Generalizing the Algebra of Throws to Rank-3 Matroids

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Abstract

The algebra of throws is a geometric construction which reveals the underlying algebraic operations of addition and multiplication in a projective plane. In Desarguesian projective planes, the algebra of throws is a well-defined, commutative and associative binary operation. However, when we consider an analogous operation in a more general point-line configuration that comes from rank-3 matroids, none of these properties are guaranteed. We construct lists of forbidden configurations which give polynomial time checks for certain properties. Using these forbidden configurations, we can check whether a configuration has a group structure under this analogous operation. We look at the properties of configurations with such a group structure, and discuss their connection to the jointless Dowling geometries.

Acknowledgements

I dedicate this thesis to the memory of my beloved grandma, Sesa Kaulamatoa. Her love, support and prayers were invaluable and I cannot express how thankful I am for her influence. Toka ā 'i hēvani 'i he nonga 'a e Eiki' siéku āngelo talavou.

Firstly, I would like to thank my wonderful supervisor, Geoff Whittle. I am hugely grateful for his understanding, guidance and generous support. Without the help of his grant I would not have been able to undertake this thesis!

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Chapter 1

Introduction

The algebra of throws is a geometric construction which recovers the underlying algebraic structure of a projective plane. This thesis is motivated by the idea of applying an operation analogous to the algebra of throws in the more general setting of rank-3 matroids, in the hope of recovering the underlying algebraic structure of the matroid. To begin, we will overview the algebra of throws construction as described in [4].

1.1 The algebra of throws

Developed in 1857 by von Staudt [6], the algebra of throws is a classical geometric way of reconstructing the underlying algebraic structure of a projective plane. If the projective plane comes from a field $\mathbb{F}$, then the geometric methods of addition and multiplication of points on a line recover the addition and multiplication of $\mathbb{F}$. We will overview these two operations as described in [4], before applying them in the context of matroids.

1.1.1 Addition

We will describe von Staudt's addition of any pair of points on a line. Let A be a line, and let p_0, p_∞ be two arbitrary, distinct fixed points on A called the *fundamental points*. In any plane through A, let B and C be any two lines through p_∞. We call the lines A, B, C the *distinguished lines*. Let l_0 be any line through p_0 meeting B and C at the points b_0 and c_0 respectively.

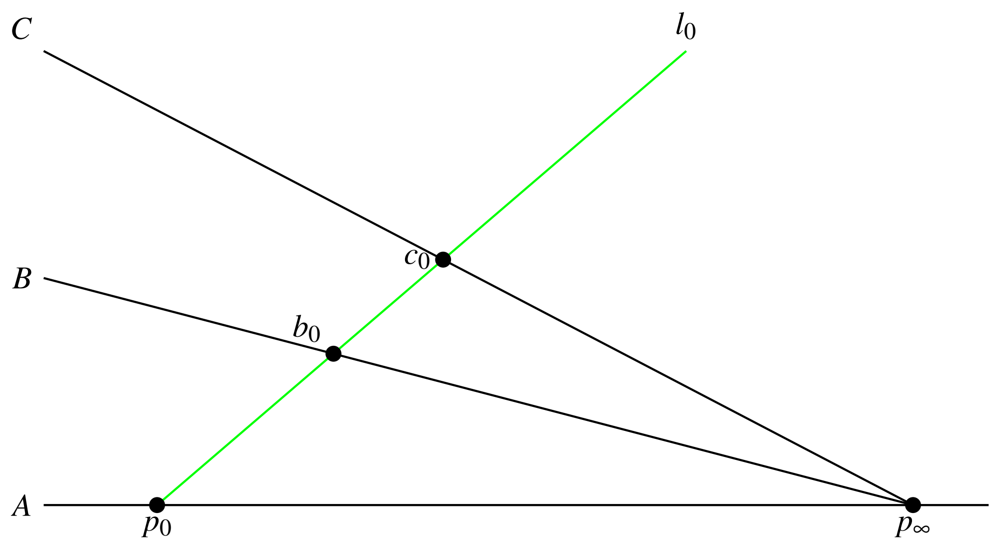


Figure 1.1: Addition of points of A

Let p_x and p_y be any two points of A. Let the lines $p_x b_0$ and $p_y c_0$ meet C and B at the points X and Y respectively.

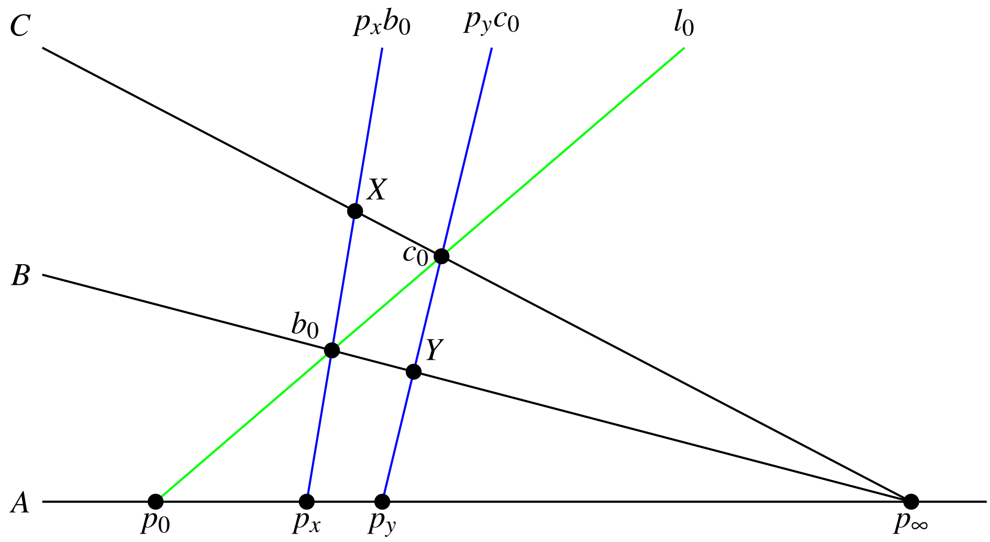


Figure 1.2: Addition of points of A

The point p_{x+y}, in which the line XY meets A, is called the *sum* of the points p_x and p_y in A. The operation of obtaining the sum of two points is called *addition*.

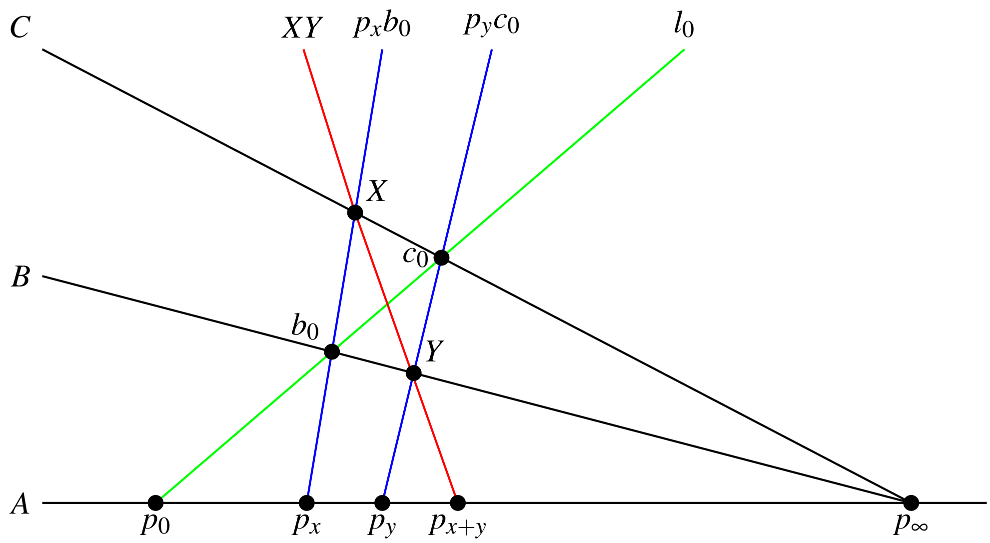


Figure 1.3: Addition of points of A

It follows from [4] that the addition of points of A is independent of both our choice of the three distinguished lines A, B, C and of our choice of l_0. We prove in Chapter 3 that if the points of A are the points of a field, then this addition of points corresponds to addition in the field. As one would expect, addition has the properties of commutativity and associativity.

1.1.2 Multiplication

We will now describe von Staudt's multiplication of any pair of points on a line. Let p_0, p_1, p_∞ be points on A called the *fundamental points*. In any plane through A, let B, l_1, C be any three lines through the p_0, p_1 and p_∞ respectively. As for addition, we call the lines A, B, C the *distinguished lines*. Let l_1 be the line which meets B and C at the points b_1 and c_1 respectively.

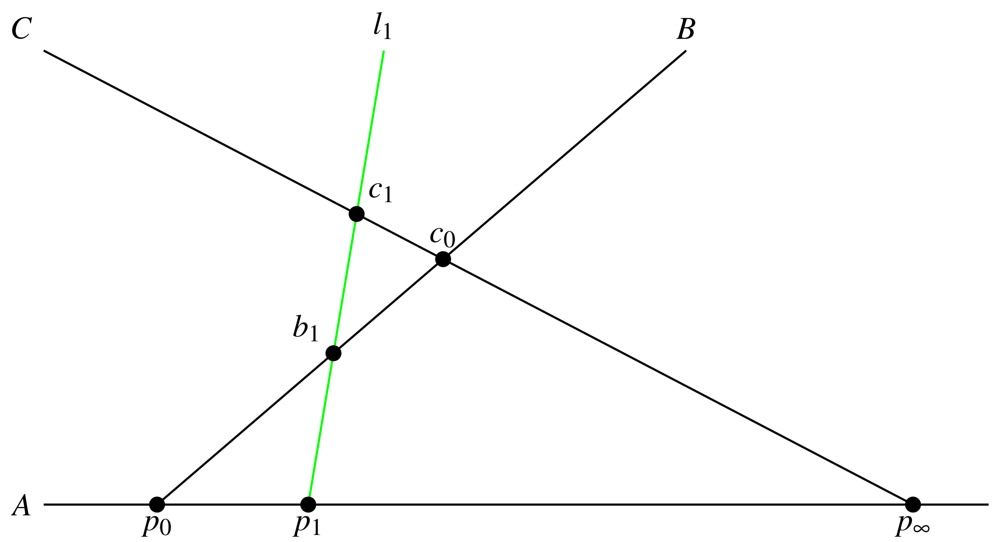


Figure 1.4: Multiplication of points of A

Let p_x, p_y be any two points of A. Let the lines $p_x b_1$ and $p_y c_1$ meet C and B in the points X and Y respectively.

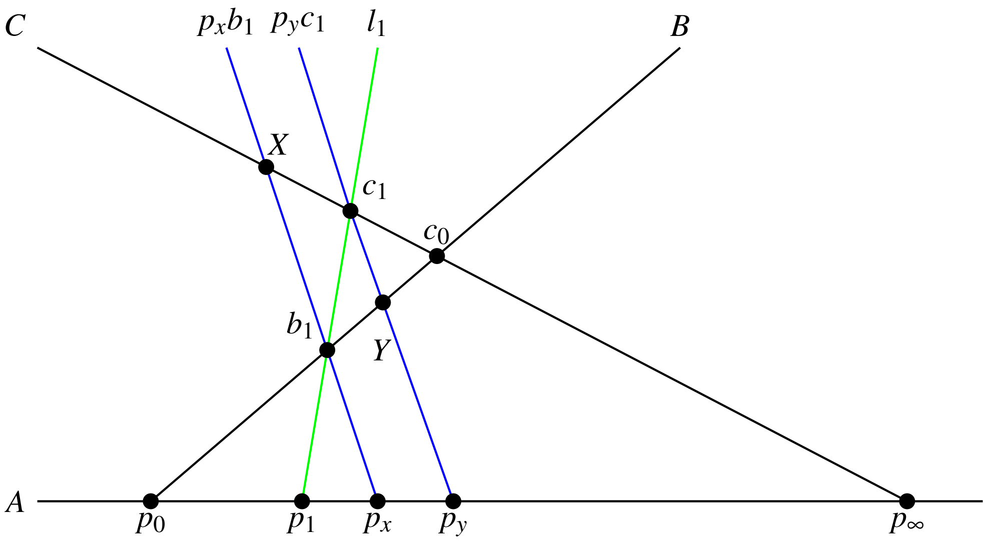


Figure 1.5: Multiplication of points of A

The point p_{xy} in which the line XY meets A is called the *product* of p_x by p_y in the scale p_0, p_1, p_∞ on A. The operation of obtaining the product of two points is called *multiplication*.

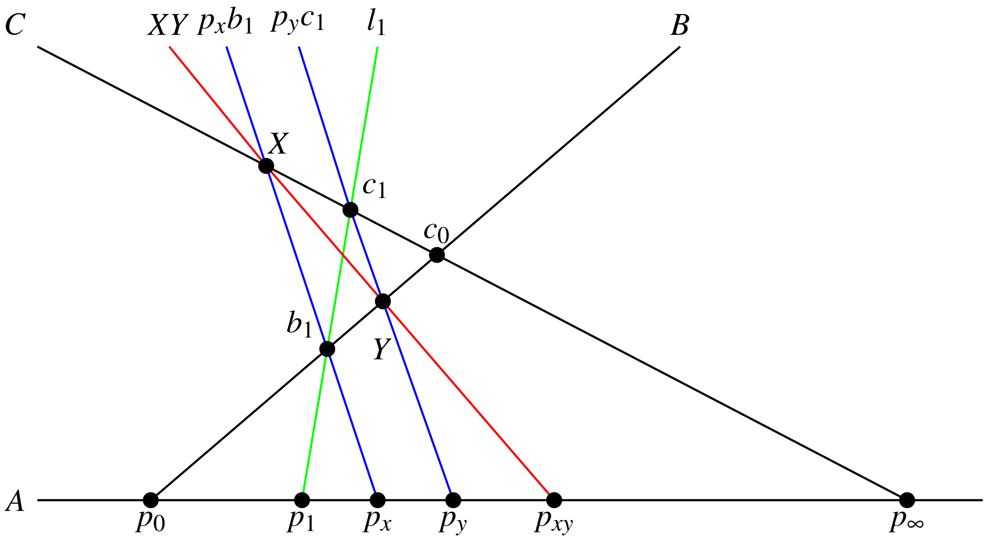


Figure 1.6: Multiplication of points of A

It follows from [4] that the multiplication of points of A is independent of both our choice of the three distinguished lines A,B,C and of our choice of l_1. We prove in Chapter 3 that if the points of A are the points of a field, then this multiplication of points corresponds to multiplication in the field. As one would expect, multiplication has the properties of commutativity and associativity.

1.1.3 The connection between addition and multiplication

If we disregard whether or not the three distinguished lines A,B,C are co-punctual, we observe that the two operations of addition and multiplication are *geometrically the same operation*. That is, if we disregard the point p_∞ in Figure 1.3 and the points p_0, c_0 in Figure 1.6, then addition and multiplication amount to same operation as shown in Figure 1.7 below. We label a point in Figure 1.7 by x/y if x and y are the equivalent points in Figure 1.3 (the additive case) and Figure 1.6 (the multiplicative case) respectively.

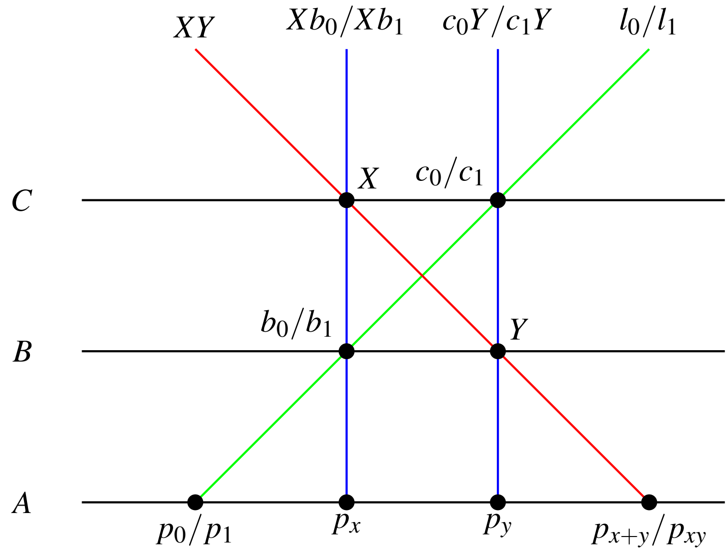


Figure 1.7: Addition and multiplication can geometrically be regarded as equivalent operations.

When we apply the algebra of throws to matroids, we will be performing this operation locally to '3-line configurations' — that is, matroid configurations partitioned by 3 lines as in Figure 1.7. Therefore we can disregard whether the 3 lines are co-punctual or not and need only consider a single geometric operation for both the addition and multiplication of points.

1.2 Overview of chapters

In Chapter 2 we generalize the algebra of throws to an analogous operation on 3-line configurations from rank-3 matroids. This analogous operation is local to three 'distinguished' lines and can give rise to either a partial binary operation or a full binary operation on one of the distinguished lines, called the 'main' line. We define this analogous operation on lines through an identity point on the main line. There may be many lines through this identity point and it is not guaranteed that we can apply this analogous operation on every line. This gives rise to this analogous operation being 'strong' if it can be applied to *all* lines through the identity point, or being 'weak' if it *cannot* be applied to all lines through the identity point. Whether we consider the strong or weak notion will hugely affect the complexity of our results.

If we apply the algebra of throws to matroids in a general setting, we want to recover the addition and multiplication of the algebraic structure. In Chapter 3, we show if the matroid configuration comes from a projective plane over a field, then addition corresponds to the field addition and similarly multiplication corresponds to the field multiplication. We also show if we have a matroid configuration from a projective plane over some algebraic structure other than a field, and we coordinatize using a classical method, then addition and multiplication corresponds to the addition and multiplication of coordinates respectively.

In Chapter 4 we consider the properties of commutativity and associativity. We construct lists of forbidden configurations, which provide a polynomial time check for these properties. We note the importance of Pappus configurations as a check for commutativity when we have a strong binary operation.

In Chapter 5 we consider matroid configurations which represent groups. We include some examples of matroid configurations of small groups and prove the uniqueness of certain group configurations when the algebra of throws defines a strong binary operation.

In Chapter 6 we overview the matroids related to biased graphs and reveal the bijection between certain group configurations and the jointless Dowling geometries.

Any undefined notation or terminology will follow [1]. Any known matroids we reference follow the notation as found in the appendix of [1].

Chapter 2

Generalizing the algebra of throws

In this chapter, we generalize the algebra of throws to rank-3 matroid configurations. Recall that within projective planes over fields, the algebra of throws defines two well-defined binary operations — namely, addition and multiplication over the given field. In the context of matroids, the analogous operation is not guaranteed to be well-defined. We construct finite lists of forbidden configurations to check for these two properties within a rank-3 matroid configuration.

2.1 3-line configurations

Recall a *simple matroid* is one with no loops or parallel elements. A *line* of a matroid M is a rank-2 flat. A *trivial line* contains exactly two points and a *non-trivial line* contains at least three points. We will only show non-trivial lines in our diagrams. A *3-line configuration* is a rank-3 simple matroid G with lines A,B,C such that $\{A,B,C\}$ partition $E(G)$. We say that A,B,C are the *distinguished lines* of G and we call the elements of $E(G)$ *points*. An *e-based 3-line configuration* is a triple $(G,\{A,B,C\},e)$ where G is a 3-line configuration with lines A,B,C and

$e \in A$. The element e is called the *identity point*. If $\{A,B,C\}$ and e are clear from context, we will abbreviate $(G,\{A,B,C\},e)$ to G and abbreviate "e-based 3-line configuration" to *configuration*.

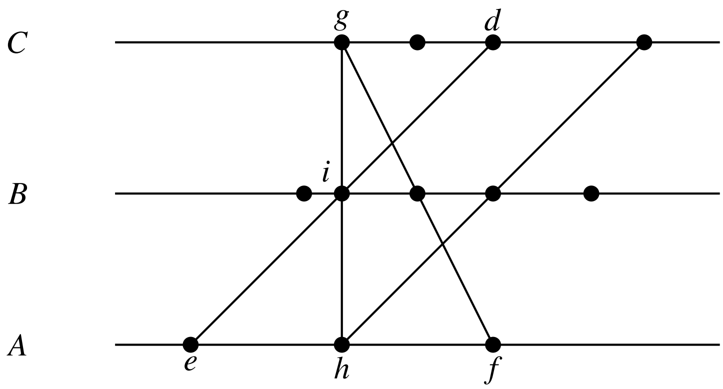


Figure 2.1: An e-based 3-line configuration

Given a configuration G, the 3-point lines we are interested in are those which contain a point from each of the distinguished lines. That is, whenever we refer to a 3-point line abc, we will assume $a \in A$, $b \in B$ and $c \in C$, and call abc a *triangle*. For example, in Figure 2.1, ehf is not a triangle as $e,h,f \in A$. However, hig is a triangle, as $h \in A, i \in B, g \in C$. For the fixed element $e \in A$, we say ebc is an *e-triangle*. For example, in Figure 2.1, eid is an e-triangle. Two configurations $(G,\{A,B,C\},e)$ and $(G',\{A',B',C'\},e')$ are *isomorphic* if there exists a bijection $\sigma : G \to G'$ such that $\sigma(A) = A'$, $\sigma(B) = B'$, $\sigma(C) = C'$, $\sigma(e) = e'$ and xyz is a triangle in G if and only if $\sigma(x)\sigma(y)\sigma(z)$ is a triangle in G'. A *sub-configuration* of a configuration G is a subset of the points and lines of G which is itself a configuration.

To ensure our 3-line configurations are indeed rank-3 matroids, we need only satisfy the following lemma:

Lemma 2.1.1. *Let E be a set and $\mathscr{L}$ be a collection of subsets of E such that if $l \in \mathscr{L}$ then $|l| \geq 3$. Then $\mathscr{L}$ is the collection of non-trivial lines of a rank-3 simple matroid if and only if $|l_i \cap l_j| \leq 1$ for all $l_i, l_j \in \mathscr{L}$.*

Recall that a *binary operation* on a set A is a function $f : A \times A \to A$. If f is not a function, but a partial function (i.e. f is defined for a subset of $A \times A$), then f is a *partial binary operation*. For example, division over $\mathbb{R}$ is a partial binary operation as division by 0 is undefined for any real number. We may refer to a binary operation as a *full binary operation* for clarity. Within projective planes, the algebra of throws defines two full binary operations — namely, addition and multiplication over the given field. However, this is not the case when we apply an analogous technique to matroid configurations. In general, it is not even the case that we have a partial binary operation. So the question is, when can we describe an analogous construction for matroid configurations, and when does this construction give rise to partial and full binary operations? We will answer this by using the algebra of throws to define a relation, $\circ$, from $A \times A$ to A. As we are defining $\circ$ on A, we call A the *main line* of the configuration. Later, we describe the conditions under which the relation $\circ$ gives rise to either a partial or full binary operation.

Let $G = (G, \{A, B, C\}, e)$ be an e-based 3-line configuration. We will define the relation $\circ$ between $A \times A$ and A as follows:

Take a pair $(x, y) \in A \times A$. We say $((x, y), z) \in \circ$, or $(x, y) \circ z$, if the following conditions hold:

(1). There exists an e-triangle, eb_1c_2, such that the triangles xb_1X and yYc_2 exist. Call the triangles xb_1X and yYc_2 the *necessary triangles* of the pair (x, y)

with respect to eb_1c_2. Note that $X \neq c_2$ and $Y \neq b_1$, by Lemma 2.1.1.

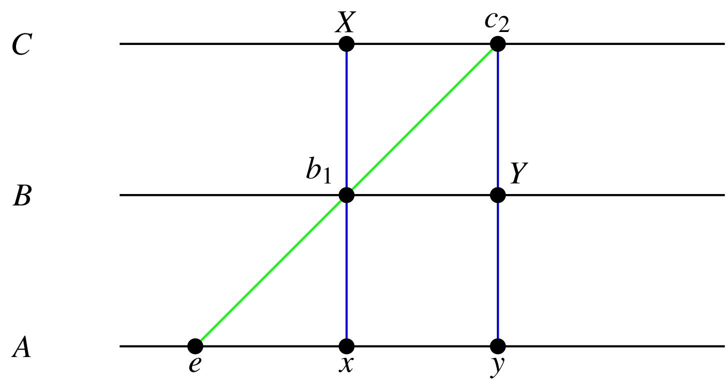


Figure 2.2: The necessary triangles (coloured blue) of the pair (x,y) with respect to the e-triangle eb_1c_2 (coloured green).

(2). There exists $z \in A$ such that zYX is a triangle, called the *relation triangle* of (x,y) with respect to eb_1c_2. Note that $z \notin \{x,y\}$ if we are to satisfy Lemma 2.1.1 and remain a matroid.

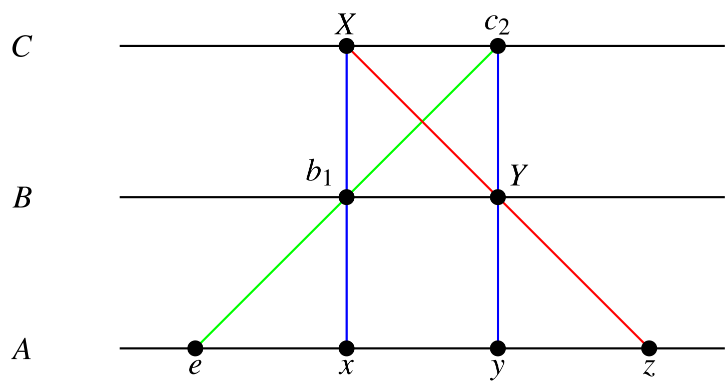


Figure 2.3: The necessary triangles and the relation triangle (coloured red) of (x,y) with respect to the e-triangle eb_1c_2. For this particular case, we have $z \notin \{x,y\}$.

If both conditions (1) and (2) hold, then $((x,y),z) \in \circ$ and we say $x \circ y$ is *defined on* eb_1c_2. We say $x \circ y$ is *undefined on* eb_1c_2 if either (1) does not hold, or if (1) holds but there does *not* exist $z \in A$ such that $((x,y),z) \in \circ$ on eb_1c_2. If there does not exist any $z \in A$ such that $((x,y),z) \in \circ$ on *any e-triangle*, then we say $x \circ y$ is *undefined*.

For some $x,y \in A$, it may be that $x \circ y$ is inconsistently defined across multiple e-triangles. That is, $x \circ y$ may be defined on eb_1c_2 and $eb'_1c'_2$, giving $((x,y),z_1) \in \circ$ and $((x,y),z_2) \in \circ$ where $z_1 \neq z_2$. Later, we will want to know when the relation $\circ$ is consistently defined. If there exists an e-triangle eb_1c_2 such that $x \circ y$ is defined on eb_1c_2, then we say $x \circ y$ is *defined*. We will say that $x \circ y$ is *well-defined* if $x \circ y$ is defined, and whenever $x \circ y$ is defined on more than one e-triangle, for any pair of e-triangles eb_1c_2 and $eb'_1c'_2$ where $x \circ y$ is defined on eb_1c_2 giving $((x,y),z_1) \in \circ$, and $x \circ y$ is defined on $eb'_1c'_2$ giving $((x,y),z_2) \in \circ$, then $z_1 = z_2$. So $x \circ y$ is well-defined if it is consistently defined. We will say that $x \circ y$ is *strongly defined* if $x \circ y$ is well-defined and for *every* e-triangle ebc in G we have $x \circ y$ defined on ebc.

The notions of strongly defined and well-defined pairs will become important later when we describe the different partial and full binary operations which may arise. For now, our focus is local — we are interested in whether pairs of $A \times A$ are defined on a *particular* e-triangle, so we are not concerned whether a pair is well-defined or strongly defined.

2.2 Complexity

We will assume basic complexity theory knowledge. As we can view rank-3 simple matroids as hypergraphs, we can use a standard model of complexity, as opposed to an Oracle model of complexity which is usually used in matroid theory. Our model will be a hypergraph, where the three hyperedges A,B,C partition the points of the hypergraph. That is, these three hyperedges correspond to the distinguished lines of the configuration. All other hyperedges will be triangles of the form abc where $a \in A, b \in B, c \in C$. The size of an instance is the number of points, n, in the hypergraph. When we say a property can be checked in polynomial time — we mean polynomial in n. All algorithms mentioned are polynomial in n.

2.3 Basic configurations

There are a finite number of configurations to check for whether there exists $z \in A$ such that $x \circ y$ is defined on an e-triangle giving $(x,y) \circ z$. We call these the *basic relation configurations*, denoted R_i. Similarly, there are a finite number of configurations to check for whether $x \circ y$ is undefined on a particular e-triangle — we call these the *basic non-relation configurations*, denoted NR_j. A *basic configuration* is a basic relation configuration or a basic non-relation configuration.

First we will list the basic relation configurations, where $\circ$ is the relation defined previously. In subsection 2.1, Figure 2.3 gives one example for which we have $(x,y) \circ z$ through the e-triangle eb_1c_2 — the following configurations make up *all* possible instances, up to isomorphism.

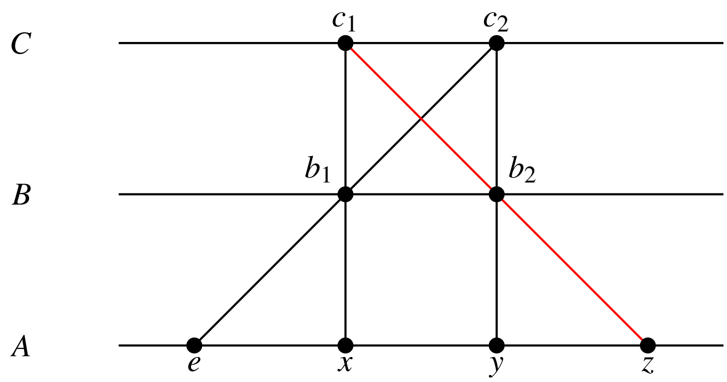


Figure 2.4: R_1 basic relation configuration.

Figure 2.4 tells us $(x,y)\circ z$ through eb_1c_2, i.e. $x\circ y$ is defined on eb_1c_2.

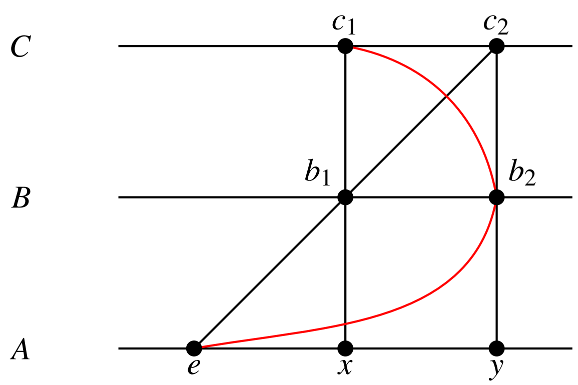


Figure 2.5: R_2 basic relation configuration.

Figure 2.5 tells us $(x,y)\circ e$ through eb_1c_2, i.e. $x\circ y$ is defined on eb_1c_2. Note that $R_2 \cong P_7$.

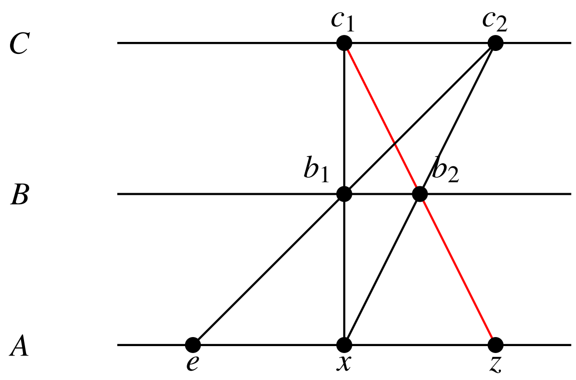


Figure 2.6: R_3 basic relation configuration.

Figure 2.6 tells us $(x,x) \circ z$ through eb_1c_2, i.e. $x \circ x$ is defined on eb_1c_2. Note that $R_3 \cong P_7$. Also note that even though the underlying matroids of both Figures 2.5 and 2.6 are the same matroid, configurations R_2 and R_3 are *not* isomorphic.

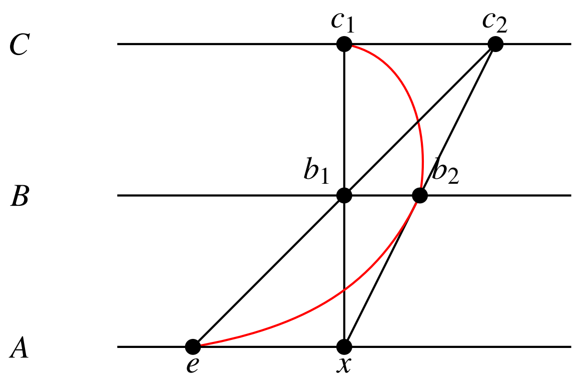


Figure 2.7: R_4 basic relation configuration.

Figure 2.7 tells us $(x,x) \circ e$ through eb_1c_2, i.e. $x \circ x$ is defined on eb_1c_2. Note that $R_4 \cong M(K_4)$.

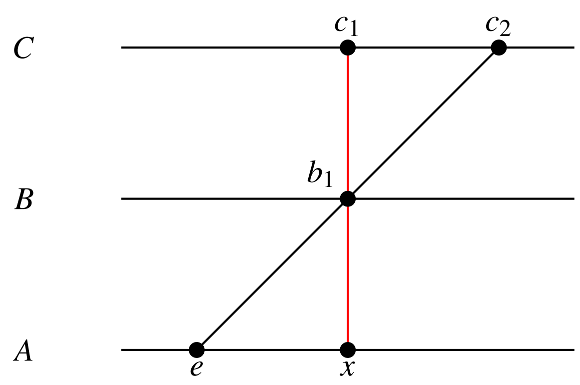


Figure 2.8: R_5 basic relation configuration.

Figure 2.8 tells us $(x,e)\circ x$ through eb_1c_2, i.e. $x\circ e$ is defined on eb_1c_2.

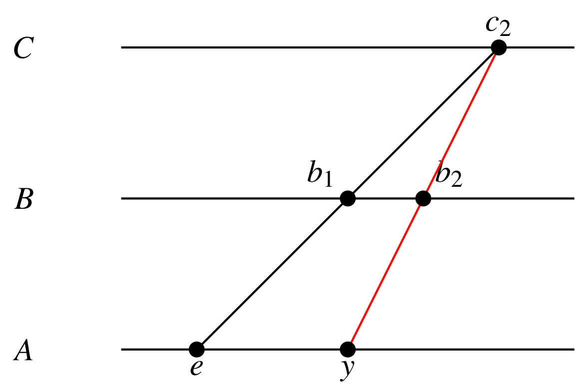


Figure 2.9: R_6 basic relation configuration.

Figure 2.9 tells us $(e,y)\circ y$ through eb_1c_2, i.e. $e\circ y$ is defined on eb_1c_2.

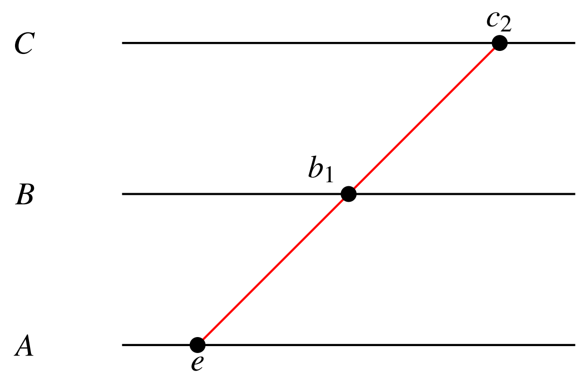


Figure 2.10: R_7 basic relation configuration.

Figure 2.10 tells us $(e,e)\circ e$ through eb_1c_2, i.e. $e\circ e$ is defined on eb_1c_2. We now prove that the above configurations $R_1 - R_7$ are all possible basic relation configurations up to isomorphism.

Lemma 2.3.1. *Given a configuration G, for $x,y,z \in A$, we have $(x,y)\circ z$ on the e-triangle eb_1c_2 if and only if there is a basic relation configuration for e,x,y,z that is isomorphic to one of $R_1 - R_7$, where the isomorphism is the identity for the points e,x,y,z,b_1,c_2.*

Proof. We will consider a case analysis of the distinct subsets of elements from $\{e,x,y,z\}$.

