, e.g., an empty number line. The student performs: • column addition and subtraction with whole numbers of up to four digits.

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Written Recording
Stage Seven: Advanced Multiplicative		The student says: • the decimal word sequences, forwards and backwards, by thousandths, hundredths, tenths, ones, tens, etc.; • the number one-thousandth, one-hundredth, one-tenth, one, ten, etc. before and after any given whole number. The student orders: • decimals to three places, e.g., 6.25 and 6.3; • fractions, including halves, thirds, quarters, fifths, tenths.	The student knows: • the groupings of numbers to 10 that are in numbers to 100 and the resulting remainders, e.g., sixes in 38, nines in 68; • the groupings of ten, one hundred, and one thousand that can be made from a number of up to seven digits, e.g., tens in 47 562, hundreds in 782 894, thousands in 2 785 671; • equivalent fractions for halves, thirds, quarters, fifths, and tenths with numbers to 100 and with 1000, e.g., 1 in 4 is equivalent to 25 in 100 or 250 in 1000.	The student recalls: • multiplications facts to 10×10 and the corresponding division facts; • fraction ↔ decimal ↔ percentage conversions for halves, thirds, quarters, fifths, and tenths, e.g., $\frac{3}{4} = 0.75 = 75\%$.	The student records: • the results of calculations using equations, e.g., $6 \times 28 = 168$, and diagrams, e.g., empty number line. The student performs: • column addition and subtraction for whole numbers; • short multiplication and division of a three-digit whole number by a single-digit number.
Stage Eight: Advanced Proportional		The student says: • the decimal word sequences, forwards and backwards, by thousandths, hundredths, tenths, ones, tens, etc., starting at any decimal number; • the number one-thousandth, one-hundredth, one-tenth, one, ten, etc. before and after any given decimal number. The student orders: • fractions, decimals, and percentages.	The student knows: • the number of tenths, hundredths, and one-thousandths that are in numbers of up to three decimal places, e.g., tenths in 45.6, hundredths in 9.03, thousandths in 8.502; • what happens when a whole number or decimal is multiplied or divided by a power of 10, e.g., 4.5×100, $67.3 \div 10$.	The student recalls: • fraction ↔ decimal ↔ percentage conversions for given fractions and decimals, e.g., $\frac{9}{8} = 1.125 = 112.5\%$.	The student records: • the results of calculations using equations, e.g., $\frac{3}{4} \times 28 = 21$, and diagrams, e.g., double number line. The student performs: • column addition and subtraction for whole numbers, and decimals to three places; • short multiplication and division of whole numbers, and decimals by a single-digit number; • multiplication of a three- or four-digit whole number by a two-digit whole number.

Ministry of Education (2002a).

Appendix 2

Descriptions of equipment in *Beginning School Mathematics*

Abacus
A device which models a place-value number system. An abacus can be open (where the beads or rings can be completely removed) or closed (where each bead or ring is placed "to count" or "not to count"). The "teaching" abacus uses ten rings on each peg. The tenth ring, when placed, is flush with the top of the peg. This model differs from the "mathematical" abacus, which takes only nine rings or beads to each peg, and is used to assist in calculation. (Cycle 11, p.140)

Concrete materials
Materials that can be used to help children understand mathematical ideas. Some examples are counters, Cuisenaire rods, rulers, and Centimo blocks. (Cycle 10, p.128)

Discrete materials
Separate or distinct objects that can be counted by ones, e.g., beads, shells. (Cycle 9, p.130)

Number line
A line (usually vertical or horizontal) with numbers placed at equal intervals. [Diagrams of both vertical and horizontal number lines are included] (Cycle 11, p.141)

Place-value/multibase blocks
A set of blocks consisting of "cubes" (1 x 1 x 1 cm), "longs" (1 x 1 x 10 cm rods), and "flats" (1 x 10 x 10 cm square). These blocks are used to model a variety of mathematical experiences, especially those involving place value (numeration). (Cycle 9, p.131)

Structured materials
Materials constructed to show mathematical ideas, e.g., Cuisenaire rods, Centimo and Multibase blocks. (Cycle 9, p.132)

References to equipment in *Beginning School Mathematics* cycle books

The beginning statements in the Cycle 8 booklet and the Cycles 9 to 12 booklets provide an insight into the intended use of equipment as part of BSM activities. The statement entitled "BSM in your classroom" in Cycle 8 states, "...BSM equipment should not be viewed in a restricted way" and goes on to suggest "be available for free exploration by the children, prior to teachers suggesting a specific use for the activity". In Cycles 9 to 12 the term "equipment" has become "materials" in the resource. These later statements in Cycles 9 to 12 expand on the use of equipment and suggest that "as children represent ideas with concrete materials and record these ideas, using signs and symbols, it is important for them to link the two actions. These concrete and visual representations will help them when they meet abstract ideas in later years".

