AN EXPERIMENTAL INVESTIGATION

INTO

CHILDREN'S MISCONCEPTIONS CONCERNING THE SYMBOLS FOR MATHEMATICAL EQUALITY.

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BEING A THESIS IN EDUCATION

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BY

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I. THE GENERAL NATURE AND SCOPE OF THE INVESTIGATION.

(a) Children's Habit of Treating the "=" sign as a symbol of Distinction.

One of the commonest faults in children's statements of an argument in arithmetic is the misuse of the sign of equality. The following examples of statements, selected from children's work by the writer during his marking of Form I arithmetic, are typical of the errors committed by children between ten and twelve years of age:

(1) $6\frac{3}{4} - 1\frac{3}{8} = \frac{6}{8} - \frac{3}{8} = 5\frac{3}{8}$.

(2) $\frac{57}{6} = 9\frac{1}{2} - 9\frac{1}{4} = \frac{1}{4}$.

(3) 1 day = $2\frac{1}{3}$ acres.

(4) $270 \div 5 = 5\overline{)270}$ 54

(5) $8\frac{1}{3} = 25$.

(6) $\frac{3}{8} + \frac{2}{8} + \frac{4}{8} + \frac{6}{8} = 1\frac{7}{8} = £39 + 1\frac{7}{8} = £40\frac{7}{8}$.

(7) 6s.8d. X 3 = £1 − 2s.6d. = 17s. 6d.

(8) 60 packets X $\frac{1}{8}$ = 2 oz. = Ans.

A study of these and numerous other instances suggests that children regard the sign of equality as a symbol of <u>distinction</u> rather than as a connecting link bringing into relation two expressions which are numerically or quantitatively equivalent: its function, to the child mind, would appear to be to separate rather than to bridge. The validity of this statement will be decided after a study of the results of the tests in this investigation.

The question is well worth investigating, owing both to the widespread occurrence of the misconception, and to the fact that the notion is of great importance in the early stages of the study of algebra, when equations are apt to lose all meaning if the pupil cannot conceive an expression as a unity and an equals-sign as something more than a punctuation mark.

Support for this contention is accorded by the "Report of Inspectors of Schools to the Education Board of the District of Wellington for 1933", published a few weeks prior to the placing of the account of this investigation in the hands of the typiste. On page 8 the report reads: "A point that is worthy of more consideration by teachers of senior classes is the logical arrangement and correct statement of the successive steps of problems. The loose and inaccurate use of the 'equals' sign is still prevalent. It is suggested that more frequent setting out of models on the blackboard by the teacher will be found helpful."

(b) <u>Effects of Such Misconceptions</u>.

The relation of equality in mathematics is the fundamental relation between quantities and is the basis of all quantitative thought and reasoning. Thus W.S. Jevons says: "The universal principle of all reasoning ... is that which allows us to substitute like for like. I have now to point out how in the mathematical sciences this principle is involved in each step of reasoning. It is in these sciences indeed that we meet with the clearest cases of substitution, and it is the simplicity with which the principle can be applied which probably led to the comparatively early perfection of the sciences of geometry and arithmetic. Euclid, and the Greek mathematicians from the first, recognised <u>equality</u> as the fundamental relation of quantitative thought ...." (1)

Thus, as the relation of equality in mathematics is fundamental, and is involved in every step of reasoning, it follows that the existence of serious misconceptions concerning the symbol for mathematical equality will render invalid the whole process of reasoning. It would seem, then, that the basis of valid mathematical reasoning and inference could be laid by eradicating any misconceptions concerning, and building up a right understanding of, the precise meaning and use of the symbol of equality in the child mind. When a child asserts that $9 \times 2 = 18 + 2 = 20$, it is evident that the working is mechanical, in the sense that the symbols of addition and multiplication are regarded merely as commands which must be executed irrespective of the relation which may exist between the quantities connected by the "=" sign. Most children would maintain that the "=" sign is used legitimately in the above example, because the 9 has been multiplied by 2

(1) W.S. Jevons: "The Principles of Science", p. 162.

to give 18, to which 2 has been added to make 20. It is remarkable that children of ten years can conceive any number, however great, written in the Arabic notation, as a unity, yet many cannot conceive a vulgar fraction or an expression containing a symbol of operation as a simple number. Thus 9 x 2 to the child mind is not a unity - like 18 - but a process to be carried out, namely, the process of multiplying 9 by 2. It would seem , then, that the existence of such misconceptions of the "=" sign robs arithmetic (and mathematics in general) of much of its meaning and value. This is true whether we follow the chief aim of mathematics as utility in solving practical problems as stated in the Syllabus of Instruction issued by the Education Department of New Zealand, or as the development of an appreciation of the value and significance of an ordered system of mathematical ideas, as stated by Sir Percy Nunn.[2]

Few will deny that an important aim of mathematics is to provide a training in logical, systematic reasoning, both inductive and deductive, yet this aim is not realized so long as the child works more or less mechanically, or when there is no logical sequence between the steps in a reasoning process. The following example illustrates an error that is typical among children of ten and twelve years of age:-

$$\begin{aligned} 136 \times 12 &= 1200 + 12 \times 30 \\ &= 360 + 6 \times 12 \\ &= 72. \end{aligned}$$

(2) T. Percy Nunn: "The Teaching of Algebra", p. 17.

When the "=" sign is misused as in this example, the reasoning process does not "hang together" - it is disjointed and illogical, inasmuch as each step does not logically follow from the previous step. The reasoning process then loses its essential unity and clarity, while the value of mathematics lies only in the development of mechanical accuracy. Too much time in mathematics seems to be devoted to the memorising of the mechanics of working various types of mathematical examples instead of trying to develop the logical aspect of the child's mind. An apt illustration of this point is offered from an investigation by Mr. E.J.G. Bradford. (3) The following is a sample of the type of problem used in the investigation:- If Henry VIII had six wives, how many had Henry II? Bradford found that roughly 60% of the children applied "the principles of numbers" without thinking about the nature of the data in which these numbers were imbedded.

The correct use of the symbol for mathematical equality minimises the possibility of error in the following ways:-
(1) Our minds, not overburdened by a mass of unnecessary detail, are thus left free to concentrate on relevant matter and on the deduction of the next step from the previous step. A comparison of a method embodying the illegitimate use of the "=" sign with one embodying the legitimate use may make this point clear.

$$
\begin{aligned}
\text{(a)} \quad 136 \times 12 &= 1200 + 12 \times 30 \\
&= 360 + 6 \times 12 \\
&= 72.
\end{aligned}
$$

(3) "Forum of Education", (Vol.3) Suggestion, Reasoning and Arithmetic: by E.J.G. Bradford.

$$\begin{aligned}\text{(b)}\quad 136 \times 12 &= 100 \times 12 + 30 \times 12 + 6 \times 12\\ &= 1200 + 360 + 72\\ &= 1632.\end{aligned}$$

In (a) it is necessary to bear in mind, while we are working the problem, several numbers that we have to add before we arrive at the answer. This may lead to oversight of one process, confused thought, or addition of the wrong numbers, in addition to inaccuracy in mechanical calculation which is the only source of error which applies equally to example (b). Moreover, when dealing with complex, involved problems the increased liability to error by employing the "=" sign incorrectly is much greater than this simple example would indicate.

(2) In the solution of a complex problem, the full reasoning process is very seldom fully and precisely formulated in the mind before putting pen to paper. Often, through more or less blind reasoning, or trial and error reasoning by the deduction of all possible relations, we may see light. This is possible only if the "=" sign has been used correctly, thus enabling us to feel confident that our reasoning from step to step is valid, although we may be on the wrong track.

The sign of equality, "=", ranks among the very few mathematical symbols that have met with universal adoption. This symbol was so admirably chosen that it survived all competitors in the long struggle for supremacy. Owing to its universality, then, it is important that the "=" sign performs always its true function, namely, the expression of relationship between two expressions which are numerically or quantitatively equivalent. Thus, for example, if the "=" sign is performing its true function, the teacher can independently and readily see any errors that may exist either in the reasoning process or in the mechanical calculations in a child's work. When the "=" sign is used illegitimately by the child in his work, the teacher is confused with a mass of figures expressing no true relation to one another.

In general, then, it is evident that misuse and loose use of the symbol of equality leads to confused thought and invalid reasoning. Miss Renwick says: "It is inconceivable that pupils who do not understand a statement that one expression is numerically equal to another can have understood the logical bases of the process of solving an equation. Their skill was due to the circumstance that they could interpret and apply certain rules. There are many children who learn mathematics in this fashion." (4)

(c) The Nature and Scope of the Investigation.

The following pages give an account of an investigation mainly experimental in character, into children's common misconceptions concerning the symbols for mathematical equality.

(4) Reported in the "British Journal of Educational Psychology" June 1932.

This topic for investigation suggested itself to the writer during his actual teaching experience. Teachers in the secondary and senior primary branches of the school frequently commented on the misuse and loose use by their pupils of the "=" sign. They maintained that the precise meaning of the symbol was not clearly understood by the majority of their pupils, with the result that many reasoning processes were erroneous and unintelligible.

This thesis embodies experiments directed to the investigation of the following problems:-

(1) To what extent misconceptions exist in the minds of children of different ages concerning the "=" sign.

(2) If, and when, the precise meaning of the symbol is clearly understood by children.

(3) If, and to what extent there is a difference in the sexes in the understanding of the symbol.

(4) Factors influencing the formation of such misconceptions in children's minds.

(5) The value (if any) of direct instruction in the removal of such misconceptions.

(6) Whether, and to what extent, inability to understand clearly the precise nature of the "=" sign has its correlative in inability to understand the correct and precise meaning of the word "equal"; or whether the word "equal" is regarded

as synonymous with the word "same".

(d) Previous Work in the Field.

As far as the writer is aware the only previous investigation in this field is that of Miss E.M. Renwick, reported in the British Journal of Educational Psychology, June 1932. Miss Renwick was more concerned to discover if misconceptions concerning the symbols for mathematical equality existed in the minds of children of a particular age only, namely, those between ten and twelve years of age. Only 121 pupils were tested and these were all girls, while the results of the tests employed in the investigation were expressed only in broad class-groups, which, even though suggestive, are too indefinite, and incapable of showing if development of the ability to understand clearly the precise meaning of the symbols varies directly with the increasing ages of children.

Realizing the limitations of Miss Renwick's investigation, the writer has endeavoured to make this investigation more definite, more comprehensive, and more convincing, -

(1) By submitting the tests to over eight hundred subjects.

(2) By testing children whose ages ranged from seven to nearly nineteen years in order to show development of ability if such development exists.

(3) By testing classes in which there was an approximately equal number of girls and boys, so that if any appreciable difference appears in the results between the sexes, then this difference is probably attributable, not to difference in teaching, but to difference of mathematical ability between the sexes. In order to administer the tests, then, it was necessary to select co-educational schools where the classes contained both sexes.

(4) By arranging the results of the tests in age-groups to discover if any relation exists between age-groups and the ability of children to understand the meaning of the "=" sign.

(5) By attempting to discover the cause (or causes) of misconceptions of the sign of equality and suggesting measures to remove such misconceptions.

(6) By modification and extension of the tests in certain directions.

(7) By attempting to discover the value of direct instruction in removing such misconceptions.

Thus, this investigation is not only more comprehensive than that conducted by Miss Renwick, but also it probes more deeply into the problem, revealing sex and age differences, possible causes and suggested remedies.

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II.

THE INVESTIGATION.

A. INVESTIGATION I.

(a) Description of the Test.

This test, designed to reveal children's misconceptions concerning the symbol of mathematical equality, was submitted to classes ranging from Standard III in the primary school to Form VI in the secondary school; that is, the test was submitted to pupils whose ages ranged from 7 to nearly 19 years.

The test is similar to that employed by Miss Renwick but the number of examples has been increased from 10 to 15 to provide for additional examples containing more common misuses of the "=" sign, e.g. Scale : 1 inch = 1 foot. The close similarity in principle between the examples devised by Miss Renwick and those employed in this investigation is due, not to slavish imitation on the writer's part, but to the fact that Miss Renwick's examples are admirably chosen. The following is the form in which the test used is this investigation was expressed: -

Girls and boys, and even grown-up people, often use wrongly the "=" sign, so we are giving <u>you</u> this simple test to find out if <u>you</u> know the correct use of the "=" sign. Read these statements carefully and if you think that the "=" sign is used correctly write "Yes": if you think it is used wrongly in any part of the statement write "No", e.g. $9 + 5 = 12$ (No).

Don't guess. If you do not know, put nothing in the brackets.