Assume all four points in $\{e,x,y,z\}$ are distinct. By inspection, $(x,y)\circ z$ if and only if we have a configuration isomorphic to R_1.

Now assume only three points in $\{e,x,y,z\}$ are distinct and $(x,y)\circ z$. There are three cases to consider. Firstly, suppose e,x,y are distinct and $z\in\{e,x,y\}$. Recall that we cannot have $z\in\{x,y\}$ and remain a matroid, so we must have $z=e$. The three points e,x,y are distinct and $z=e$ if and only if we have a configuration isomorphic to R_2. Secondly, suppose e,x,z are distinct and $y \in \{e,x\}$ (recall we cannot have $y = z$). Then $y = e$ if and only if $z = x$, contradicting our assumption that x,z are distinct, so we must have $y = x$. The three points e,x,z are distinct and $y = x$ if and only if we have a configuration isomorphic to R_3. Finally, suppose e,y,z are distinct and $x \in \{e,y\}$ (recall we cannot have $x = z$). Then $x = e$ if and only if $z = e$, contradicting our assumption that e,z are distinct, so we must have $x = y$. The three points e,y,z are distinct and $x = y$ if and only if we have a configuration isomorphic to R_3.

Now assume only two points in $\{e,x,y,z\}$ are distinct and $(x,y) \circ z$. There are two cases to consider. Firstly, suppose e and x are distinct. Then $y = e$ if and only if $z = x$, if and only if we have a configuration isomorphic to R_5. On the other hand, $y = x$ if and only if $z = e$ if and only if we have a configuration isomorphic to R_4. For the second case, suppose e and y are distinct. Then $x = e$ if and only if $z = y$, if and only if we have a configuration isomorphic to R_6. On the other hand, $x = y$ if and only if $z = e$, if and only if we have a configuration isomorphic to R_4.

Finally, assume $e = x = y = z$. By inspection, $(x,y) \circ z$ if and only if we have a configuration isomorphic to R_7. $\square$

These basic relation configurations show when $x \circ y$ *is* defined on a particular e-triangle. Recall that the basic non-relation configurations, denoted NR_j, show when $x \circ y$ is *not* defined on a particular e-triangle. We now list these configurations, denoting non-collinearities by dashed lines. When we have an unlabelled red point on A, which forms a dashed-line triangle with two points $c \in C, b \in B$ (eg. Figures 2.11 and 2.12), this means that $\{a,b,c\}$ is independent for *all* $a \in A$. In other words, there is *no* point in A which forms a triangle with bc. If we have dashed-line triangle *abc* where $a \in A$ is a black filled point (eg. Figures 2.13 and 2.14), this means for the *specific* point a, a is independent from bc. There may (or may not) exist some point $a' \neq a$ which is collinear with bc.

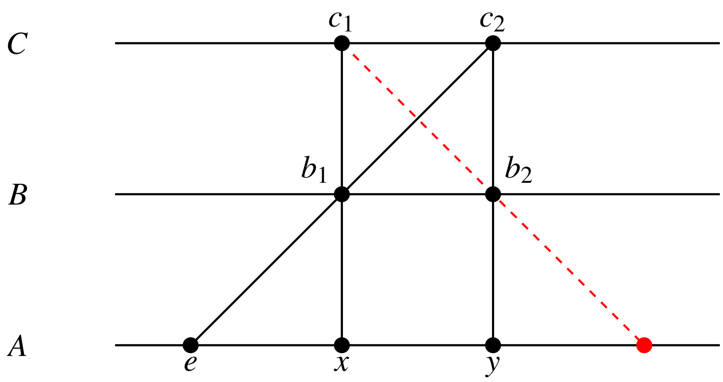


Figure 2.11: NR_1 basic non-relation configuration.

In Figure 2.11, the red dashed line tells us there is *no* point on A that is collinear with c_1 and b_2, i.e. $x \circ y$ is undefined on eb_1c_2.

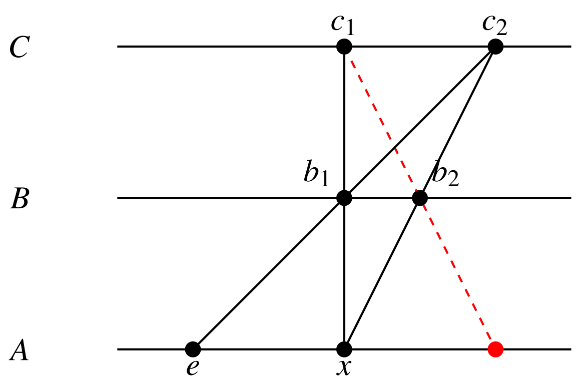


Figure 2.12: NR_2 basic non-relation configuration.

In Figure 2.12, the red dashed line tells us there is *no* point on A that is collinear with c_1 and b_2, i.e. $x \circ x$ is undefined on eb_1c_2.

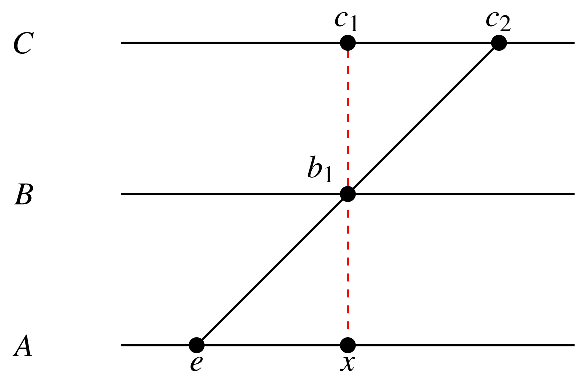


Figure 2.13: NR_3 basic non-relation configuration.

In Figure 2.13, the red dashed line says for the *specific* point x, that $x \circ e$ is undefined on eb_1c_2.

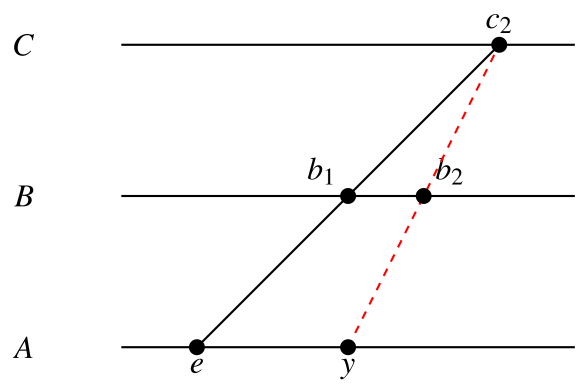


Figure 2.14: NR_4 basic non-relation configuration.

In Figure 2.14, the red dashed line says for the *specific* point y, that $e \circ y$ is undefined on eb_1c_2.

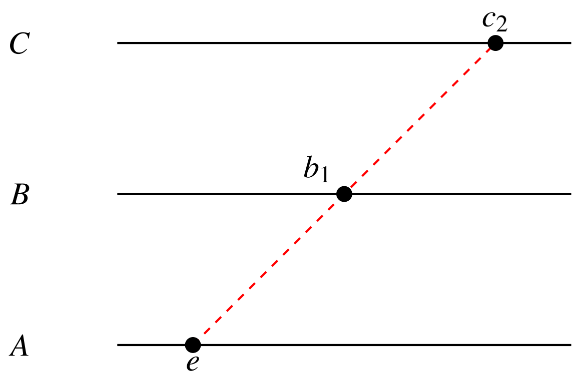


Figure 2.15: NR_5 basic non-relation configuration.

We now prove that the above configurations $NR_1 - NR_5$ are all possible basic non-relation configurations.

Lemma 2.3.2. *Given a configuration G, for $x, y \in A$, then $x \circ y$ is undefined on the e-triangle eb_1c_2 if and only if there is a basic non-relation configuration for e, x, y that is isomorphic to one of $NR_1 - NR_5$, where the isomorphism is the identity for the points e, x, y, z, b_1, c_2.*

Proof. As for Lemma 2.3.1, our proof considers a case analysis of the distinct subsets of elements of $\{e, x, y\}$.

Assume all three elements from $\{e, x, y\}$ are distinct. By inspection, $x \circ y$ is undefined on eb_1c_2 if and only if we have a configuration isomorphic to NR_1.

Now assume two elements from $\{e, x, y\}$ are distinct and $x \circ y$ is undefined on eb_1c_2. Suppose e, x are distinct. By inspection, $y = x$ if and only if we have a configuration isomorphic to NR_2 and $y = e$ if and only if we have a configuration isomorphic to NR_3. Suppose e, y are distinct. By inspection, $x = e$ if and only if we have a configuration isomorphic to NR_4 and $x = y$ if and only if we have a configuration isomorphic to NR_2. Suppose x,y are distinct. By inspection, $e = x$ if and only if we have a configuration isomorphic to NR_4 and $e = y$ if and only if we have a configuration isomorphic to NR_3.

Finally, assume $e = x = y$. Then $x \circ y$ is undefined on eb_1c_2 if and only if we have a configuration isomorphic to NR_5. $\square$

To check whether a pair is defined on an e-triangle, we need only check for five forbidden configurations. As each of these configuration has no more than eight points, for a configuration with n points, we can check this in no more than $5\binom{n}{7}\binom{8}{3}$ steps, which is polynomial in n. This proves the following corollary.

Corollary 2.3.2.1. *Let $(G, \{A,B,C\}, e)$ be an e-based configuration with n points. There is an algorithm, which is polynomial in n, to check whether a pair of points is defined on an e-triangle.*

2.4 Forbidden configurations for binary operations

The basic relation and basic non-relation configurations consider the relation $\circ$ on a *single* e-triangle at a time. Now, we want to check when $\circ$ is consistent, so we must consider when $\circ$ is defined on *pairs* of e-triangles. We say $\circ$ is *consistent* if for all $x,y \in A$, if $(x,y) \circ z_1$ and $(x,y) \circ z_2$, then $z_1 = z_2$. In other words, $\circ$ is consistent if every defined pair $(x,y) \in A \times A$ is well-defined. Now define the operation $\circ$ by $x \circ y = z$ if there exists an e-triangle eb_1c_2 such that $(x,y) \circ z$. Recall that $\circ$ is a partial binary operation if $\circ : A \times A$ is a partial function. The next lemma follows immediately from the definition of a partial binary operation.

Lemma 2.4.1. *The operation $\circ$ is a partial binary operation if and only if $\circ$ is consistent.*

We want to test for when a configuration has a partial binary operation, $\circ$. We only need to check that $\circ$ is consistent, that is, we need to check that every defined pair in $A \times A$ is well-defined. The basic configurations test when $x \circ y$ is defined for a *particular* e-triangle. If $x \circ y$ is defined for *multiple* e-triangles, we want to check that $x \circ y$ is well-defined, i.e. gives the same answer on all e-triangles. There is a small number of configurations, each with no more than twelve points, which show that $x \circ y$ is not well-defined for a pair $(x, y) \in A \times A$. We will construct these forbidden configurations by a case analysis of the distinct points of $\{e, x, y, z\}$. This list will consist of exactly the forbidden configurations for being a partial binary operation. The following eight configurations make up part of this list.

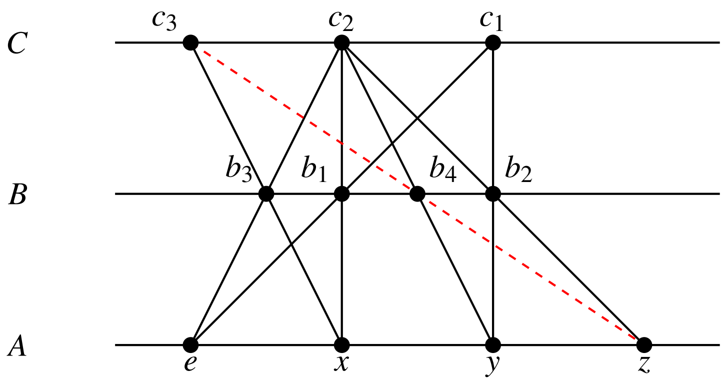


Figure 2.16: W_1 forbidden configuration

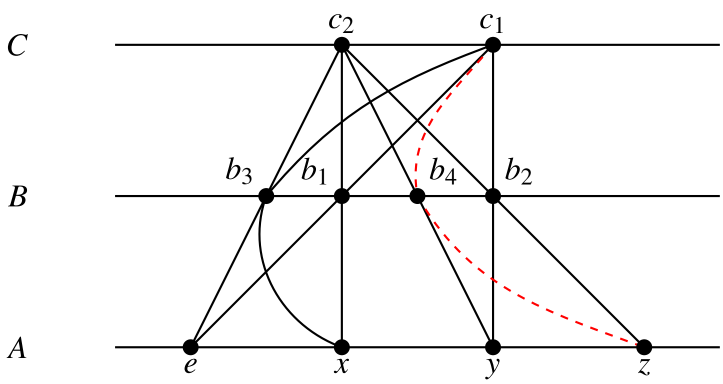


Figure 2.17: W_2 forbidden configuration

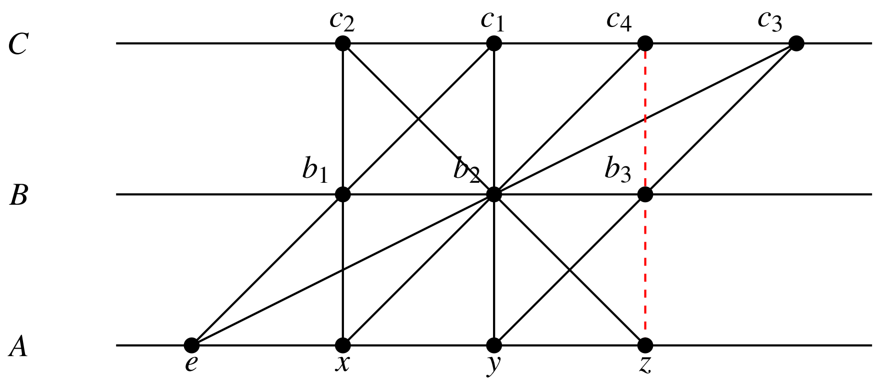


Figure 2.18: W_3 forbidden configuration

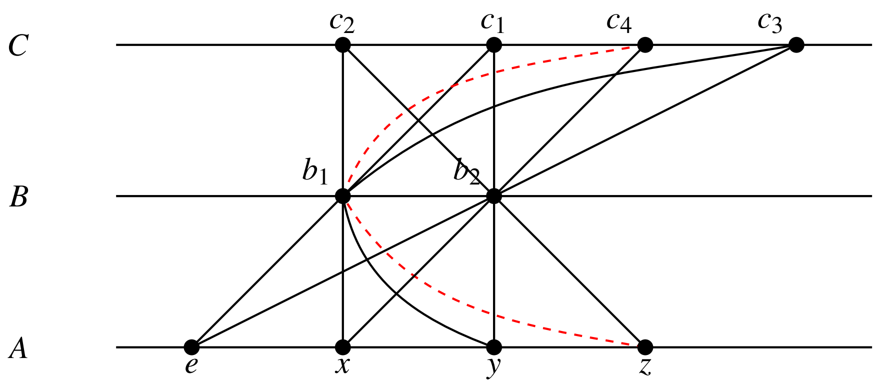


Figure 2.19: W_4 forbidden configuration

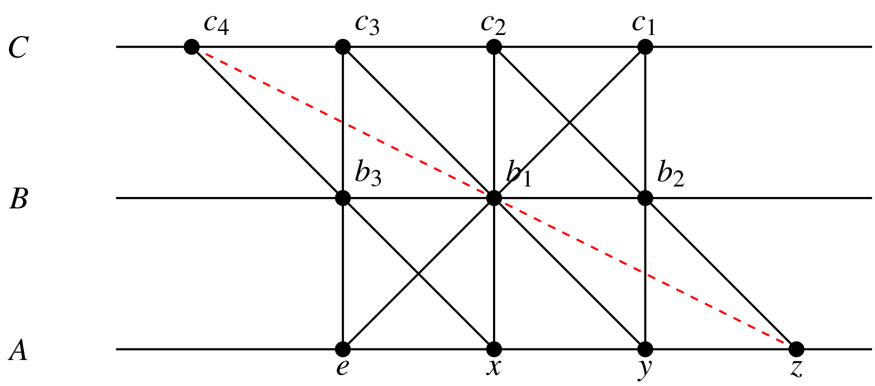


Figure 2.20: W_5 forbidden configuration

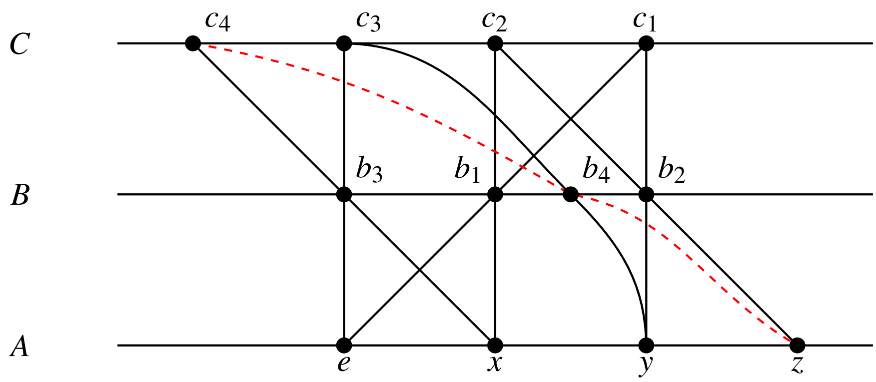


Figure 2.21: W_6 forbidden configuration

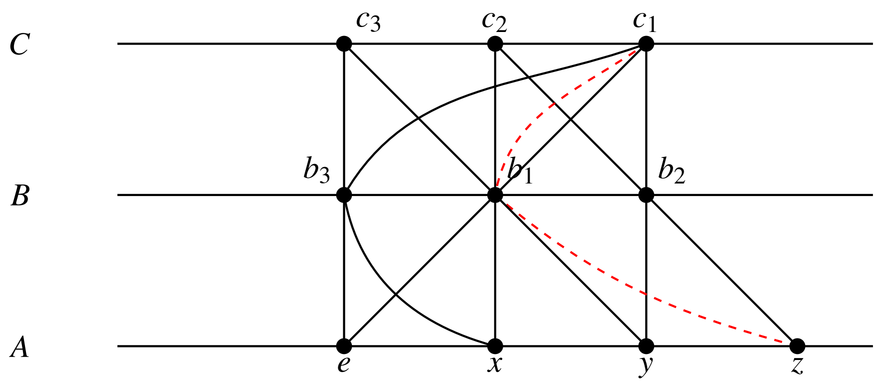


Figure 2.22: W_7 forbidden configuration

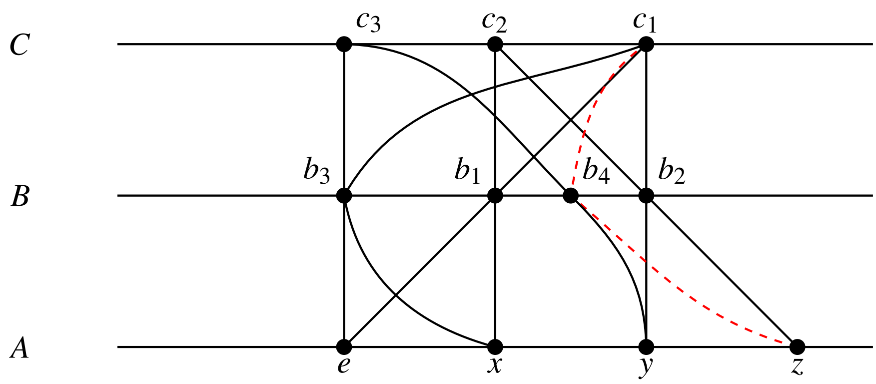


Figure 2.23: W_8 forbidden configuration

We will prove that the configurations $W_1 - W_8$ are all possible forbidden configurations for the pair (x,y) being well-defined when e,x,y,z are distinct points. First, a quick note. For the rest of this thesis, we will employ a particular colour coding in an attempt to make the myriad of configurations more digestible. We will colour the e-triangles we are working with green, colour the necessary triangles blue and finally colour the relation triangle red.

Lemma 2.4.2. *For a configuration G, suppose $e,x,y \in A$ are distinct points and there exists $z \in A$ such that $(x,y) \circ z$ and $z \notin \{e,x,y\}$. Then $x \circ y$ is well-defined and $x \circ y = z$ if and only if G has no sub-configuration isomorphic to any of $W_1 - W_8$, where the isomorphism is the identity for the points e,x,y,z.*

Proof. We will consider a case analysis of the configurations which define $x \circ y = z$ on two distinct e-triangles. We will begin with the basic relation configuration R_1, which defines $x \circ y = z$. We will then consider how to extend R_1 so that $x \circ y$ is defined on two distinct e-triangles. Given R_1, there are three other structurally different e-triangles on which we can define $x \circ y = z$:

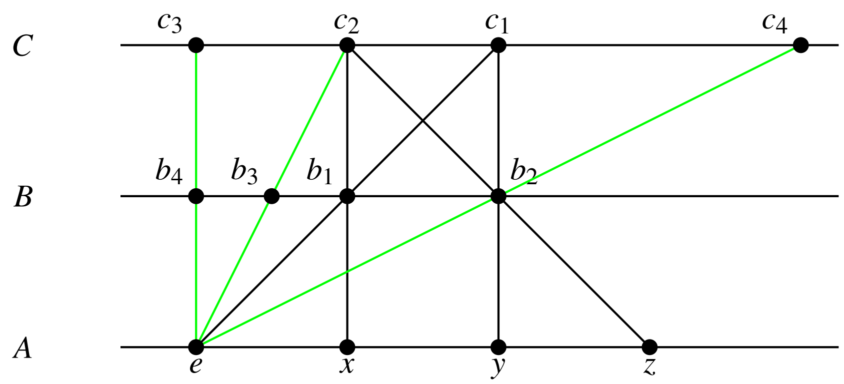


Figure 2.24:

Case (i). Suppose we define $x \circ y = z$ on the e-triangle eb_3c_2:

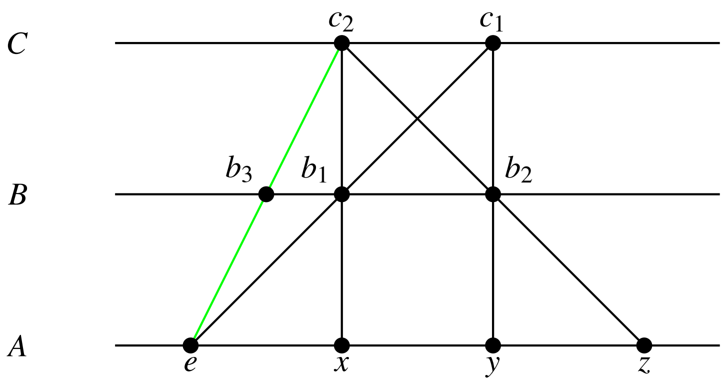


Figure 2.25:

By inspection, we must have the necessary triangle yb_4c_2, where $b_4 \neq b_i$ for $i \in \{1,2,3\}$, and either one of the two following sub-cases:

Sub-case (a). The necessary triangle xb_3c_3 exists, where $c_3 \neq c_i$ for $i \in \{1,2\}$. In this case $x \circ y$ is not well-defined if and only if the relation triangle zb_4c_3 does *not* exist — as the configuration shows both $x \circ y = z$ and $x \circ y \neq z$. So in this case $x \circ y$ is not well-defined if and only if we have a configuration isomorphic to W_1.

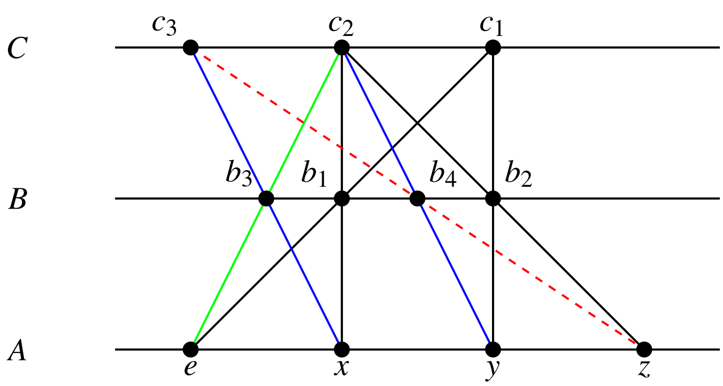


Figure 2.26: The configuration forced by defining $x \circ y$ on the e-triangle eb_3c_2, with the necessary triangles xb_3c_3 and yb_4c_2.

Sub-case (b). The necessary triangle xb_3c_1 exists. In this case $x \circ y$ is not well-defined if and only if the relation triangle zb_4c_1 does not exist, if and only if we have a configuration isomorphic to W_2.

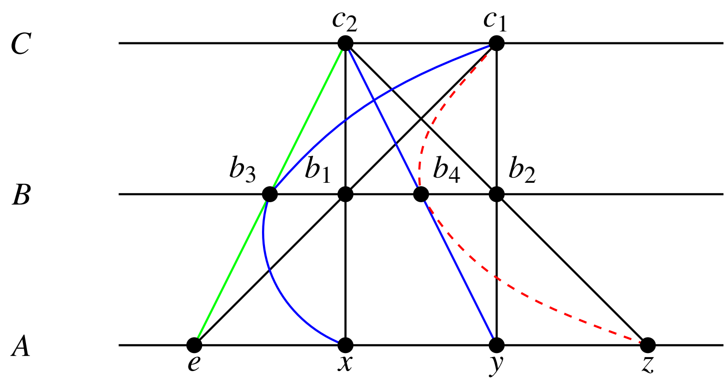


Figure 2.27: The configuration forced by defining $x \circ y$ on the e-triangle eb_3c_2, with the necessary triangles xb_3c_1 and yb_4c_2.

Case (ii). Suppose we define $x \circ y = z$ on the e-triangle eb_2c_4, where $c_4 \neq c_i$ for $i \in \{1,2\}$.

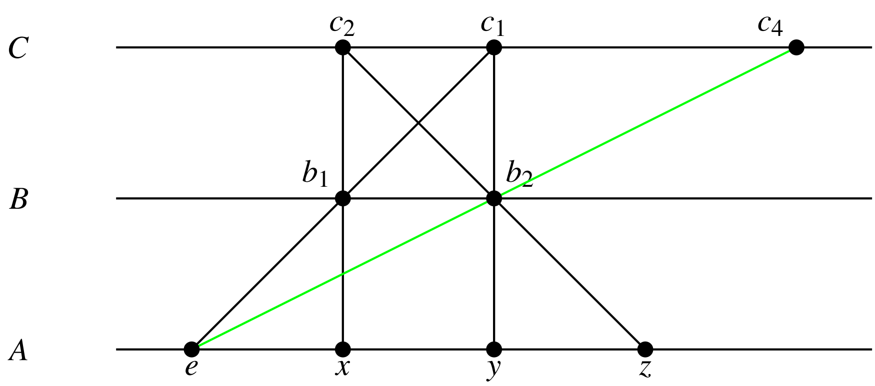


Figure 2.28:

By inspection, we must have the necessary triangle xb_2c_3, where $c_3 \neq c_i$ for $i \in \{1,2,4\}$, and either one of the two following sub-cases:

Sub-case (a). The necessary triangle yb_3c_4 exists, where $b_3 \neq b_i$ for $i \in \{1,2\}$. In this case $x \circ y$ is not well-defined if and only if the relation triangle zb_3c_3 does not exist, if and only if we have a configuration isomorphic to W_3.

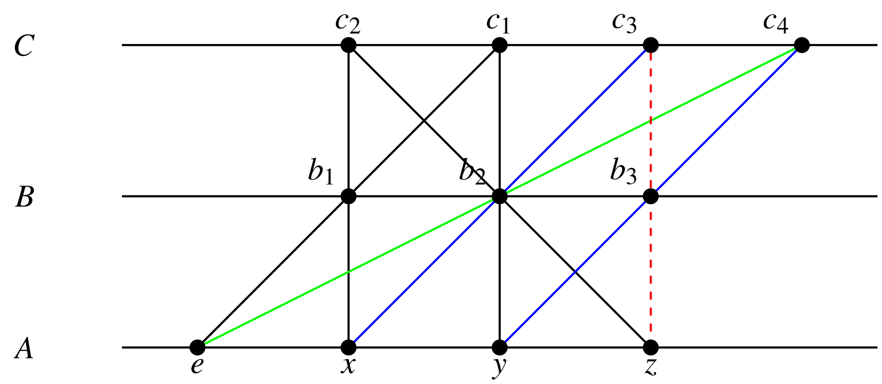


Figure 2.29: The configuration forced by defining $x \circ y$ on the e-triangle eb_2c_4, with the necessary triangles xb_2c_3 and yb_3c_4

Sub-case (b). The necessary triangle yb_1c_4 exists. In this case $x \circ y$ is not well-defined if and only if the relation triangle zb_1c_3 does not exist, if and only if we have a configuration isomorphic to W_4.

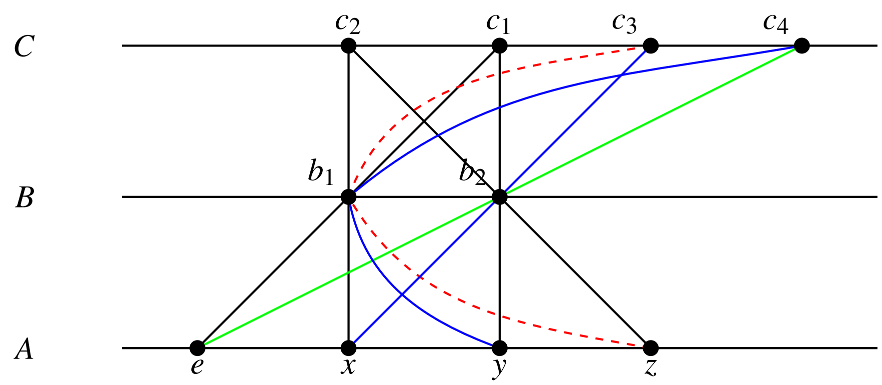


Figure 2.30: The configuration forced by defining $x \circ y$ on the e-triangle eb_2c_4, with the necessary triangles xb_2c_3 and yb_1c_4.

Case (iii). Suppose we define $x \circ y = z$ on the e-triangle eb_3c_3, where $b_3 \neq b_i$, $c_3 \neq c_i$ for $i \in \{1,2\}$.

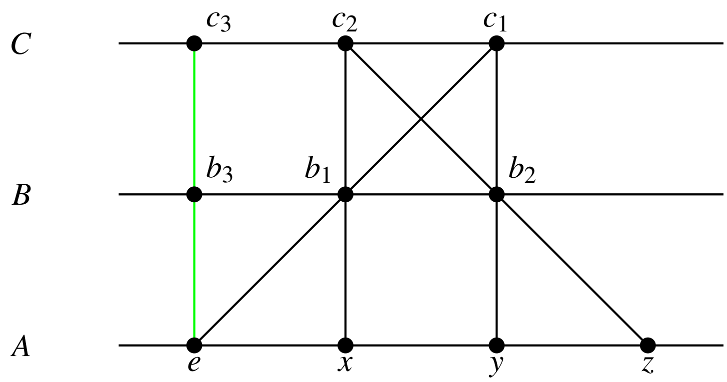


Figure 2.31:

By inspection, we must have one of the three following sub-cases:

Sub-case (a). Suppose the necessary triangles xb_3c_4 and yb_1c_3 exist, where $c_4 \neq c_i$ for $i \in \{1,2,3\}$. In this case $x \circ y$ is not well-defined if and only if the relation triangle zb_1c_4 does not exist, if and only if we have a configuration isomorphic to W_5.

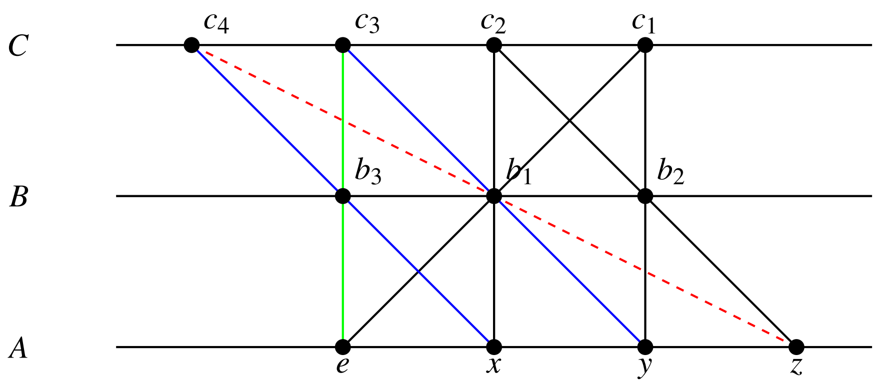


Figure 2.32: The configuration forced by defining $x \circ y$ on the e-triangle eb_3c_3, with the necessary triangles xb_3c_4 and yb_1c_3

Sub-case (b). Suppose the necessary triangles xb_3c_4 and yb_4c_3 exist, where $b_4 \neq b_i$ for $i \in \{1,2,3\}$ and $c_4 \neq c_i$ for $i \in \{1,2,3\}$. In this case $x \circ y$ is not well-defined if and only if the relation triangle zb_4c_4 does not exist if and only if we have a configuration isomorphic to W_6.

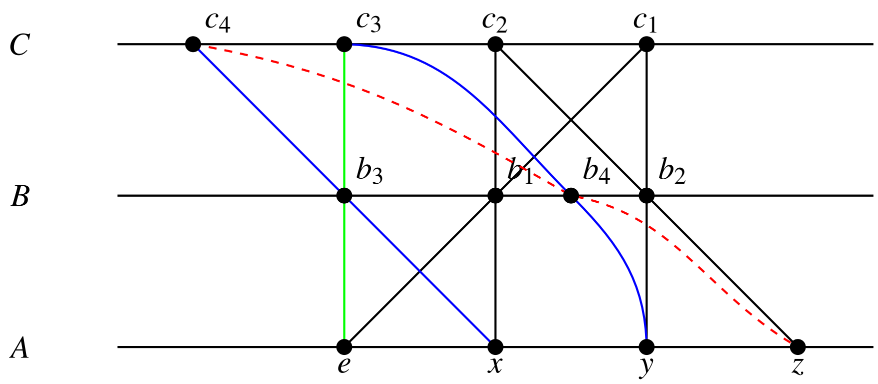


Figure 2.33: The configuration forced by defining $x \circ y$ on the e-triangle eb_3c_3, with the necessary triangles xb_3c_4 and yb_4c_3

Sub-case (c). Suppose the necessary triangles xb_3c_1 and yb_1c_3 exist. In this case $x \circ y$ is not well-defined if and only if the relation triangle zb_1c_1 does not exist, if and only if we have a configuration isomorphic to W_7. Note that the triangle zb_1c_1 cannot exist — if it did, we would have two lines meeting at more than one point, giving a non-matroid configuration.

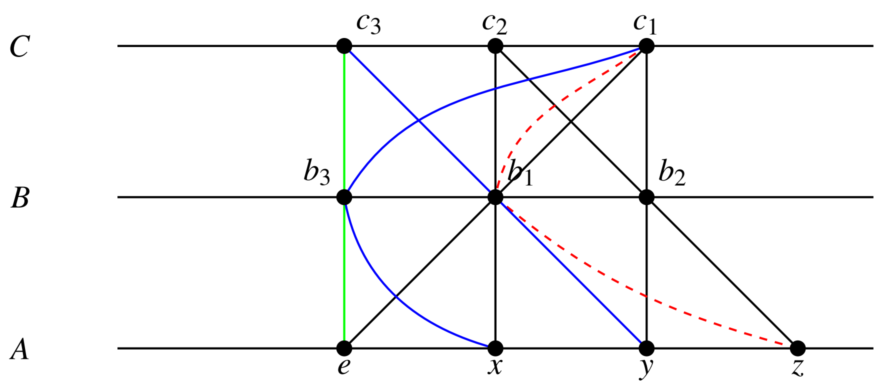


Figure 2.34: The configuration forced by defining $x \circ y$ on the e-triangle eb_3c_3, with the necessary triangles xb_3c_1 and yb_1c_3

These are all possible extensions of R_1 such that $x \circ y$ is not well-defined, completing the proof. □

The following figures show further forbidden configurations.