There are further references to equipment through the BSM cycles booklets under "Notes about mathematical ideas".

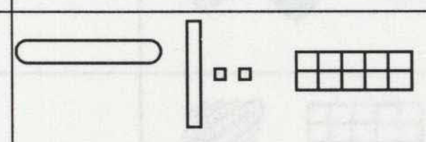
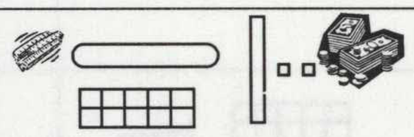
Higgins, J. (2001a). *Developing numeracy: Understanding place value.* (Report to the Ministry of Education.) Wellington: Ministry of Education.

Appendix 3

Equipment References in Key Documents

The following three tables give an overview of the inclusion in key mathematics documents of the six pieces of equipment that were the focus of this study. To some degree the information shown here disguises the fact that certain pieces of equipment are not specifically referred to in some documents. Instead, they might be included by the use of such general phrases as 'structured materials' (Department of Education, 1985, p.24; Ministry of Education, 1992, p.33) and 'grouped discrete objects' (Ministry of Education, 1992, p. 34).

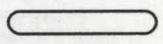
Table 7: Equipment referred to in the syllabus *Mathematics: Junior classes to standard four* (Department of Education, 1985)

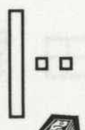
	Years 1 & 2	Years 3 & 4	Years 5 & 6
Number			
Money[11]			

Key for each of the three tables x, y, and z:

Number strips/lines

Calculators

Bundled ice-block sticks 

Place value blocks 

Money (incl. notes)

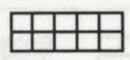
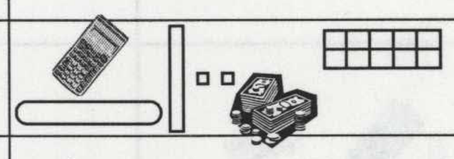
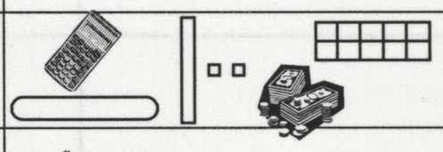
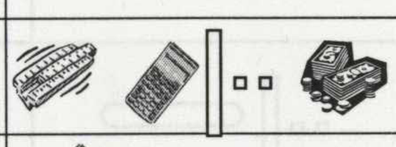
Tens frames 