(1) $9 \times 2 = 14 + 4$ ( )

(2) $6 + 3 = 9 \div 3 = 3.$ ( )

(3) 1s. x 12 = 12d. ( )

(4) 6d. x 2 = 1s. ( )

(5) $12 - 4 = 16 \div 2 = 8.$ ( )

(6) $18 \div 2 = \begin{array}{r} 2\underline{)18} \\ 9 \end{array}$ ( )

(7) $6 + 2 + 1 = 8 + 1 = 9$ ( )

(8) $4 \times 6 = 3 \times 8$ ( )

(9) $3 \times 5 = 14 + 1$ ( )

(10) $3 \times 5 = 15 + 2 = 17$ ( )

(11) $9 + 4 = 13 - 2 = 11$ ( )

(12) $6 \times 3 = 2 \times 9$ ( )

(13) $3 \times 4 + 4 = 4 \times 5 - 4 = 16$ ( )

(14) $2\frac{3}{4} - \frac{1}{4} = \frac{3}{4} - \frac{1}{4} = 2\frac{1}{2}$ ( )

(15) Scale: 1 inch = 1 foot. ( )

An examination of the test reveals that no effort was made to "catch" the children by means of freak examples. The examples displaying an illegitimate use of the "=" sign are representative of the ones the writer has encountered during his actual teaching experience. In addition, the children had their attention drawn to the purpose of the test, namely, to discover the ability of each individual child to distinguish between the correct and incorrect uses of the sign of equality. Primarily, it was not the writer's intention to reveal to the children the purpose of the test, nor to specify the nature of the error to be sought. Had this been the case the results displaying misconceptions of the symbol would have been more arresting because the child's attention would have been distributed over a much wider range of objects. However, the results the writer has secured are "pure", in the sense that provision was made to eliminate the possibility of extraneous factors affecting their validity. Besides, the children were instructed not to guess, so any errors noted are errors in the understanding of the correct and precise meaning of the sign. Also, examples involving small and easy numbers were chosen so that any errors would result not from faulty calculation but from a faulty knowledge of the principles involved in the examples. Hence, every precaution was taken to ensure that errors in children's judgments would be due to no other factor than misconceptions concerning the meaning of the "=" sign.

(b) Technique of Marking.

The technique of marking employed in this test has been followed in all the tests of this investigation. Briefly, it is as follows:- One mark was subtracted from the number of examples attempted for every incorrect judgment. Thus a paper which contained no errors was credited with 100%, even though the child had not passed his judgment on every example. It is well to recall that the instructions on the test stated: "Don't guess. If you do not know, put nothing in the brackets." The investigation is not concerned with the guessing ability of children but with their misconceptions concerning the symbol of equality. The purpose of this investigation is to discover to what extent children are definite in supporting a wrong judgment. Hence, in this test, any errors that arise are due to definite misconceptions in the child's mind, and not to inability to understand the example. Thus, standard III pupils may never have heard of "scale" in arithmetic, or their knowledge of fractions may be very meagre, indeed, so to obtain valid results (probably conservative) we cannot subtract marks for inability to form a judgment on the correct or incorrect use of the "=" sign in such examples.

A practical illustration may make the plan of marking clear. If a child formulated judgments on 14 of the 15 examples, we must assume, taking into consideration the nature of the instructions, that the child was quite decided in his own mind that his 14 judgments were valid. If, say, 4 of his judgments were incorrect his score would be assessed as ($\frac{10}{14} \times \frac{100}{1}$)%, that is, 71%. Every child's individual score was assessed first on a percentage basis, then it was translated into a fraction of 15. In the above example the child's new score would be ($\frac{71}{100} \times \frac{15}{1}$), that is, 10.6. In the total scores for each age-group, only whole numbers were employed.

There were comparatively few cases in which all the examples were not attempted. Over 90% of the pupils passed definite judgments on all the examples, so apparently the tests were not over-difficult. Thus, the meaning of the examples seems to have been clear so any errors are due entirely to misconceptions concerning the symbol of equality. This statement is supported by the technique of marking which ignored any example whose meaning or use of the "=" sign perplexed the child.

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(c) Results of Test and Discussion of Results.

The following table shows the average mark for each age-group: -

Years	Total Marks	No. of pupils	Average mark
7 - 8	10	1	10
8 - 9	66	6	11
9 - 10	271	32	8.5
10 - 11	368	40	9.2
11 - 12	539	58	9.3
12 - 13	741	72	10.3
13 - 14	838	78	10.7
14 - 15	650	55	11.8
15 - 16	565	44	12.8
16 - 17	375	29	12.9
17 - 18	122	9	13.5
18 - 19	28	2	14.0

Graphic representation of this result may be more instructive.

It is not possible to compare the results of this test with those of Miss Renwick because the latter recorded her results, not in age-groups, but in broad class-groups, while her classes comprise girls only, of differing ages.

The following are the groups tested by Miss Renwick:

Class A : 29 girls, average age 10 years 11 months, ages ranging from 8 years 11 months to 13 years, and including 16 girls over 11.

Class B : 32 girls of medium ability, average age 11 years 8 months.

Class C : 34 girls classified as of superior intelligence, average age 13 years.

Class D : 26 girls, of whom about half were very weak in mathematics, average age 14 years.

The following are the results obtained by Miss Renwick: -

(A dash indicates that the class did not consider the statement.)

Statement	Class A (29)	Class B (32)	Class C (34)	Class D (26)
	Number of correct answers.			
(1) $17 \times 2 = 30 + 4$	13	23	34	22
(2) $12 + 2 = 14 \div 2 = 7$	9	19	32	22
(3) 6d. x 2 = 1s.	27	31	34	25
(4) £1 x 20 = 20s.	15	30	31	22
(5) $13 - 4 = 18 \div 2 = 9$	11	18	28	21
(6) $26 \div 2 = 2\overline{)26}$ 13	8	6	8	1
(7) $6 + 2 + 1 = 8 + 1 = 9$	16	-	-	-
(8) $2 \times 4 \times 3 = 8 \times 3$	13	-	-	-
(9) $7\frac{3}{4} - 4\frac{1}{8} = \frac{6}{8} - \frac{1}{8} = \frac{5}{8} = 3\frac{5}{8}$	-	25	28	14
(10) $\frac{1}{4}$ of $12 = \frac{1}{3}$ of $9 = 3$	-	32	29	24

Thus Miss Renwick's results do not show any relation between the age of children and the ability to understand the true meaning of the symbol of equality. The tabulated, statistical results, while suggestive, are too indefinite and convey very little.

The results of the test in this investigation seem to show that after the age of 9 there is a definite, steady increase in the ability of children with increasing age to understand more clearly the precise meaning of the symbol. One's eye is immediately captured by the high marks - probably unexpected - secured by the children in the two lowest age-groups. However, the results in these groups must be discarded owing to the insufficient number of pupils tested. The writer
* proffers the following explanation of the high marks in
* these groups. The lowest class tested was Standard III,
* so these pupils of seven and eight years of age were at least
* in Standard III. This means that these pupils are gifted with more than average intelligence. This fact, together with the fact of paucity of cases tested, will probably account for the phenomena. The writer feels convinced that were a sufficient number of cases taken of children between the age of 7 and 9 years, not only in Standard III and higher classes, but also in Standard II and lower classes, then the average of the marks would be below the average of the marks attained by

children whose ages range from 9 years to 10 years.

The result that there is a direct relation between increase of age in children and their ability to understand the true meaning of the symbol of equality is what we would naturally expect for increase in age normally means increase in experience, increase in school instruction, and an increase in the development of mental functions. This should mean that even the symbol of mathematical equality should take a clearer and more definite shape in the child's mind.

It may appear that an average mark of 8.5 from a possible average of 15 is a good mark for children of nine years of age but it must be remembered that if a sufficient number of cases were taken, an average mark of approximately 7.5 would be obtained simply by chance or guessing. About one half of the judgments of the children of nine years of age were wrong. This shows that the precise and true meaning of the "=" sign is not by any means clear to the child of 9, nor even to the child of greater years.

To be consistent with our decision to discard the results of the tests on the seven and eight year old children through insufficiency of cases, we should do the same with the results from the seventeen and eighteen year old pupils for the same reason. In the opinion of the writer the results of the test applied to these students of seventeen and eighteen years are probably higher than they should be, for the reason that most of the students are boys, and boys - as will be pointed out later in this thesis - after the age of 13 or 14, show definite superiority over the girls of the same age in the tests of this investigation. Hence it seems that if the results of the tests applied to students of seventeen and eighteen were truly representative of both sexes, they would not be so high. However, the results are not so high as to indicate that the real meaning of the symbol is accurately conceived. Apparently doubt still exists in the minds of students of eighteen years of age as one student, for example, asserted that the use of the "=" sign in $18 \div 2 = 2\overline{)18}$ with quotient 9 was quite legitimate. Doubt not only exists in the minds of our primary and secondary school pupils, but also in the minds of educated adults and fully-qualified teachers as will be pointed out during the course of this investigation. Miss Renwick says: "The misconception is not unknown among educated adults, as the following extract from a modern book on heredity testifies:" (1)

$$(44 - 30)^2 = 14^2 = 196 \times 2 = 392.$$

$$(44 - 34)^2 = 10^2 = 100 \times 7 = 700.$$

(1) Reported in the "British Journal of Educational Psychology", June 1932.

The results of the test may be presented in a different tabulated form to show the number of pupils in each age-group who were correct in their judgments concerning each example. This method makes the results illuminating as increase (or decrease) in the ability of children with increasing age in any particular example may be traced. The figures in the following table give the number of children in each group who were successful in their judgments concerning each example. The numbers in brackets indicate the number of children in each age-group.

Statement	7–8 (1)	8–9 (6)	9–10 (32)	10–11 (40)	11–12 (58)	12–13 (72)	13–14 (78)	14–15 (55)	15–16 (44)	16–17 (29)	17–18 (9)	18–19 (2)
(1) 9 x 2 = 14+4	-	6	26	28	42	50	62	50	42	26	8	2
(2) 6+3 = 9 ÷ 3 = 3	-	4	15	20	29	45	50	48	35	26	9	2
(3) 1s. x 12 = 12d.	1	3	13	23	39	64	67	51	43	27	8	2
(4) 6d. x 2 = 1s.	-	5	29	35	51	72	77	50	42	26	9	2
(5) 12 - 4 = 16 ÷ 2 = 8	-	5	15	21	34	40	50	40	39	23	8	2
(6) 18 ÷ 2 = 2)18 9	1	-	5	4	6	10	9	13	19	12	4	2
(7) 6+2+1 = 8+1 = 9	1	4	18	34	45	60	64	45	41	26	9	2
(8) 4 x 6 = 3 x 8	-	5	24	32	35	54	63	49	42	26	9	2
(9) 3 x 5 = 14+1	1	6	22	24	44	53	66	50	41	27	9	1
(10) 3 x 5 = 15+2 = 17	1	5	16	23	33	51	59	49	39	29	9	2
(11) 9+4 = 13 - 2 = 11	1	4	10	20	26	47	58	46	37	27	9	2
(12) 6 x 3 = 2 x 9	1	6	23	26	44	54	59	48	41	28	9	2
(13) 3 x 4 4 = 4 x 5 - 4 = 16	1	5	21	30	38	49	60	42	41	26	9	2
(14) $3\frac{3}{4} - \frac{1}{4} = \frac{3}{4} - \frac{1}{4} = 2\frac{1}{2}$	1	3	8	15	29	49	49	44	43	28	7	2
(15) Scale: 1 inch = 1 foot	1	5	21	21	44	48	44	25	18	18	6	2

As already pointed out the results of the tests submitted to pupils of 7, 8, 17 and 18 years of age must be discarded owing to insufficiency of cases tested.