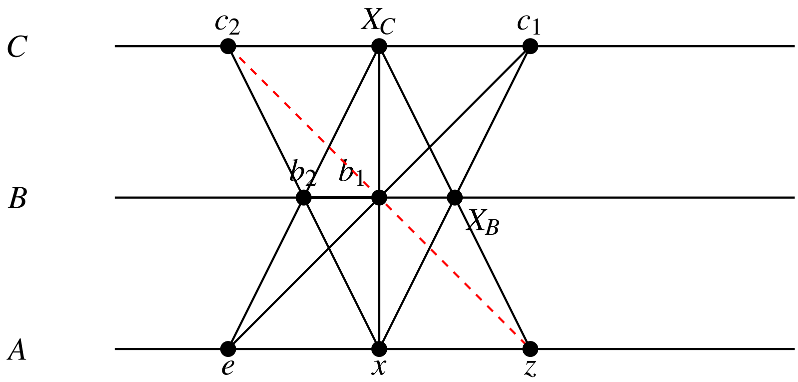


Figure 2.35: W_9 forbidden configuration

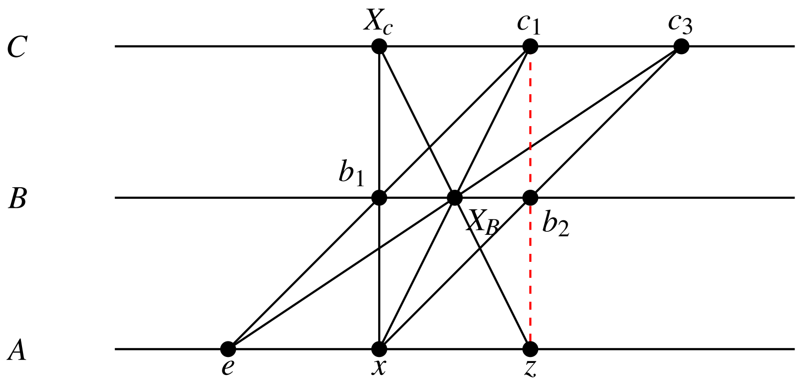


Figure 2.36: W_{10} forbidden configuration

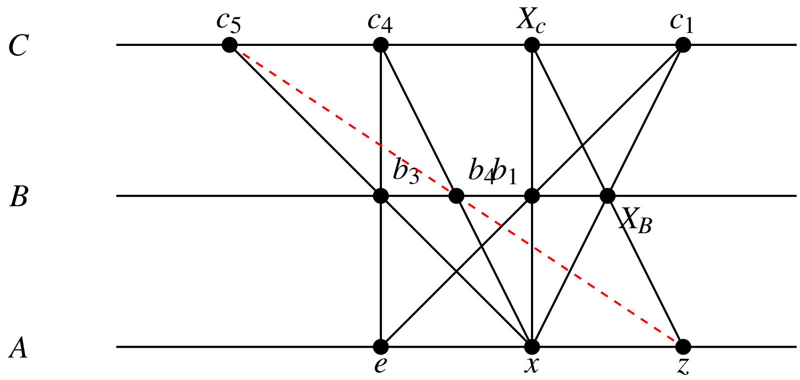


Figure 2.37: W_{11} forbidden configuration

We will prove that above configurations $W_9 - W_{11}$ are all possible forbidden configurations for the pair (x,y) being well-defined when e,x,z are distinct and $y = x$.

Lemma 2.4.3. *For a configuration G, suppose $e,x,\in A$ are distinct points, and there exists $z \in A$ where $z \notin \{e,x\}$ such that $(x,x)\circ z$. Then $x \circ x$ is well-defined and $x \circ x = z$ if and only if G has no sub-configuration isomorphic to any of $W_9 - W_{11}$, where the isomorphism is the identity for the points e,x,z.*

Proof. As for Lemma 2.4.2, we will begin with a basic relation configuration. For this case, we will begin with R_3, which defines $x \circ x = z$, and consider a case analysis of possible extensions of this configuration so that $x \circ x$ is defined on two distinct e-triangles. Given R_3, there are three other e-triangles on which we can define $x \circ x = z$:

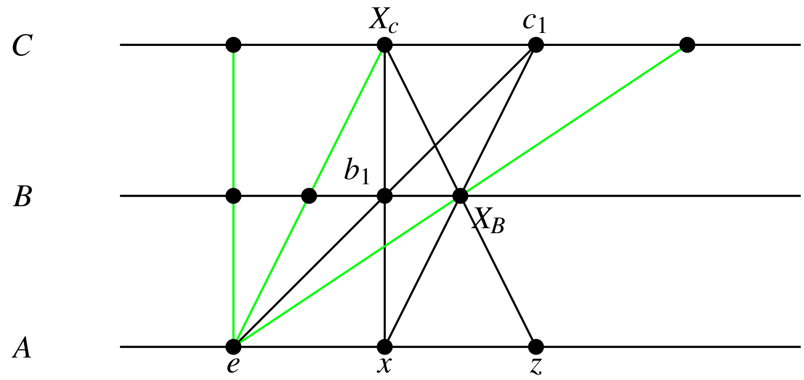


Figure 2.38:

Case (i). Suppose we define $x \circ x = z$ on the e-triangle eb_2X_C:

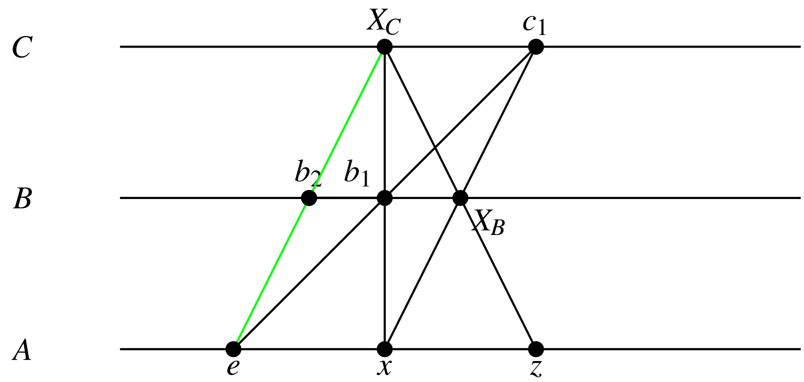


Figure 2.39:

As the necessary triangle xb_1X_C already exists, for the remaining necessary triangle we must have either one of the two following sub-cases:

Sub-case (a). Suppose the triangle xb_2c_1 exists:

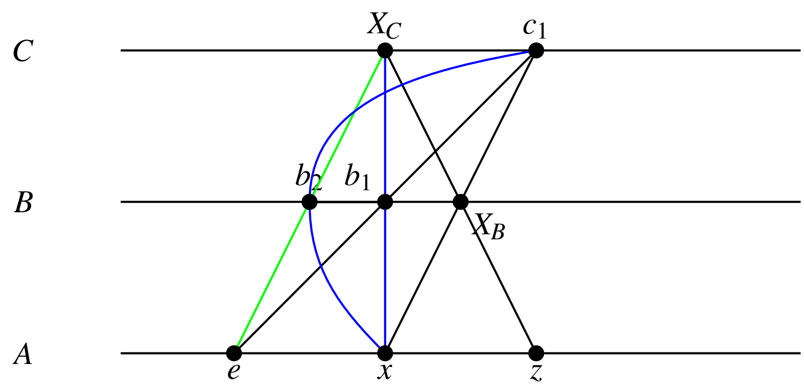


Figure 2.40: Not a matroid

But this forces a non-matroid configuration.

Sub-case (b). So we must have the triangle xb_2c_2, where $c_2 \notin \{c_1, X_C\}$. In this case $x \circ x$ is not well-defined if and only if the relation triangle zb_1c_2 does not exist, if and only if we have a configuration isomorphic to W_9:

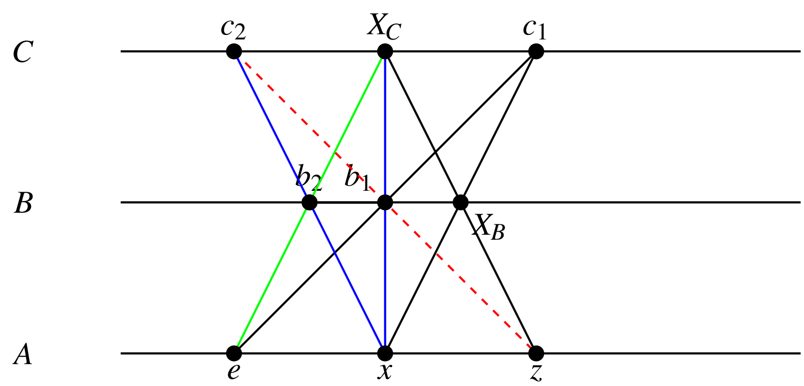


Figure 2.41: The configuration forced by defining $x \circ x$ on the e-triangle eb_2X_C, with the necessary triangles xb_2c_2 and xb_1X_C.

Case (ii). Suppose we define $(x,x) \circ z$ on the e-triangle eX_Bc_3:

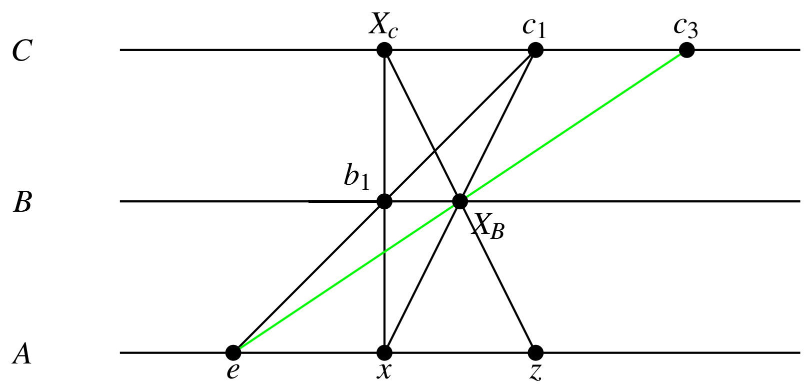


Figure 2.42:

Note that the necessary triangle xX_Bc_1 already exists. By inspection, the remaining necessary triangle must be xb_2c_3. In this case $x \circ x$ is not well-defined if and only if the relation triangle zb_2c_1 does not exist, if and only if we have a configuration isomorphic to W_{10}:

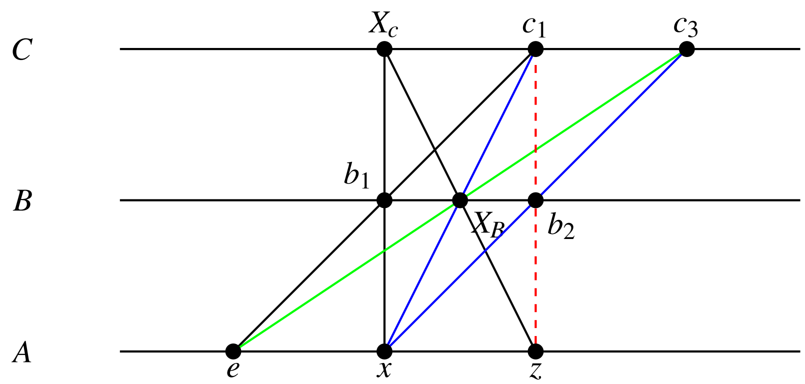


Figure 2.43: The configuration forced by defining $x \circ x$ on the e-triangle eX_Bc_3, with the necessary triangles xX_Bc_1 and xb_2c_3.

Note that this configuration is isomorphic to Figure 32.

Case (iii). Finally, suppose we define $(x,x) \circ z$ on the e-triangle eb_3c_4:

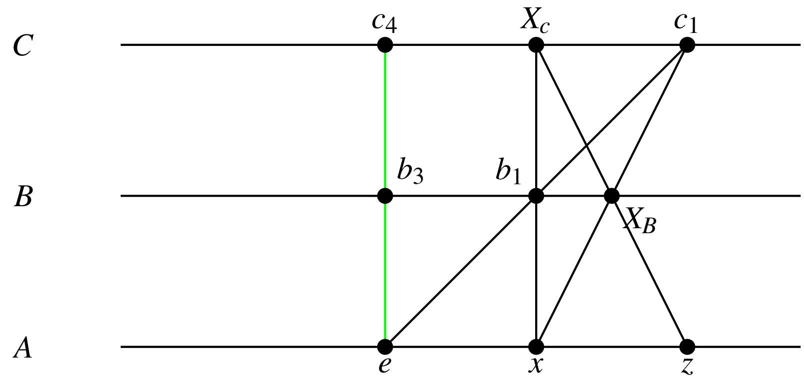


Figure 2.44:

By inspection, the necessary triangles must be xb_3c_5 and xb_4c_4, where $c_5 \notin \{c_1, c_4, X_c\}$ and $b_4 \notin \{b_1, b_3, X_B\}$. In this case $x \circ x$ is not well-defined if and only if the relation triangle zb_4c_5 does not exist, if and only if we have a configuration isomorphic to W_{11}:

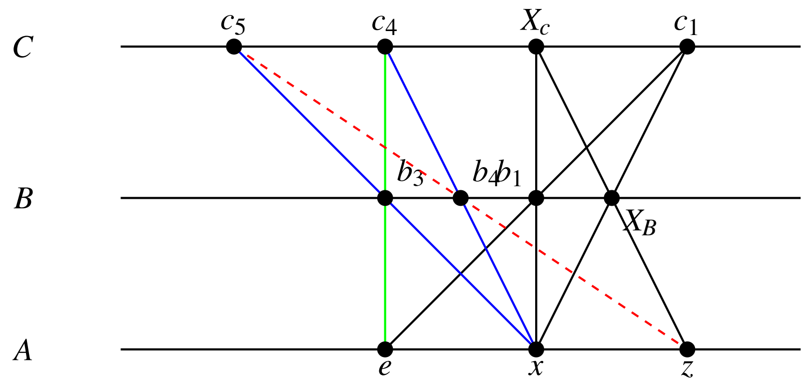


Figure 2.45: The configuration forced by defining $x \circ x$ on the e-triangle eb_3c_4, with the necessary triangles xb_3c_5 and xb_4c_4.

These are all possible extensions of R_3 such that $x \circ x$ is not well-defined, completing the proof. □

The following figure is another forbidden configuration.

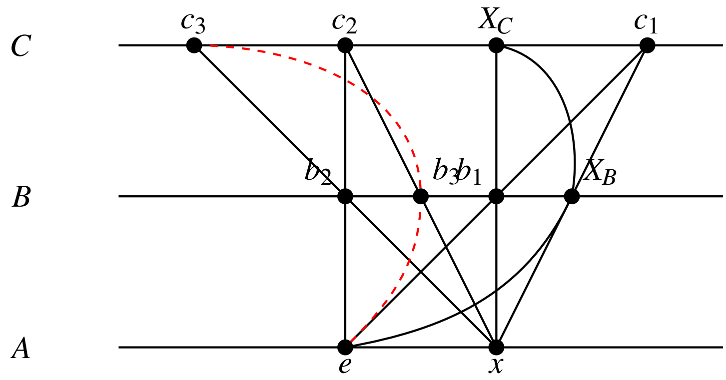


Figure 2.46: W_{12} forbidden configuration

We will now prove that for $x \circ x = e$ to be well-defined, W_{12} is the only forbidden configuration.

Lemma 2.4.4. *For a configuration G, suppose $e, x \in A$ are distinct points such that $(x,x) \circ e$. Then $x \circ x$ is well-defined and $x \circ x = e$ if and only if G has no sub-configuration isomorphic to W_{12}, where the isomorphism is the identity for the points e, x.*

Proof. As for the previous two lemmas, we will begin with a basic relation configuration. Given R_3, which defines $x \circ x = e$, there is only one other e-triangle on which $x \circ x = e$ may be defined — the e-triangle eb_2c_2, where $b_2 \notin \{b_1, X_B\}$ and $c_2 \notin \{c_1, X_C\}$:

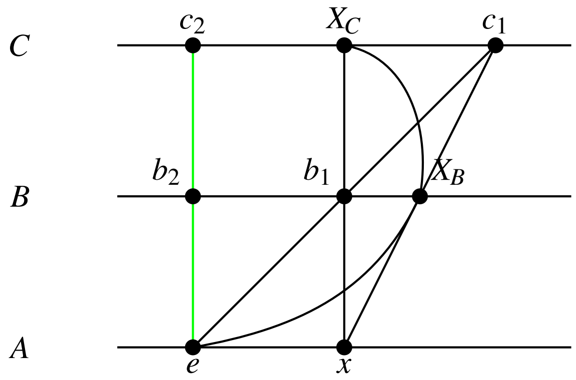


Figure 2.47:

By inspection, the necessary triangles must be xb_2c_3 and xb_3c_2 where $c_3 \notin \{c_1, c_2, X_C\}$ and $b_3 \notin \{b_1, b_2, X_B\}$. In this case $x \circ x$ is not well-defined if and only if the relation triangle eb_3c_3 does not exist, if and only if we have a configuration isomorphic to W_{12}:

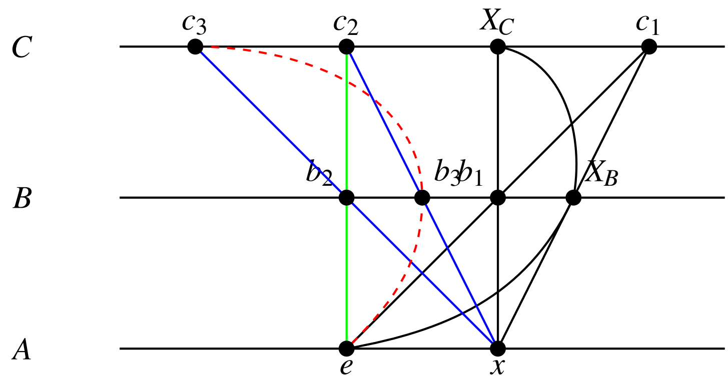


Figure 2.48: The configuration forced by defining $x \circ x$ on the e-triangle eb_2c_2, with the necessary triangles xb_2c_3 and xb_3c_2.

These are all possible extensions of R_3 such that $x \circ x = e$ is not well-defined, completing the proof. □

Finally, we list the remaining forbidden configurations for a pair being well-defined.

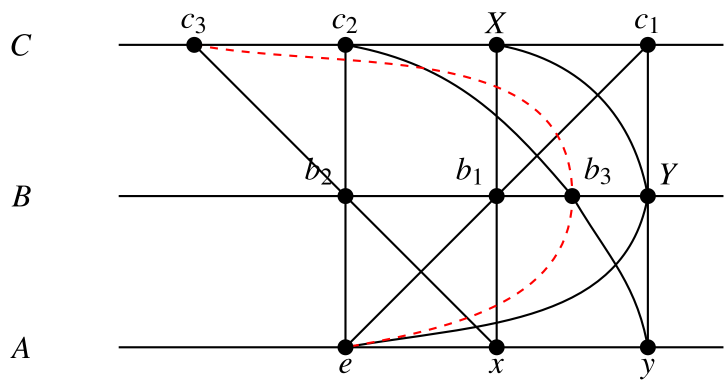


Figure 2.49: W_{13} forbidden configuration

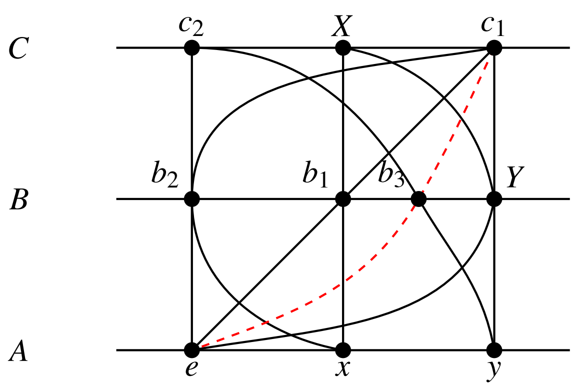


Figure 2.50: W_{14} forbidden configuration

Note that in Figure 2.50, the red, dashed triangle eb_3c_1 is not really necessary — it cannot exist as it would violate Lemma 2.1.1.

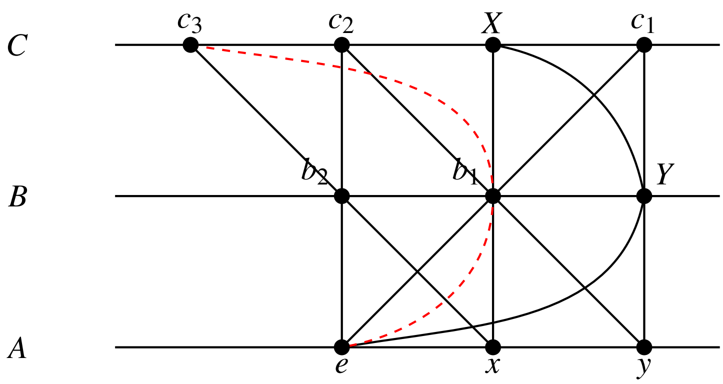


Figure 2.51: W_{15} forbidden configuration

Similarly, note that in Figure 2.51, the non-existent, red, dashed triangle eb_1c_3 is not really necessary — it cannot exist as it would violate Lemma 2.1.1.

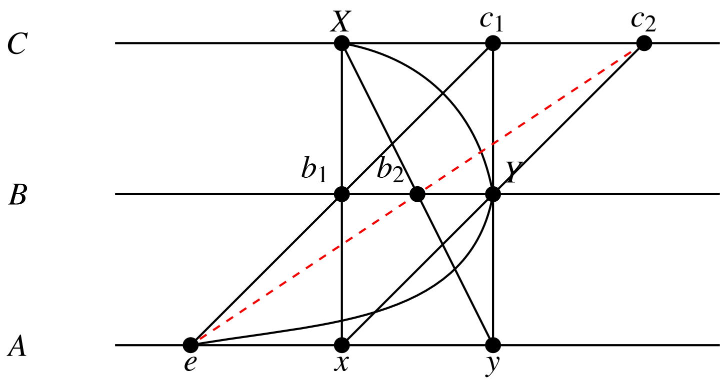


Figure 2.52: W_{16} forbidden configuration

We will now prove $W_{13} - W_{16}$ are the only forbidden configurations for $x \circ y$ being well-defined when $x \circ y = e$.

Lemma 2.4.5. *For a configuration G, suppose $e, x, y \in A$ are distinct points such that $(x, y) \circ e$. Then $x \circ y$ is well-defined and $x \circ y = e$ if and only if G has no sub-configuration isomorphic to any of $W_{13} - W_{16}$, where the isomorphism is the identity for the points e, x, y.*

Proof. Starting with the basic relation configuration R_2, which defines $x \circ y = e$, the only two other e-triangles on which we can define $x \circ y = z$ are eb_2c_2 and eYX where $b_2 \notin \{b_1, Y\}$ and $c_2 \notin \{c_1, X\}$:

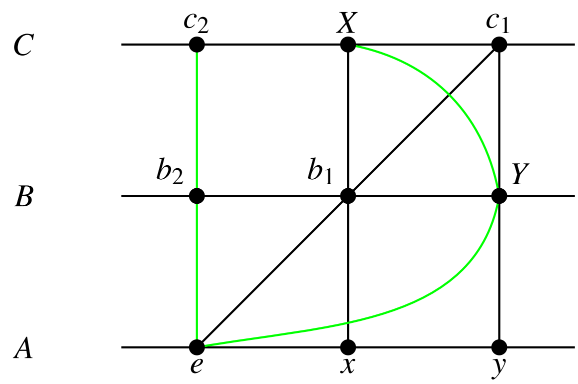


Figure 2.53:

Suppose we define $x \circ y$ on the e-triangle eb_2c_2. By inspection we must have one of the following four possibilities for our choice of necessary triangles:

Case (i). The triangles xb_2c_1 and yb_1c_2 exist if and only if $x \circ y$ is well-defined, as the relation triangle eb_1c_1 already exists.

Case (ii) Suppose we have the necessary triangles xb_2c_1 and yb_3c_2, where $b_3 \notin \{b_1, b_2, Y\}$. In this case $x \circ y$ is not well-defined if and only if the relation triangle eb_3c_1 does not exist, if and only if we have a configuration isomorphic to W_{14}:

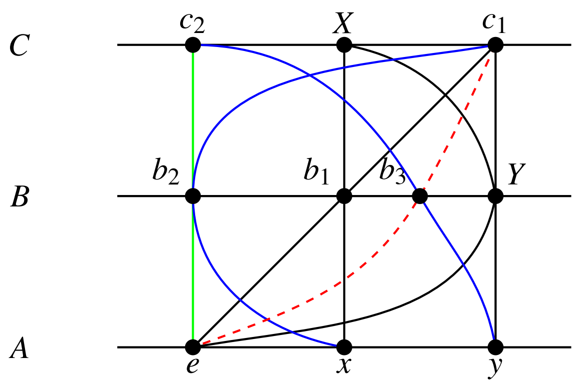


Figure 2.54: The configuration forced by defining $x \circ y = e$ on the e-triangle eb_2c_2, with the necessary triangles xb_2c_1 and yb_3c_2.

Case (iii). Suppose we have the necessary triangles xb_2c_3 and yb_1c_2, where $c_3 \notin \{c_1, c_2, X\}$. In this case $x \circ y$ is not well-defined if and only if the relation triangle eb_1c_3 does not exist, if and only if we have a configuration isomorphic to W_{15}:

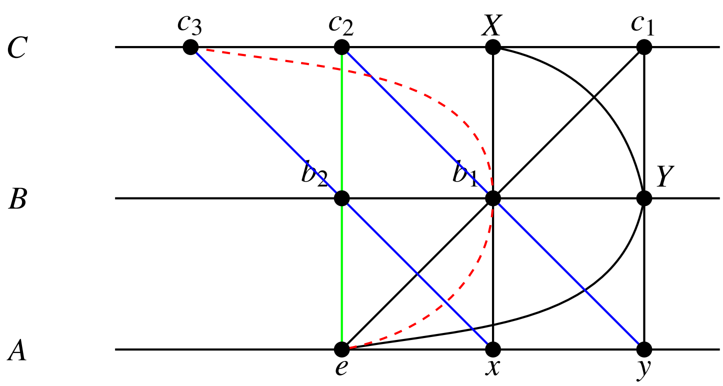


Figure 2.55: The configuration forced by defining $x \circ y = e$ on the e-triangle eb_2c_2, with the necessary triangles xb_2c_3 and yb_1c_2.

Case (iv). Suppose we have the necessary triangles xb_2c_3 and yb_3c_2, where $c_3 \notin \{c_1, c_2, X\}$ and $b_3 \notin \{b_1, b_2, Y\}$. In this case $x \circ y$ is not well-defined if and only if the relation triangle eb_3c_3 does not exist, if and only if we have a configuration isomorphic to W_{13}:

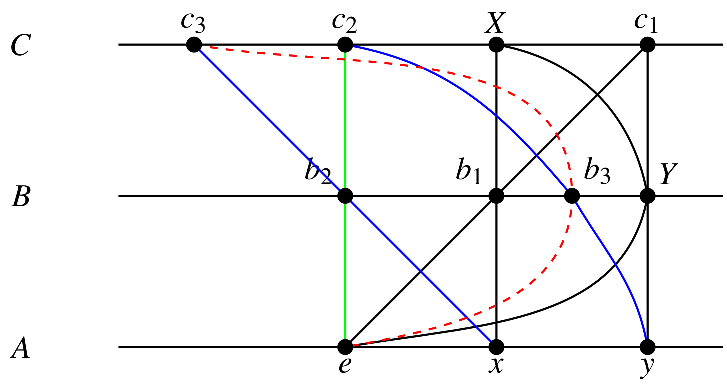


Figure 2.56: The configuration forced by defining $x \circ y = e$ on the e-triangle eb_2c_2, with the necessary triangles xb_2c_3 and yb_3c_2.

Now suppose we define $x \circ y$ on the e-triangle eYX. By inspection we must have the necessary triangles xYc_2 and yb_2X, where $c_2 \notin \{X, c_1\}$ and $b_2 \notin \{b_1, Y\}$. In this case $x \circ y$ is not well defined if and only if the relation triangle eb_2c_2 does not exist, if and only if we have a configuration isomorphic to W_{16}:

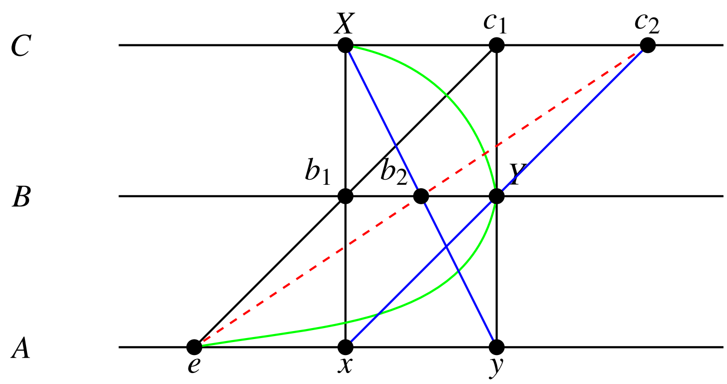


Figure 2.57: The configuration forced by defining $x \circ y = e$ on the e-triangle eYX, with the necessary triangles xYc_2 and yb_2X.

These are all possible extensions of R_2 such that $x \circ y = e$ is not well-defined, completing the proof.

□

Now we have proved Lemmas 2.4.2 – 2.4.5, we have the complete list of forbidden configurations for any pair of points of A being well-defined, which we prove in the next theorem.

Theorem 2.4.6. *Let $(G,\{A,B,C\},e)$ be an e-based configuration. Then the relation $\circ$ is a partial binary operation if and only if G does not contain any subconfiguration isomorphic to any of $W_1 - W_{16}$, where the isomorphism is the identity for the point $e \in A$.*

Proof. It follows from Lemmas 2.4.2, 2.4.3, 2.4.4 and 2.4.5 that if G contains any of $W_1 - W_{15}$, then $\circ$ is not a well-defined operation and therefore not a partial binary operation. If G doesn't contain any of $W_1 - W_{16}$, then $\circ$ is a well-defined operation and therefore a partial binary operation. □

Corollary 2.4.6.1. *Let $(G,\{A,B,C\},e)$ be an e-based configuration. Then the relation $\circ$ is a full binary operation if and only if $\circ$ is a partial binary operation and every pair $(x,y) \in A \times A$ is defined.*

Observe that to check whether a pair $(x,y) \in A \times A$ is well-defined, we need only check for 16 forbidden configurations. As each forbidden configuration has no more than 12 points, for a configuration with n points, we can check this in less than $16\binom{n}{11}\binom{12}{3}$ steps, which is polynomial in n. This proves the following corollary.

Corollary 2.4.6.2. *Let $(G,\{A,B,C\},e)$ be an e-based configuration with n points. There is an algorithm, which is polynomial in n, to check whether $\circ$ is a partial binary operation.*

The next corollary follows from Corollaries 2.4.6.1 and 2.4.6.2.

Corollary 2.4.6.3. *Let $(G,\{A,B,C\},e)$ be an e-based configuration with n points. There is an algorithm, which is polynomial in n, to check whether $\circ$ is a full binary operation.*

2.5 Strong and weak binary operations

We conclude this chapter by describing strong and weak binary operations, which exist in addition to full and partial binary operations.

Recall that $x \circ y$ is well-defined if $x \circ y$ is defined and when eb_1c_2 and $eb_1'c_2'$ are e-triangles such that $x \circ y$ is defined on eb_1c_2 giving $x \circ y = z_1$ and $x \circ y$ is defined on $eb_1'c_2'$ giving $x \circ y = z_2$, then $z_1 = z_2$. Also recall that $x \circ y$ is strongly defined if it is well-defined and for *every* e-triangle ebc we have $x \circ y$ defined on ebc. We say G has a *weak partial binary operation* $\circ$ if $\circ$ is a partial binary operation and at least one defined pair of $A \times A$ is well-defined but not strongly defined. We say G has a *strong partial binary operation* $\circ$ if $\circ$ is a partial binary operation and every defined pair $(x,y) \in A \times A$ is strongly defined. That is, $\circ$ is a strong partial binary operation if for every defined pair (x,y), then $x \circ y$ is defined on every e-triangle.

Similarly, we say G has a *weak full binary operation* $\circ$ if $\circ$ is a full binary operation and at least one well-defined pair $(x,y) \in A \times A$ is not strongly defined. In other words, $\circ$ is a weak full binary operation if there exists some pair (x,y) such that $x \circ y$ is not defined on at least one e-triangle. We say G has a *strong full binary operation* $\circ$ if $\circ$ is a full binary operation and every pair of $A \times A$ is strongly defined. That is, $\circ$ is a strong full binary operation if for every pair (x,y), then $x \circ y$ is defined on every e-triangle. We say G has a *weak full binary operation* $\circ$ if $\circ$ is a full binary operation and at least one pair $(x,y) \in A \times A$ is well-defined but not strongly defined. In other words, $\circ$ is a weak full binary operation if there exists some pair (x,y) such that $x \circ y$ is not defined on at least one e-triangle.

From now on, though we will specify when necessary, we generally assume that $\circ$ is a full binary operation. Whether we consider the strong or weak binary operation will hugely impact the complexity of our results.

Chapter 3

Connections with the projective plane

In this chapter we will explore the connections between the algebra of throws within matroid configurations and the algebraic structure of the projective plane. If we have a matroid configuration from a projective plane representable over a field, we show that the addition and multiplication defined by the algebra of throws corresponds to addition and multiplication respectively in the field. On the other hand, if we have matroid configuration from a projective plane over some other algebraic structure, using a classical coordinatization of the projective plane, we show that the algebra of throws corresponds to the addition and multiplication over the given structure.

3.1 Coordinatizing the projective plane

There are various classical, equivalent methods of coordinatizing the projective plane. We will paraphrase the most commonly used method by Hughes and Piper, as described in Chapter 5 of [2].

Let $\mathscr{P}$ be a projective plane of order n and let R be any set of n symbols such that $0,1 \in R$, $0 \neq 1$ and $\infty \notin R$. Choose any line of $\mathscr{P}$ and label it l_∞. Choose any two other lines of $\mathscr{P}$, which we label l_1, l_2, with the condition that l_1, l_2, l_∞ form the sides of a non-degenerate triangle. We will label the points of this triangle by X, Y, O, where $X = l_2 l_\infty$, $Y = l_1 l_\infty$ and $0 = l_1 l_2$. Pick any point I which is not incident with any of the lines l_1, l_2, l_∞. Finally, we will label three more points as follows. Let A be the intersection of the lines XI and l_1; let B be the intersection of the lines YI and l_2 and let J be the intersection of the lines AB and l_∞. We will use the elements of $R \cup \{\infty\}$ to coordinatize $\mathscr{P}$ with respect to the quadrangle $0, X, Y, I$.