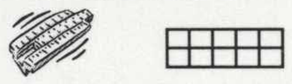
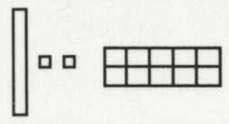
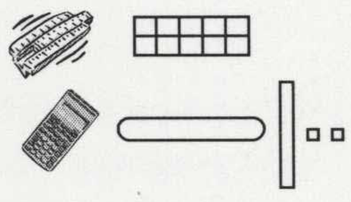
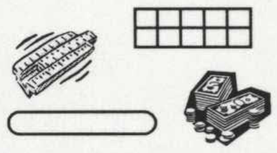
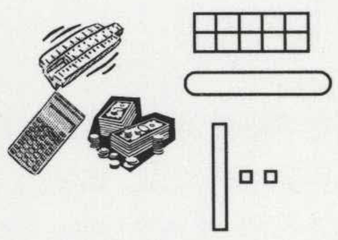
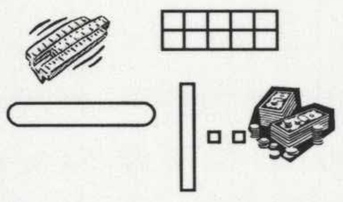
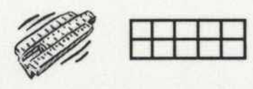
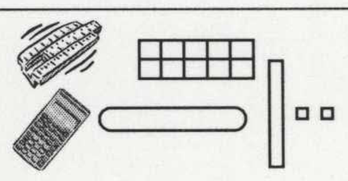
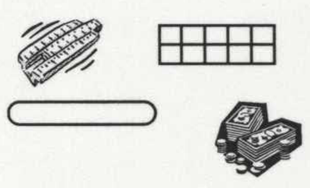
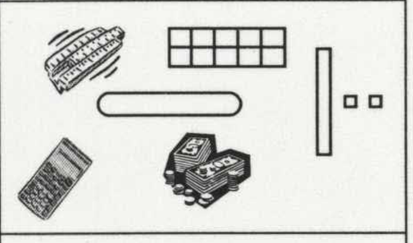
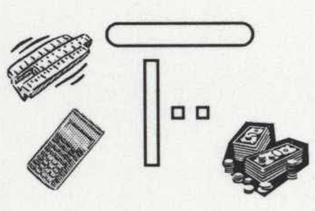
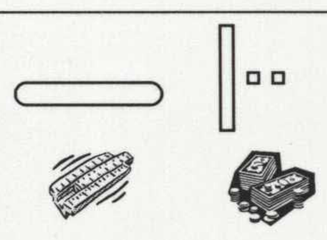
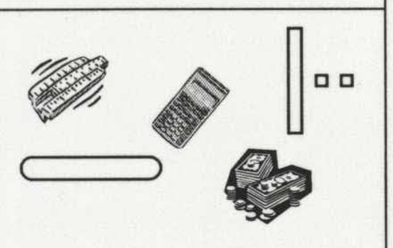
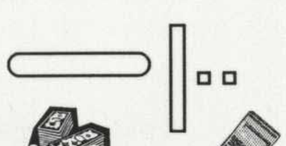
Table 8: Equipment referred to in Levels 1 to 3 of the curriculum statement *Mathematics in the New Zealand Curriculum* (Ministry of Education, 1992a)

	Level 1	Level 2	Level 3
Number			
Algebra			
Measurement			
Problem solving			

[11] Money is treated as one of nine topics in this document; the 1992 curriculum statement included money in the Measurement strand.

In *Mathematics in the New Zealand Curriculum* (Ministry of Education, 1992) place value blocks in Number Level 3 are referred to in the context of decimal place value, rather than whole numbers (see Table 8). The number strips/lines included here in Measurement Levels 2 and 3 represent rulers for measuring in centimetres and metres. None of these pieces of equipment are referred to in Levels 1 to 3 of the Geometry and Statistics strands; neither do they appear in the Mathematical Processes: Developing Logic and Reasoning and Communicating Mathematical Ideas.

Table 9: Equipment referred to in the resources *Numeracy Development Project Draft Teachers' Materials* (Ministry of Education, 2002c, d, e, & f)

Stage	Number Knowledge	Addition & Subtraction	Multiplication & Division
0: Emergent			
1-3: Counting All			
4: Advanced Counting			
5: Early Additive			
6: Advanced Additive			
7: Advanced Multiplicative			
8: Advanced Proportional			

In the NDP draft teachers' materials, ice-block sticks are used singly, as discrete materials, in Addition & Subtraction at the Emergent/Counting All stages, and are not included in Table 9. Similarly, $1 coins are used to share $10 between 2 people in Multiplication & Division at the Advanced Counting/Early Additive stages, and have not been included. A variety of number strips and number lines are recommended for use in the NDP materials, providing children with different levels of support and direction at the various strategy stages. (None of these six pieces of equipment is used in the booklet *Teaching Fractions and Decimals* (Ministry of Education, 2002f.)

Appendix 4

Dear _____

Mathematics research

As part of my Master of Education studies at Wellington College of Education, I am undertaking a small research project to explore Year 3 children's ideas about the mathematics equipment they find the most helpful when solving number problems, and why this is. This research involves interviewing around 30 children from 3 schools, as well as interviewing their teachers.