This table enables one to see at a glance the examples producing most errors in the children's judgments in each age-group. Thus, in each age-group we can arrange the examples in order of difficulty from the child's point of view. However, the following method of presentation of results is probably more illuminating and lucid as the graph shows, at a glance,

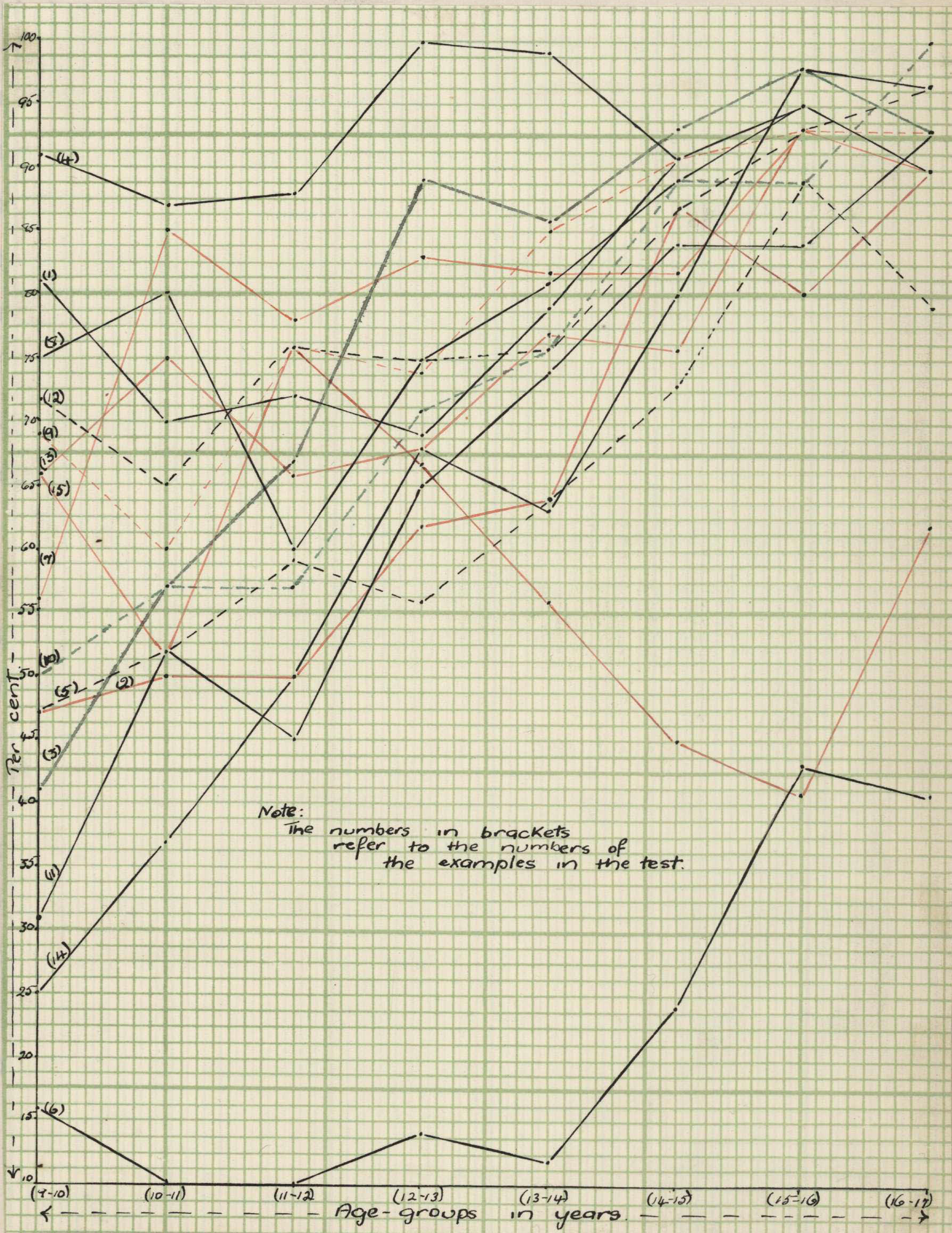
(1) The order in which the examples were judged most correctly in each age-group.

(2) The increase (or decrease) in the ability of children of successive age-groups to form correct judgments concerning the use of the "=" sign in each example.

To make possible this method of presenting the results, the figures showing the number of correct judgments had to be expressed on a percentage basis to secure a common working-unit for purposes of comparison and contrast. For example, 24 of the 40 pupils between 10 and 11 years of age were successful in their judgment of example 9. This result would be expressed by $(\frac{24}{40} \times \frac{100}{1})$, that is, 60. Similarly with all the other numbers in the above table. Hence we get the following table on a percentage basis:-

	Years. 9–10	10–11	11–12	12–13	13–14	14–15	15–16	16–17
(1) 9 x 2 = 14 + 4	81	70	72	69	79	91	95	90
(2) 6 + 3 = 9 ÷ 3 = 3	47	50	50	62	64	87	80	90
(3) 1s. x 12 = 12d.	41	57	67	89	86	93	98	93
(4) 6d. x 2 = 1s.	91	87	88	100	99	91	95	90
(5) 12 - 4 = 16 ÷ 2 = 8	47	52	59	56	64	73	89	79
(6) 18 ÷ 2 = 2)18 / 9	16	10	10	14	12	24	43	41
(7) 6+2+1 = 8 + 1 = 9	56	85	78	83	82	82	93	90
(8) 4 x 6 = 3 x 8	75	80	60	75	81	89	95	90
(9) 3 x 5 = 14 + 1	69	60	76	74	85	91	93	93
(10) 3 x 5 = 15 + 2 = 17	50	57	57	71	76	89	89	100
(11) 9 + 4 = 13 - 2 = 11	31	50	45	65	74	84	84	93
(12) 6 x 3 = 2 x 9	72	65	76	75	76	87	93	97
(13) 3 x 4+4 = 4 x 5 - 4 = 16.	66	75	66	68	77	76	93	90
(14) $2\frac{3}{4} - \frac{1}{4} = \frac{3}{4} - \frac{1}{4} = 2\frac{1}{2}$	25	37	50	68	63	80	98	97
(15) Scale: 1 inch = 1 foot	66	52	76	67	56	45	41	62

These results can be graphically represented as follows:



(d) Interpretation of Certain Results.

(1) General Results.

All the children seemed to weigh their decisions very carefully, some of the pupils checking their answers two or three times, making alterations and erasures, and appearing to find the task difficult. There were in spite of this, inconsistencies in the judgments of a few of the best pupils and this fact serves to emphasize the indefiniteness and want of clearness of their ideas. It is worthy of note and very significant that only 10 of the 426 papers contained no mistakes.

Most of the pupils tested were unknown to the writer, hence it is not possible to ascertain any correlation that may exist between success in the test and general intelligence. However, the writer, two years ago, conducted an intelligence test (Stanford revision) on one of the pupils - Betty. Betty's I.Q. was then assessed at 130, so she would be classed as of superior intelligence. Her age, now, is 10 years 2 months so that her misconceptions in this test may be regarded as affording evidence of the point of view of the ordinary child of eleven or twelve. Betty is a very careful and conscientious worker and she decided as follows:-

Incorrect.	Correct.
$12 - 4 = 16 \div 2 = 8.$	$6 + 3 = 9 \div 3 = 3.$
$3 \times 4 + 4 = 4 \times 5 - 4 = 16.$	$18 \div 2 = 2\underline{)18}$ 9
	$2\frac{3}{4} - \frac{1}{4} = \frac{3}{4} - \frac{1}{4} = 2\frac{1}{2}$

It seems clear that she could not conceive a mathematical expression as a unity; to her, it was not a number but an instruction. Her "correct" statement is correct because the instructions in it have been carried out - the 6 has been added to the 3 to make 9, while the 9 has been divided by the 3 to give a quotient of 3. In her "incorrect" statements, some of the instructions may seem to have been ignored, hence their rejection.

Children seem to find it very difficult to conceive a vulgar fraction or an expression containing a symbol of operation as a simple number. They are in the habit of translating these symbols of operation into verbs in the imperative mood; thus "12 + 15" is rendered as "12 add 15". The child does not think of 12 + 15 as a simple number, namely 27, but as a process to be executed. Children's inability to think of an expression as a number is illustrated by their difficulty in solving the following easy problems: $\frac{25+24}{7}$, $(400 - 1) \div 7$. In the latter example, the child, knowing very well the function of the brackets, is perplexed, and does not know whether to subtract 1 from 400 and find how many sevens in 399 or to find how many sevens in 400 and then subtract one, that is, one seven.

Miss Renwick states that she is in the habit of introducing to her pupils the ordinary conception of any expression by setting an exercise in which they are required to insert in an equation the symbols of operation, thus:

$$17 \wedge 2 = 5 \wedge 3.$$

$$17 - 2 = 5 \times 3.$$

The statements always provoke a protest: "17 - 2 = 15 and you have written 5!"(2) This use of the sign is regarded as unnatural. Their habit is to write down expressions, conceived as instructions, <u>in the order in which they arise in the mind</u>. For example, a simple problem which is solved by means of the series of calculations: -

$$12 \times 3 = 36, \quad 36 - 6 = 30, \quad 30 \div 6 = 5.$$

is explained in writing by the child thus:-

(a) $12 \times 3 = 36 - 6 = 30 \div 6 = 5$ nuts each.

or

(b)
```
   12
    3
   36
    6
 6)30
    5 nuts each.
```

where (a) and (b) are regarded as equivalent modes of stating the argument.

(2) Reported in the British Journal of Education al Psychology," June 1932.

In this connection it is interesting to note the following extract [3] from an American arithmetic book, "The Columbian Arithmetician", published in 1811:

$$1+6, = 7, \times 6 = 42, \div 2 = 21.$$

A rather unusual use [4] of equality signs is found in a work of Deidier in 1740, viz. -

$$\frac{0+1+2}{2+2+2} \begin{matrix} = \\ = \end{matrix} \frac{3}{6} = \frac{1}{2}; \quad \frac{0.\ 1.\ 4.}{4,\ 4,\ 4,} \begin{matrix} = \\ = \end{matrix} \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$$

A curious use, [5] in the same expressions, of =, the comma, and the word "aequalis" is found in a Tacquet-Whiston edition of Euclid, where one reads, for example,

"erit $8 \times 432 = 3456$ aequalis

$8 \times 400 = 3200, + 8 \times 30 = 240, + 8 \times 2 = 16.$"

(ii) Results of Specific Examples.

Example No. 6: $18 \div 2 = \begin{array}{l} 2\underline{)18} \\ \quad 9 \end{array}$ The graph shows far more clearly than do the marks, the relative number of misconceptions of the "=" sign in connection with this type of example. It is evident that approximately 90% of children between the ages of 9 and 14 think that the use of the "=" sign is legitimate in examples of which

(3) Quoted by Cajori: "A History of Mathematical Notations," Vol. I, p. 307

(4) Quoted by Cajori: "A History of Mathematical Notations," Vol. I, p308

(5) Quoted by Cajori: "A History of Mathematical Notations," Vol. I, p. 308

$18 \div 2 = 2\overline{)18}$ with 9 below is a type. Why does this misconception exist? $18 \div 2$ is equal to 9 or $\frac{18}{2}$, but is certainly not equal to 2 in front of a right angle in which is 18, and below which is 9. The writer suggests that the trouble lies, not in the child's own misconceptions but in those of his teachers and the writers of his arithmetic books. This point will be elaborated later in the thesis but it is relevant here to note the widespread existence of this type of misconception among authors and teachers. Even at the age of 17, only 40% of children are free from this misconception.

Example No. 15: Scale 1 inch = 1 foot.

The graphical representation of the decisions of children in successive age-groups in connection with this example is very interesting. It shows that children of 9 years of age are not inferior, but slightly superior, to pupils of 12 - 17 years in their decisions <u>re</u> this example. The writer suggests that the reason may be found in the fact that prior to the age of 10 or 11, the word "scale" may never have been used in this connection, hence the children rely for their decisions on common sense. However, after the age of 10 or 11 years examples of this type are found in current text-books and in teachers' assignments of work and model answers, hence an erroneous idea is conceived by the pupil. Never can 1 inch = 1 foot. One inch may <u>represent</u> one foot.

The example in connection with which the greatest number of correct decisions were given, was the simple one, namely, 6d. x 2 = 1s. It is rather surprising that a single incorrect decision should exist, yet only 90% of children at the age of 17 tendered correct judgments.

Example No. 14: $2\frac{3}{4} - \frac{1}{4} = \frac{3}{4} - \frac{1}{4} = 2\frac{1}{2}$.

The results of this example, particularly in the age-groups above 12 years, are higher than the writer's experience in teaching would lead him to expect for this is a very common type of error found in the class-rooms of both primary and secondary school pupils The graph shows that pupils from 9 - 12 years frequently commit this type of error. The whole reasoning process is not grasped as a unity, probably because of the relatively immature nature of the minds of children under 12 years of age with regard to abstract thought. Children show a strong tendency to write down expressions <u>in the order in which they are evaluated</u>. There is no doubt that children regard this use of the sign as legitimate, the middle expression being, as it were, placed in parenthesis, by means of the signs separating them from the end expressions.