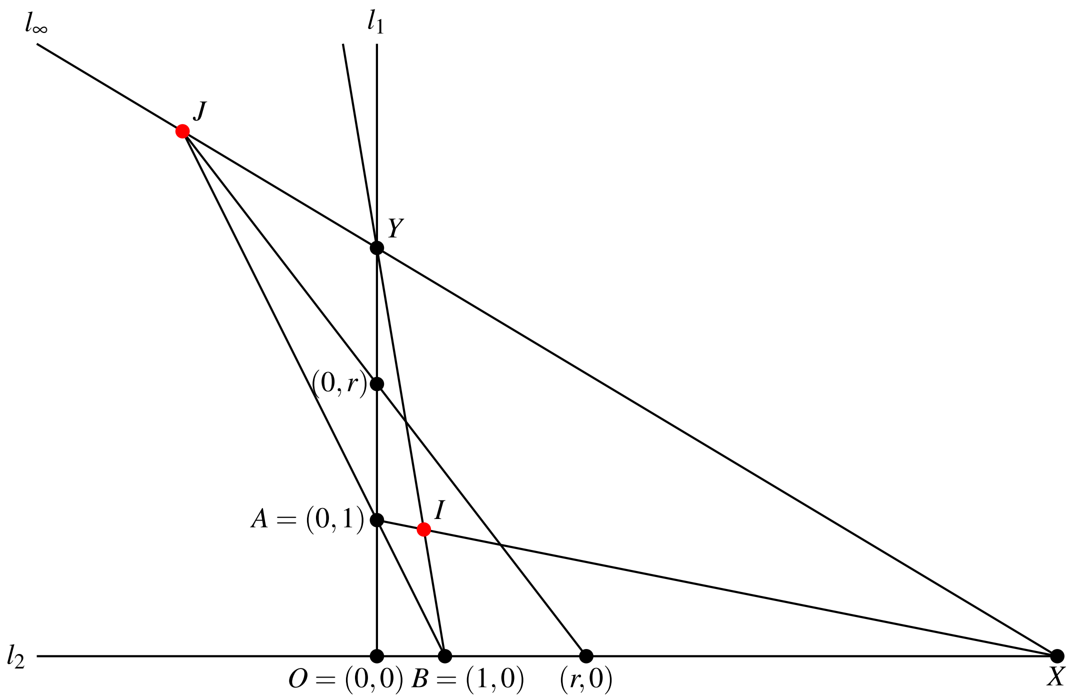


Figure 3.1:

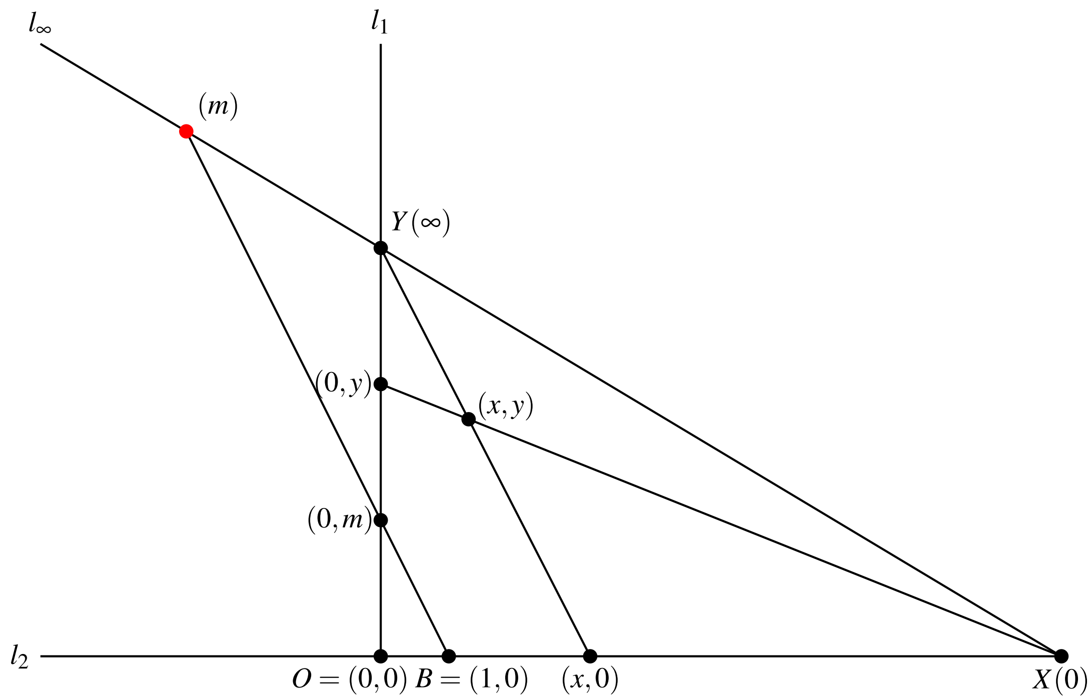


Figure 3.2:

First, we arbitrarily assign elements of R to the points of the line $l_1 \backslash Y$, with the condition that 0 is assigned to O and 1 is assigned to A. If $c \in R$ is assigned to the point $C \in l_1$, then we give C the coordinate $(0,c)$. For $D \in l_2$ such that $D \neq X$, let D' be the intersection of the lines JD and l_1. Then if D' has coordinate $(0,d)$, we say D has coordinate $(d,0)$. This means that O has the coordinate $(0,0)$. For any point $E \notin l_\infty$, if $XE \cap l_1$ is the coordinate $(0,g)$ and $YE \cap l_2$ is the coordinate $(f,0)$, then E is given the coordinates (f,g). Excluding the points on the line l_∞, every point has been given unique coordinates (x,y) where $x,y \in R$. Now we will coordinatize points of l_∞. Suppose $M \in l_\infty \backslash Y$ and the line joining M to $(1,0)$ meets l_1 at the point $(0,m)$. Then we give M the coordinate (m). Finally, coordinatize M by giving it the coordinate (m). Now every point of $\mathscr{P}$ has been coordinatized, and depends only on our initial choice of the quadrangle O,X,Y,I and the way in which we assigned the elements of R to the points of $l_1 \backslash Y$.

We will now coordinatize the lines. If l is any line where $Y \notin l$, then if $l \cap l_\infty = (m)$ and $l \cap l_1 = (0,k)$, then we give l the coordinates $[m,k]$. If l is a line where $Y \in l$ and $l \neq l_\infty$, then we label the line $[k]$ where $l \cap c_1 = (k,0)$. Finally, call l_∞ the line $[\infty]$. Now we have coordinatized every point and every line of $\mathscr{P}$.

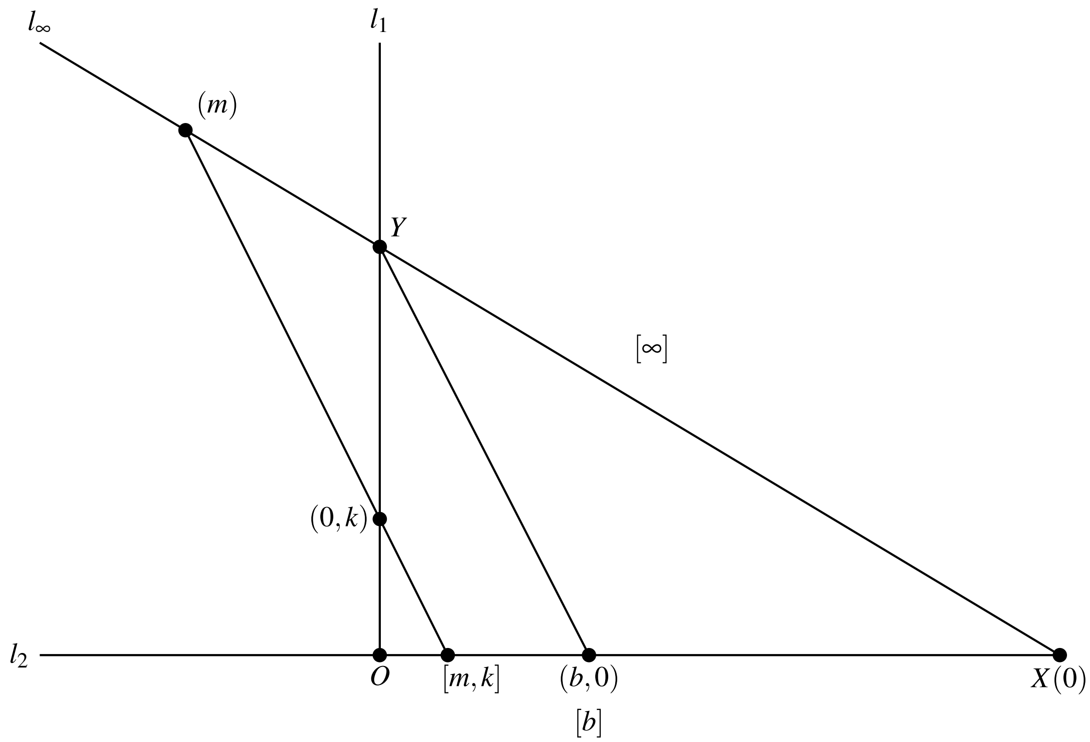


Figure 3.3:

It follows from [2] that we can think of l_2 and l_1 as the x-axis and y-axis respectively, and l_∞ as the line at infinity. Any line with slope m and y-intercept k is labelled $[m,k]$ and meets l_∞ at the point (m).

We will show that the algebra of throws constructions defining addition and multiplication is equivalent to the addition and multiplication of points along the line l_1. Before we do so, we define some notation. When we apply the algebra of throws operation $a \circ b$, if this corresponds to the addition of points, we denote the sum of the points a and b by $a \oplus b$. On the other hand, if $a \circ b$ corresponds to the multiplication of points, we denote the product of the points a and b by $a \otimes b$. We let $(0, a+b)$ be the sum of the points $(0,a), (0,b) \in l_1$, as described in [2], where $(0, a+b)$ is defined to be on the line through the points (1) and (a,b); i.e. the line $[1, a+b]$ with slope 1 and y-intercept $a+b$. We let $(0, a \times b)$ be the product of the points $(0,a), (0,b) \in l_1$, as described in [2], where $(0, a \times b)$ is defined to be on the line through the points (a) and $(b, 0)$; i.e. the line $[a, a \times b]$ with slope a and y-intercept $(b, 0)$.

3.1.1 Addition

Suppose we are given three lines l_1, l_2, l_3, which are incident at a point labelled (∞); i.e. this corresponds to the point Y as in Figure 3.1. Choose any point on l_1 and call it $(0,0)$; i.e. this corresponds to the point O as in Figure 3.1. We will label this point with coordinate $(0,0)$ by e. Choose any line except l_1 through the point e and call it l_e. Let $l_e \cap l_2 = (1,0)$; i.e. this corresponds to the point B as in Figure 3.1. We will label this point with coordinate $(1,0)$ by b_1. Let $l_e \cap l_3 = (0)$; i.e. this corresponds to the point X as in Figure 3.1. We will label the point (0) by c_1.

Choose any point on $l_1 \backslash c_1$ and call it $(0,1)$. Then the point on the intersection of the lines $(0,1)(0)$ and l_2 is the point I in the coordinatization of the plane. Note, however, that we do not use the point I doing addition.

We will show that if we apply the algebra of throws to any pair of points $(0,\alpha), (\beta, 0) \in l_1$, then $(0, \alpha \oplus \beta) = (0, \alpha + \beta)$, where $(0, \alpha + \beta)$ is defined to be on the line through the points (1) and (α,β).

Note that the lines l_1,l_2,l_3 correspond to the distinguished lines A,B,C (as described in the previous chapters) respectively. Pick points $(0,\alpha),(0,\beta)\in l_1$, which we denote by a and b respectively. We now apply the algebra of throws to the points a and b to obtain their sum, the point $(0,\alpha\oplus\beta)\in l_1$. Let us consider the necessary triangles for the operation $a\circ b$ with respect to the e-triangle eb_1c_1. The line through the points $(0,\alpha)$ and $(1,0)$ has slope α, so meets l_3 at the point (α). The line through $(0,\beta)$ and c_1 meets l_2 at the point labelled $b'=(1,\beta)$.

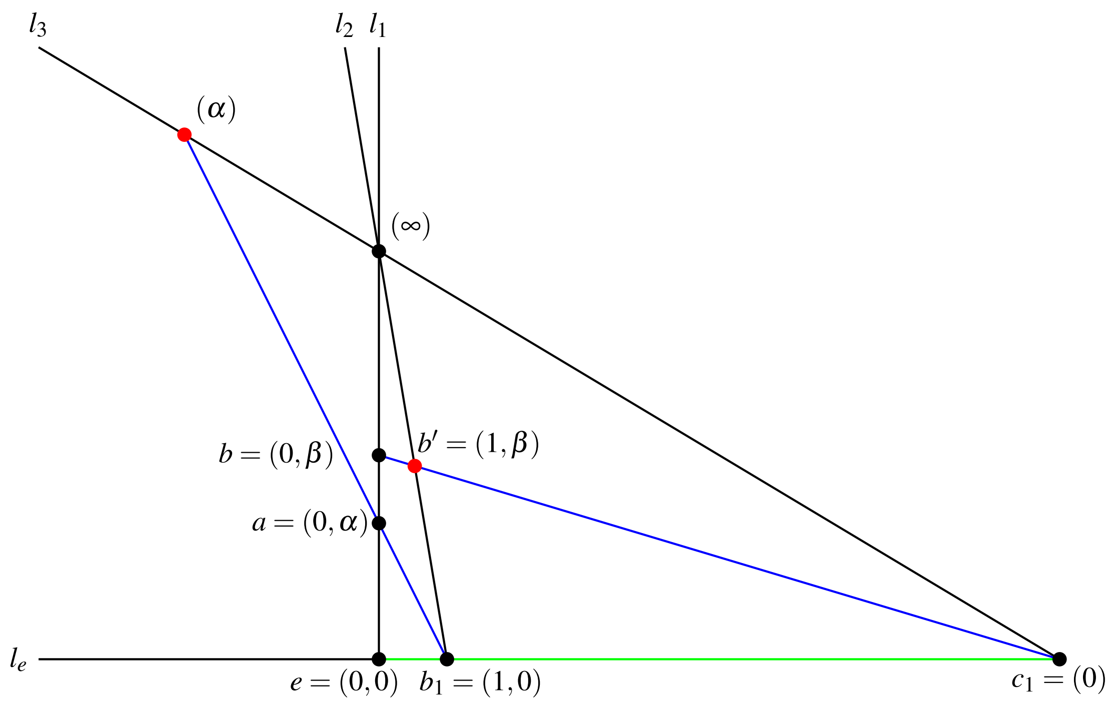


Figure 3.4: The blue triangles $ab_1(\alpha)$ and $bb'c_1$ are the necessary triangles of $a\circ b$ with respect to the green e-triangle eb_1c_1.

The line through (α) and b' meets l_1 at some point $(0,y)$. As the line through (α) and $(1,\beta)$ has slope α, then $y-\beta=\alpha$ and so $y=\alpha+\beta=\alpha\oplus\beta$. That is, $(0,\alpha\oplus\beta)=(0,\alpha+\beta)$ therefore the addition of points using the algebra of throws is equivalent to the addition of points on l_1 defined by the coordinatization of the projective plane.

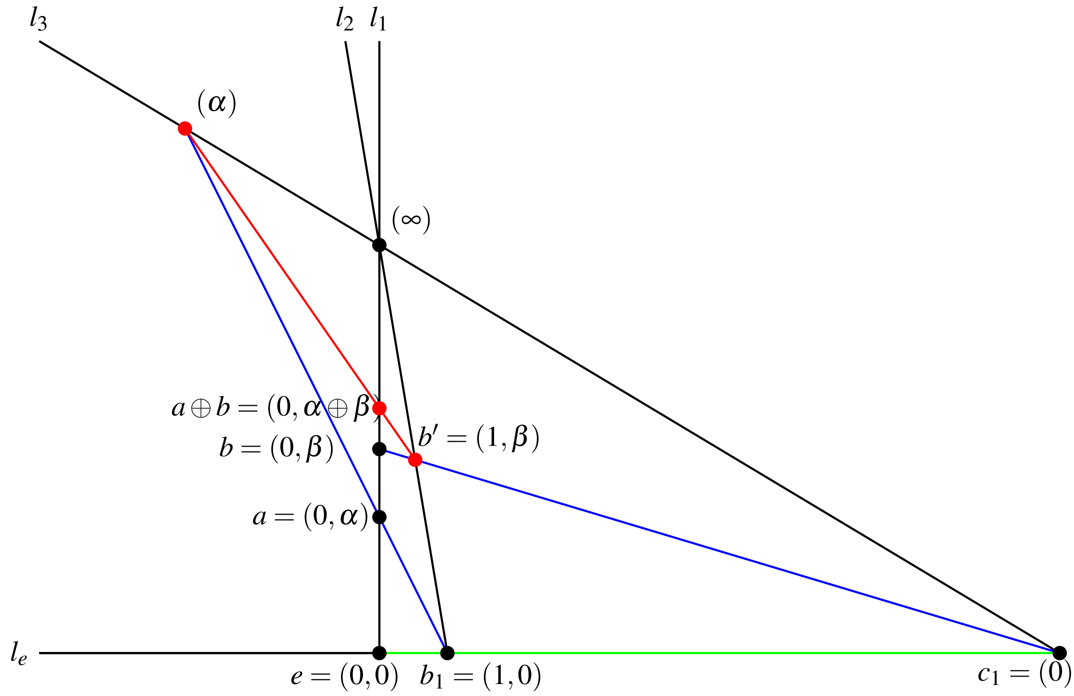


Figure 3.5: The red triangle $(a \oplus b)b'(\alpha)$ is the relation triangle of $a \circ b$.

3.1.2 Multiplication

Suppose we are given three lines l_1, l_2, l_3 which are not co-punctual. Choose any point on l_1 and call it $e = (0,1)$; i.e. this corresponds to the point A as in Figure 3.1. Choose any line through e and call it l_e. Let $l_e \cap l_2 = (1,0)$; i.e. this corresponds to the point B as in Figure 3.1. We denote the point with coordinate $(1,0)$ by b_1. Let $l_e \cap l_3 = (1)$; i.e. this corresponds to the point J as in Figure 3.1. We denote the point (1) by the label c_1. We will show that if we apply the algebra of throws to any pair of points $(0,\alpha),(0,\beta) \in l_1$, then $(0, \alpha \otimes \beta) = (0, \alpha \times \beta)$, where $(0, \alpha \times \beta)$ is defined to be on the line through the points (α) and $(\beta, 0)$. Note that we do not use the points I, X, Y, ∞ when doing multiplication.

As was the case for addition, note that the lines l_1, l_2, l_3 correspond to the distinguished lines A, B, C (as described in the previous chapters) respectively. Pick points $(0,\alpha), (0,\beta) \in l_1$, which we denote by a and b respectively. We now apply the algebra of throws to the points a and b to obtain their product, the point $(0, a \otimes b) \in l_1$. Let us consider the necessary triangles for the operation $a \circ b$ with respect to the e-triangle eb_1c_1. The line through the points $(0,\alpha)$ and $(1,0)$ has slope α, so meets l_3 at the point labelled (α). The line through $(0,\beta)$ and c_1 has slope 1, so meets l_2 at the point $(\beta, 0)$, which we will denote by b'.

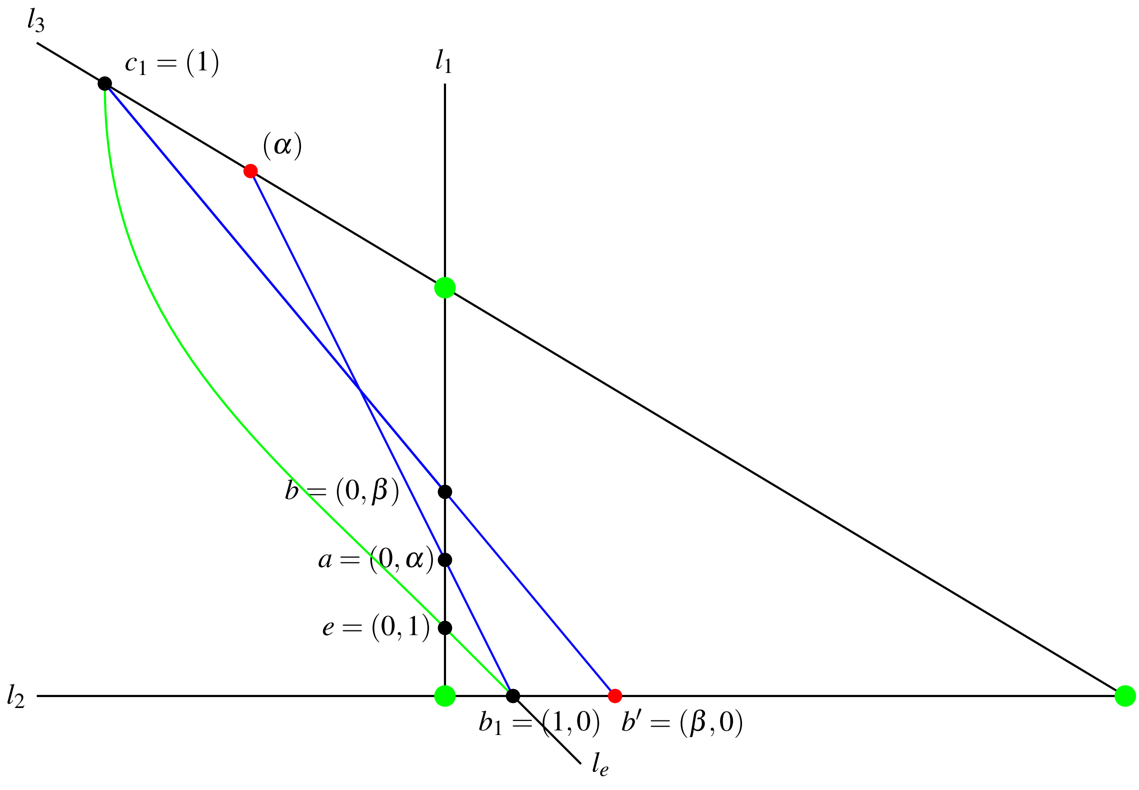


Figure 3.6: The blue triangles $ab_1(\alpha)$ and $bb'c_1$ are the necessary triangles of $a \circ b$ with respect to the green e-triangle eb_1c_1.

The line through (α) and $(\beta, 0)$ meets l_1 at the point $(0, \alpha \otimes \beta)$. As the line through (α) and $(\beta, 0)$ has slope α, it must be the line $[\alpha, \alpha \otimes \beta]$ and we must have $\frac{\alpha\otimes\beta}{\beta} = \alpha$, therefore $\alpha\otimes\beta = \alpha\times\beta$. That is, $(0,\alpha\otimes\beta) = (0,\alpha\times\beta)$ and therefore the multiplication of points using the algebra of throws is equivalent to the multiplication of points on l_1 defined by the coordinatization of the projective plane.

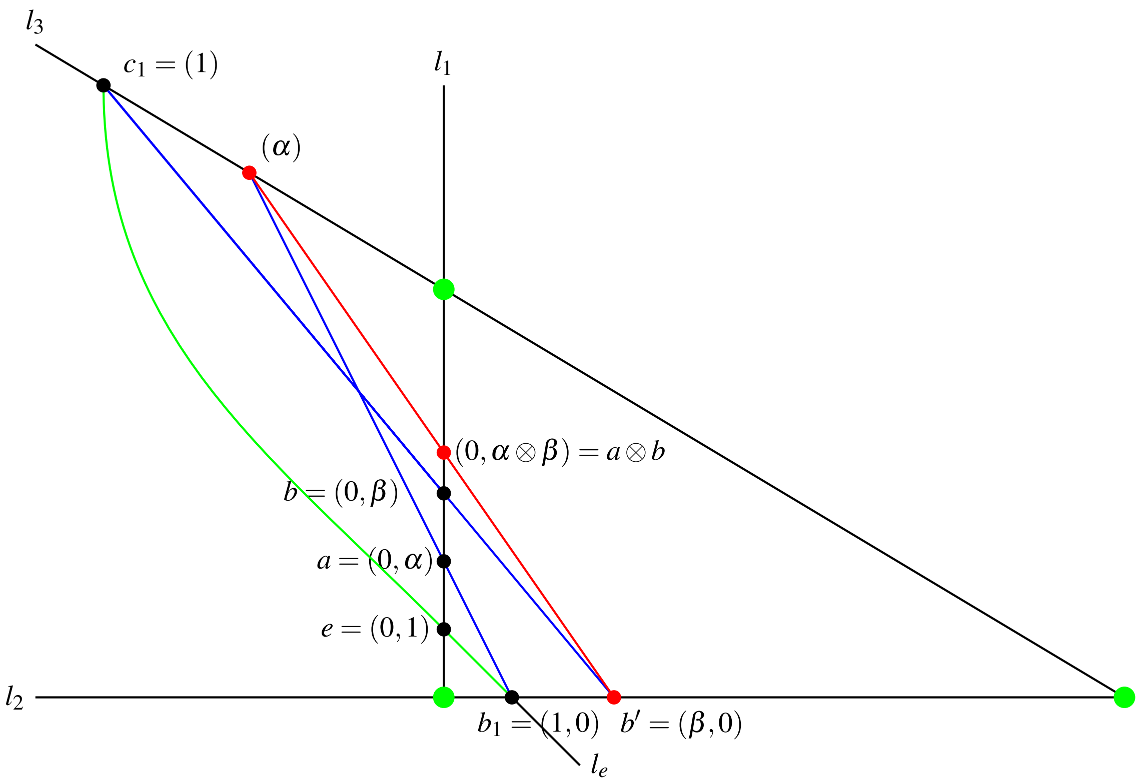


Figure 3.7: The red triangle $(a\otimes b)b'(\alpha)$ is the relation triangle of $a\circ b$.

3.2 Projective planes over fields

We will now prove algebraically that given a projective plane over a field, the addition and multiplication of points using the algebra of throws corresponds to addition and multiplication respectively over the given field. Note that for this section, when we refer to *xyz*, this need not be a triangle such that $x\in A, y\in B, z\in C$, as we defined in Chapter 2. Rather, *xyz* are simply *any* three collinear points.

In all future chapters, we will return to the notation as described previously in Chapter 2.

3.2.1 Addition

Recall the following diagram defining the addition of points using the algebra of throws.

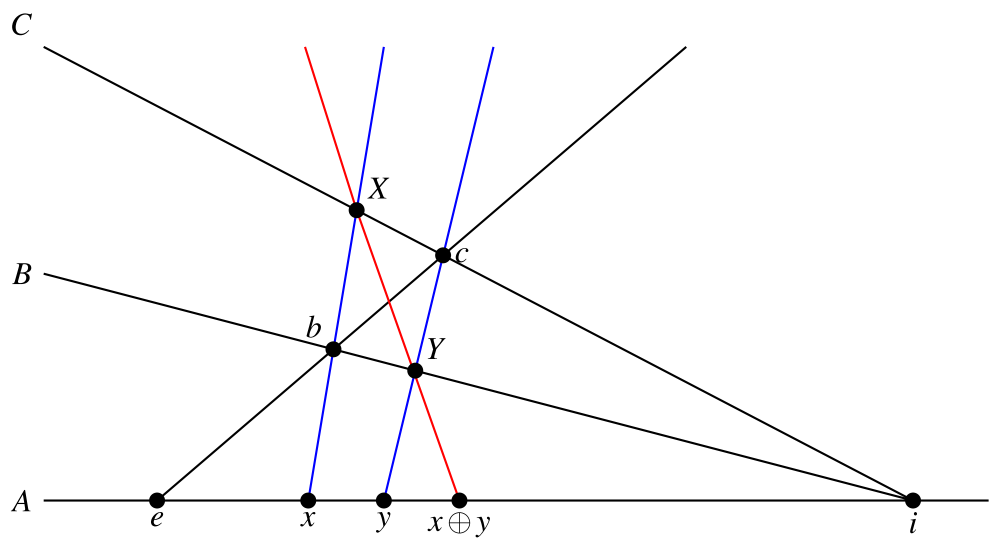


Figure 3.8:

The following lemma will use the same notation as in Figure 3.8.

Lemma 3.2.1. *Suppose G is representable over a field $\mathbb{F}$ and we scale the basis (e,i,c) such that $\vec{e}=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\vec{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\vec{c}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ and scale x,y such that $\vec{x}=\begin{pmatrix}x\\1\\0\end{pmatrix}$ and $\vec{y}=\begin{pmatrix}y\\1\\0\end{pmatrix}$. Then $\overrightarrow{x\oplus y}=\begin{pmatrix}x+y\\1\\0\end{pmatrix}$.*

Proof. Let $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, $\vec{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$, $\vec{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$ and $\overrightarrow{x \oplus y} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$.

As bce is a circuit, $\begin{vmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 1 \\ b_3 & 1 & 0 \end{vmatrix} = 0$ if and only if $b_1 = 0$.

Therefore, after scaling, $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ b_3 \end{pmatrix}$. As Xci is a circuit, $\begin{vmatrix} X_1 & 0 & 1 \\ X_2 & 0 & 0 \\ X_3 & 1 & 0 \end{vmatrix} = 0$ if and only if $X_2 = 0$. Therefore $\vec{X} = \begin{pmatrix} X_1 \\ 0 \\ X_3 \end{pmatrix}$. As Xbx is a circuit, $\begin{vmatrix} X_1 & 0 & x \\ 0 & 1 & 1 \\ X_3 & b_3 & 0 \end{vmatrix} = 0$ if and only if

$X_1(-b_3) + x(-X_3) = 0$ if and only if $x = \frac{-X_1 b_3}{X_3}$ if and only if $X_1 = \frac{-xX_3}{b_3}$.

As cYy is a circuit, $\begin{vmatrix} 0 & Y_1 & y \\ 0 & Y_2 & 1 \\ 1 & Y_3 & 0 \end{vmatrix} = 0$ if and only if $Y_1 - yY_2 = 0$

if and only if $Y_1 = yY_2$. As bYi is a circuit, $\begin{vmatrix} 0 & Y_1 & 1 \\ 1 & Y_2 & 0 \\ b_3 & Y_3 & 0 \end{vmatrix} = 0$ if and only if $Y_3 - b_3 Y_2 = 0$ if and only if $Y_3 = b_3 Y_2$.

As $e(x \oplus y)i$ is a circuit, $\begin{vmatrix} 0 & z_1 & 1 \\ 1 & z_2 & 0 \\ 0 & z_3 & 0 \end{vmatrix} = 0$ if and only if $z_3 = 0$.

As XYz is a circuit, $\begin{vmatrix} \frac{-xX_3}{b_3} & yY_2 & z_1 \\ 0 & Y_2 & z_2 \\ X_3 & b_3Y_2 & 0 \end{vmatrix} = 0$ if and only if

$\frac{-xX_3}{b_3}(-z_2b_3Y_2) - yY_2(-z_2X_3) + z_1(-X_3Y_2) = 0$ if and only if

$xX_3z_2Y_2 + yz_2X_3Y_2 - z_1X_3Y_2 = 0$ if and only if $xz_2 + yz_2 - z_1 = 0$

if and only if $z_1 = z_2(x+y)$.

Therefore $\overrightarrow{x \oplus y} = \begin{pmatrix} z_2(x+y) \\ z_2 \\ 0 \end{pmatrix}$, which we can scale to $\begin{pmatrix} x+y \\ 1 \\ 0 \end{pmatrix}$.

Note that $\vec{x}_c$ and $\vec{y}_b$ scale to $\begin{pmatrix} \frac{-x}{e_3} \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} y \\ 1 \\ e_3 \end{pmatrix}$ respectively. $\square$

3.2.2 Multiplication

Recall the following diagram defining the multiplication of points using the algebra of throws.

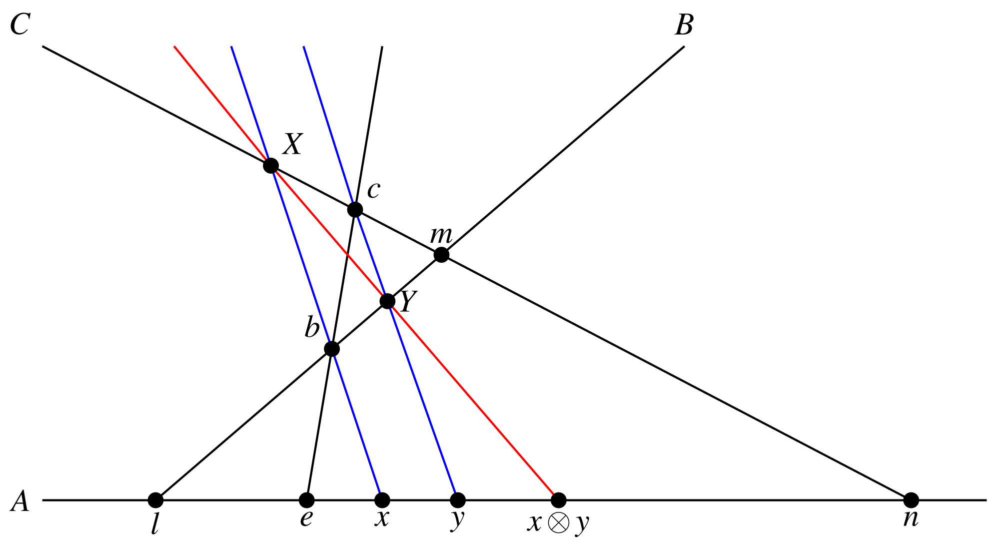


Figure 3.9:

The following lemma will use the same notation as in Figure 3.9.

Lemma 3.2.2. *Suppose G is representable over a field $\mathbb{F}$ and we scale the basis (n,l,m) such that $\vec{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{l} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{m} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and scale x, y such that $\vec{x} = \begin{pmatrix} x \\ 1 \\ 0 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y \\ 1 \\ 0 \end{pmatrix}$. Then $\overrightarrow{x \otimes y} = \begin{pmatrix} xy \\ 1 \\ 0 \end{pmatrix}$.*

Proof. Scale $\vec{e}$ and $\overrightarrow{x \otimes y}$ respectively so that $\vec{e} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{x \otimes y} = \begin{pmatrix} z_1 \\ 1 \\ 0 \end{pmatrix}$.

Let $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, $\vec{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$, $\vec{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.

As lmb is a circuit, $\begin{vmatrix} 1 & 0 & b_1 \\ 0 & 0 & b_2 \\ 0 & 1 & b_3 \end{vmatrix} = 0$ if and only if $-b_2 = 0$.

Therefore $\vec{b} = \begin{pmatrix} b_1 \\ 0 \\ b_3 \end{pmatrix}$, which we can scale to $\begin{pmatrix} 1 \\ 0 \\ b_3 \end{pmatrix}$.

Similarly, as lYb is a circuit, $\vec{Y} = \begin{pmatrix} Y_1 \\ 0 \\ Y_3 \end{pmatrix}$, which we can scale to $\begin{pmatrix} 1 \\ 0 \\ Y_3 \end{pmatrix}$.

As mnc is a circuit, $\begin{vmatrix} 0 & 0 & c_1 \\ 0 & 1 & c_2 \\ 1 & 0 & c_3 \end{vmatrix} = 0$ if and only if $c_1 = 0$. Therefore $\vec{c} = \begin{pmatrix} 0 \\ c_2 \\ c_3 \end{pmatrix}$, which we can scale to $\begin{pmatrix} 0 \\ 1 \\ c_3 \end{pmatrix}$. Similarly, as Xmn is a circuit, we have $\vec{X} = \begin{pmatrix} 0 \\ 1 \\ X_3 \end{pmatrix}$. As xbX is a circuit, $\begin{vmatrix} x & 1 & 0 \\ 1 & 0 & 1 \\ 0 & b_3 & X_3 \end{vmatrix} = 0$ if and only if $-xb_3 - X_3 = 0$ if and only if $X_3 = -xb_3$. As ebc is a circuit, $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & b_3 & c_3 \end{vmatrix} = 0$ if and only if $-b_3 - c_3 = 0$ if and only if $-b_3 = c_3$. As yYc is a circuit, $\begin{vmatrix} y & 1 & 0 \\ 1 & 0 & 1 \\ 0 & Y_3 & c_3 \end{vmatrix} = 0$ if and only if $-yY_3 - c_3 = 0$ if and only if $Y_3 = \frac{-c_3}{y}$.

Finally, as XYz is a circuit, $\begin{vmatrix} 0 & 1 & z_1 \\ 1 & 0 & 1 \\ X_3 & Y_3 & 0 \end{vmatrix} = 0$ if and only if $X_3 - z_1 Y_3 = 0$

if and only if $z = \frac{X_3}{Y_3} = -xb_3 \frac{-y}{c_3} = \frac{xc_3y}{c_3} = xy$. Therefore $\overrightarrow{x \otimes y} = \begin{pmatrix} xy \\ 1 \\ 0 \end{pmatrix}$ □

Given a projective plane over a field, the algebra of throws corresponds to the addition or multiplication of the given field. Therefore, we have the properties of commutativity (i.e. we have a Desarguesian projective plane) and associativity. These properties are not guaranteed in the context of matroids, as we will explore in the next chapter.

Chapter 4

Commutativity and associativity

In Desarguesian projective planes over finite fields, the algebra of throws is both a commutative and associative operation. However, in the context of matroids, it is not necessarily true that $\circ$ is either commutative or associative. We construct the forbidden configurations for commutativity and associativity by a detailed analysis.

Recall that in Chapter 2 we showed we are able check in polynomial time whether $\circ$ is a full binary operation. Within this chapter, we assume $\circ$ is a full binary operation. However, it is important to note that the forbidden configurations are exactly the same for both full and partial binary operations — the only difference being that it clearly takes longer to check for the commutativity or associativity of a full binary operation than for a partial binary operation. We simply assume $\circ$ is a full binary operation to be consistent with later chapters. We will clarify when necessary whether $\circ$ is a weak binary operation or a strong binary operation. Checking for both commutativity and associativity can be done in polynomial time, regardless of whether $\circ$ is weak or strong. Having said this, if $\circ$ is a weak binary operation, matters become more complicated than if $\circ$ is a full binary operation, particularly when we are checking for associativity.