I need your consent to be interviewed about your views on equipment use in your teaching of number. This will take between half an hour and an hour. I will record all my interviews with teachers and with children, using a cassette recorder, and transcribe the interviews for analysis. I may also make some notes during the interviews. You are entitled to withdraw from the interview at any time.

Individual teachers, children and schools will not be identified in the report of the interview results; pseudonyms will be used, and any identifying characteristics will be excluded. A copy of the final report will be made available to you on request.

I also need your help in order to gain consent for the children in your class to be interviewed. I attach a copy of a letter to parents and caregivers, outlining my research, with a tear-off slip for them to complete and return to you.

Sincerely

Linda Bonne

I agree to participate in this research under the conditions outlined above.

_____ _____

(Signature) (Date)

Appendix 5

Dear Parents and Caregivers,

Interviews with children

As part of my Master of Education studies at Wellington College of Education, I am undertaking a small research project to explore Year 3 children's ideas about the mathematics equipment they find the most helpful when solving number problems, and why this is. This research involves interviewing around 30 children from 3 schools, as well as interviewing their teachers.

I need your consent to interview your child about their ideas. The interviews will be conducted at school during normal school hours during Term 3, in a space arranged by your child's teacher. The interview will take approximately twenty minutes. Your child can withdraw from the interview at any time.

Individual children and schools will not be identified in the report of the interview results; pseudonyms will be used, and any identifying characteristics will be excluded. A copy of the final report will be made available to your child's teacher on request.

Should you have any questions, you are welcome to ring me on [phone number].

Sincerely

Linda Bonne

I give / do not give (please delete one) my consent for my child

_____ to be interviewed for the purposes, and under the conditions, outlined above.

_____ _____

(Signature) (Date)

Appendix 6

Teacher Interview Questions

- Tell me about how you see the role of equipment in your overall mathematics programme.

- What is the role of equipment, then, in your teaching of number concepts?

- Thinking specifically about addition and subtraction, which equipment do you find children use with success?

- Apart from the equipment you find children use successfully for addition and subtraction, what are the other materials that you find helpful for teaching other number concepts?

- Are there particular pieces of equipment that you find helpful when trying to help children progress from one strategy stage to the next?

- Are there pieces of equipment that you've found less helpful than you'd expected them to be?

- I'm going to show you the equipment I talked to the children about. What I'd like to know is, do you use this piece of equipment in your number programme? Why / why not? And how is it used? *(Present each manipulative used in the interviews with children.)*

- Is there anything else you'd like to add, relating to numeracy equipment?

THANK YOU!

Appendix 7

Children's Interview Questions

Operational Strategy Questions: Addition & Subtraction

(i) **Get 9 counters for me.**

(Yes, go to (ii))

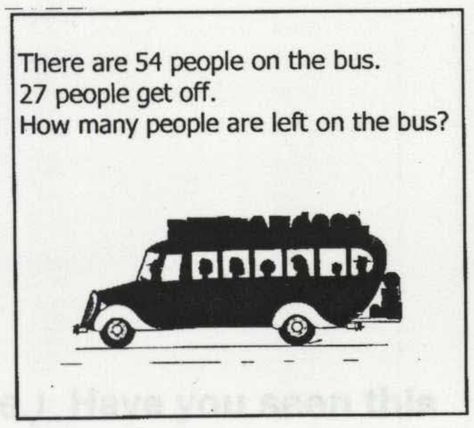
(ii) **Five counters in one hand and three in the other. How many is that altogether?**

(Yes, by counting all – go to Assessment A; solves by knowing the fact, counting from one by imaging or counting on – go to (iii))

(iii) **I have nine counters under here and six counters under here** *(masked).* **How many is that altogether?**

(No – go to Assessment A. Yes, by counting on using fingers – go to Assessment B. Yes, by knowing the fact, counting on by imaging, or using a part-whole strategy – go to (iv))

(iv) *(Present the number problem on a piece of card.)* **54 people are on the bus. 27 people get off. How many people are left on the bus?**
(No – go to Assessment B. Yes, by counting on or back mentally or using fingers – go to Assessment B. Yes, by using a part-whole strategy – go to Assessment C)