In general, ability in judging correctly the use of the symbol in such examples as $6+3 = 9 \div 3 = 3$, 1s. x 12 = 12d., $12 - 4 = 16 \div 2 = 8$, and $9+4 = 13 - 2 = 11$ seems to show gradual development through the age-groups, with a slight accentuation in the development in the groups comprising children of thirteen and fourteen years of age. This accentuation may be accounted for in two ways. Firstly, interest in abstract science usually begins to awaken about this age, that is, at the onset of adolescence, and secondly, at about this age most children are embarking upon higher mathematical studies, algebra in particular. In such studies the child's mathematical ideas take more definite shape in his mind. He begins to see reason in many processes which before were unintelligible and mechanical to him. However, an analysis of the decisions in the 13-14 year age-group reveals that 38 of the 78 pupils tested in this group incorrectly considered the use of the "=" sign in "$12 - 4 = 16 \div 2 = 8$" to be illegitimate, as likewise did 15 of the 55 pupils tested in the 14-15 year age-group. This shows that misconceptions of earlier years had persisted in spite of the fact that most of these pupils had studied simple equations for two years in the case of some pupils and over one year in the case of most pupils. Their skill in solving simple equations had obviously been acquired as any mechanical skill is acquired, by imitation and practice. There are many children who learn mathematics in this fashion.

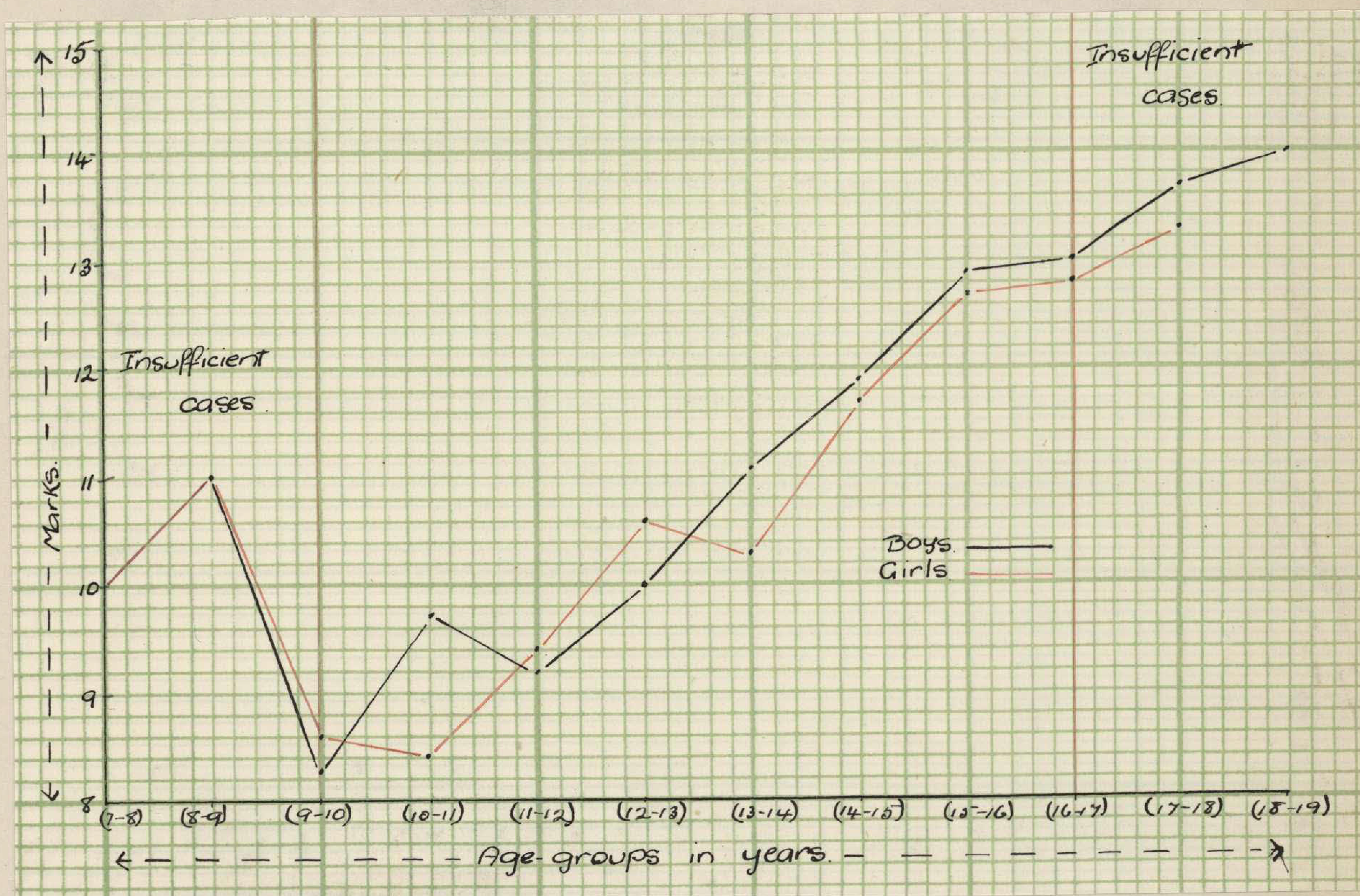
(e) Sex-differences.

(i) Results.

Co-educational schools with classes containing as far as possible an equal number of both girls and boys were selected for testing, to discover whether and to what extent there existed a difference in the ability of the sexes in regard to the test. The following table presents a summary of the results:-

Years	Total Marks		No. of pupils.		Average	
	Boys	Girls	Boys	Girls	Boys	Girls.
7 - 8	-	10	-	1		10
8 - 9	11	55	1	5	11	11
9 - 10	141	130	17	15	8.3	8.6
10 - 11	233	135	24	16	9.7	8.4
11 - 12	276	263	30	28	9.2	9.4
12 - 13	339	402	34	38	10.0	10.6
13 - 14	468	370	42	36	11.1	10.3
14 - 15	392	258	33	22	11.9	11.7
15 - 16	425	140	33	11	12.9	12.7
16 - 17	260	115	20	9	13.0	12.8
17 - 18	82	40	6	3	13.7	13.3
18 - 19	28	-	2	-	14.0	—

Graphical representation of the results may be more lucid and illuminating. Owing to the insufficiency of cases tested in the two lowest and the two highest age-groups we must discard the results.



(ii) <u>Discussion of Results.</u>

The general trend of the graph shows that up to and including the age of 12 years, boys and girls are credited with practically equal ability in performing the test. Admittedly, between the ages of 9 and 12 years the lines on the graph representing the scores of each sex do not coincide, but they cross and re-cross, thus showing that neither sex, as yet, has marked superior ability.

However, after the age of 13 years boys seem to have superior ability - though slight, it is definite and consistent - in the tests. These results cannot be regarded as definitely conclusive as insufficient cases of each sex were tested while the ratio of the number of girls to boys tested is 3 : 4. However, it cannot be denied that the results are very suggestive and seem to agree with the popular belief that boys are superior to girls in higher mathematics. This opens up a very wide field and raises the question whether difference in ability is attributable to innate ability, to practice or even to attitude. Our local press reported quite recently that as the outcome of extensive investigations Dr. Cyril Burt is convinced that there is no difference in intelligence between man and woman. However, mathematical ability may quite easily be a special ability - 's' in Spearman's phraseology - which may vary in individuals and in the sexes. In such a case the importance of general intelligence in determining ability in mathematics would be relatively small.

It is interesting and significant to note that the average of the marks of all the boys between the ages of 9 and 17 is 10.9 while the average for the girls of corresponding age is only 10.4.

An explanation of the high marks recorded for the groups containing pupils of 7 and 8 years of age has already been proffered. (See * * * p. 19).

B.

INVESTIGATION II.

B. INVESTIGATION II.

(a) The Nature and Scope of the tests.

The purpose of this investigation is to discover:
(1) if failure to assign the correct meaning to the symbol "=" has its correlative in failure to use correctly the word "equal";
(2) if the meaning of "equal" tends to become assimilated to that of "same".

Miss Renwick observes: "This difficulty (the difficulty in using the word 'equal') is observable in girls of twelve and thirteen years of age. Their uncertainty as to the meaning of the word may be inferred both from the reluctance to use it which some girls exhibit, and from the tendency on the part of others to regard it as interchangeable with 'same' or 'alike'."(6)

Beginners in geometry are often vague in their interpretation of the simplest of statements. Thus, in the test, the pupils had to compare two equal lines, DA and DB. Many of their statements were expressed as follows: "From D to A is the same as from D to B."

The meaning of the word "equal" is often assimilated in ordinary class-room practice to that of the familiar word "same" with consequent error of using it to describe any kind of resemblance, as "equal shapes",

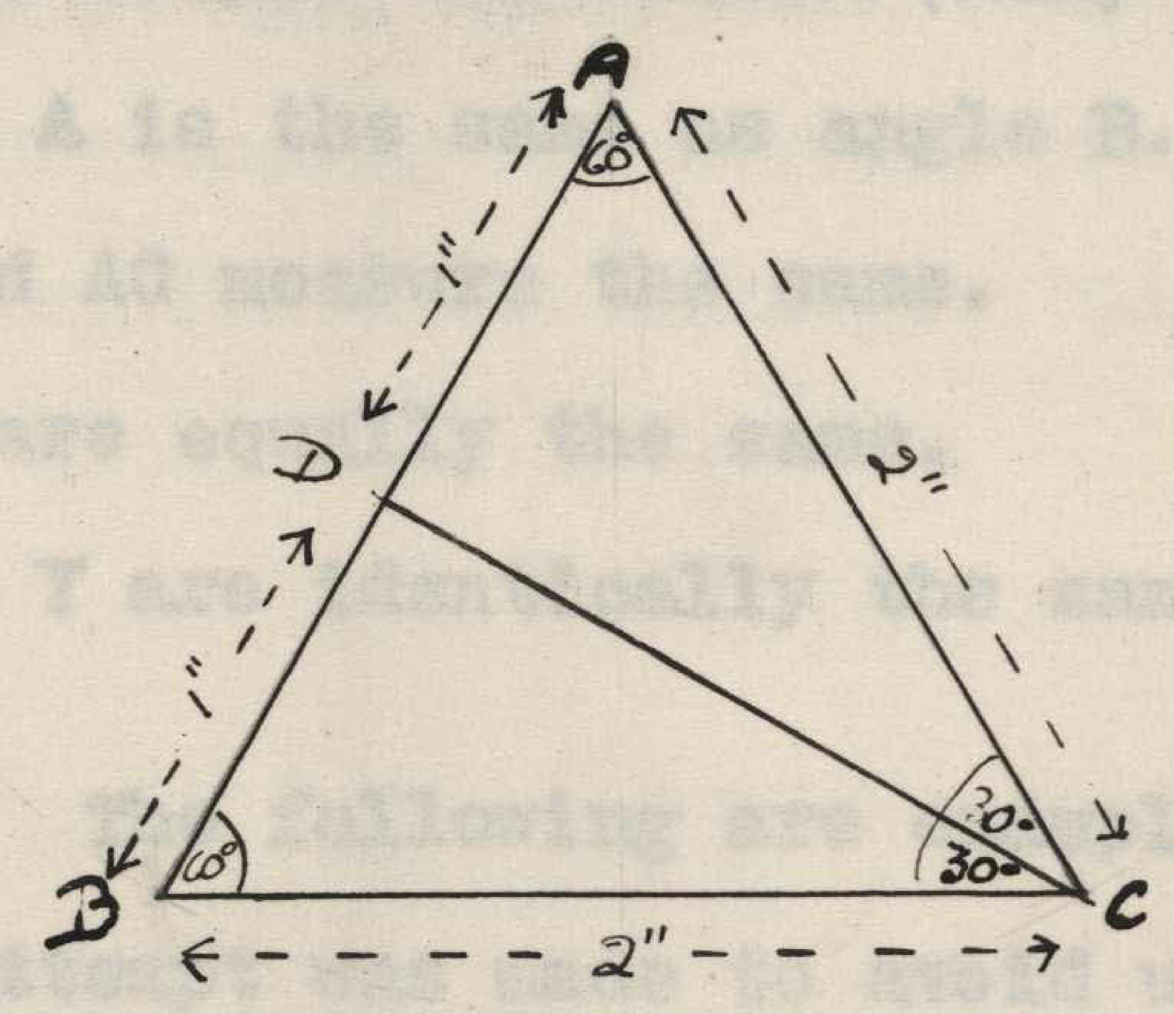
(6) Reported in the "British Journal of Educational Psychology," June 1932.

"equal directions" and "equal colours". The quantitative concept seems to develop very slowly as the results recorded below indicate.