4.1 Commutativity

Given a configuration G, for $x,y \in A$ where $x \circ y = z$ and $y \circ x = z'$, we say the pair (x,y) is a *commuting pair* if $z = z'$. On the other hand, if $z \neq z'$, then we say the pair (x,y) is a *non-commuting pair*. If a configuration G has a strong binary operation $\circ$ and all pairs $(x,y) \in A \times A$ are commuting pairs, then we say $\circ$ a *commutative strong binary operation*. Similarly, if a configuration G has a weak binary operation $\circ$ and all defined pairs $(x,y) \in A \times A$ are commuting pairs, then we say $\circ$ is a *commutative weak binary operation*. A configuration G is a *commutative configuration* if it has either a commutative strong binary operation or a commutative weak binary operation. We want to be able to check when a configuration is commutative, so we will construct a list of forbidden configurations for commutativity. Before doing so, we will briefly touch on the relevance of Non-Pappus configurations with respect to projective planes over non-commutative division rings.

4.1.1 Pappus configurations and commutative projective planes

The following fundamental theorem is attributed to Pappus of Alexandria (circa 340 AD).

Theorem 4.1.1 (Pappus' hexagon theorem). *Let a,b,c be three points on a straight line and let x,y,z be three points on another line. If the lines ay, bz, cx intersect the lines bx, cy, az respectively at the points l,n,m, then the three points l,n,m are collinear.*

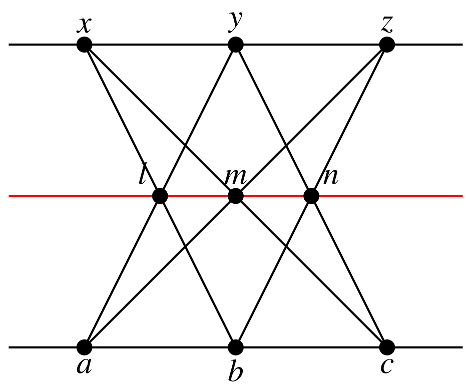


Figure 4.1: A Pappus configuration

We call such a configuration a *Pappus configuration*, and call the line containing the points l, m, n the *Pappus line*. Projective planes satisfying this theorem are called *Pappian planes*. If a projective plane satisfies Pappus' theorem, this means the underlying coordinate system is commutative. For example, a projective plane over any field is clearly a Pappian plane. However, projective planes over any non-commutative division ring (or skew-field) are not Pappian planes. We will see that the Non-Pappus matroid plays a similar role in revealing the commutativity of a configuration under a strong binary operation.

Note that in the appendix of [1], there is only one Pappus matroid and one Non-Pappus matroid. However, there are other matroids satisfying Pappus' hexagon theorem which are not isomorphic to the Pappus matroid. For example, in Figure 4.1, the collinearities *bmy*, *xna* and *clz* may exist. We will still call a configuration satisfying Pappus' theorem with any of these added collinearities a Pappus configuration. Similarly, we can have a Non-Pappus configuration with any of these collinearities.

4.1.2 Forbidden configurations for commutativity when $\circ$ is a strong binary operation

For this subsection, we will assume that $\circ$ is a strong binary operation. We will prove that the following Figure 4.2 is the *only* forbidden configuration for a commutative configuration.

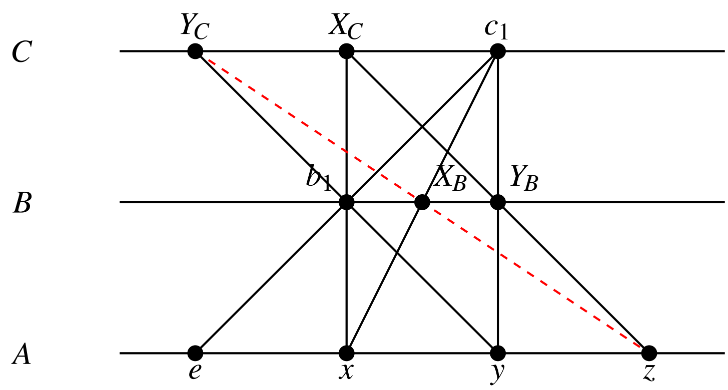


Figure 4.2: C_1 forbidden configuration.

Theorem 4.1.2. *Suppose G is an e-based 3-line configuration with a strong binary operation $\circ$. Then G is a commutative configuration and $\circ$ is a commutative binary operation if and only if G does not contain a sub-configuration isomorphic to C_1, where the isomorphism is the identity for the point e.*

Proof. We will consider all possible commuting pairs $(x,y) \in A \times A$. If $x = e$, then (e,y) is a commuting pair; if $y = e$ then (x,e) is a commuting pair and if $x = y = e$ then (e,e) is a commuting pair. If $x = y$, then (x,x) is clearly a commuting pair. So suppose $x \neq y$. There are two cases to consider, the first being the case when $x \circ y = e$ and the second being when $x \circ y \neq e$. For the first case, $x \circ y = e$ if and only if $y \circ x = e$, so (x,y) is a commuting pair. Therefore we are left with the second case, for which e,x,y,z are all distinct points. As $x \circ y = z$, there must be a sub-configuration of G which is isomorphic to the basic relation configuration R_1.

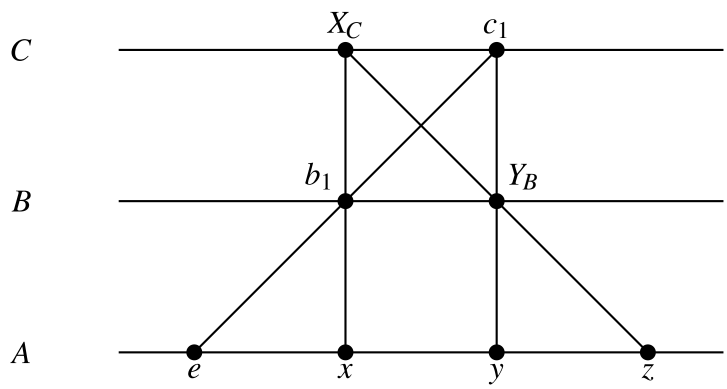


Figure 4.3: R_1 basic relation configuration. Note that the labelling is slightly different to that in Chapter 2.

As $\circ$ is a strong binary operation, we know $y \circ x$ must also be defined on the e-triangle eb_1c_1. By inspection, we must have the necessary triangles yb_1Y_C and xX_Bc_1, where $Y_C \notin \{X_C, c_1\}$ and $X_B \notin \{b_1, Y_B\}$.

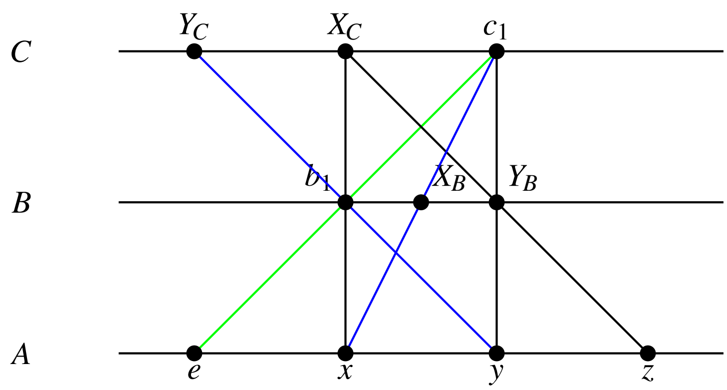


Figure 4.4:

If (x,y) is a commuting pair, then the relation triangle zX_BY_C must exist, and we have the following configuration:

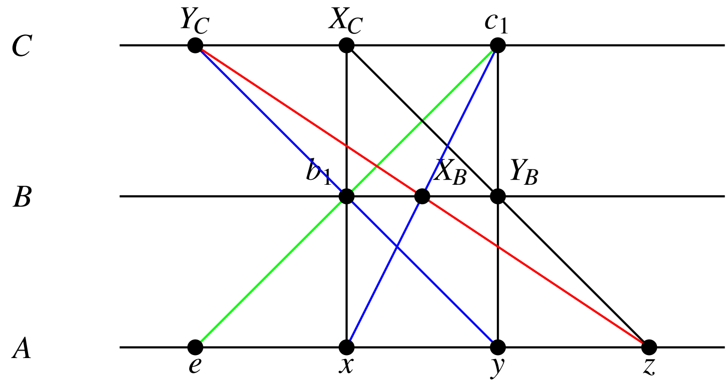


Figure 4.5: The commutative configuration of the pair (x,y), denoted $Com_{x,y}$.

So by inspection, we see that (x,y) is a non-commuting pair if exactly one of the relation triangles zY_BX_C or zX_BY_C exists. If the relation triangle zY_BX_C exists but zX_BY_C doesn't, then we have a configuration isomorphic to C_1 where the isomorphism is the identity for the points e,x,y,z. On the other hand, if the relation triangle zY_BX_C does *not* exist and the relation triangle zX_BY_C *does*, then we have a configuration isomorphic to C_1 where the isomorphism is the identity for the points e,z (but not for the points x,y). Therefore (x,y) is a non-commuting pair if and only if G has a sub-configuration isomorphic to C_1.

Therefore $\circ$ is commutative if and only if every pair (x,y) is a commuting pair if and only if G does not contain a configuration isomorphic to C_1. $\square$

Observe that to check whether a configuration is commutative, we need only check for one forbidden sub-configuration, C_1, which has 10 points. For a configuration G with n points, we can do this in less than $\binom{n}{9}\binom{10}{3}$ steps, which is polynomial in n. This proves the following corollary.

Corollary 4.1.2.1. *Let $(G,\{A,B,C\},e)$ be an e-based configuration with n points, where $\circ$ is a strong binary operation. There is an algorithm, which is polynomial* *in n, to check whether $\circ$ is a commutative binary operation.*

Let M be the underlying matroid of C_1, the forbidden configuration for commutativity as shown in Figure 4.2. Notice that $M \backslash e$ is isomorphic to a Non-Pappus configuration, where the dashed red triangle zX_BY_C corresponds to the non-existent Pappus line. Consider the commutative configuration $Com_{x,y}$ as in Figure 4.5, which we call the *commutative configuration with respect to* x, y, z. Let M' be the underlying matroid of the configuration $Com_{x,y}$. Then $M' \backslash e$ is isomorphic to a Pappus configuration. This implies the following corollary.

Corollary 4.1.2.2. *Let G be an e-based 3-line configuration with a strong binary operation $\circ$. Then G is a commutative configuration and $\circ$ is commutative if and only if for every distinct triple of points $x, y, z \in A$ where $x \circ y = z$, the commutative configuration with respect to x, y, z is isomorphic to a Pappus configuration, where the Pappus line is the relation triangle of $y \circ x$.*

Therefore when $\circ$ is strong, the Pappus line exists if and only if (x, y) is a commuting pair. That is, the Pappus line is a necessary condition for commutativity when $\circ$ is strong. However, this is not the case when $\circ$ is weak.

4.1.3 Forbidden configurations for commutativity when $\circ$ is a weak binary operation

For this subsection we will assume $\circ$ is a weak binary operation and construct the forbidden configurations for commutativity. Along with Figure 4.2 in subsection 4.1.2, the following four figures consider the case when the points e, x, y, z are distinct.

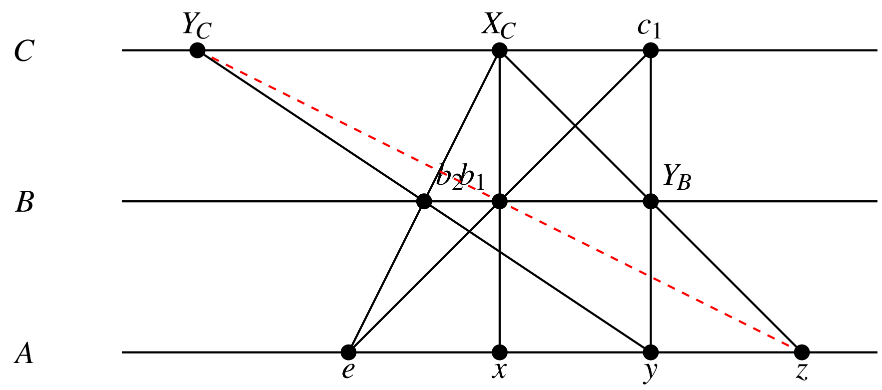


Figure 4.6: C_2 forbidden configuration.

Let M be the underlying matroid in Figure 4.6. Note that $M\backslash x$ is isomorphic to a Non-Pappus matroid. Furthermore, the non-existent relation triangle zb_1Y_C corresponds to the non-existent Pappus line.

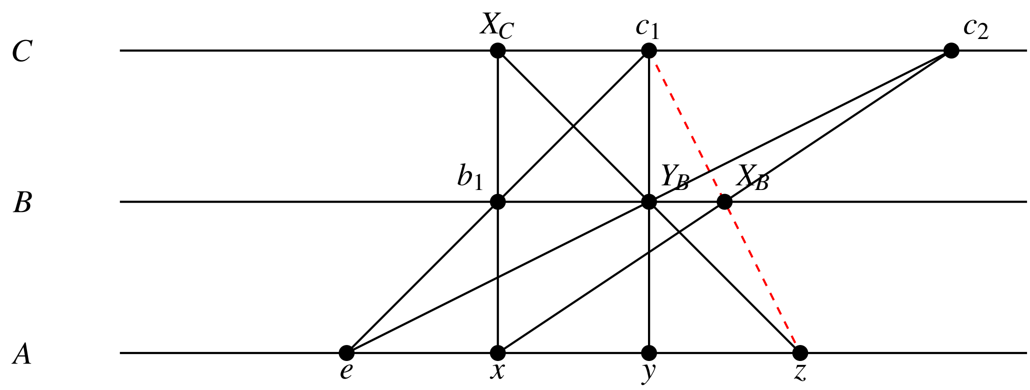


Figure 4.7: C_3 forbidden configuration.

Let M be the underlying matroid in Figure 4.7. Note that $M\backslash y$ is isomorphic to a Non-Pappus matroid. Furthermore, the non-existent relation triangle zX_Bc_1 corresponds to the non-existent Pappus line.

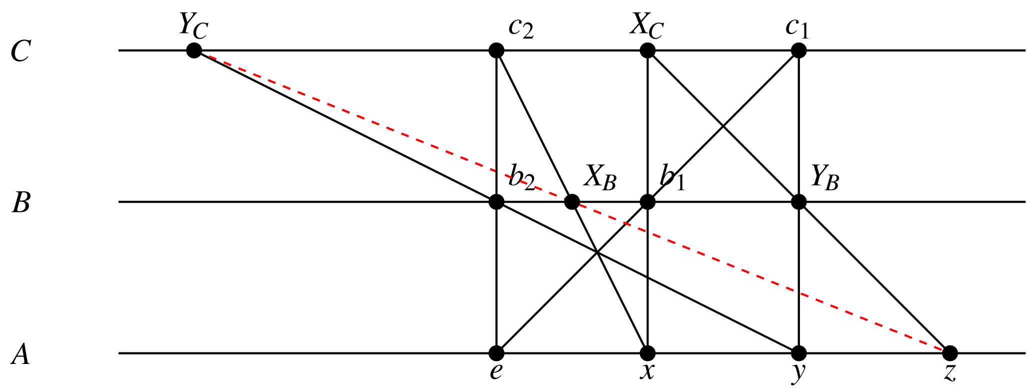


Figure 4.8: C_4 forbidden configuration.

Note that Figure 4.8 does *not* contain a sub-configuration isomorphic to a Non-Pappus configuration.

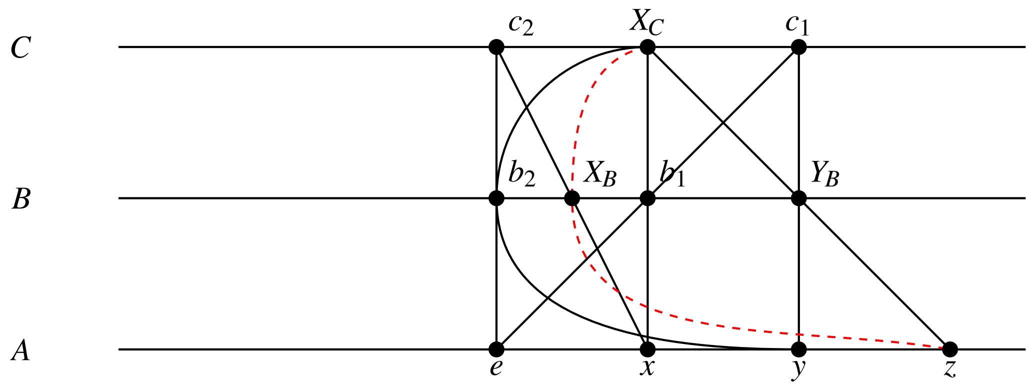


Figure 4.9: C_5 forbidden configuration.

Note that Figure 4.9 does *not* contain a sub-configuration isomorphic to a Non-Pappus configuration. The next lemma proves that Figures 4.2 – 4.9 are the only forbidden configurations for commutativity when the points e,x,y,z are distinct.

Lemma 4.1.3. *For a configuration G, suppose $e,x,y,z,z' \in A$ are distinct points such that $x \circ y = z$ and $y \circ x = z'$. Then (x,y) is a commuting pair if and only if G does not contain a sub-configuration isomorphic to any of $C_1 - C_5$, where the* *isomorphism is the identity for the points* e, z.

Proof. We will start with the basic relation configuration of $x \circ y$ and consider the different e-triangles on which $y \circ x$ may be defined. Note that if we began with the basic relation configuration of $y \circ x$, the analysis would be the same, up to the labelling of the points x and y — which is shown in our statement of the lemma, when we state the isomorphism need only be the identity for the points e and z.

So, given the basic relation configuration R_1 which defines $x \circ y$, there are four possible e-triangles on which to define $y \circ x$:

Case (i). Define $y \circ x$ on eb_1c_1. By inspection we must have the triangles yb_1Y_C and xX_Bc_1, where $Y_C \notin \{X_C, c_1\}$ and $X_B \notin \{b_1, Y_B\}$. In this case the pair (x,y) is a non-commuting pair if and only if the relation triangle zX_BY_C does not exist if and only if we have a configuration isomorphic to C_1:

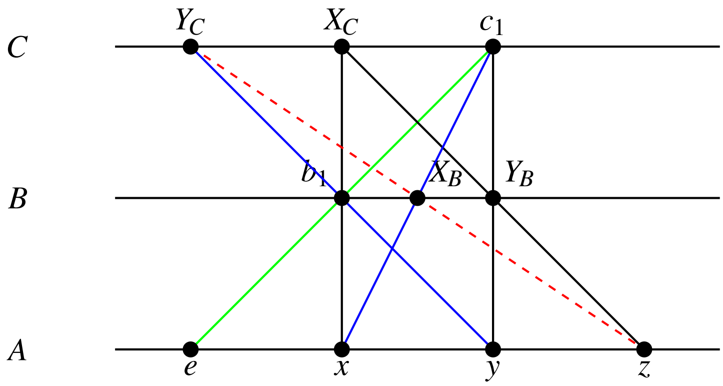


Figure 4.10:

Case (ii). Now suppose we define $y \circ x$ on the e-triangle eb_2X_C. By inspection there is only one option for our choice of necessary triangles, as the necessary triangle xb_1X_C already exists and by inspection the second necessary triangle must be yb_2Y_C where $Y_C \notin \{c_1, X_C\}$. In this case the pair (x,y) is a non-commuting pair if and only if the relation triangle zb_1Y_C does not exist if and only if we have a configuration isomorphic to C_2:

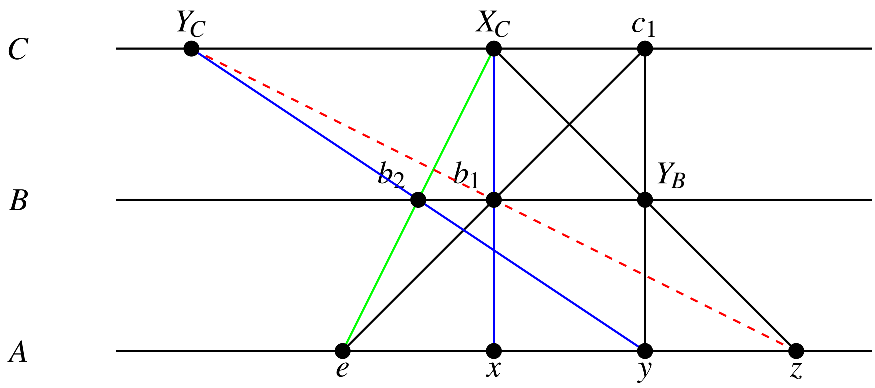


Figure 4.11:

Case (iii). Suppose we define $y \circ x$ on the e-triangle eY_Bc_2, where $c_2 \notin \{c_1, X_C\}$. By inspection the necessary triangle yY_Bc_1 already exists and by inspection the remaining necessary triangle must be xX_Bc_2, where $X_B \notin \{b_1, Y_B\}$. In this case the pair (x,y) is a non-commuting pair if and only if the relation triangle zX_Bc_1 does not exist if and only if we have a configuration isomorphic to C_3:

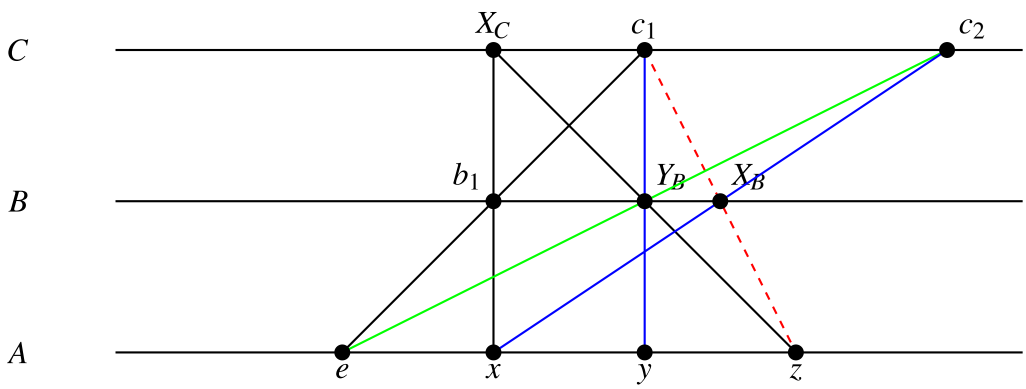


Figure 4.12:

Case (iv). Finally, suppose we define $y \circ x$ on the e-triangle eb_2c_2, where $b_2 \notin$

$\{b_1, X_B, Y_B\}$ and $c_2 \notin \{c_1, X_C, Y_C\}$. Note that if the necessary triangles yb_2X_C and xY_Bc_2 existed, then this would force (x,y) to be a commuting pair. So by inspection, we must have the necessary triangle xX_Bc_2, where $X_B \notin \{b_1, Y_B\}$, and either one of the following two sub-cases:

Sub-case (a). Suppose the necessary triangles xX_Bc_2 and yb_2Y_C exist, where $Y_C \notin \{c_1, c_2, X_C\}$ and $X_B \notin \{b_1, b_2, Y_B\}$. In this case the pair (x,y) is a non-commuting pair if and only if the relation triangle zX_BY_C does not exist if and only if we have a configuration isomorphic to C_4:

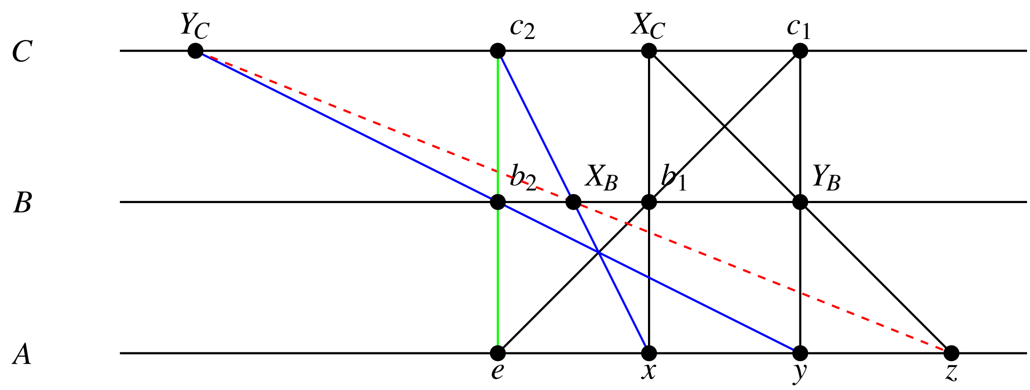


Figure 4.13:

Sub-case (b). Suppose we have the necessary triangles xX_Bc_2 and yb_2X_C where $X_B \notin \{b_1, b_2, Y_B\}$. In this case the pair (x,y) is a non-commuting pair if and only if the relation triangle zX_BX_C does not exist if and only if we have a configuration isomorphic to C_5:

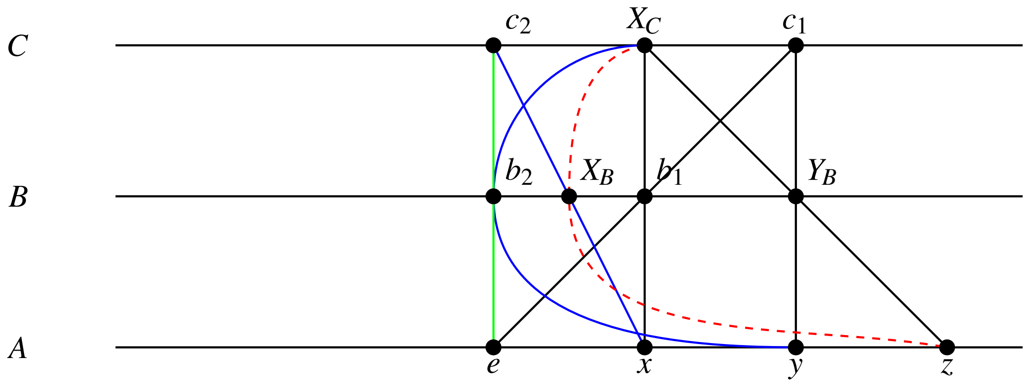


Figure 4.14:

We have considered all non-commuting pairs for the case when e,x,y,z are distinct points, completing the proof. □

It is clear that $C_1 - C_5$ are the *only* forbidden configurations for commutativity, as we now prove.

Theorem 4.1.4. *Suppose G is an e-based 3-line configuration with a weak binary operation* $\circ$*. Then G is a commutative configuration with a commutative binary operation* $\circ$ *if and only if G does not contain a sub-configuration isomorphic to any of* C_1, C_2, C_3, C_4, C_5*, where the isomorphism is the identity for the point e.*

Proof. As in the proof of Theorem 4.1.2, (e,e), (e,y) and (x,e) are all commuting pairs. For any $x \in A$, the pair (x,x) is a commuting pair as $\circ$ is well-defined. For any $x,y \in A$, then $x \circ y = e$ if and only if $y \circ x = e$, in which case (x,y) is a commuting pair. For the remaining cases — the only cases which may not be commutative — the points e,x,y,z are distinct. It follows from Lemma 4.1.3 that the forbidden configurations for these cases are $C_1 - C_5$ and the theorem follows. □

Therefore to check whether a configuration is commutative, we need only check for five forbidden sub-configurations, each of which has no more than 16 points. For a configuration G with n points, we can do this in less than $5\binom{n}{15}\binom{16}{3}$ steps, which is polynomial in n. This proves the following corollary.

Corollary 4.1.4.1. *Let $(G,\{A,B,C\},e)$ be an e-based configuration with n points, where $\circ$ is a weak binary operation. There is an algorithm, which is polynomial in n, to check whether $\circ$ is a commutative binary operation.*

4.2 Associativity

Given a configuration G, for $x,y,z \in A$, we say (x,y,z) is an *associative triple* if $(x \circ y) \circ z = x \circ (y \circ z)$. On the other hand, we say (x,y,z) is a non-associative triple if $(x \circ y) \circ z \neq x \circ (y \circ z)$. For a configuration G with a binary operation $\circ$, we say $\circ$ is an *associative binary operation* if every ordered triple $(x,y,z) \in A \times A \times A$ is an associative triple. We say a configuration G is an *associative configuration* if it has an associative binary operation.

4.2.1 Forbidden configurations for associativity when $\circ$ is a strong binary operation

For this subsection we will assume that $\circ$ is a strong binary operation. Despite there being only one forbidden configuration for commutativity, there are an overwhelming 54 forbidden configurations for associativity. We construct this list by a case analysis of all possible non-associative triples. During this construction, some of the configurations are easily seen to contradict $\circ$ being well-defined. It is possible that upon more detailed inspection, other configurations from this list may also contradict $\circ$ being well-defined. However, as we can check for a well-defined operation first, such configurations will never appear during our checks for associativity.

The next lemma follows immediately from the definition of $\circ$ and the point e.

Lemma 4.2.1. *For $x,y,z \in A$, if any of x,y,z equals e, then (x,y,z) is an associative triple.*

Note that the set $\{e,x,y,z,x\circ y,y\circ z,x\circ(y\circ z),(x\circ y)\circ z\}$ is the set of *necessary points* for an associative configuration. For convenience, we will denote this set by $\mathscr{P}$. Constructing all forbidden configurations for associativity requires a case analysis of the distinct subsets of $\mathscr{P}$. We will first consider the forbidden configurations for the case when all points in $\mathscr{P}$ are distinct. These configurations will be the maximal sized forbidden configurations, from which all other forbidden configurations can be obtained through 'compression'. We can think of compression as a sequence of merging pairs of points, where each pairs of points is contained on the same distinguished line. Merging points may force related triangles to merge. For example, suppose we merge the points $x,y \in A$ in Figure 4.15 so that $x=y$. Consider the triangles xb_1x_c and yb_1y_c. If $x=y$, in order to remain a matroid this forces $y_c = x_c$ and $y_b = x_b$, merging the triangles xb_1x_c and yb_1y_c and leaving us with a *compression* of Figure 4.15. The following are the two maximal sized forbidden configurations for associativity.

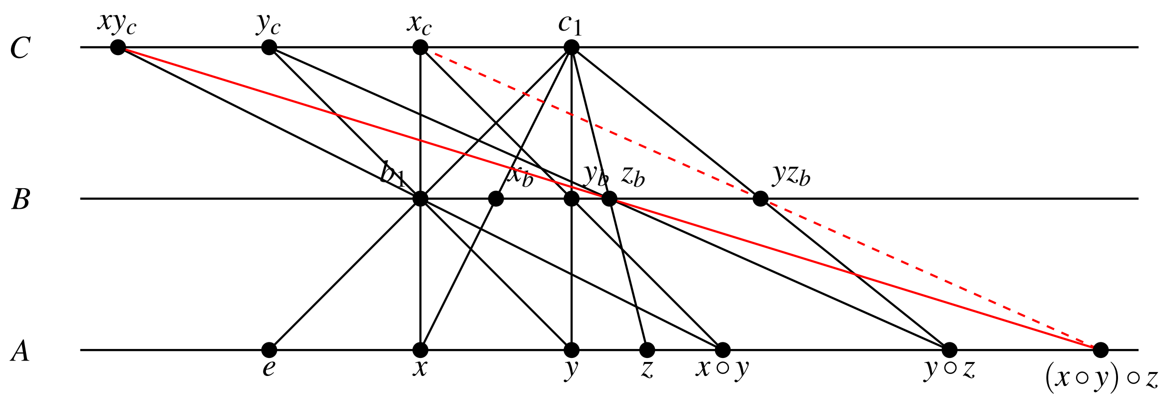


Figure 4.15: A_1 forbidden configuration

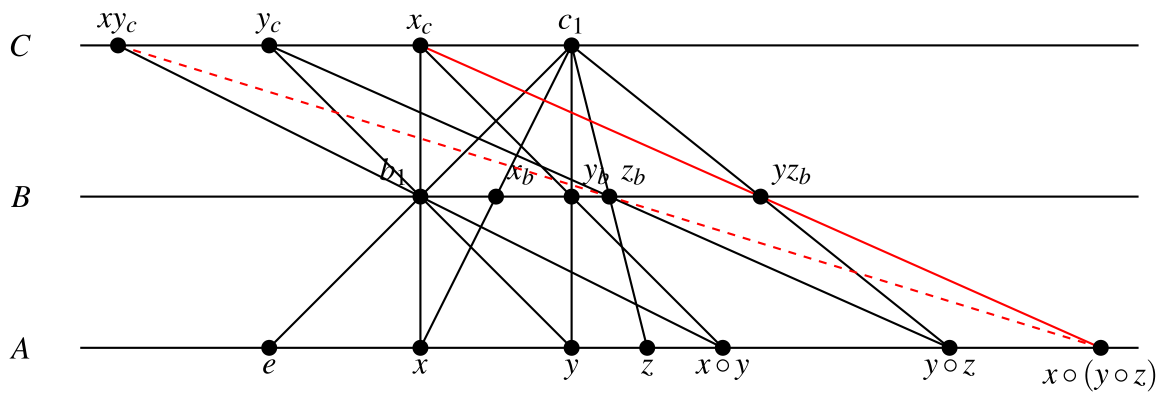


Figure 4.16: A_2 forbidden configuration

Note that Figures 4.15 and 4.16 are isomorphic up to the relation triangles defining $x \circ (y \circ z)$ and $(x \circ y) \circ z$. We will say any two configurations which are isomorphic up to the relation triangles defining $x \circ (y \circ z)$ and $(x \circ y) \circ z$ are a *non-associative pair*. One configuration of the pair will include the relation triangle defining $(x \circ y) \circ z = m$, and will have $x \circ (y \circ z) \neq m$; i.e. Figure 4.15. The other configuration of the pair will include relation triangle defining $x \circ (y \circ z) = m$, and will have $(x \circ y) \circ z \neq m$; i.e. Figure 4.16. Clearly, many of the forbidden configurations for associativity will be part of a non-associative pair.

The next lemma proves that A_1 and A_2 are the only forbidden configurations for the case when all points in $\mathscr{P}$ are distinct.

Lemma 4.2.2. *Let G be a configuration and suppose all points in the set $\mathscr{P}$ of necessary points are distinct. Then (x,y,z) is an associative triple if and only if G does not contain a sub-configuration isomorphic to either A_1 or A_2, where the isomorphism is the identity for the points in $\mathscr{P}$.*

Proof. Given an e-triangle eb_1c_1, we must have necessary triangles through b_1 and the points x,y and $x \circ y$. These must be the triangles xb_1x_c, yb_1y_c and $(x\circ y)b_1(xy_c)$ respectively, where the points x_c, y_c and xy_c are distinct from one another and from c_1. Similarly, we must have necessary triangles through c_1 and the points y,z and $y \circ z$. These must be the triangles yy_bc_1, zz_bc_1 and $(y\circ z)(yz_b)c_1$ respectively, where the points y_b, z_b and yz_b are distinct from one another and from b_1. By inspection, once we define $x\circ y, y\circ z$ and $(x\circ y)\circ z$, then (x,y,z) is an associative triple if and only if we do not have a configuration isomorphic to A_1. Similarly, once we define $x \circ y, y \circ z$ and $x \circ (y \circ z)$, then (x,y,z) is an associative triple if and only if we do not have a configuration isomorphic to A_2. $\square$

For the remaining cases, we will consider the possibilities for associative triples. Due to the large number of forbidden configurations for associativity, we will not list the corresponding forbidden configurations before each lemma.

The next lemma considers the forbidden configurations for the associative triple (x,y,z) where $\{e,x,y,z\}$ is a distinct set of points.

Lemma 4.2.3. *Let G be a configuration where $\mathscr{P}$ is the set of necessary points and $\{e,x,y,z\} \in \mathscr{P}$ is a distinct set of points. Then (x,y,z) is an associative triple if and only if G does not contain a sub-configuration isomorphic to one of $A_3 - A_{32}$,* *where the isomorphism is the identity for the points in $\mathscr{P}$.*

Proof. As we assume all points in $\{e,x,y,z\}$ are distinct, we will consider all possible distinct subsets of points from $\{x\circ y, y\circ x, x\circ(y\circ z),(x\circ y)\circ z\}$, the remaining points of $\mathscr{P}$.