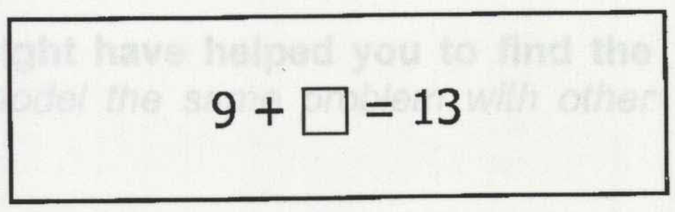


Assessment A

1. **Here are 7 counters.** *(Briefly display and then screen.)* **Here are 5 more counters.** *(Briefly display and then screen.)* **How many counters are there altogether?**
2. **I have 9 marbles. I give 4 to a friend. How many marbles do I have left?**
3. **I have 14 lollies and I eat 6 of them. How many lollies do I have now?**

Assessment B

1. **I have 9 counters under here, and I'm putting some more counters under here** *(screen the counters).* **Altogether there are 13 counters now. How many are under here?** *(circling above unknown collection. Present the number problem on a piece of card.)*



2. **You have 36 lollies and you eat 8 of them. How many have you got left?** *(Present the number problem on a piece of card.)*

3. **In the bush, there are 43 tui and 28 kiwi. How many birds are there altogether?** *(Present the number problem on a piece of card.)*

In the bush, there are 43 tui and 28 kiwi.
How many birds are there altogether?

Assessment C

1. **Donna has 276 stamps. She gets another 89 stamps from her best friend. How many stamps does she have then?** *(Present the number problem on a piece of card.)*

Donna has 276 stamps. She gets another 89 stamps from her best friend.
How many stamps does she have then?

2. **Tony has $504 in his bank account. He takes out $96 to buy a new skateboard. How much money is left in his account?** *(Present the number problem on a piece of card.)*

Tony has $504 in his bank account.
He takes out $96 to buy a new skateboard.
How much money is left in his account?

Equipment Questions:

NB. Children will be encouraged to use their choice of equipment to solve a problem before yielding to any requests to use pencil and paper.

The equipment that will be presented to the child includes:

- Tens frames & counters
- A number line
- Bundled ice-block sticks
- Place value blocks
- Play money
- Calculator

1. *(Show the child the various pieces of equipment, one by one.)* **Have you seen this before? Have you used it before? How?**

2. **Which of these pieces of equipment would you choose to help you find the answer to this problem: There were 41 seagulls on the beach. 18 of the seagulls flew away. How many seagulls were left on the beach?** *(Present the number problem on a piece of card. When the child has chosen their equipment, hand it to them and allow them time to solve the problem.)*

There were 41 seagulls on the beach. 18 of the seagulls flew away.
How many seagulls were left on the beach?

3. *(When the child has finished working on the problem:)* **Tell me how you used the [equipment name] to help you solve the problem?**

4. **Which of the other pieces of equipment might have helped you to find the answer?** *(Give the child the opportunity to model the same problem with other equipment.)*

5. **Why did you think the [name of the piece/s of equipment] was good to use?**

6. **Apart from the equipment I've shown you, what else do you like to use to help you solve problems like these?**

Appendix 8

Operational Strategy Questions – Addition & Subtraction

Child's name _____ **Date** _____

(i) Get 9 counters for me. Y / N

(ii) Five counters in one hand and three in the other. How many is that altogether?

(iii) I have nine counters under here and six counters under here (masked). How many is that altogether?

(iv) 54 people on the bus. 27 people get off. How many people are left on the bus?

Assessment A

4. Here are 7 counters. *(Briefly display and then screen.)* Here are 5 more counters. *(Briefly display and then screen.)* How many counters are there altogether?

5. I have 9 marbles. I give 4 to a friend. How many marbles do I have left?

6. I have 14 lollies and I eat 6 of them. How many lollies do I have now?

Assessment B

4. I have 9 counters under here, and I'm putting some more counters under here *(screen the counters)*. Altogether there are 13 counters now. How many are under here? *(circling above unknown collection)*

5. You have 36 lollies and you eat 8 of them. How many have you got left?

6. In the bush, there are 43 tui and 28 kiwi. How many birds are there altogether?

Assessment C

3. Donna has 276 stamps. She gets another 89 stamps from her best friend. How many stamps does she have then?

4. Tony has $504 in his bank account. He takes out $96 to buy a new skateboard. How much money is left in his account?

References

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