The tests were submitted to Forms I and II in the primary schools and to all classes in the secondary schools.

(b) Description of Tests and Results.

(i) Test I. The following is the form in which the test was expressed:



Note the above diagram. Write a sentence about each of the following. (The word "same" must **NOT** be used.)

(1) Length of DA compared with length of DB.

(2) Length of BC compared with length of DA.

(3) Length of BC compared with length of AC.

(4) Size of angle A compared with size of angle B.

(5) Size of angle X compared with size of angle Y.

(6) Size of angle A compared with size of angle Y.

In spite of the prohibition many pupils used the word "same", in most cases incorrectly. Some of these cases are quoted: (All spelling errors have been corrected)

(1) When BC and DA are compared they are found to be the same.

(2) BC and AC are the same. (Many cases).

(3) Angle A is the same as angle B.

(4) BC and AC measure the same.

(5) They are equally the same.

(6) X and Y are identically the same.

The following are examples of answers in which an attempt was made to avoid using the word "same". Some of these answers illustrate the children's vague ideas concerning the true meaning of "equal" (or "equally")

(1) The length of BC is exactly twice as much as DA. They are equal.

(2) DA is equal to the angle of DB.

(3) It is equal.

(4) The two sides are equal in degrees.

(5) A is equally the equal length of B.

(6) The lengths are similar.

(7) Each angle is equal.

(8) The length of DA is equally like DB.

(9) The length of BC is equally compared to that of AC.

(10) DA is equally as long as DB.

(11) They correspond equally.

(12) DA is twice as far as DB.

These statements, made by pupils even of 17 and 18 years of age, illustrate the difficulty children have in understanding the correct meaning of the word "equal". Very few wrote simply: "DA and DB are equal."

(ii) <u>Test II</u>.

The same diagram as employed in Test I of this investigation applies also in this test. The following are the instructions: (Write another sentence about each of the following - do <u>NOT</u> use the word "equal")

(1) Length of DA compared with length of DB.
(2) Length of BC compared with length of DA.
(3) Length of BC compared with length of AC.
(4) Size of angle A compared with size of angle B.
(5) Size of angle X compared with size of angle Y.
(6) Size of angle A compared with size of angle Y.

The following are examples of answers in which an attempt was made to avoid using the word "equal":

(1) DA and DB are both the same. (Numerous cases).
(2) Length of BC is identical to DA.
(3) BC and AC are like lines and have a common vertex.
(4) They are both alike. (Several cases).
(5) Angle X is similar to angle Y.
(6) Angle A is even to angle B.
(7) BC is the alike with AC.
(8) They are not corresponding.
(9) BC is the exact length of AC. (Several cases.)

(10) DA and DB are even in length.

(11) DA is the same to DB.

(12) A's angle is like B's angle.

An analysis of these answers reveals the fact that when the word "equal" is avoided either "alike" or "same" is substituted for it in formulating comparisons of equal quantities. During the marking of the tests the writer found that the answers to tests I and II were practically identical, only "equal" was used in the answers of the first test and "same" was used in the answers of the second test. For example:-

BC and AC are equal.

BC and AC are the same.

Thus the two words are regarded by most children as synonymous and therefore interchangeable.

It is surprising to note the high percentage of pupils even at the age of 18 years who apparently find nothing wrong in such statements as: "Angles A and B are the same." They do not realize that the only point of similarity is the <u>size</u> of the angles. Just as two individuals can never be the same, except in respect, for example, of height or weight, so no two lines or angles can be the same except in respect of length of line or size of angle.

(iii) <u>Test III</u>.

The purpose of this test is to discover if children used the appropriate word by which to express the particular type of similarity concerned in certain comparisons.

The instructions informed the pupils that they were to reconstruct the given sentences using the word "equal" (or "equally") in one reconstructed sentence, <u>if it could be used legitimately</u>, and the word "same" in another, with the same proviso. The following illustrative example was given:

A is just as heavy as B. (1)(A's weight is equal to B's
(A and B are equally heavy.
(2) A's weight is the same as B's.

Some of the given sentences had no close relation to their exercises in mathematics; they were included in the test because they provided examples of conventional uses of the words. Thus example (c) deals with a comparison of the direction in which two main streets in the local city run.

In the following account each given sentence is followed by samples of the children's attempts to reconstruct it, illustrating the difficulty they experienced in choosing between singular and plural nouns and verbs, in inserting suitable prepositions and conjunctions, and in selecting the appropriate word by which to express the particular type of similarity concerned in the comparison.

(a) A is just as clever as B.

(1) A is equal to B.

(2) A is the same as B. (Several cases)

(3) A is as equally clever as B.

(4) They are equally the same.

(5) A is equal clever to B.

(6) A is just the same clever as B.

(7) A's cleverness is the same as B's.

(b) The field is just as big as the paddock.

(1) The field is the same as the paddock. (Many cases, even among children of 17 years of age)

(2) The field is equal to the paddock.

(3) The field is equally as the paddock.

(4) The field is as equal as the paddock.

(5) The field is the same largeness as the paddock.

(6) The field is nearly the same paddock.

(7) The size of the field and paddock are the same.

(c) Willis Street runs due north and Cuba Street runs due north.

(1) Willis Street and Cuba Street go in equal directions. (Many cases).

(2) The direction of Willis Street is equal to that of Cuba Street. (Several cases, even among children of 16 years of age.)

(3) Willis Street is equal to Cuba Street.

(4) Willis Street is the same as Cuba Street.

(5) Willis Street runs equally the same as Cuba Street.

(6) Willis Street and Cuba Street run the same.

(d) There are just as many peas as there are pencils.

(1) There are the same amount of peas and pencils

(2) The peas equal the pencils. (Several cases, even among children of 15 years of age.)

(3) Peas are the same as pencils.

(4) There are equal peas and pencils.

(5) The peas and pencils are equally the same.

(6) There are just an equal size of peas as of pencils.

(7) There are same peas as there are pencils.

(e) A runs as fast as B.

(1) A and B run equal speeds.

(2) A and B are the same.

(3) A is equal as B.

(4) B's running is the same as A's.

(5) A and B run equally in fastness.

(6) A runs equally as fast as B.

(7) A and B are equally the same.

(f) A takes just as long to do it as B does.

(1) A and B take equal times to do it.

(2) A equals B. (Several cases among children of 14 years of age.)

(3) A and B run the same.

(4) A takes equally as long as B does.

(5) A's longness is equal to B's.

(6) A and B are equally the same.

(g) A is egg-shaped and B is egg-shaped.

(1) A and B are equal in shape. (Several cases, even among children of 17 years of age.)

(2) A and B are equally alike.

(3) A and B are equal.

(4) A's shape is equal to B's shape.

(5) A and B are equally shaped.

(6) The perimeter of A equals that of B at similar points.

(7) A is egg-shaped as equal as B is.

(8) A is equal to B.

(9) A is the same as B.

(h) A is 9 inches high and B is 9 inches high.

(1) A and B are equally the same height.

(2) The height of A and B are equal.

(3) A and B are equally 9 inches.

(4) A and B are equal.

(5) A is the same as B.

(6) A and B are of equal height.

(i) The cup is just as hot as the plate.

(1) The cup equals the plate in being hot.

(2) The cup is just as equal in heat as the plate.

(3) The cup is equal hotness to the plate.

(4) The cup and the plate are the same hotness as each other.

(5) The cup is equal to the plate.

(6) The cup is the same as the plate.

(7) The cup is as equally hot as the plate.

(j) A is blue and B is blue also.

(1) A and B are of equal colour.

(2) A and B are equally blue. (Several cases, even among pupils of 16 and 17 years of age.)

(3) A's colour is equal to B.

(4) A and B are of equal blueness.

(5) A and B are equally coloured.

(6) A and B are equally colours.

(7) A equals blue, B equals blue.

(8) A and B is an equal blue.

(9) A is equal to B.

(10) A is the same as B.

(11) A is equally like B in their bluedness.

(k) A is just as good as B.

(1) A's goodness is the same as B's.

(2) A is the same in behaviour as B.

(3) The goodness of A and B are the same.

(4) A is the same as B.

(5) A is equal to B.

(6) A is the same betterness as B.

(7) A and B's goodness is the same.

(8) A's credit is the same as B's.

(l) The cup holds one pint and the mug holds one pint.

(1) The capacity of the cup and mug is equal.

(2) The 1 pint cup equals the 1 pint mug.

(3) The cup is equal to the mug.

(4) The cup holds a equal pint like the mug.

(5) The cup and mug holds equally the same amount.

(6) The cup is the same as the mug.

These samples of the children's attempts to reconstruct certain given sentences illustrate their difficulty in choosing between singular and plural, nouns and verbs, in inserting suitable prepositions and conjunctions, <u>and in selecting the appropriate word by which to express the particular type of similarity concerned in the comparison</u>. Thus, we find "equal" being used in describing resemblances where neither degree nor quantity is involved. The words "equal" and "same" are regarded as synonymous for children think it is just as correct to say "equal colour" and "equal shape" as "same colour" and "same shape". These types of error are not peculiar solely to children for we find the word "equal" used incorrectly, for example, in a text-book on Education, as follows:-

> "One class gives the following results in a dictation test: ....
>
> Another equal class gives the following:.."(7)

The only respect in which the two classes are equal is in <u>the number of pupils</u>, and this factor is not stated.

(7) Hayward: "The Psychology of Educational Administration and Criticism", p.245.

C.

INVESTIGATION III.

C. INVESTIGATION III.

(a) The Nature and Scope of the Test.

It was thought that the tests in Investigation II were too difficult for pupils in the primary schools, or, at least, for pupils in Standards III and IV. However, it is just as important, and probably more important to obtain results from younger children as from the older pupils. Hence, a simplified summary of the tests of Investigation II was embodied in the form of this investigation - Investigation III - which was submitted to children of ages ranging from 7 to nearly 19 years. In addition, a brief summary - in the form of one question- of Investigation I was included in this test.

Another purpose of this investigation is to obtain results of Investigation II which may be expressed statistically. The form in which the tests in Investigation II were expressed made it impossible to obtain such statistical results.

Thus, in general, the aims of this investigation are to discover:

(1) The number of children at different age-levels who understand the true use of the "=" sign.

(2) The number of children at different age-levels who can distinguish between the correct and the incorrect uses of the words "equal" (or "equals") and "same".

(b) <u>Description of the Test.</u>

The children were asked to cross out the wrong statement in each of the following pairs: -

* (a) The "=" sign is used : (1) To separate two quantities,
  <u>or</u>,
  (2) To join together two equal quantities. *

* (b) Which is correct: (1) 9 + 5 equals 14,
  <u>or</u>
  (2) 9 + 5 is the same as 14?

* (c) Which is correct: (1) All squares are equal in shape
  <u>or</u>
  (2) All squares have the same shape?