Case (i). Firstly, suppose that $x\circ y \neq y\circ x$ and both $x\circ y, y\circ z \notin \{e,x,y,z\}$. By inspection, we must have a configuration isomorphic to the following:

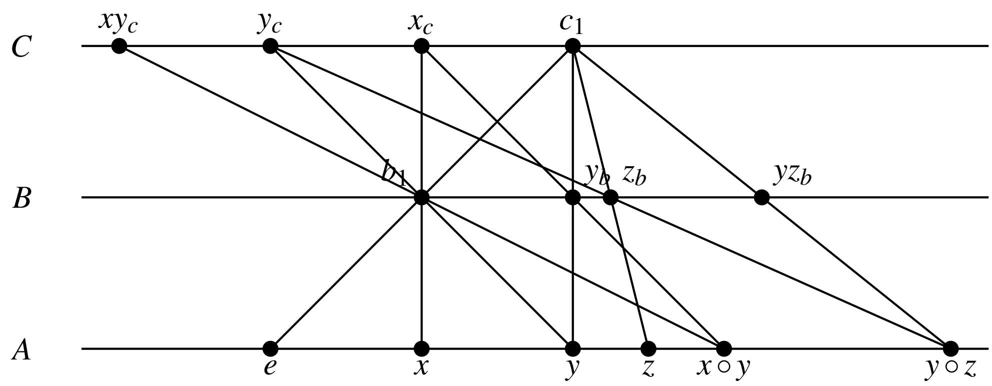


Figure 4.17:

Now consider the possibilities for $x\circ(y\circ z)$ and $(x\circ y)\circ z$. If $x\circ(y\circ z)$ and $(x\circ y)\circ z$ are not distinct from the existing points of A in Figure 4.17, then $x\circ(y\circ z)\in\{e,y,z,x\circ y\}$ and $(x\circ y)\circ z\in\{e,x,y,y\circ z\}$. Then we must have one of the following four sub-cases:

Sub-case (a). Suppose either $x\circ(y\circ z)=e$ or $(x\circ y)\circ z=e$. In this case (x,y,z) is an associative triple if and only if G does not contain a configuration isomorphic to either one of A_3 or A_4:

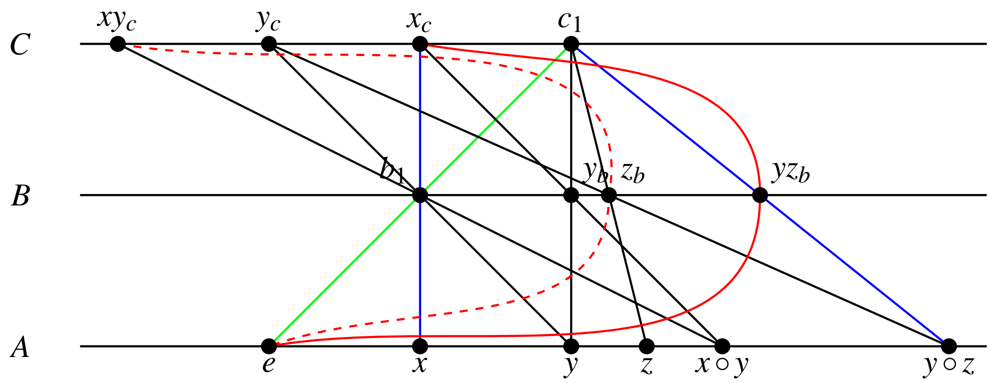


Figure 4.18: A_3 forbidden configuration

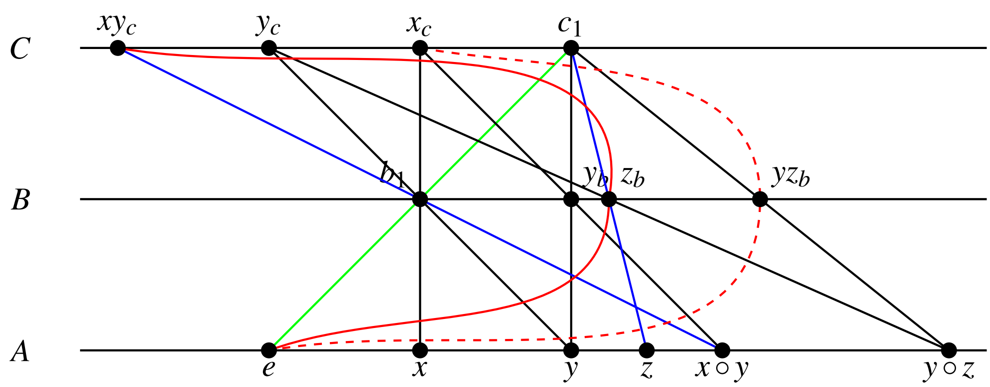


Figure 4.19: A_4 forbidden configuration

Sub-case (b). Suppose we have $(x \circ y) \circ z = x$. In this case (x, y, z) is an associative triple if and only if G does not contain a configuration isomorphic to A_5:

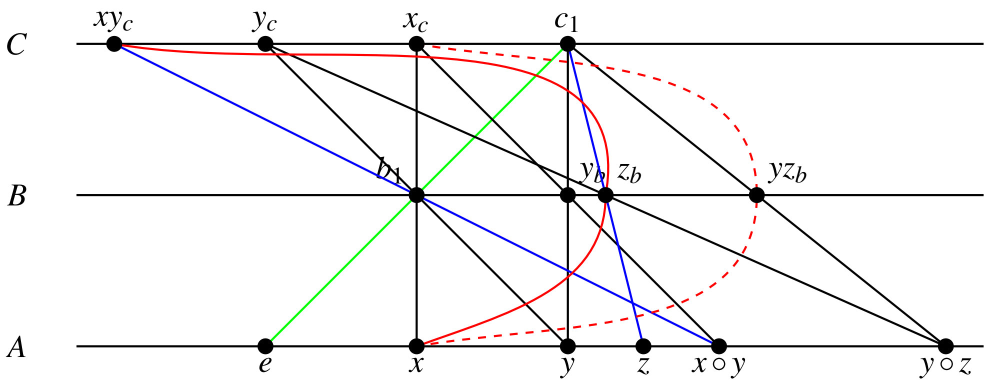


Figure 4.20: A_5 forbidden configuration

Note that we cannot have $(x \circ y) \circ z = x$, because if the triangle $x(yz_b)x_c$ in Figure 4.20 above existed, we would have two lines meeting at more than one point — giving a non-matroid configuration.

Sub-case (c). Suppose either $x \circ (y \circ z) = y$ or $(x \circ y) \circ z = y$. In this case (x,y,z) is an associative triple if and only if G does not contain a configuration isomorphic to either one of A_6 or A_7:

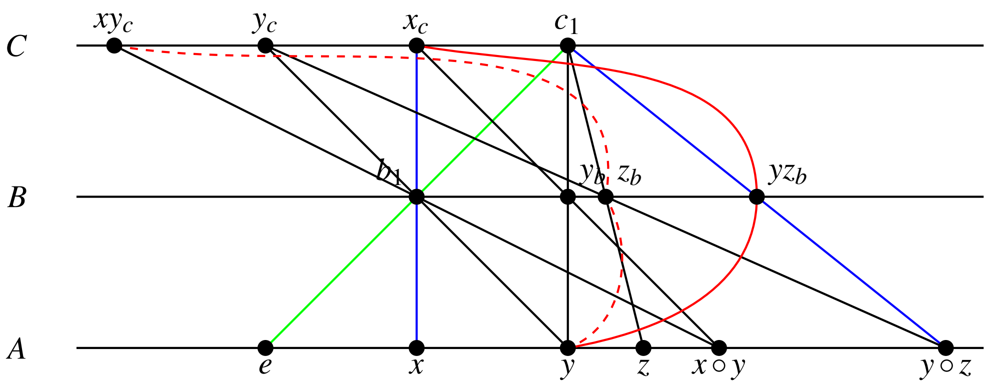


Figure 4.21: A_6 forbidden configuration

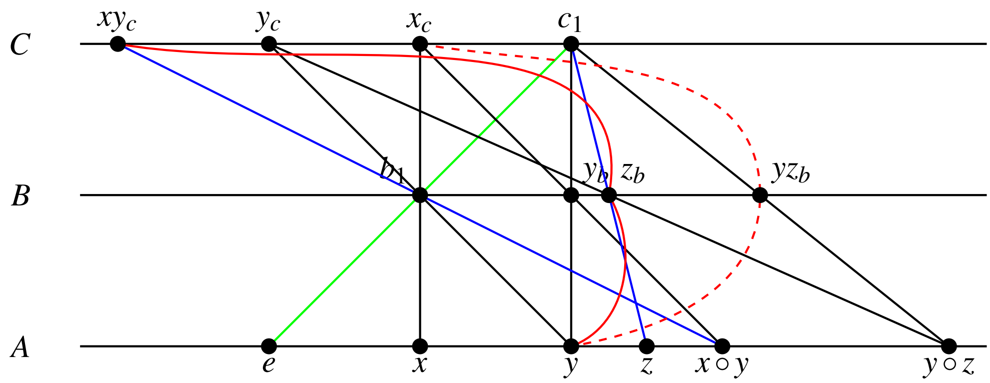


Figure 4.22: A_7 forbidden configuration

Sub-case (d). Suppose we have $x \circ (y \circ z) = z$. In this case (x, y, z) is an associative triple if and only if G does not contain a configuration isomorphic to A_8:

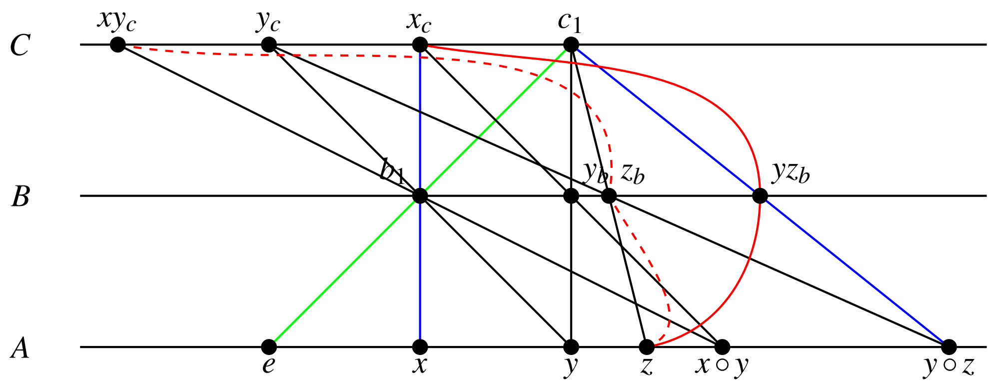


Figure 4.23: A_8 forbidden configuration

Note that we cannot have $x \circ (y \circ z) = z$, because if the triangle $zz_b(xy_c)$ in Figure 4.23 above existed, we would have two lines meeting at more than one point — giving a non-matroid configuration.

These are all possible cases when $x \circ y \neq y \circ x$ and both $x \circ y, y \circ z \notin \{e, x, y, z\}$.

Case (ii). Now we will consider the cases when $x \circ y \notin \{e, z\}$ and $y \circ z \in \{e, x, x \circ y\}$. By inspection we must have the following sub-configuration:

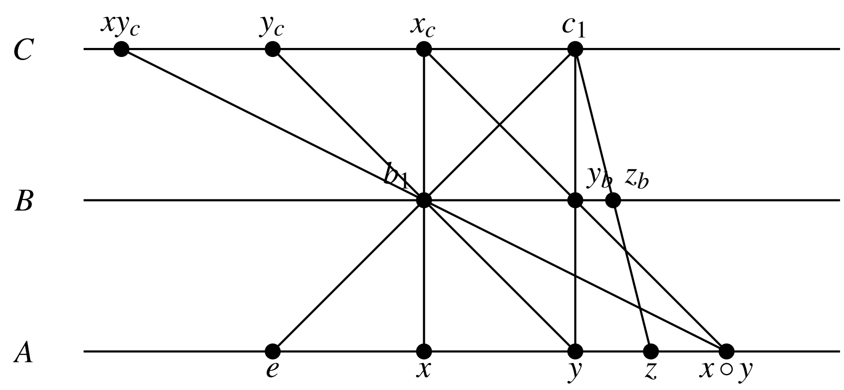


Figure 4.24:

Now we will consider the possibilities for the remaining points of $\mathscr{P}$; the points $y \circ z$, $x \circ (y \circ z)$ and $(x \circ y) \circ z$. We must have one of the three following sub-cases:

Sub-case (a). Suppose $y \circ z = e$. Therefore we must have $x \circ (y \circ z) = x$ and the following configuration:

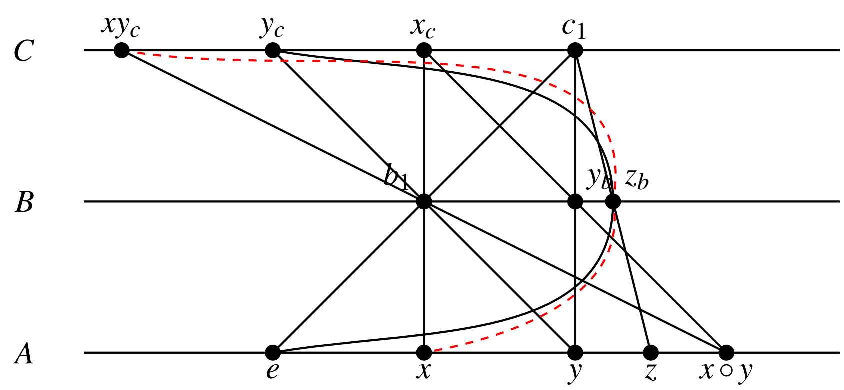
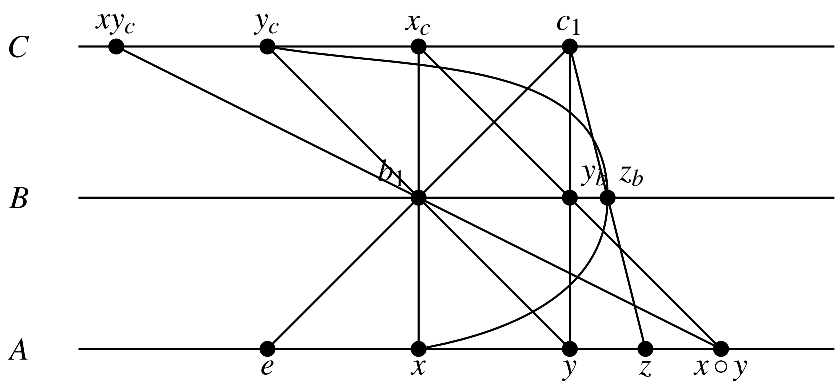


Figure 4.25:

However, this is not a forbidden configuration for associativity. Note that as $x \circ y$ must be defined on the e-triangle ez_by_c, the triangle $xz_b(xy_c)$ must actually exist (contrary to the depiction by a dashed red line as in Figure 4.2.4) in order for $x \circ y$ to be well-defined.

Sub-case (b). Suppose $y \circ z = x$. By inspection we must have the following configuration and one of the four following sub-subcases:



* Suppose either $x \circ (y \circ z)$ or $(x \circ y) \circ z$ is distinct from all existing points. In this case (x, y, z) is an associative triple if and only if we have do not have a configuration isomorphic to either one of A_9 or A_{10}:

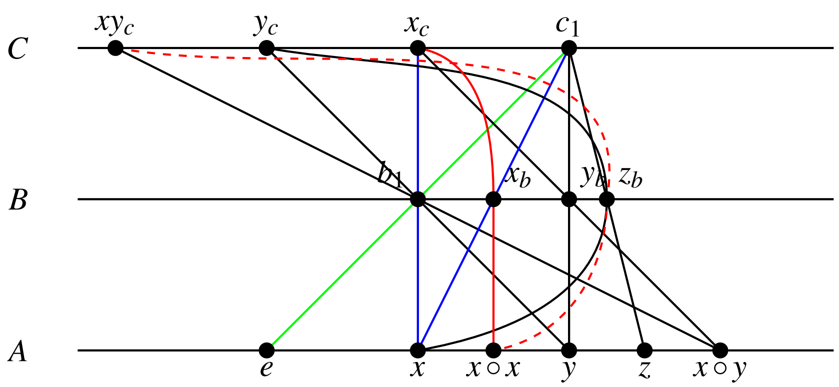


Figure 4.26: A_9 forbidden configuration

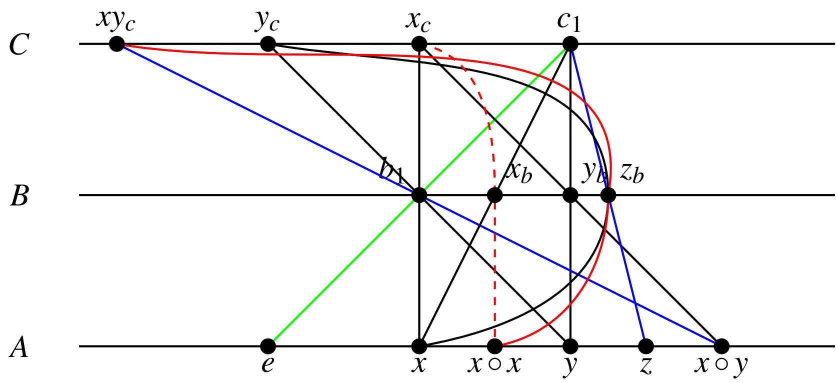


Figure 4.27: A_{10} forbidden configuration

* Suppose either $(x \circ y) \circ z = y$ or $x \circ (y \circ z) = y$. In this case (x, y, z) is an associative triple if and only if we do not have a configuration isomorphic to either one of A_{11} or A_{12}:

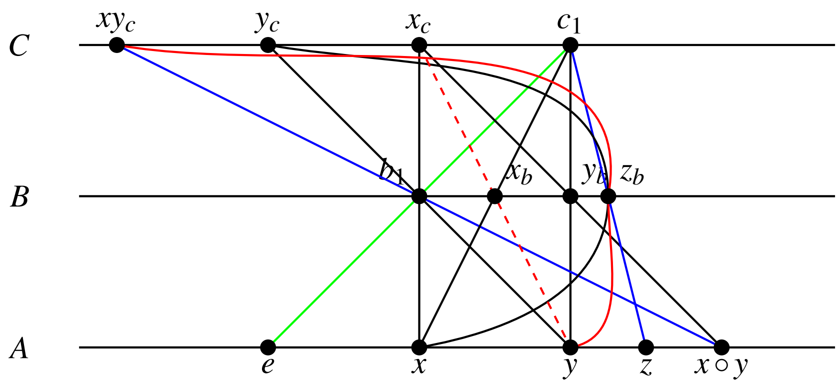


Figure 4.28: A_{11} forbidden configuration

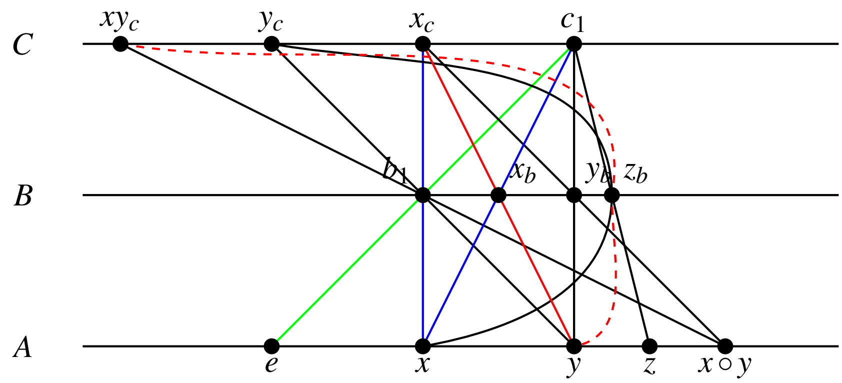


Figure 4.29: A_{12} forbidden configuration

Note that we cannot have $(x \circ y) \circ z = x$, as $y \circ z = x$, as either case forces a non-matroid configuration. Similarly, we cannot have $x \circ (y \circ z) = x$.

* Suppose $x \circ (y \circ z) = z$. In this case (x, y, z) is an associative triple if and only if we do not have a configuration isomorphic to A_{13}:

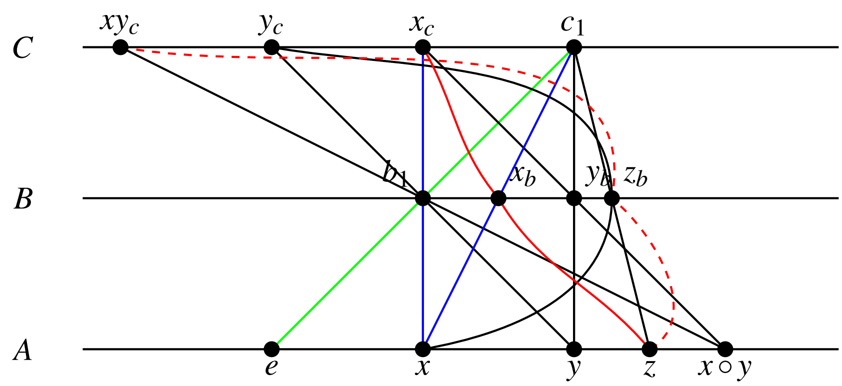


Figure 4.30: A_{13} forbidden configuration

Note that we cannot have $(x \circ y) \circ) = z$, because if triangle $zz_b(xy_c)$ in Figure 4.30 above existed, we would have two lines meeting at more than one point — giving a non-matroid configuration.

* Suppose either $(x \circ y) \circ z = e$ or $x \circ (y \circ z) = e$. In this case (x, y, z) is an associative triple if and only if we do not have a configuration

isomorphic to either A_{14} or A_{15}:

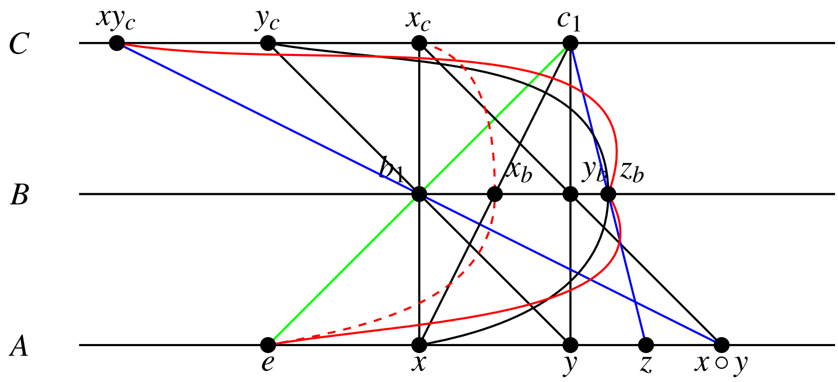


Figure 4.31: A_{14} forbidden configuration

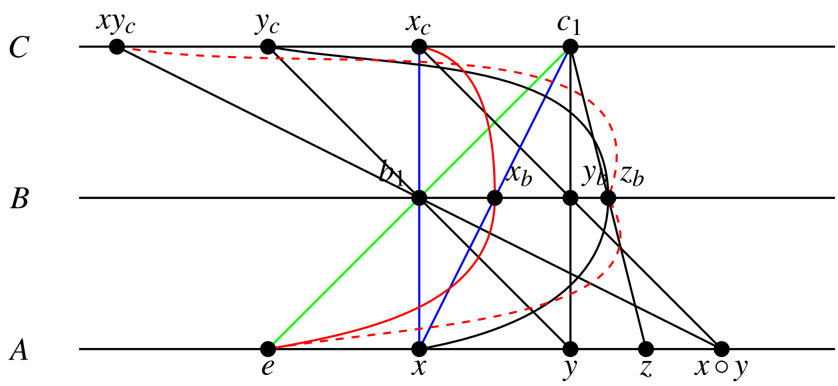


Figure 4.32: A_{15} forbidden configuration

Note that we cannot have $x \circ (y \circ z) = x \circ y$ or $(x \circ y) \circ z = x \circ y$, so these are all possibilities for the case when $y \circ z = x$.

Sub-case (c). Now suppose $y \circ z = x \circ y$. Then we must have the following configuration:

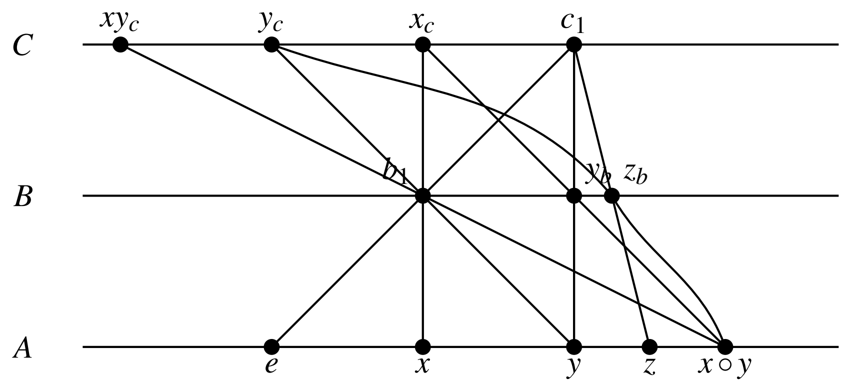


Figure 4.33: A_{16} forbidden configuration

As the point z_b must be contained in any relation triangle defining $x \circ (y \circ z)$ and $(x \circ y) \circ z$, this is a forbidden configuration for associativity and we need not consider the possibilities for $x \circ (y \circ z)$ and $(x \circ y) \circ z$.

These are all cases for when $x \circ y \notin \{e, x, y, z\}$ and $y \circ z \in \{e, x, x \circ y\}$.

Case (iii). Now we will consider the cases when $y \circ z \notin \{e, x\}$ and $x \circ y \in \{e, z, y \circ z\}$. The we must have one of the five following sub-cases.

Sub-case (a). Suppose $x \circ y = e$. Then we must have $(x \circ y) \circ z = z$ and the following configuration.

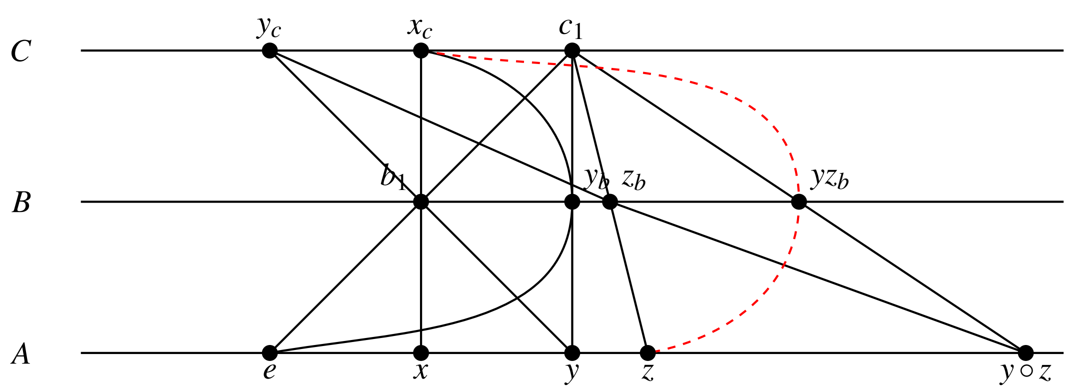
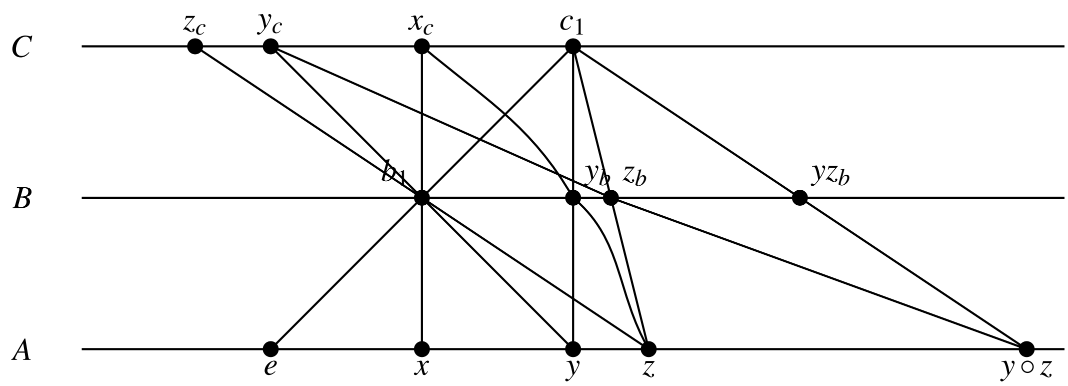


Figure 4.34:

Note that as $y \circ z$ must be defined on ey_bx_c, the triangle $z(yz_b)x_c$ must actually exist (contrary to being depicted by a red dashed line in Figure 4.33) in order for $y \circ z$ to be well defined. So this case is always associative and Figure 4.34 is not a forbidden configuration for associativity.

Sub-case (b). Now suppose $x \circ y = z$. By inspection we must have the following configuration:



Now we will consider the different possibilities for $(x \circ y) \circ z$ and $x \circ (y \circ z)$. We must have one of the five following sub-subcases:

* Suppose either $(x \circ y) \circ z = e$ or $x \circ (y \circ z) = e$. In this case (x, y, z) is an associative triple if and only if we do not have a configuration isomorphic to either A_{18} or A_{19}:

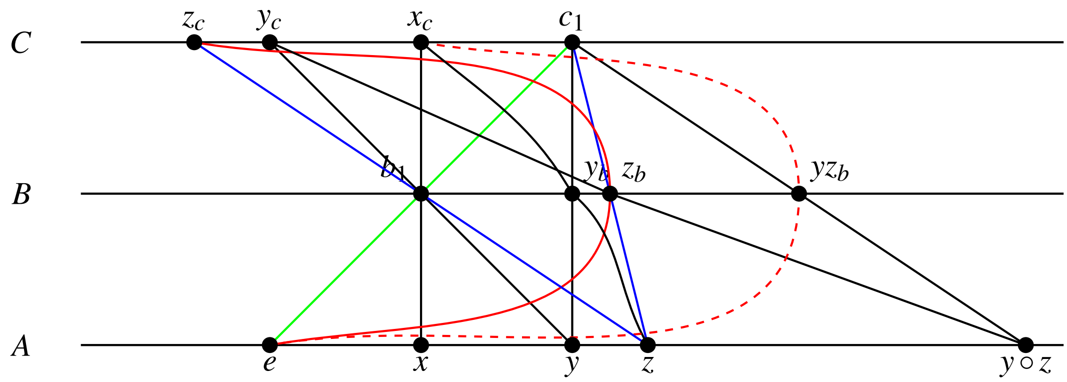


Figure 4.35: A_{18} forbidden configuration

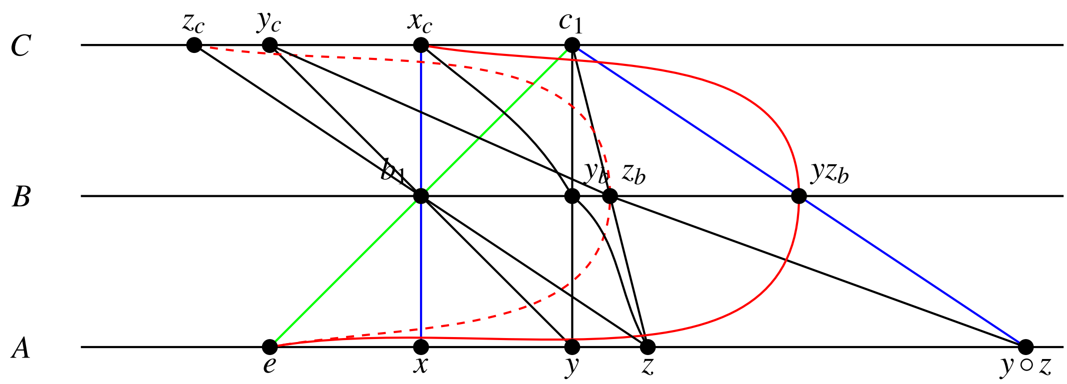


Figure 4.36: A_{19} forbidden configuration

* Suppose $(x \circ y) \circ z = x$. In this case (x, y, z) is an associative triple if and only if we do not have a configuration isomorphic to A_{20}:

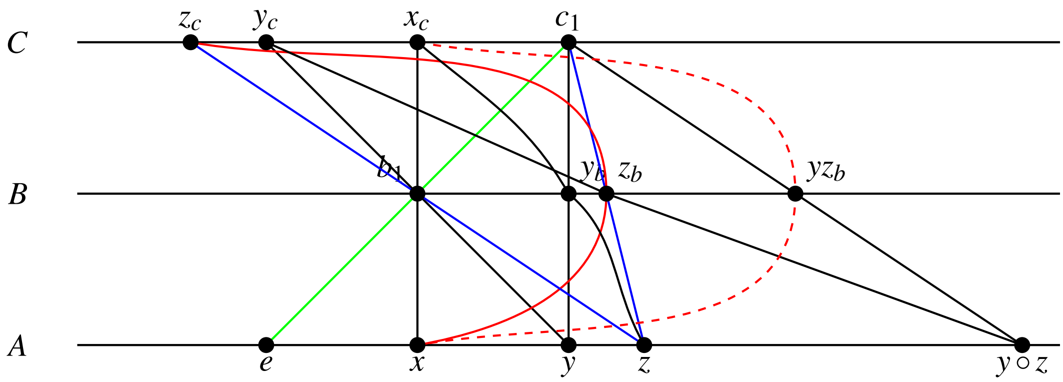


Figure 4.37: A_{20} forbidden configuration

Note that we cannot have $x \circ (y \circ z) = x$, because if the triangle $x(yz_b)x_c$ in Figure 4.37 above existed, we have two lines meeting at more than one point — giving a non-matroid configuration.

* Suppose either $(x \circ y) \circ z = y$ or $x \circ (y \circ z) = y$. In this case (x, y, z) is an associative triple if and only if we do not have a configuration isomorphic to either A_{21} or A_{22}:

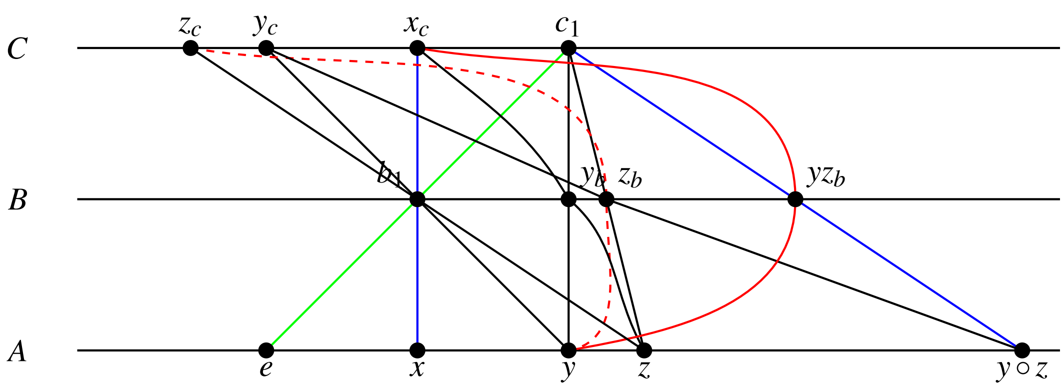


Figure 4.38: A_{21} forbidden configuration

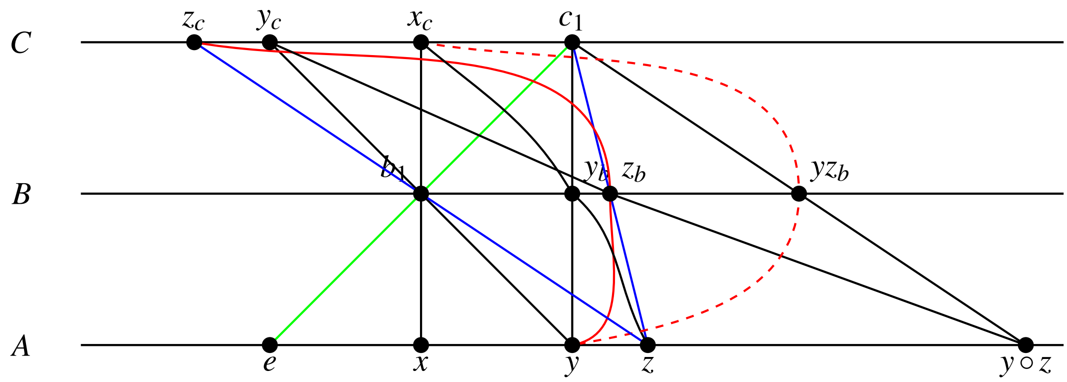


Figure 4.39: A_{22} forbidden configuration

* Suppose $x \circ (y \circ z) = z$. In this case (x, y, z) is an associative triple if and only if we do not have a configuration isomorphic to A_{23}:

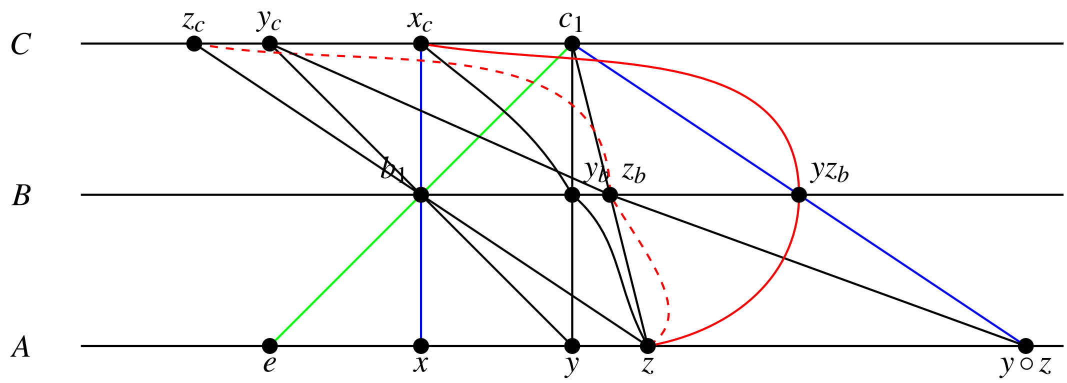


Figure 4.40: A_{23} forbidden configuration

Note that we cannot have $(x \circ y) \circ z = z$, because if the triangle zy_bz_c in Figure 4.40 above existed, we would have two lines meeting at more than one point — giving a non-matroid configuration. Also note that we cannot have $(x \circ y) \circ z = y \circ z$, as $x \circ y = z$.