(c) <u>Results of the Test.</u>

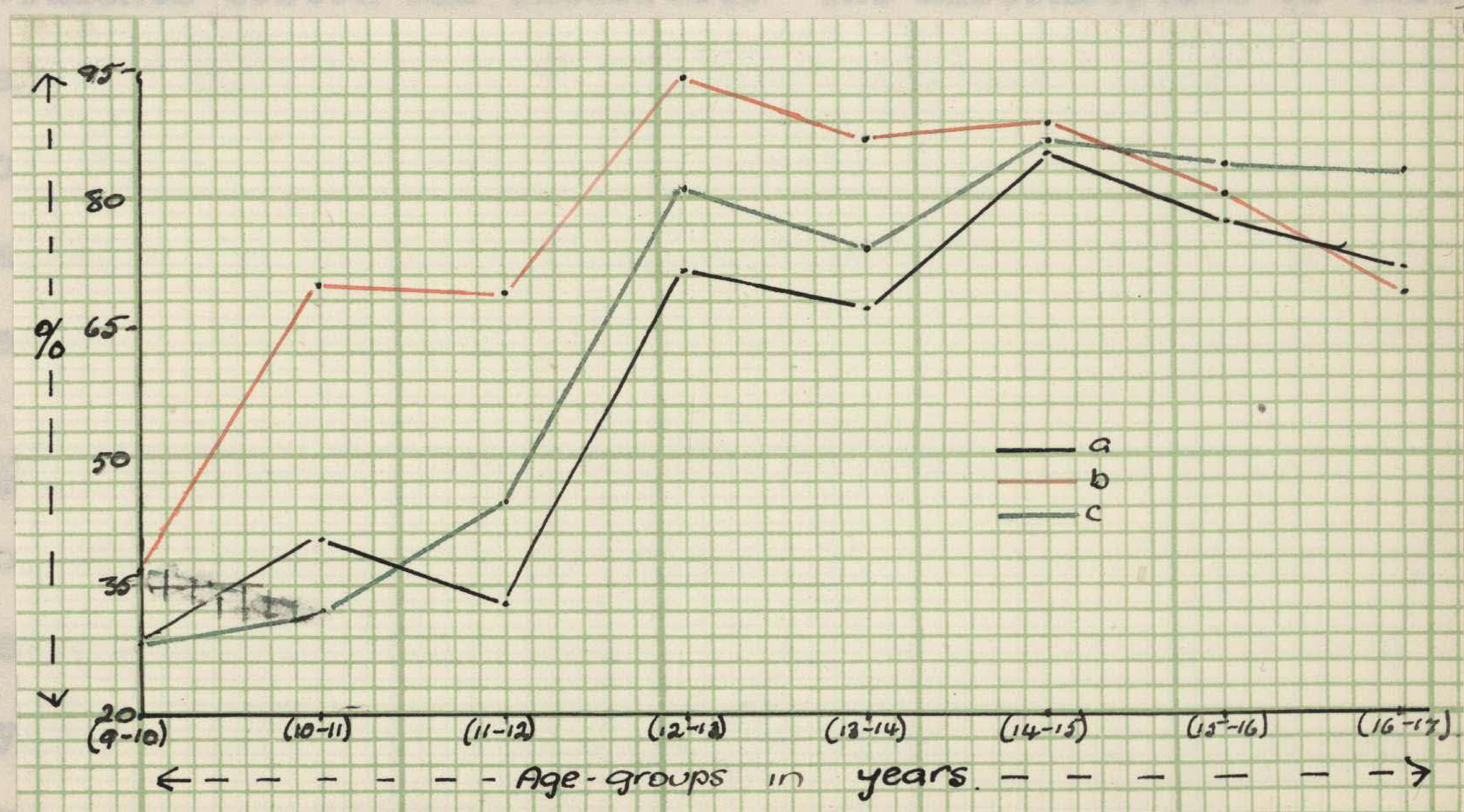
The following table indicates the number of pupils in each age-group who were correct in their decisions for each example. The number of pupils in each age-group is shown in brackets.

	7	8	9	10	11	12	13	14	15	16	17	18 19
Example	(1)	(6)	(32)	(40)	(58)	(72)	(78)	(55)	(44)	(29)	(9)	(2)
A	0	1	9	16	19	40	52	47	34	21	7	1
B	0	5	12	28	40	60	68	49	35	20	8	2
C	1	2	9	13	26	42	58	48	37	24	9	2

For the purpose of showing increase or decrease in the ability of children of different age-levels to judge correctly in the test, the following table is presented. In this table the results of the test are expressed on a common basis, namely, a percentage basis. Owing to insufficiency of numbers we must ignore the results prior to the age of 9 years and after the age of 16 years.

Example	9	10	11	12	13	14	15	16	17 years.
A	28	40	33	72	67	85	77	72	
B	37	70	69	94	87	89	80	69	
C	28	32	45	81	74	87	84	83	

The following is a graphical representation of these results.



(d) Interpretation of Results.

Example (a). - It is evident that up to the age of 12 years, at least, most children regard the "=" sign as a symbol of distinction. Its use, from the child's point of view, is merely to separate two quantities, or to separate an expression from its answer. These results support the contention that pupils who do not understand the true use of the "=" sign cannot understand the logical bases of most mathematical or arithmetical reasoning, particularly of the process of solving an equation. Any skill they have acquired in the subject has been the result of the process by which any mechanical skill is acquired, namely, by imitation and practice.

It is interesting and significant to note that the judgment in this example of one of the two 18-year-old students tested was incorrect. The misconception of this student's earlier years had therefore persisted, in spite of the fact that this pupil was in Form VI, had studied algebra for at least three, and probably four or more years, and was credited with the University Entrance Examination.

Example (b). - This example is undoubtedly the easiest of the three, yet it is surprising to note the number of erroneous decisions made. Only 37% of the pupils of 9 years of age were correct in branding "9 + 5 is the same as 14" as "wrong" and "9 + 5 equals 14" as "correct", while it is more significant to note that even at the age of 16 years 31% of the children were still incorrect in their decisions. This shows that in school, teachers have a great tendency to take too much for granted. The simplest of facts are sometimes not clearly understood even by students who are on the eve of entering a University.

Re this example, there seems to be little appreciable development of ability in children after the age of 10 years.

Example (c). - Incorrect judgments in this example are due to misunderstanding of the true meaning of the word "equal", which tends to become assimilated to that of the familiar word "same". In such cases the children fall into the error of using "equal" to describe any kind of resemblance, be it colour, shape or direction. The quantitative concept seems to develop very slowly and even at the age of 16 years the judgments of one-fifth of the pupils are incorrect.

In general, we note that each graph shows periods of great progress followed by "plateaus" or even periods of slight retrogression. Initial progress seems to be made at about the age of 10 years, after which follows a plateau in two of the three examples. Then comes the greatest period of progress at about the age of 12 when each graph shows a phenomenal rise. Then comes a period of slight retrogression followed by a further period of progress at about the age of 14 years. After this there seems to be no improvement, but rather retrogression in each case, till after the age of 16. At this juncture we should expect another wave of progress, and the results from children of 17 years of age, though inconclusive through insufficiency of cases, are definitely suggestive in this direction. Hence, it seems that the development of the ability to make correct judgments in this test follows the normal curve of learning, that is, periods of progress alternate with periods during which no apparent improvement takes place.

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III

FACTORS INFLUENCING THE FORMATION OF SUCH MISCONCEPTIONS IN CHILDREN'S MINDS.

III. FACTORS INFLUENCING THE FORMATION OF SUCH MISCONCEPTIONS IN CHILDREN'S MINDS.

Very little is achieved, if, having found that misconceptions of the symbols for mathematical equality exist in the minds of children, we do not follow the investigation to its logical conclusion by an attempt to discover possible causes of such misconceptions and to suggest possible remedies. This chapter, then, is concerned with the discovery of factors influencing the formation of such misconceptions in children's minds, and, for this purpose, the writer has carefully examined:

(a) The current arithmetic text-books in use in New Zealand.

(b) The "Syllabus of Instruction" issued by the Education Department of New Zealand.

(c) Assignments of work, and

(d) Model answers, compiled by the teachers of the Education Department's Correspondence School.

(a) The Current Arithmetic Text-Books in Use in New Zealand.

An examination of these books reveals the following illegitimate uses of the "=" sign, thus signifying that the minds of educated adults are not free from such misconceptions. One example, only, of each type of error committed in each class text-book will be mentioned. The current arithmetic text-books in use in the New Zealand schools are titled "The New Progressive".

Standard II.
(1) p.95.

$$69 \text{ inches} = 69 \div 12 = 5 \times 12 + 9 = 5\text{ft.}9\text{in.}$$

In the first place "69 inches" does not equal "69 ÷ 12", which is certainly unequal to "5 x 12+ 9", which is not equal to "5ft.9in." This use of the "=" sign leads to confusion.

Standard III.
(1) p.39. 27 + 12 = 39 oranges.

(2) p.56.

$$\begin{array}{r} 94 \\ \underline{4} \\ 376 \\ \underline{9} \\ 3384 \end{array} \begin{array}{l} \\ \\ = 4 \text{ times.} \\ \\ = 36 \text{ times.} \end{array}$$

4 times ? = 376. If this use of the "=" sign is allowed, there is nothing to prevent its being used, as it frequently is used by children, thus:-

(1) 224 x 16 =

$$\begin{array}{r} 224 \\ \underline{16} \\ 1344 \\ \underline{2240} \\ 3584 \end{array} = \text{Ans.}$$

(2) 3584 ÷ 224 =

$$\begin{array}{r} 224)3584(\ 16 = \text{Ans.} \\ \underline{224} \\ 1344 \\ \underline{1344} \\ \cdots\cdot \end{array}$$

Standard IV.

(1) p.14. 5 bales weigh (548 x 5) lb. = 2740 lb.

This is one instance of the many slipshod uses of the sign which occur fairly frequently in modern text-books.

This is a composite sentence obtained by running together the two statements: "5 bales weigh (548 x 5)lb.," and "(548 x 5) lb. = 2740 lb." It will be seen to bear a close resemblance to the statement "4 + 3 = 7 x 2 = 14," which most children would class as correct, believing that "4 + 3 = 7" and "7 x 2 = 14" could be combined to form a single statement.

This misuse of the sign is very common and is found in the text-books of every class. Throughout the remainder of this chapter such misuses will be denoted by "composite statement".

(2) p.14. 1 bale = 548 lb.

The **weight** of one bale may equal 548 lb. but "1 bale" certainly does not equal "548 lb."

(3) p.19. 2 times 7 = 14, and 7 = 21.

In this connection it is interesting to note the following extract [1] from an American arithmetic book, "The Columbian Arithmetician", published in 1811: 1 + 6, = 7, x 6 = 42, ÷ 2 = 21.

(1) Quoted by Cajori: "A History of Mathematical Notations", Vol.I, p.307.

(4) p.20. 1 bale contains 134260 ÷ 245 = 548 lb. (composite statement).

(5) p.115. 56 lb. = 1 bushel of maize.

Standard V.

(1) p.11. Giving change: 3s.6d. + 6d. = 4s. and 6s. = 10s. and 10s. = £1.

If this use of the sign is allowed, there is nothing to prevent its being used, as it frequently is used by children thus: -

$$123 \times 12 = 12 \times 100 = 1200 + 12 \times 20 = 240 + 12 \times 3 = 36.$$

(2) p.77 $\therefore$ 15s. will buy 15s. $\div \frac{21s}{7} = \frac{15 \times 7}{21}$ lb. $= 5$lb.

(a) Composite statement.

(b) How can shillings equal lbs?

(15s. $\div \frac{21s.}{7} = \frac{15 \times 7}{21}$ lb)

(3) p.63. How many pieces of ribbon $\frac{5}{8}$yd. long can be cut from $7\frac{1}{2}$yd.? (= 60 eighths ÷ 5 eighths).

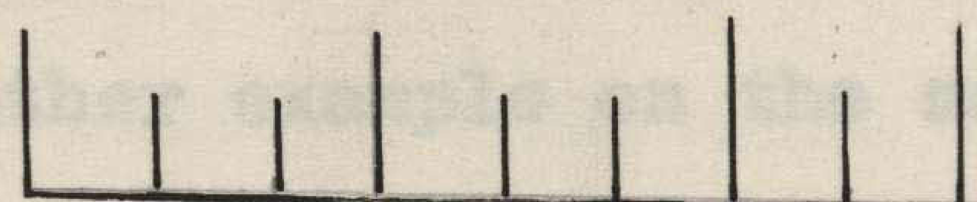
We may justifiably ask for what the "=" stands. Should we marvel at a child not understanding the legitimate use of the sign when his text-books display such a loose use of the sign?

(4) p.95. D.C.M. of 8, 9, 12 is $2^3 \times 3^2 = 72$.
(Composite statement).

Standard VI.

(1) p.26. Fraction of C.P. gained $= \frac{6d.}{2s.6d.} = \frac{1}{5} = \frac{5}{100} \neq 5\%$

This must be an error either of calculation or of printing for $\frac{1}{5} = \frac{20}{100} = 20\%$.

(2)  A.

A = 48 cwt. A = 60 ch.

A, a figure, may represent 48 cwt. but it can hardly equal 48 cwt.

(3) p.53. In 16 hours he would earn ? of £3.6.0. = ?
(Composite statement)

(4) p.98. ·5" = £1,000,000.

(5) p.106. Let x = son's age.

x can stand for a number only and as the statement stands, it means that x equals (say) 12 years. To be correct it should be (1) Let x years = son's age, then x years = (say) 12 years, so x = 12.

or (2) Let x = son's age in years. In this case x = (say) 12.

(6) p.116. 60 = 75%. We may justifiably ask: 75% of what is equal to 60?

(7) p.126. £56 per ton $= 5\frac{5+6+3}{14}$ d. = 6d. per lb.

This example bears a very close resemblance to the statement, "$2\frac{3}{4} - \frac{1}{4} = \frac{3}{4} - \frac{1}{4} = 2\frac{1}{2}$" where the two end expressions are equal but neither are equal to their connecting expression. Here is another example on the same page:

$$£75 = (7\tfrac{7}{14}\text{d.} + \frac{5+2\frac{1}{2}}{14})\text{d.} = 7\frac{7+5+2\frac{1}{2}}{14}\text{d. per lb.}$$
$$= 8\tfrac{1}{28}\text{d. per lb.}$$

(8) p.129. A triangle has its base = 25 chains, and perpindicular = 14 chains.

What a triangle! The sentence really means that the base <u>represents</u> 25 chains.

The following are examples, found in this book, of the misuse of the sign in connection with "scale":

p.134. On x axis 1" = 1 year.

On Y axis 1" = 10 lb.

p.135. Use as scale: 1" = 1 yd. or 1" = 2 yards.

After an examination of such cases it is not surprising we note the number of children whose judgments concerning "scale" were incorrect. What is more surprising is the number of educated adults who commit this type of error.

(b) The "Syllabus of Instruction".

An examination of the "syllabus" reveals the following errors:

(1) p.24. Drawing a post and its shadow to scale of 1in. = 1 ft., or $\frac{1}{2}" = \frac{1}{2}'$.

(2) p.120. 1 ch. cost $\frac{£112\frac{3}{4}}{205} = \frac{451}{4} \times \frac{1}{205}$.

(i) This is an example of a composite sentence for it contains the two statements, "1 ch. cost $\frac{£112\frac{3}{4}}{205}$" and "$\frac{£112\frac{3}{4}}{205} = \frac{451}{4} \times \frac{1}{205}$"

(ii) As the above statement stands the two expressions are not equal for brackets and the "£" sign are omitted from the expression on the right hand side of the "=" sign. The expression should read thus: $£(\frac{451}{4} \times \frac{1}{205})$.

(c) Assignments of Work.

These assignments of work, compiled by the ~~fully~~-qualified teachers of the Education Department's Correspondence School, may be taken as truly representative of the quality of the work of the majority of teachers in New Zealand. An examination of these assignments, with the purpose of discovering misuse and loose use of the symbol of mathematical equality, reveals the following errors. As the errors are numerous, only one example of each type is stated.

Standard II.

(1) How many pence in 4s.2d. =

(2) 3s.4d. ÷ 2.

Say: Twos in three = 1 + 1 shilling over

Put 1s. in answer and carry a shilling =

12d. + 4d. = 16d. Twos in 16d. = 8d.

There are several errors in this example. Here is one error: How can " put 1s. in answer and carry a shilling" equal 12d. + 4d.?

Standard III.

(1) 8 + 7 = 15.

18 + 7 = 25.

28 + 7 = 35 up to 98 + 7.

Standard IV.

(1) 2 times 3 = 6 and 2 carried = 8.

This bears a close resemblance to "3 x 5 = 15 + 2 = 17" which many pupils classed as correct.

Standard V.

(1) 34 boys get 34 quarter-oranges = $\frac{34}{4}$ oranges.
(composite sentence)

(2) £7.17.8 x 2 = 2 x £7 = £14; + 2 x 17s. = 34s; +
2 x 8d. = 1s. 4d.

In the writer's opinion this is a most serious error and gives the child a totally incorrect idea of the true meaning of the symbol. We cannot marvel at the strange uses to which the symbol is put by children when their teachers put the symbol to such an illegitimate use. If this use of the "=" sign is permitted there is nothing to prevent its being used - as happens occasionally - by children, thus: -

£7. 17. 8d. x 2 = £7 x 2
= £14.
= 17s. x 2
= £1.14.0.
= 8d. x 2
= 1s.4d.

(3) 15s. ÷ 20s. = $\begin{array}{r} 00.75 \\ 20\overline{)15.00} \\ \underline{140} \\ 100 \\ \underline{100} \end{array}$

This statement is very similar to the one, namely, "26 ÷ 2 = $2\underline{)26}$ / 13," which was employed in the test in Investigation I. As this error was committed by one teacher in the only school investigated we should be justified in assuming that this type of error is not infrequently committed by teachers. If such is the case we have a probable explanation of the fact that only 85 of the 426 pupils tested were correct in their judgments re this statement.

Standard VI.

(1) $2\frac{1}{2}$ cwt. = $\frac{1}{8}$.

One-eighth of what we might well ask.

(2) Commission on £100 = $6\frac{1}{3}\%$.

The rate of commission may be $6\frac{1}{3}\%$, but the commission on £100 must be expressed in terms of £. s. d.

(d) Model Answers.

These are compiled by the teachers of the Correspondence School and despatched to their pupils. A study of the answers reveals the following errors:

Standard III.

(1) 6 jars cost 1s.6d. x 6 = 9s.
(Composite sentence).

(2) 15 yards = 15 x 3
= 45 ft. + 1ft.
= 46 ft.

(3) 7lb.9oz. = 7 x 16
= 112 + 9oz.
= 121 oz.

Standard IV.

(1) 162 + 5s. = 20)167s.
£8. 7s.

Compare with the test example: "18 ÷ 2 = 2)18 "
9.

(2) Find the cost of 49 books at 1s.4d. each =
1s. + 4d. each.

(3) $\frac{3}{4}$ of 32 = £24.

(In other terms, 24 = £24.)

(4) 50 times = £$\frac{1}{4}$ = £12. 10. 0.

(5) 70 ch. + 70 ch. x 2 = 140 ch. x 2 = 280 ch.

Brackets should have been inserted in this statement to read as follows: (70ch. + 70ch.) x 2 = etc.

(6) $\frac{8}{12} = \frac{2}{3}$ = 24 inches.

(7) 3s.2d. x 4 = 12s.8d. + 1s.7d. for the $\frac{1}{2}$lb. = 14s.3d.

(8) ∴ 319 pints cost $3\frac{1}{2}$d. x 319 = $3\frac{1}{2}$d.

$$\begin{array}{r} 319 \\ \hline \underline{£4.13.0\frac{1}{2}d.} \end{array}$$

Standard V.

(1) 1 pint costs 2s.8d. ÷ 8 = 4d. (Compound sentence)

(2) 112 threepences ÷ 4 = 28 shillings.
(But 112 threepences = 28 shillings.)

(3) ∵ 93 cwt.2qr. ÷ 72 will be one-half of answer.

to b = $\begin{array}{r} 2\underline{)2cwt.\ 2qr.} \\ 1cwt.\ 1qr. \end{array}$

It will be seen that this example is very similar in principle to the statement, "18 ÷ 2 = $\begin{array}{r} 2\underline{)18} \\ 9. \end{array}$"

(4) $\frac{60}{8}$ = $7\frac{1}{2}$lb.

(5) $\frac{5}{12}$ of £18. 12. 6. = £18. 12. 6.

$$\begin{array}{r} 5 \\ \hline 12\underline{)£93.\ 2.\ 6.} \\ 7.\ 15.\ 2\frac{1}{2}d. \end{array}$$

(6) Gain = £30. 6. 8.

$$\begin{array}{r} - \underline{19.\ 3.\ 6.} \\ \underline{£11.\ 3.\ 2.} \end{array}$$ (The gain is £11. 3. 2.)

(7) 28 times $7\frac{3}{4}$s. = 28 times 7s. = 196s. + 28 times $\frac{3}{4}$s.

(8) £·95 = 9 florins = 18s. + 1s. = 19s.

(9) $\frac{1}{2}$ dozen at 42s.6d. = £2. 2. 6.

<u>Standard VI</u>.

(1) 12 doz. eggs @ $\frac{7}{8}$ of £1 = 17s. 6d.

Obviously there is an error in this statement. It should read thus:

Cost of 12 doz. eggs @ $\frac{7}{8}$ of 20d. per doz. = 17s.6d.

(2) $24\frac{1}{2}$ = ·245.

(3) 94·185 = 94·185 x 8 = 753·48 pints.

(4) 20% = $\frac{1}{5}$%.

(5) $\frac{1}{6}$ of 100 = $16\frac{2}{3}$ %.

As the statement stands it means that $\frac{1}{6}$ of 100 = $\frac{1}{6}$.

(6) Loss on 10d. = 1d. = $\frac{1}{10}$.

(7) 7 = 7 x 7 = 49.

(8) Tax = 8d. = £$\frac{1}{30}$ in £1 = £24. 3. 4.

(9) ·5 inches = 50 miles.

(10) $11\frac{1}{2}$d. = 11 x 4 + 2 + 1 = £·047.

(11) $\frac{49}{16}$ = $\frac{7}{4}$ = 1·75.

Obviously the statement should read thus:

$$\sqrt{\frac{49}{16}} = \frac{7}{4} = 1\cdot75.$$

(12) 10% = $\frac{1}{10}$ of 5s. = 6d.

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These are some of the most frequently committed types of errors found in the teachers' assignments of work and model answers. In most classes only one example of each type of error has been recorded because a full list of all examples exhibiting illegitimate uses of the sign would take up too much space. It must be remembered that the teachers' assignments of work and model answers have not been prepared hastily, but have been revised or re-drafted annually by the teachers for probably several years. Hence, we are justified in assuming that errors such as are noted above are due entirely to the existence in the teachers' minds of misconceptions concerning the "=" sign.

If such is the case, should we wonder if pupils have such misconceptions? It is significant to note that the teacher of nearly every class in the one school investigated, committed every type of error, as revealed by the tests, found in the children's decisions concerning the use - legitimate or illegitimate - of the "=" sign. Can we not conclude that children's misconceptions may be due to faulty teaching? To what extent will direct instruction remove such misconceptions? This problem will engage our attention in the next chapter.

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IV.

AN INVESTIGATION INTO THE VALUE OF INSTRUCTION IN REMOVING SUCH MISCONCEPTIONS.

IV. AN INVESTIGATION INTO THE VALUE OF INSTRUCTION IN REMOVING SUCH MISCONCEPTIONS.

A. The Nature and Scope of the Investigation.

It has thus been found by experimentation that there exists in children's minds only a very hazy and indefinite conception of the true meaning and legitimate use of the symbol of equality. We have also noted that the existence of such misconceptions greatly detracts from the value of, and goodness to be derived from, a study of mathematics. What can be done, then, to remove these misconceptions, in order to derive the utmost goodness from mathematical studies? We note that even text-book editors and teachers are not free from such misconceptions and as these two classes of people have such influence upon the developing mind of the child, it follows that our editors and teachers should be our first pupils in the lesson whose purpose is to build up a true conception of the strict meaning and legitimate use of the "=" sign. As a result, we should expect a lessening and a final eradication of children's misconceptions. However, as it is not possible, for the purpose of this thesis, to "teach" our editors and teachers, we shall inquire to what extent children's misconceptions can be removed through direct instruction. This, then, is the purpose of this investigation.

B. The Investigation.

(1) Test I.

(a) Description of the Test.

The same test as was used in Investigation I was used in this inquiry and was submitted to an ordinary mixed class of approximately 50 Form I pupils over 95% of whose ages ranged from 11 to 14 years. The technique of marking employed in this inquiry was the same as that described in Investigation I, so a comparison of the results of the two investigations could quite legitimately be made.

The following is a brief outline of the form taken by this inquiry:

(1) Before instruction the test was submitted to the pupils and their marks recorded.

(2) Three days later a comparatively short lesson of approximately 25 minutes was taken on the meaning and use of the symbol of equality. The lesson was conducted on the following lines:

(a) What is the correct use of this sign (=)?

Many strange answers were given. For example, (i) The "=" sign is used in the answer. Thus, Ans. = 25.

(ii) It is used instead of "are".

Apparently the child was thinking of his tables. (7 x 8 = 56, seven eights are fifty-six.)

(iii) The "=" sign is used to separate two lots of figures.

Only one child hinted at the true meaning of the symbol, namely, to join together two expressions that are quantitatively equivalent.

(b) Several examples, similar in principle to those employed in the test, were written on the blackboard. In such examples the children were asked if the symbol was used legitimately, and their reasons for their decisions were requested. Effort was made by theteacher to make the children understand that in the example "12 - 4 = 6 + 2," "12 - 4" is a <u>unit</u>, namely, 8, while "6 + 2" is similarly a unit and not just two figures joined by an addition sign. Most of the principles represented in the original test were discussed, and the legitimate and illegitimate uses of the "=" sign revealed.

(c) The children with the help of the teacher formulated the following definition of the function of the "=" sign: To join together two expressions quantitatively equivalent.

(d) Applications of this principle to specific cases illustrating both legitimate and illegitimate uses of the sign were then made.

(3) Five days later the essentials of the lesson were recapitulated.

(4) The same test was submitted to the same pupils three days after the recapitulatory lesson, that is, 8 days after the original lesson and 11 days after their first acquaintanceship with the test. The technique of marking employed was the same in the children's two attempts, that is, the same as that adopted in Investigation I.