* Suppose either $(x \circ y) \circ z$ or $x \circ (y \circ z)$ is distinct from all existing points. In this case (x, y, z) is an associative triple if and only if we do not have a configuration isomorphic to either A_{24} or A_{25}:

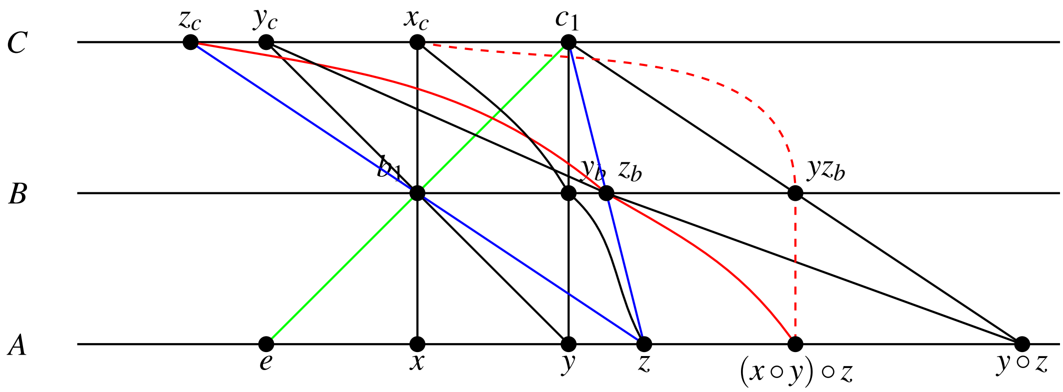


Figure 4.41: A_{24} forbidden configuration

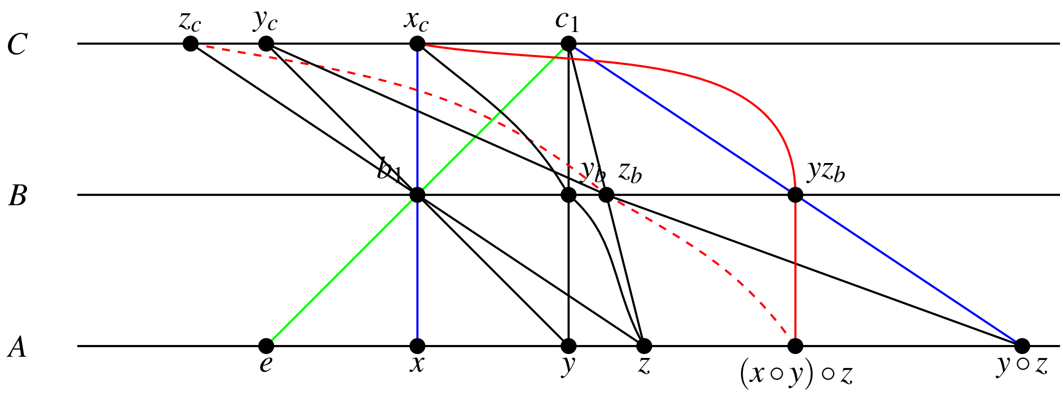


Figure 4.42: A_{25} forbidden configuration

These are all possibilities for the case when $y \circ z \notin \{e, x\}$ and $x \circ y \in \{e, z, y \circ z\}$.

Case (iv). Now suppose $x \circ y \in \{e, z\}$ and $y \circ z \in \{e, x\}$. Then we must have one of the following four sub-cases.

Sub-case (a). Suppose $y \circ z = x$ and $x \circ y = e$. By inspection we must have the following configuration:

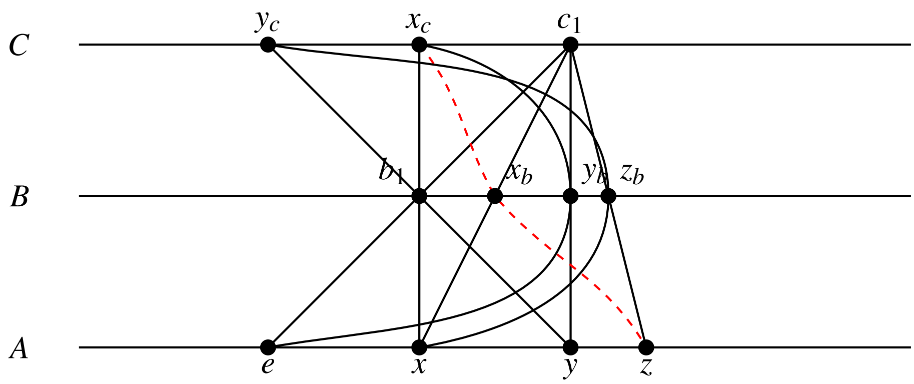


Figure 4.43:

Note that as $y \circ z$ must be defined on ey_bx_c, the triangle zx_bx_c must actually exist (contrary to being depicted by a red dashed line above in Figure 4.42) in order for $y \circ z$ to be well defined. So this case is always associative and Figure 4.43 is not a forbidden configuration for associativity.

Sub-case (b). Suppose $y \circ z = x \circ y = e$. This forces $x \circ (y \circ z) = x$ and $(x \circ y) \circ z = z$. In this case (x, y, z) is an associative triple if and only if we do not have a configuration isomorphic to A_{26}:

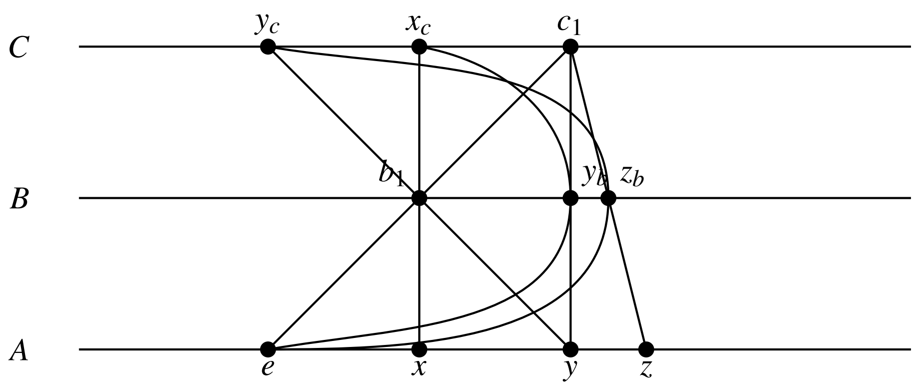
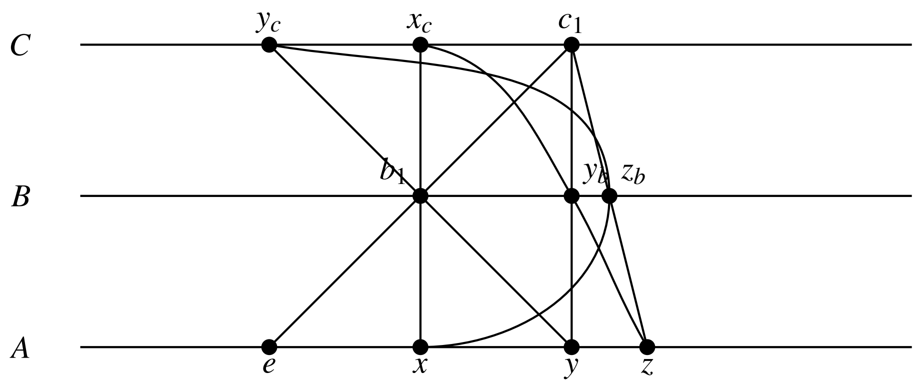


Figure 4.44: A_{26} forbidden configuration

Sub-case (c). Now suppose $y \circ z = x$ and $x \circ y = z$. By inspection we must have the following configuration and one of the three following sub-subcases:



* Suppose either $(x \circ y) \circ z = e$ or $x \circ (y \circ z) = e$. In this case (x, y, z) is an associative triple if and only if G does not contain a configuration isomorphic to either A_{27} or A_{28}:

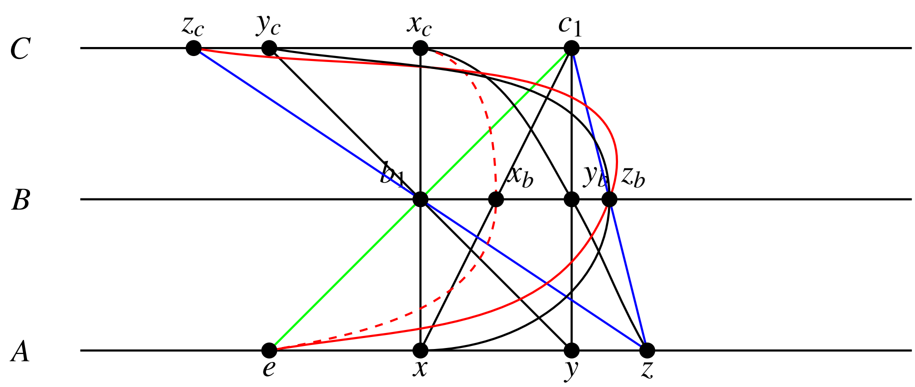


Figure 4.45: A_{27} forbidden configuration

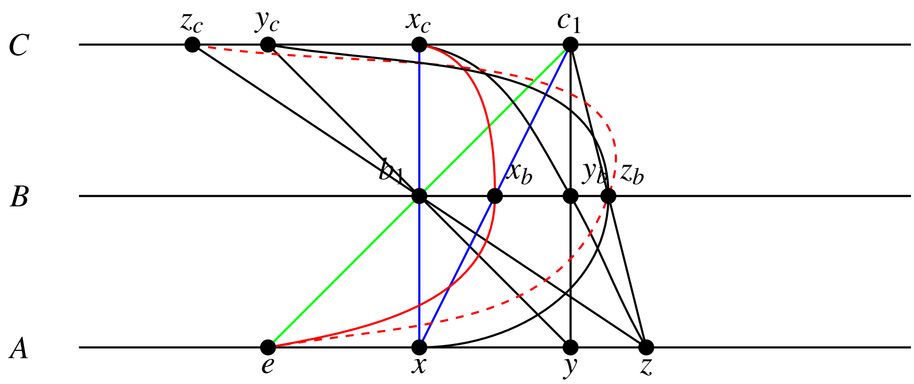


Figure 4.46: A_{28} forbidden configuration

Note that we cannot have $(x \circ y) \circ z = x$ or $x \circ (y \circ z) = x$, as either case forces a non-matroid configuration.

* Suppose either $(x \circ y) \circ z = y$ or $x \circ (y \circ z) = y$. In this case (x, y, z) is an associative triple if and only if G does not contain a configuration isomorphic to either A_{29} or A_{30}:

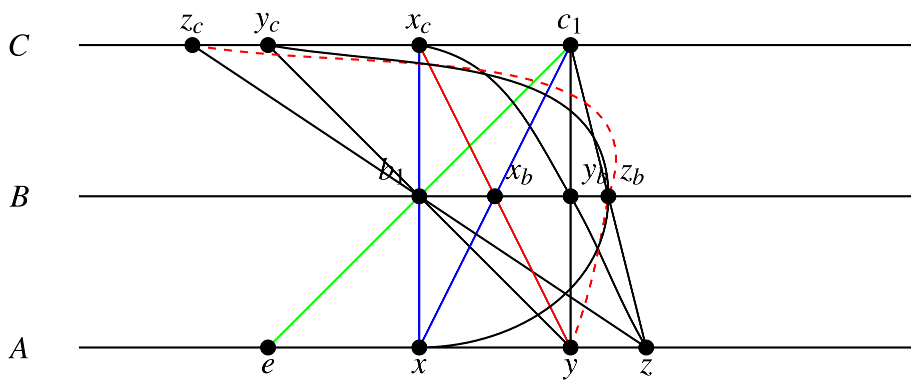


Figure 4.47: A_{29} forbidden configuration

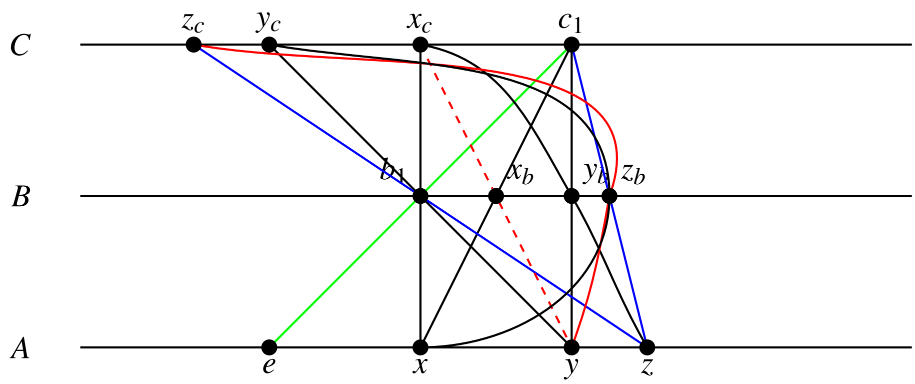


Figure 4.48: A_{30} forbidden configuration

Note that we cannot have $(x \circ y) \circ z = z$ or $x \circ (y \circ z) = z$, as either case forces a non-matroid configuration.

$*$ Suppose either $(x \circ y) \circ z \notin \{e, x, y, z\}$ or $x \circ (y \circ z) \notin \{e, x, y, z\}$. In this case (x, y, z) is an associative triple if and only if G does not contain a configuration isomorphic to either A_{31} or A_{32}:

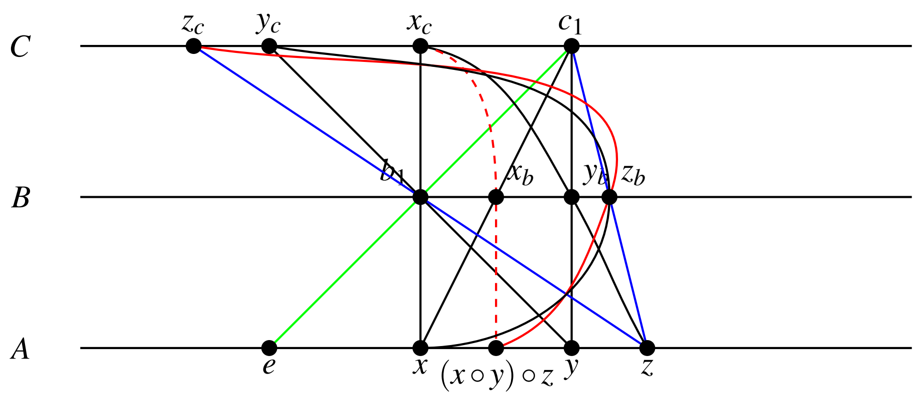


Figure 4.49: A_{31} forbidden configuration

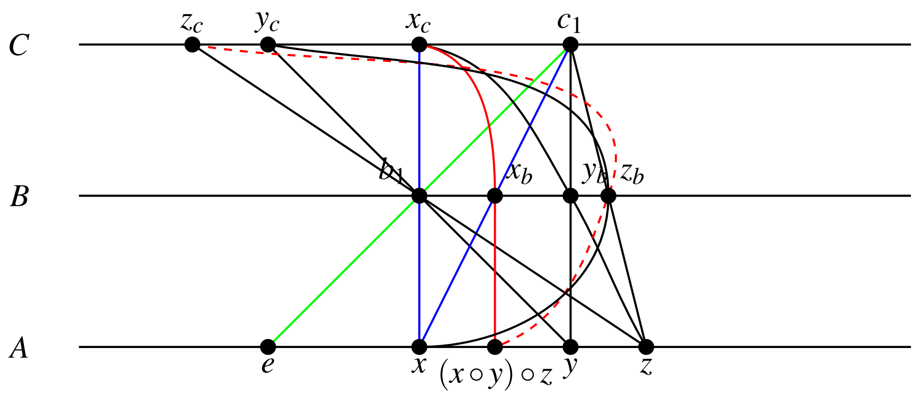
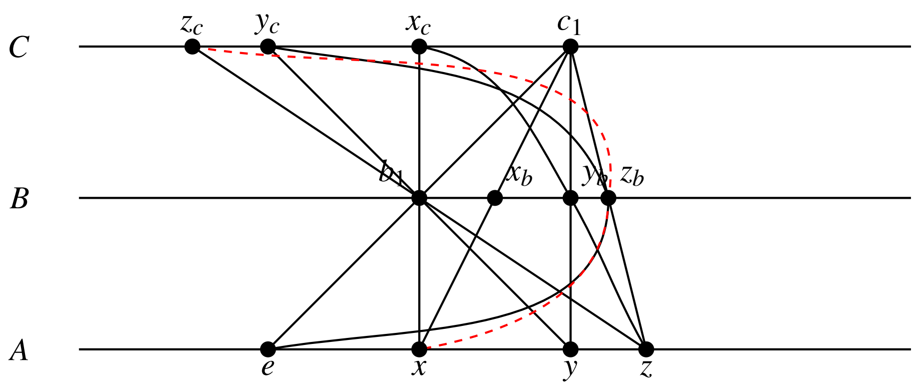


Figure 4.50: A_{32} forbidden configuration

Sub-case (d). Suppose $y \circ z = e$ and $x \circ y = z$. This forces $x \circ (y \circ z) = x$ and the following configuration:



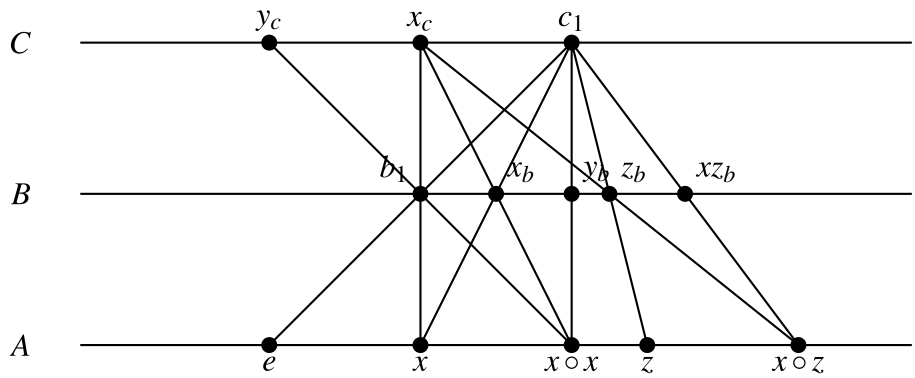
Note that $x \circ y$ must be defined on ez_by_c, so the triangle xz_bz_c must actually exist (contrary to being depicted by a red dashed line in the above Figure) in order for $x \circ y$ to be well-defined. So this case is always associative and this is not a forbidden configuration for associativity.

We have constructed all forbidden configurations for associative triples for the case when $\{e,x,y,z\}$ are distinct points, completing the proof. $\square$

The next lemma considers the forbidden configurations for the associative triple (x,x,z) where $\{e,x,z\}$ is a distinct set of points.

Lemma 4.2.4. *Let G be a configuration where $\mathscr{P}$ is the set of necessary points of A, where $\{e,x,z\} \in \mathscr{P}$ is a distinct set of points and $y = x$. Then (x,x,z) is an associative triple if and only if G does not contain a sub-configuration isomorphic to one of $A_{33} - A_{37}$, where the isomorphism is the identity for the points in $\mathscr{P}$.*

Proof. Case (i). Suppose $x \circ x, x \circ z \notin \{e,z\}$. Then by inspection we must have the following configuration:



Now consider the possibilities for $(x \circ x) \circ z$ and $x \circ (x \circ z)$.

Sub-case (a). Suppose either $(x \circ x) \circ z = e$ or $x \circ (x \circ z) = e$. Then (x,x,z) is an associative triple if and only if G does not have a configuration isomorphic to either A_{33} or A_{34}:

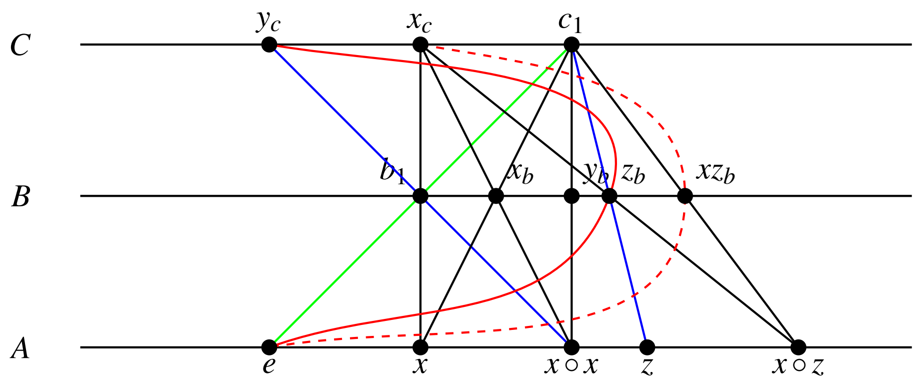


Figure 4.51: A_{33} forbidden configuration

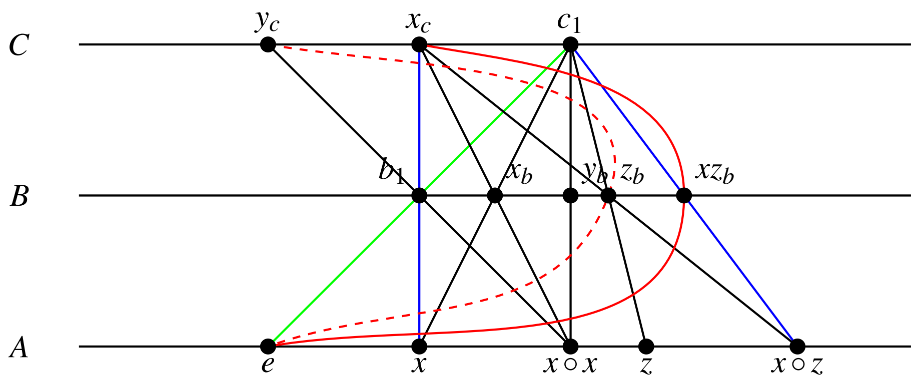


Figure 4.52: A_{34} forbidden configuration

Sub-case (b). Suppose $(x \circ x) \circ z = x$. Then (x, x, z) is an associative triple if and only if G does not have a configuration isomorphic to A_{35}:

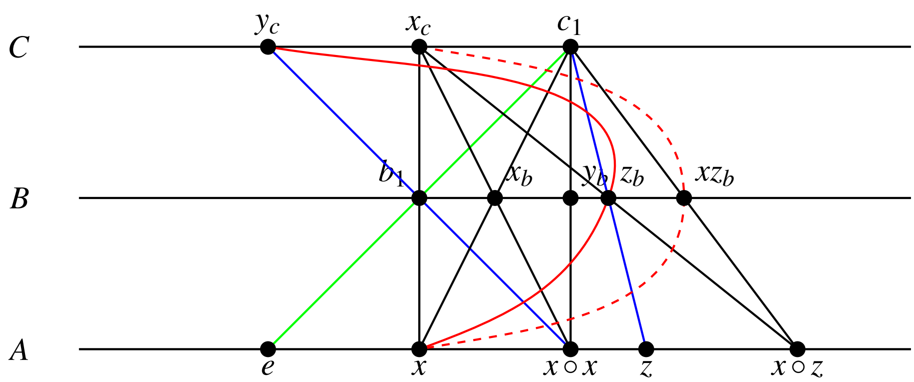


Figure 4.53: A_{35} forbidden configuration

Note that we cannot have $x \circ (x \circ z) = x$, because if the triangle $x(xz_b)x_c$ in Figure 4.53 above existed, we would have two lines meeting at more than one point — giving a non-matroid configuration.

Sub-case (c). Note that we cannot have either $(x \circ x) \circ z = x \circ x$ or $x \circ (x \circ z) = x \circ x$.

Sub-case (d). Suppose $x \circ (x \circ z) = z$. Then (x, x, z) is an associative triple if and only if G does not have a configuration isomorphic to A_{36}:

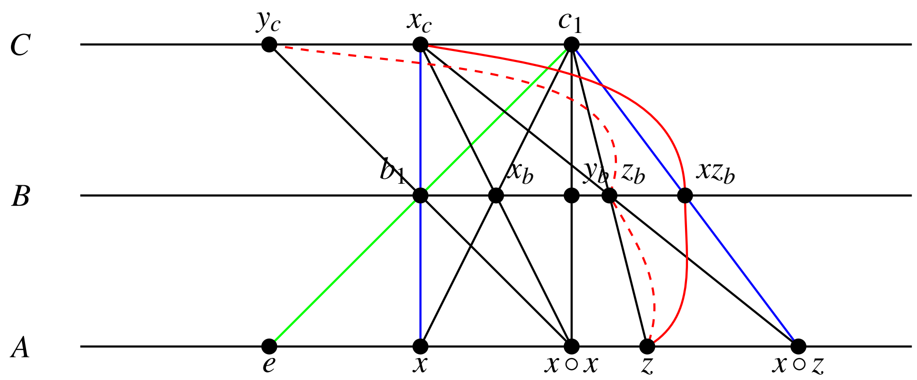


Figure 4.54: A_{36} forbidden configuration

Note that we cannot have $(x \circ x) \circ z = z$, because if the triangle $z(xz_b)y_c$ in Figure 4.54 above existed, we would have two lines meeting at more than one point — giving a non-matroid configuration.

Sub-case (e). Suppose either $(x \circ x) \circ z \notin \{e, x\}$ or $x \circ (x \circ z) \notin \{e, z\}$. Then (x, x, z) is an associative triple if and only if G does not have a configuration isomorphic to either A_{37} or A_{38}:

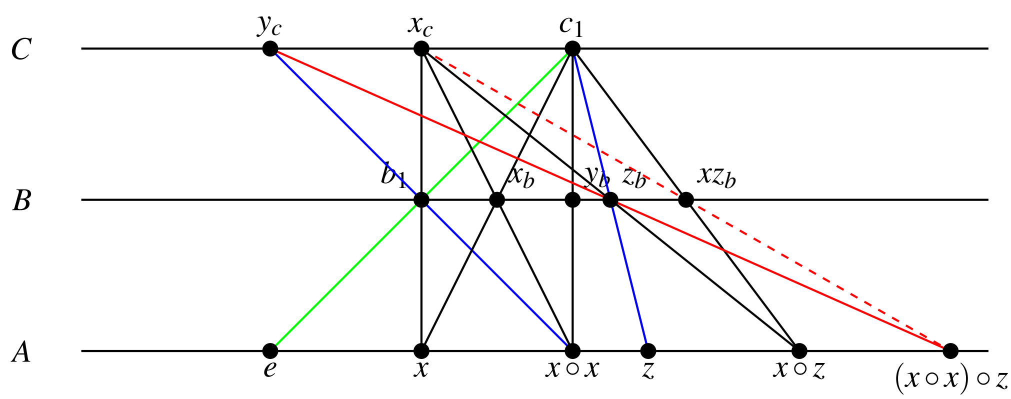


Figure 4.55: A_{36} forbidden configuration

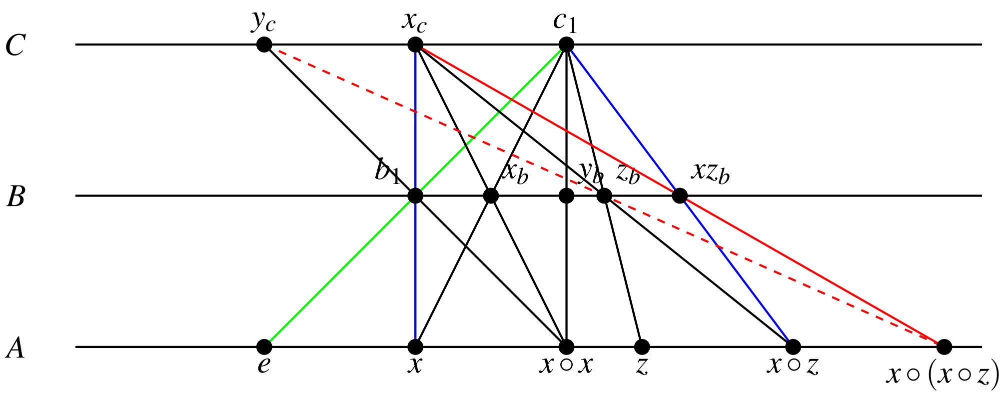


Figure 4.56: A_{38} forbidden configuration

Sub-case (f). Note that we cannot have both $x \circ x = e$ and $x \circ z = e$ as their respective relation triangles would meet at more than one point.

Sub-case (g). Now suppose $x \circ x = z$ and $x \circ z = e$. This forces $x \circ (x \circ z) = x$. Then we have the following configuration:

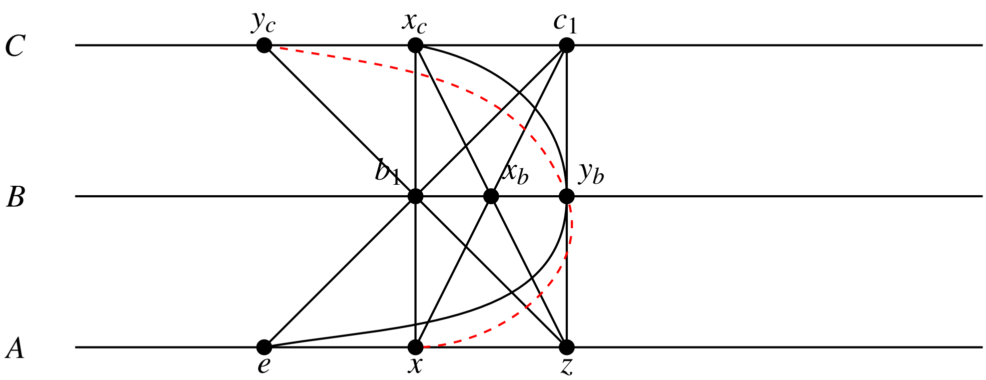


Figure 4.57:

Note that as $x \circ x$ must be defined on the e-triangle ey_bx_c, the dashed red triangle xy_by_c must exist in order for $x \circ x$ to be well-defined. Therefore Figure 4.57 is not a forbidden configuration for associativity.

These are all possible forbidden configurations for the associative triple (x,x,z).

□

The next lemma considers the forbidden configurations for the associative triple (x,y,y) where $\{e,x,y\}$ is a distinct set of points.

Lemma 4.2.5. *Let G be a configuration and $\mathscr{P}$ is the set of necessary points of A where $\{e,x,y\} \in \mathscr{P}$ are a distinct set of points and $z = y$. Then (x,y,y) is an associative triple if and only if G does not contain a sub-configuration isomorphic to one of $A_{39} - A_{44}$, where the isomorphism is the identity for the points in $\mathscr{P}$.*

Proof. First, we will consider the possibilities for the points $x \circ y$ and $y \circ y$.

Case (i). Suppose $x \circ y, y \circ y \notin \{e,x,y\}$. Then by inspection we must have the following configuration:

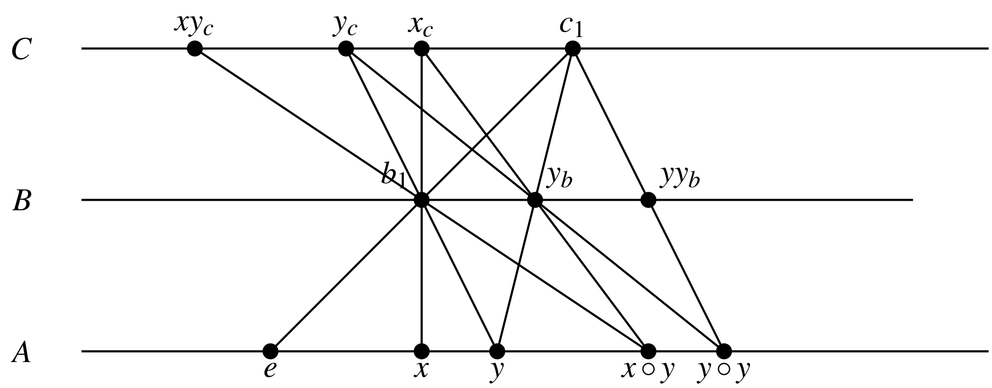


Figure 4.58:

Now we consider the five possibilities for the points $(x \circ y) \circ y$ and $x \circ (y \circ y)$.

Sub-case (a). Suppose either $(x \circ y) \circ y = e$ or $x \circ (y \circ y) = e$. Then (x,y,y) is an associative triple if and only if G does not contain a configuration isomorphic to either one of A_{39} or A_{40}:

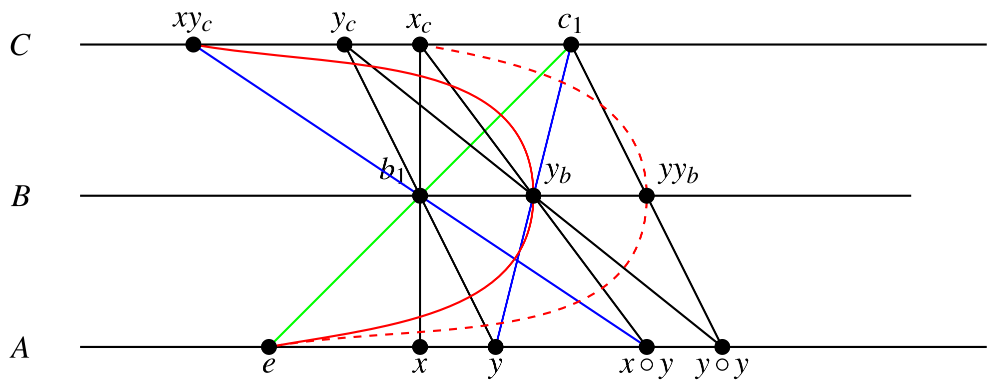


Figure 4.59: A_{39} forbidden configuration

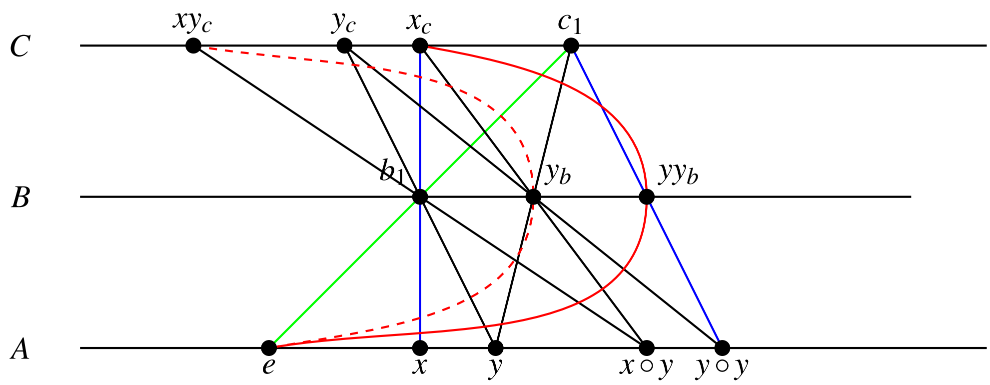


Figure 4.60: A_{40} forbidden configuration

Sub-case (b). Suppose $(x \circ y) \circ y = x$. Then (x, y, y) is an associative triple if and only if G does not contain a configuration isomorphic to A_{41}:

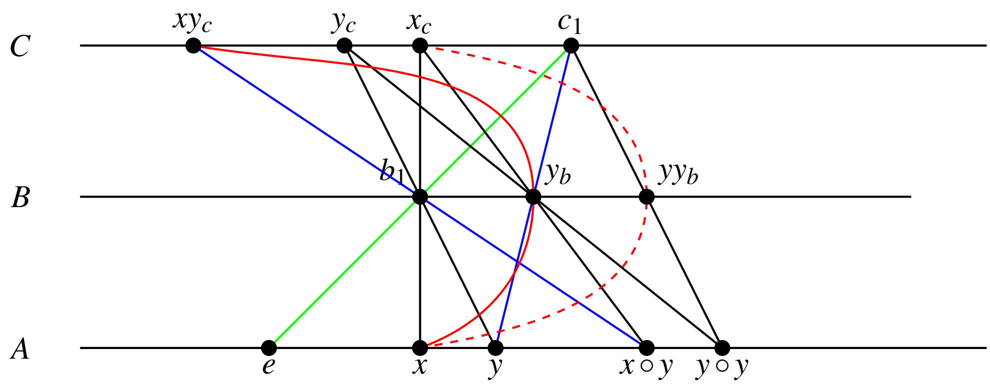


Figure 4.61: A_{41} forbidden configuration

Note that we cannot have $(x \circ y) \circ y = x$, because if the triangle $x(yy_b)x_c$ in Figure 4.61 above existed, we would have two lines meeting at more than one point — giving a non-matroid configuration.