(b) Results of the Tests.

The following table gives the results of the test submitted to the class before instruction:

The average mark is 10.

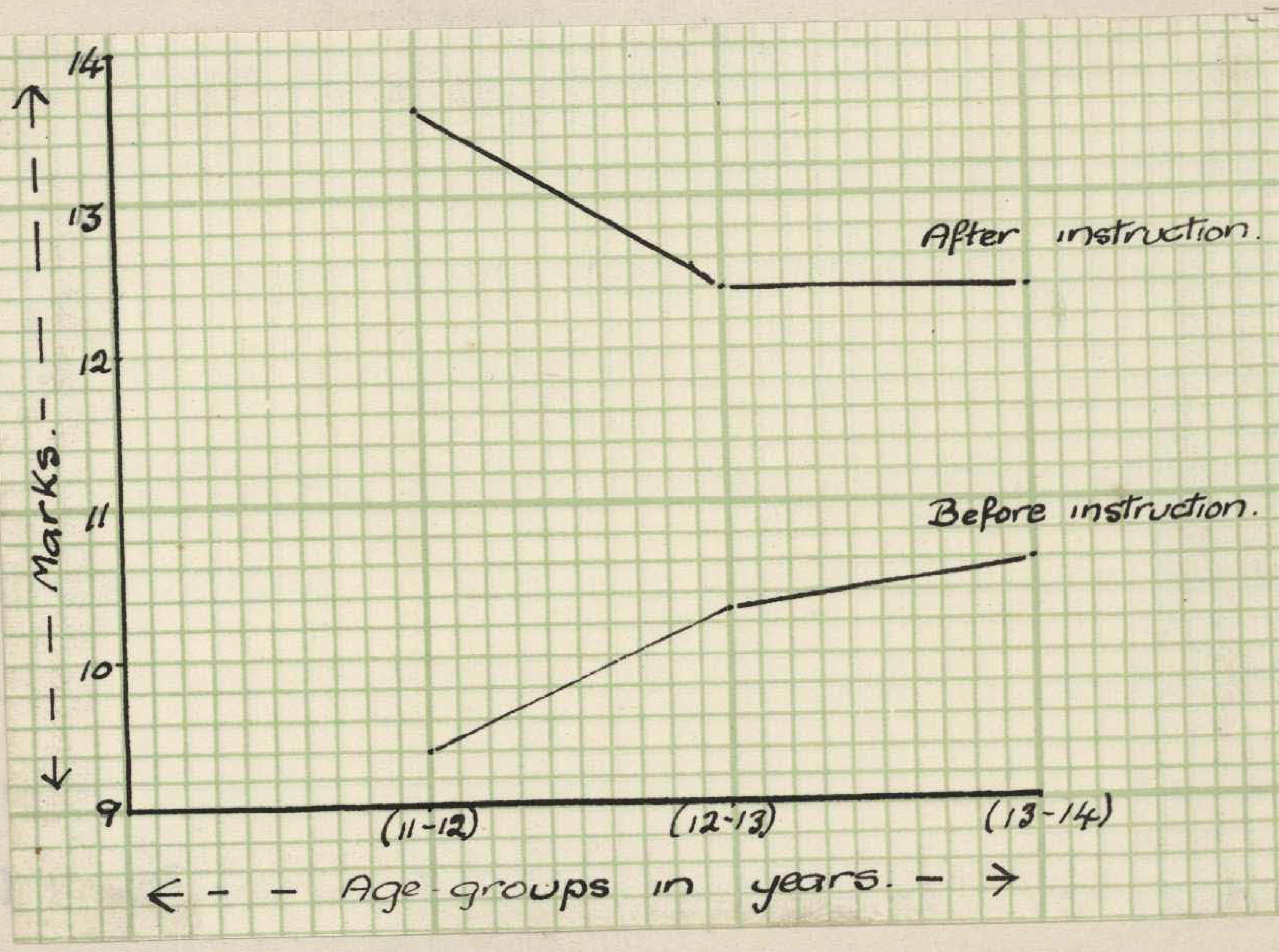
Age-groups	No. of pupils	Total marks	Average.
11 - 12 years	20	188	9.4
12 - 13 "	21	217	10.3
13 - 14 "	10	106	10.6

The following table gives the results of the same test submitted to the same class after instruction:

Age-groups	No. of pupils	Total marks	Average
11 - 12 years	20	272	13.6
12 - 13 "	21	261	12.4
13 - 14 "	10	124	12.4

The average mark after instruction is 12.9.

The following is a graphical representation of the results before and after instruction:



(c) Discussion of the Results.

An average improvement of 2.9 marks is effected in the scores of the pupils in the test as a result of instruction. This amount of improvement expressed in percentage terms is 29 and has been accomplished in the space of one short lesson while such improvement without direct instruction would have taken approximately 4 years to be effected, for the average mark obtained from the testing of uninstructed pupils of

16 years of age is the same as the average mark of this class after instruction, namely, 12.9.

A closer analysis reveals the fact that through the instruction the pupils of 11, 12 and 13 years of age gained respectively approximately 6 years, $2\frac{1}{2}$ years and $1\frac{1}{2}$ years. The graph shows very strikingly, that while the older pupils in the class were superior before instruction they were inferior to the younger members ofthe class after instruction. Thus the range of marks before and after instruction for the 11, 12 and 13 year-old groups were respectively 9.4 - 13.6, 10.3 - 12.4, and 10.6 - 12.4. This phenomenon is rather startling and can probably be explained on the following lines. These younger pupils of 11 years of age, to be in Form I, must be fairly bright and have more than average intelligence while the pupils of 13 years of age are, generally speaking, not so bright and intelligent, otherwise we should expect to find them in a higher class, probably Form III. Thus, it seems reasonable to suppose that the younger and brighter children, though behind the older and duller pupils in educational age, yet are more rapid learners. The amount of progress or improvement effected by the pupils of 11, 12 and 13 years of age was respectively 45%, 20% and 17%.

That the 12 and 13-year-old groups contained some dullards is borne out by the fact that some marks in these groups, even after instruction, were as low as 5.

An analysis of the results reveals a decided improvement in the number of correct decisions re the use - legitimate or illegitimate - of the "=" sign in each example. Example number 6, "$18 \div 2 = 2\overline{)18}$," (quotient 9) as was the case prior to instruction, produced the most erroneous decisions, about half the class regarding the use of the sign in this example as legitimate.

(ii) Test II.

(a) Description of the Test.

The same test as was used for Investigation III was employed in this enquiry. (see p.52 * * *)

(b) Results of the Test.

The following table presents statistical results of the test submitted to the class before instruction. The numbers indicate the number of children in each age-group who were correct in their decisions re each example. The numbers in brackets indicate the number of pupils in each age-group.

Example	11 years 12 (20)	12 13 (21)	13 14 (10)
A	7	10	6
B	13	15	8
C	9	10	6

The following table presents the results after instruction:

Example	11 years 12 (20)	12 13 (21)	13 14 (10)
A	17	17	9
B	19	20	9
C	15	13	7

For the purpose of comparison it is necessary to have the results expressed on a common basis - a percentage basis.

Before Instruction:

Example	11 years 12	12 13	13 14
A	35	48	60
B	65	71	80
C	45	48	60

Average mark: 57.

After Instruction:

Example	11 years 12	12 13	13 14
A	85	81	90
B	95	95	90
C	75	62	70

Average mark: 83.

(c) Discussion of Results:

The difference of 26 in the averages of the test before and after instruction represents an improvement of 46% due to the value of instruction. As in the first test of this investigation we find that the amount of improvement recorded by the younger pupils in the class is far greater than that recorded by the older, and probably duller, members of the class. However, every child benefited considerably from the instruction.

C. CONCLUSION.

We can justifiably conclude that the benefits accruing from instruction re the meaning and use of the symbols for mathematical equality are unquestionably substantial as the improvement effected by means of one short lesson is equivalent to that produced in approximately 4 years by indirect means. It is not difficult to imagine how much confusion and vagueness in the child's mind could be remedied and how much trouble and effort on the teacher's part could be avoided by means of direct instruction. It would be interesting, and undoubtedly significant, if one could watch the progress of these pupils who received instruction to note if their mathematical reasoning reflected their greater knowledge and understanding of the real meaning and legitimate use of the symbol of equality. This study would involve observation over a considerable period of time so is impossible of accomplishment for the purpose of this thesis.

In spite of misuse for years of the symbol by children and encouraged by similar misuses on the part of editors and teachers, direct instruction has proved sufficiently potent to effect an improvement of 29% in the case of Test I and 46% in the case of Test II during the course of one short lesson. Does it not seem reasonable, then, to suppose that direct teaching, in a very short time, could remove all misconceptions concerning the symbols? Despite the impracticability of destroying at once misconceptions in the minds of our editors and teachers,we claim that an immediate solution of our problem, lies through direct instruction the potent effects of which are undeniable.

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V.

RESULTS. CONCLUSIONS AND SUGGESTIONS.

V. RESULTS, CONCLUSIONS AND SUGGESTIONS.

(a) Summary of Results and Conclusions.

(1) Among children up to the age of 17 years there are many who find unintelligible a statement that one numerical expression is equal to another. This appears to be due to the fact that their early training in arithmetic leads them to look upon any symbol of operation as an imperative, and this prevents them from conceiving a numerical expression as a unity. They use the "=" sign simply to separate an expression from its answer.

(2) Up to the age of 17 there is a direct relation between the increase of age in the pupils and development of ability to understand the meaning and use of the symbols for mathematical equality. Insufficient cases were treated after the age of 17 to give reliable results but up to this age, at no period were the symbols properly understood - misconceptions existed at 16, 17 and 18 years of age although in a slightly decreasing degree.

(3) The tests reveal that after the age of 12 boys seem to display definite superior ability to girls of corresponding age, yet prior to this age the sexes have approximately equal ability. This difference is hardly due to difference in teaching as the tests were submitted to classes containing both sexes in co-educational schools. It seems likely

then, that the difference in ability is attributable to difference in innate qualities which determine the attitudes, interests and capacities peculiar to each sex.

(4) Children's misconceptions of the symbols seem to be due chiefly to faulty teaching per medium of text-books and teacher. An analysis of current Arithmetic text-books and teachers' assignments of work and model answers reveals the extent to which reputedly educated people - editors and teachers - are not free from misuses of the symbols of equality. The trouble seems to lie in faulty teaching because the inquiry into the value of direct instruction revealed striking results, some children raising their scores by even as much as 50% as a result of a short lesson on the meaning and use of the symbols. The removal of misconceptions first in the minds of our editors and teachers - as this is the root of the trouble - should be tackled first, but this would be impracticable for the purpose of this thesis as a long period of time is required. However, the great effects of direct instruction are undeniable and this seems to be the immediate solution of our problem.

(5) Failure to assign a meaning to the symbol "=" has its correlative in failure to use correctly the word "equal", the meaning of which tends to become assimilated to that of "same". This is made evident when: (a) The word is avoided, "alike" or "same" being substituted for it in formulating comparisons of equal quantities;

(b) It is used in describing resemblances where neither degree nor quantity is involved.

(b) Suggestions.

(1) As a sequel to this investigation, a further investigation re the extent and nature of misconceptions of the symbol for mathematical equality existing in the minds of adolescents over 16 years of age would be of value. The University and Training Colleges would be the fields for such an investigation and by this inquiry we should hope to discover if, and to what extent, University and Training College students are free from misconceptions re the meaning and use of the symbols. Up to the age of 16 and 17 years students' conceptions are still vague.

(2) The writer suggests that direct instruction should be given in the schools as the unquestionable value of such

instruction has been indicated. This would raise further problems for solution, namely, (1) At what age instruction is most opportune and beneficial;

(2) Whether, and to what extent, mathematics gain in value and understanding as a result of this instruction.

(3) The results seem to indicate that boys are superior to girls in these tests, and as these tests involve the relation of equality which is the fundamental relation in mathematics and is involved in all reasoning processes such as the solution of equations, it does not seem unreasonable to presume that as a result of the boys' superiority in these tests, they are similarly superior in general mathematical exercises. This problem calls for further investigation, however.

(4) Finally, but not negligible, is the fact that the root of the error should be eradicated by definite instruction of our teachers both in Training Colleges and Summer Schools. If misconceptions in children's minds are cleared up now we can feel sure that there will be little cause for anxiety in this respect in a few years' time because some of our present pupils will become teachers and will not be insensitive to misconceptions in the minds of their

pupils, as are our teachers today.

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VI.

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