Sub-case (c). Suppose $x \circ (y \circ y) = y$. Then (x, y, y) is an associative triple if and only if G does not contain a configuration isomorphic to A_{42}:

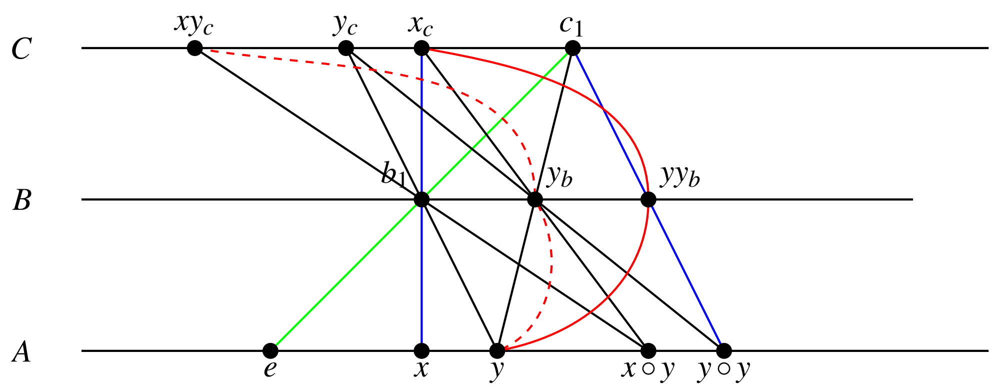


Figure 4.62: A_{42} forbidden configuration

Note that we cannot have $(x \circ y) \circ y = y$, because if the triangle $y(y_b)(xy_c)$ in Figure 4.62 above existed, we would have two lines meeting at more than one point — giving a non-matroid configuration.

Sub-case (d). Note that we cannot have either $(x \circ y) \circ y = x \circ y$ or $x \circ (y \circ y) = x \circ y$ as these force non-matroid configurations. Similarly, we cannot have either $(x \circ y) \circ y = y \circ y$ or $x \circ (y \circ y) = y \circ y$.

Sub-case (e). Suppose both $(x \circ y) \circ y$ and $x \circ (y \circ y)$ are distinct from the existing points of A in Figure 4.58. Then (x, y, y) is an associative triple if and only if G does not contain a configuration isomorphic to either one of A_{43} or A_{44}:

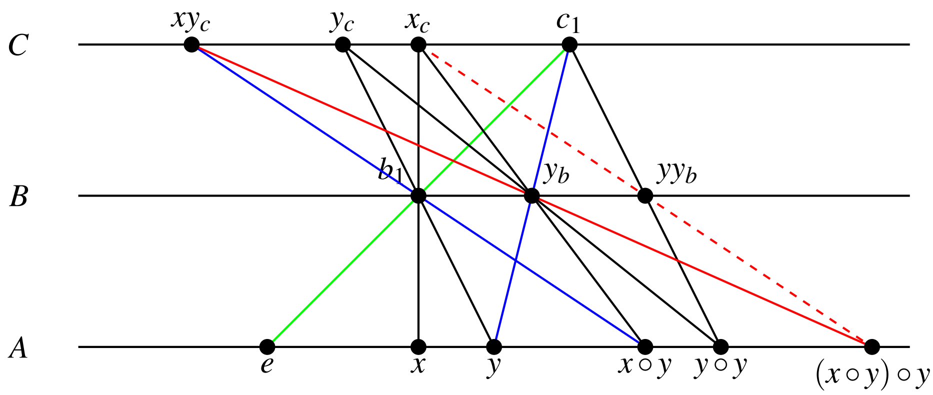


Figure 4.63: A_{43} forbidden configuration

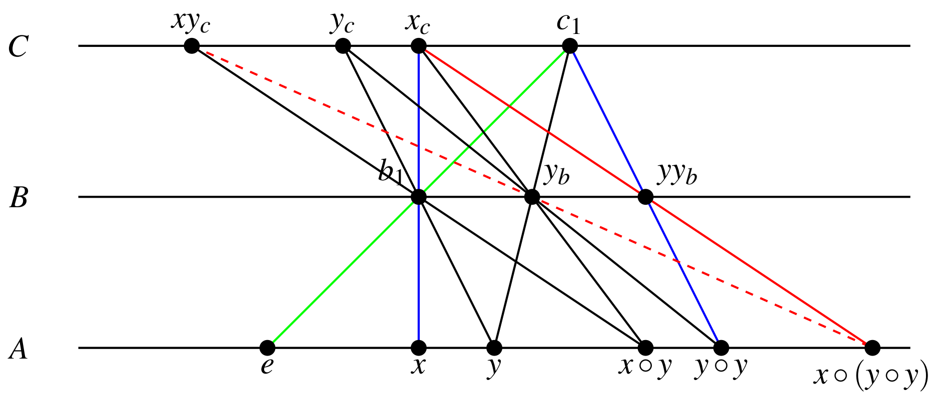
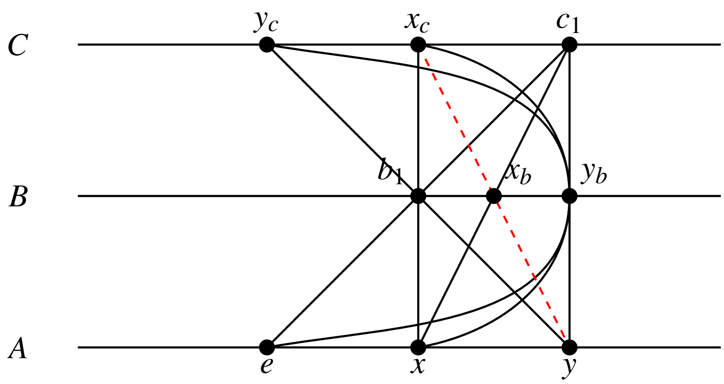


Figure 4.64: A_{44} forbidden configuration

These are all possibilities for the points $x \circ (y \circ y)$ and $(x \circ y) \circ y$ given our choice of $x \circ y$ and $y \circ y$.

Case (ii). Now suppose $x \circ y = e$ and $y \circ y = e$. Then $x \circ (y \circ y) = x$ and $(x \circ y) \circ y = y$, forcing $x = y$ and contradicting the assumption that they are distinct points.

Case (iii). Now suppose $x \circ y = e$ and $y \circ y = x$. This forces $(x \circ y) \circ y = y$. Then we have the following configuration:



However, note that the triangle yx_bx_c must actually exist (contrary to being depicted as a dashed red line in the above Figure) in order for $y \circ y$ to be well-defined, as $y \circ y$ must be defined on the e-triangle ey_by_c. Therefore the above figure is not a forbidden configuration for associativity.

These are all possible forbidden configurations for the associative triple (x,y,y).

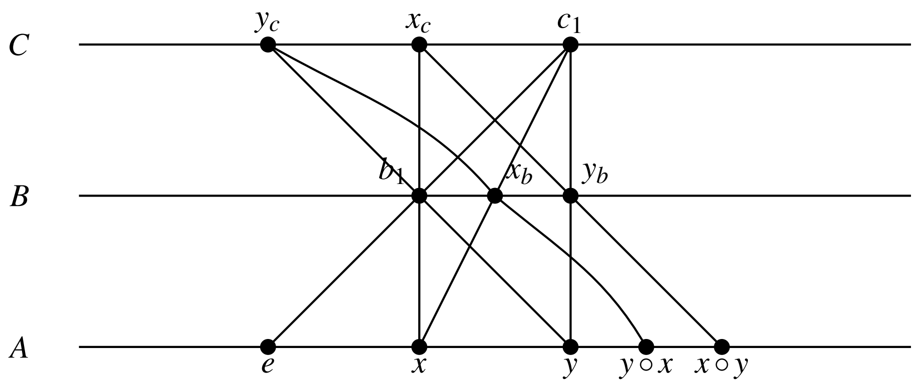
□

The next lemma considers the forbidden configurations for the associative triple (x,y,x) where $\{e,x,y\}$ is a distinct set of points.

Lemma 4.2.6. *Let G be a configuration and $\mathscr{P}$ is the set of necessary points of A where $\{e,x,y\} \in \mathscr{P}$ are a distinct set of points and $x = z$. Then (x,y,x) is an associative triple if and only if G does not contain a sub-configuration isomorphic to one of $A_{45} - A_{50}$, where the isomorphism is the identity for the points in $\mathscr{P}$.*

Proof. It is clear that if $\circ$ is a commutative operation, then $\circ$ is also associative. Therefore we assume that $\circ$ is not commutative.

Case (i). Suppose $x \circ y \neq e$ and $y \circ x \neq e$. Then by inspection we must have the following configuration:



Now we consider the five possibilities for the points $(x \circ y) \circ x$ and $x \circ (y \circ x)$.

Sub-case (a). Suppose either $(x \circ y) \circ x = e$ or $x \circ (y \circ x) = e$. Then (x, y, x) is an associative triple if and only if G does not contain a configuration isomorphic to either A_{45} or A_{46}:

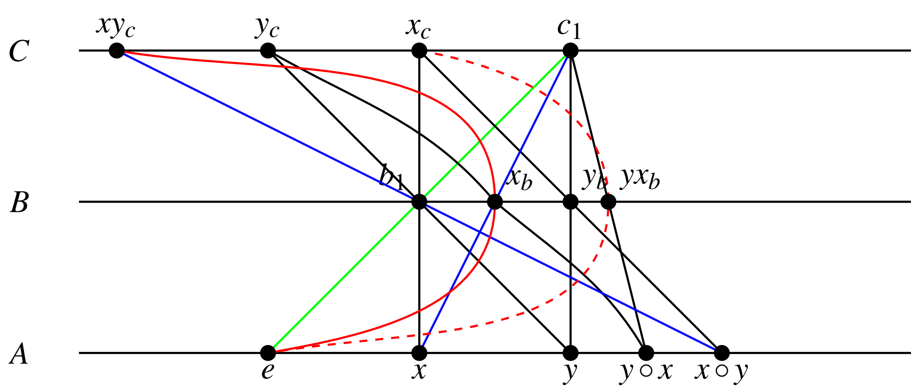


Figure 4.65: A_{45} forbidden configuration

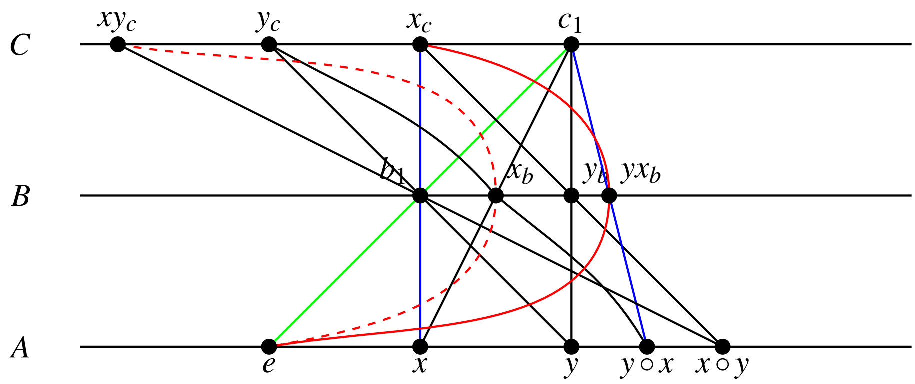


Figure 4.66: A_{46} forbidden configuration

Sub-case (b). Note that we cannot have either $(x \circ y) \circ x = x$ or $x \circ (y \circ x) = x$, as either case would force a non-matroid configuration.

Sub-case (c). Suppose either $(x \circ y) \circ x = y$ or $x \circ (y \circ x) = y$. Then (x, y, x) is an associative triple if and only if G does not contain a configuration isomorphic to either A_{47} or A_{48}:

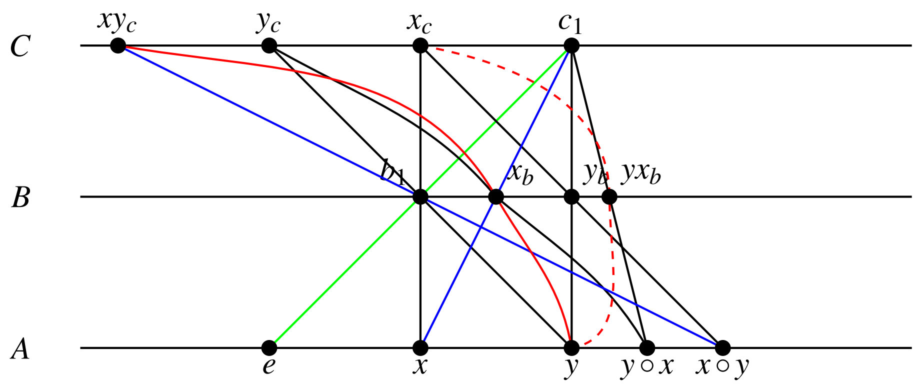


Figure 4.67: A_{47} forbidden configuration

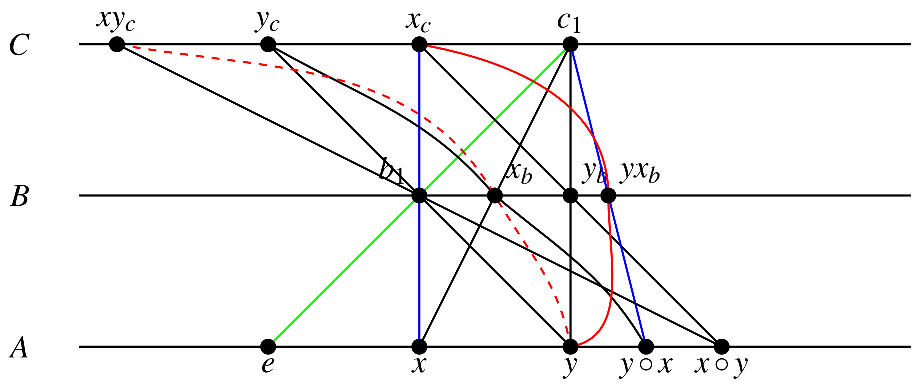


Figure 4.68: A_{48} forbidden configuration

Sub-case (d). Note that we cannot have either $(x \circ y) \circ x \in \{x \circ y, y \circ x\}$ or $x \circ (y \circ x) \in \{x \circ y, y \circ x\}$, as either case would force a non-matroid configuration.

Sub-case (e). Suppose either $(x \circ y) \circ x \notin \{e, x, y, x \circ y, y \circ x\}$ or $x \circ (y \circ x) \notin \{e, x, y, x \circ y, y \circ x\}$. Then (x, y, x) is an associative triple if and only if G does not contain a configuration isomorphic to either A_{49} or A_{50}:

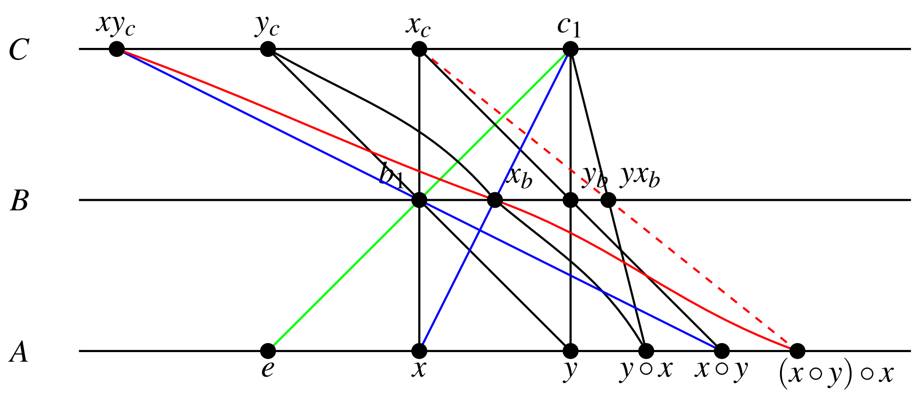


Figure 4.69: A_{49} forbidden configuration

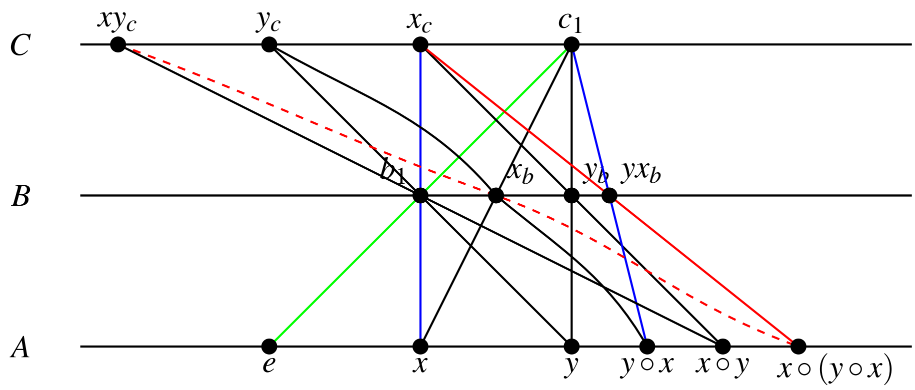


Figure 4.70: A_{50} forbidden configuration

These are all possibilities for the points $(x \circ y) \circ y$ and $x \circ (y \circ x)$ given our choice of the points $x \circ y$ and $y \circ x$.

Case (ii). Note that we have $x \circ y = e$ if and only if $y \circ x = e$ if and only if (x,y,x) is an associative triple.

These are all possible forbidden configurations for the associative triple (x,y,x). □

The next lemma considers the forbidden configurations for the associative triple (x,x,x) where $\{e,x\}$ is a distinct set of points.

Lemma 4.2.7. *Let G be a configuration and let $\mathscr{P}$ be the set of necessary points of A where $\{e,x\} \in \mathscr{P}$ are a distinct set of points and $x = y = z$. Then (x,x,x) is an associative triple if and only if G does not contain a sub-configuration isomorphic to one of $A_{51} - A_{54}$, where the isomorphism is the identity for the points in $\mathscr{P}$.*

Proof. Case (i). Suppose either $(x \circ x) \circ x = e$ or $x \circ (x \circ x) = e$. By extending the basic relation configuration R_3 (which defines $x \circ x$ on eb_1c_1) so that either $(x \circ x) \circ x = e$ or $x \circ (x \circ x) = e$ is defined, by inspection we see that

(x,x,x) is an associative triple if and only if G does not have a configuration isomorphic to either A_{51} or A_{52}:

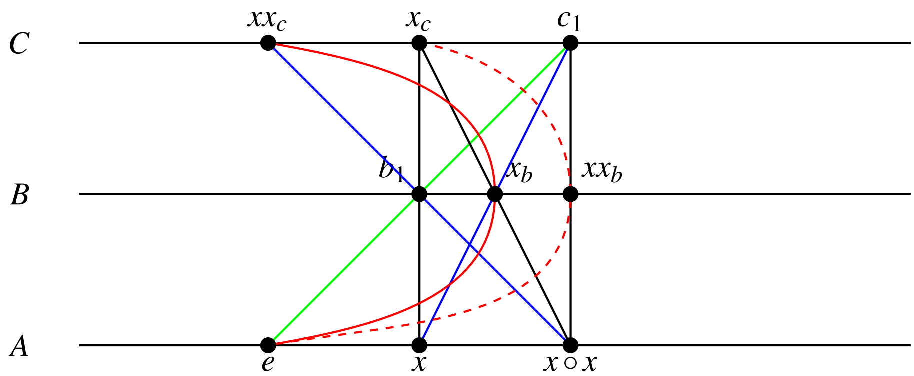


Figure 4.71: A_{51} forbidden configuration

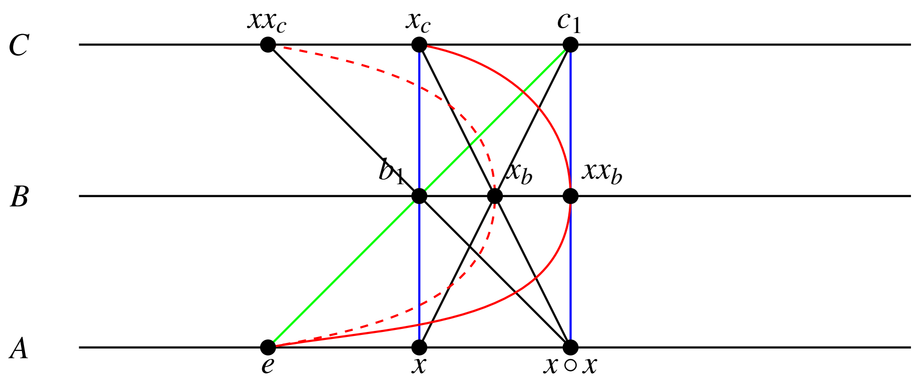


Figure 4.72: A_{52} forbidden configuration

Case (ii). Suppose either $(x \circ x) \circ x \neq e$ or $x \circ (x \circ x) \neq e$. Then (x,x,x) is an associative triple if and only if G does not have a configuration isomorphic to either A_{53} or A_{54}:

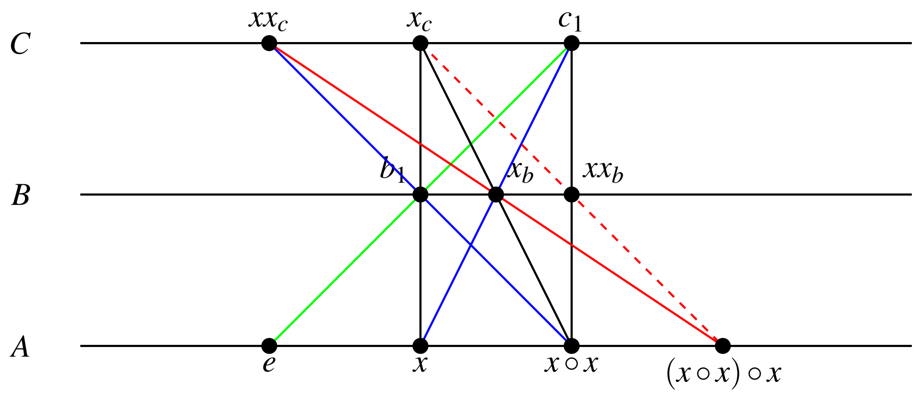


Figure 4.73: A_{53} forbidden configuration

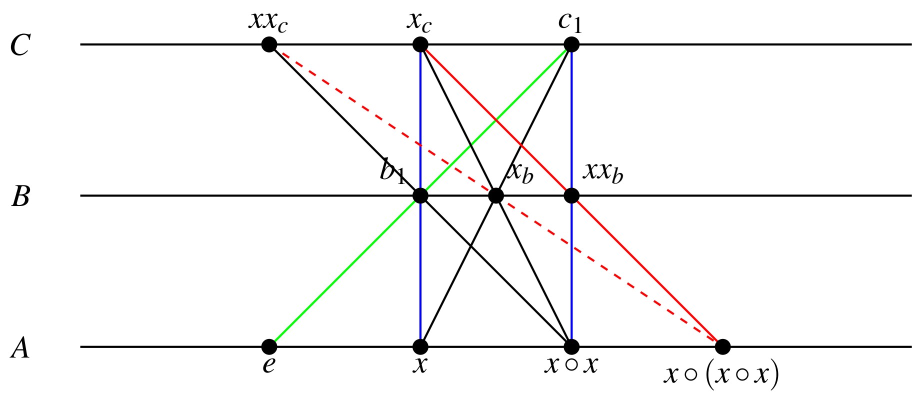


Figure 4.74: A_{54} forbidden configuration

These are all possible forbidden configurations for the associative triple (x,x,x).

□

Finally, Lemmas 4.2.1 – 4.2.7 allow us to prove the following theorem, which states there are at most 54 forbidden configurations for associativity.

Theorem 4.2.8. *Suppose G is an e-based, 3-line configuration configuration where $\circ$ is a strong binary operation. Then G is an associative configuration with an associative binary operation $\circ$ if and only if G does not contain a sub-configuration* *isomorphic to one of* $A_1 - A_{54}$*, where the isomorphism is the identity for the point* e*.*

Proof. We will consider the different possibilities for associative triples. If any of x, y, z equals e, it follows from Lemma 4.2.1 that (x, y, z) is an associative triple. So we will assume for the rest of the proof that none of x, y, z is equal to e. Suppose x, y, z are distinct from one another. Then it follows from Lemmas 4.2.2 and 4.2.3 that (x, y, z) is an associative triple if and only if G does not contain a sub-configuration isomorphic to any of $A_3 - A_{32}$. Suppose $x = y$ and x, z are distinct. It follows from Lemma 4.2.4 that (x, x, z) is an associative triple if and only if G does not contain a sub-configuration isomorphic to any of $A_{33} - A_{38}$. Suppose $y = z$ and x, y are distinct. It follows from Lemma 4.2.5 that (x, y, y) is an associative triple if and only if G does not contain a sub-configuration isomorphic to any of $A_{39} - A_{44}$. Now suppose $x = z$ and x, y are distinct. It follows from Lemma 4.2.6 that (x, y, x) is an associative triple if and only if G does not contain a sub-configuration isomorphic to any of $A_{45} - A_{50}$. Finally, suppose $x = y = z$. It follows from Lemma 4.2.7 that (x, x, x) is an associative triple if and only if G does not contain a sub-configuration isomorphic to any of $A_{51} - A_{54}$. Therefore $\circ$ is associative if and only if every triple of A is an associative triple if and only if G does not contain a sub-configuration isomorphic to any of $A_1 - A_{54}$. $\square$

Therefore to check whether a configuration is associative, we need to check for 54 forbidden sub-configurations, each of which has no more than 16 points. For a configuration G with n points, we can do this in less than $54\binom{n}{15}\binom{16}{3}$ steps, which is polynomial in n. This proves the following corollary.

Corollary 4.2.8.1. *Let* $(G, \{A, B, C\}, e)$ *be an e-based configuration with n points, where* $\circ$ *is a strong binary operation. There is an algorithm, which is polynomial* *in n, to check whether ∘ is an associative binary operation.*

4.2.2 Forbidden configurations for associativity when ∘ is a weak binary operation

We will now assume ∘ is a weak binary operation. Compared with the case analysis for when ∘ was strong, the case analysis when ∘ is weak explodes. However, the list of forbidden configurations is still finite and each forbidden configuration is relatively small, containing no more than 23 points. We will not list them all, rather, we will construct the two maximal sized forbidden configurations. Every other forbidden configuration will be a compression of one of these maximal sized forbidden configurations.

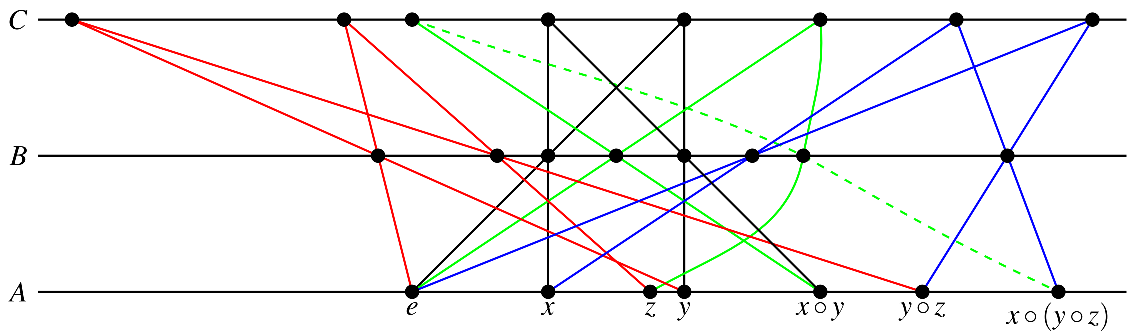


Figure 4.75: A_1' maximal forbidden configuration

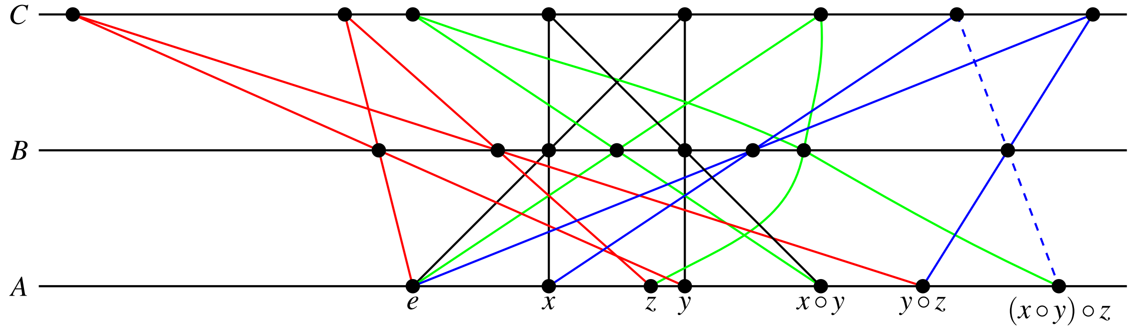


Figure 4.76: A_2' maximal forbidden configuration

Recall that we denote the set $\{e, x, y, z, x \circ y, y \circ z, x \circ (y \circ z), (x \circ y) \circ z\}$ of points in A by $\mathscr{P}$.

Lemma 4.2.9. *Suppose G is an e-based 3-line configuration with a weak binary operation $\circ$. Then G is a maximal sized forbidden configuration for associativity if and only if G is isomorphic to either A_1' or A_2', where the isomorphism is the identity for all points in $\mathscr{P}$.*

Proof. The maximal associative forbidden configuration will use a distinct e-triangle to define each necessary pair — that is, it will use four distinct e-triangles to define each $x \circ y$, $y \circ z$, $x \circ (y \circ z)$ and $(x \circ y) \circ z$. This results in a basic relation configuration for each pair. If we are maximal, then every pair of such basic relation configurations will be disjoint on the lines B and C. In other words, we will have four copies of the R_1 basic relation configuration as in Figure 5.1. The configuration G is not associative if either the relation triangle defining $x \circ (y \circ z)$ or the relation triangle defined $(x \circ y) \circ z$ does not exist. So G must be isomorphic to either one of A_1' or A_2'. □

Corollary 4.2.9.1. *Let G be an e-based 3-line configuration with a weak binary operation $\circ$. Then there are a finite number of forbidden configurations to check whether $\circ$ is an associative binary operation on G.*

Corollary 4.2.9.2. *Let $(G,\{A,B,C\},e)$ be an e-based configuration with n points, where $\circ$ is a weak binary operation. There is an algorithm, which is polynomial in n, to check whether $\circ$ is an associative binary operation.*

In conclusion, after some lengthy case analysis, we have finite lists of forbidden configurations for commutativity under both weak and strong binary operations, and for associativity under a strong binary operation. However, it is important to note that this detailed analysis was not necessary to show that the list of forbidden configurations is finite, or that each forbidden configuration is of bounded size. These properties follow from the fact that we can construct the maximal forbidden configurations, which are finite. All other forbidden configurations can be obtained by a finite sequence of compressions of these maximal configurations — therefore the complete list of forbidden configurations is finite and each of these configurations is of bounded size.

We apply this argument when considering associativity under a weak binary operation. Though it is possible to construct the complete list of forbidden configurations for this case, as we painstakingly realized, doing so is a substantial undertaking. The salient point is that the list of such configurations is indeed finite and each forbidden configuration is of bounded size, allowing a polynomial time check for associativity under a weak binary operation.

Chapter 5

Group configurations

Given a configuration G, we can check whether $\circ$ is a full binary operation, associative and closed under inverses. Therefore we can consider configurations which represent groups. Before defining such configurations in detail, we will define some notation.

In Chapter 2, given an e-based 3-line configuration $(G, \{A, B, C\}, e)$, we define the operation $\circ$ to be based on the point e. Later in this chapter, we will consider the same geometric operation $\circ$, but based on points of A other than e. If $\circ$ is an e-based operation, we will either denote this as usual by $\circ$ or, for added clarity, by $\circ_e$. However, if $\circ$ is an x-based configuration, for $x \in A$ where $x \neq e$, we will denote this by $\circ_x$. For example, in the basic relation configuration R_1 as shown below, if we consider the x-based binary operation $\circ_x$, we have $e \circ_x z = y$. Similarly, if we consider the y-based binary operation $\circ_y$, we have $z \circ_y e = x$.

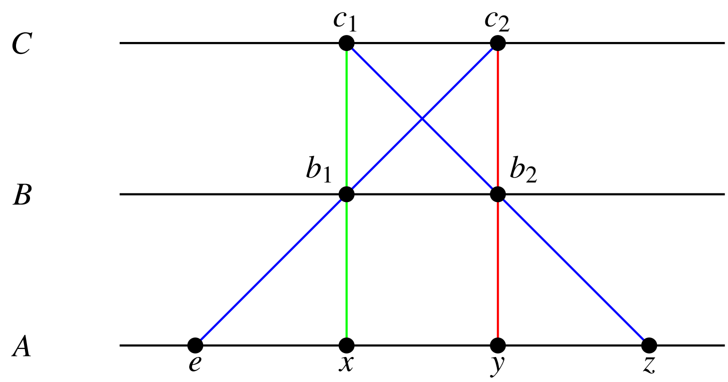


Figure 5.1: R_1 basic relation configuration. Note that the blue and red triangles are the necessary and relation triangles respectively for $e \circ_x z = y$.

Given an e-based group configuration G, recall that e is the *identity*. For $p \in A$, if there exists $q \in A$ such that $p \circ_e q = q \circ_e p = e$, we say q and p are *inverses* and denote this by $q = p^{-1}$ and $p = q^{-1}$. For example, in the basic relation configuration R_2 in Figure 2.5, the points x and y are inverses. If $p = p^{-1}$, we say p is *self-inverse*. The point x in the basic relation configuration R_4 in Figure 2.7 is an example of a self-inverse point.

We will now define the configurations which represent groups. Given an e-based 3-line configuration $(G, \{A, B, C\}, e)$ (which, as in the previous chapter, we will abbreviate to G when the context is clear), we can check in polynomial time whether the following properties are satisfied:

- The points of A are closed under the e-based binary operation $\circ_e$. That is, we can check whether $\circ_e$ is a full binary operation.
- Every element of A has an inverse in A.
- The binary operation $\circ_e$ is associative.

If G satisfies these three properties, then the points of A form a group $(H, \circ_e)$

(which we may also refer to as H if the context is clear, or $(H_e, \circ_e)$ for clarity when we are not referring to a specific group), with identity e, under the full e-based binary operation $\circ_e$. We say G is an *e-based group configuration* of H, which we will abbreviate to *group configuration* or *H-configuration* when the context is clear. Note that $\circ_e$ may be a strong or weak binary operation — we will clarify which when necessary. For an example, let us consider the following e-based group configuration of $(C_2, \circ_e)$, the cyclic group of order 2.

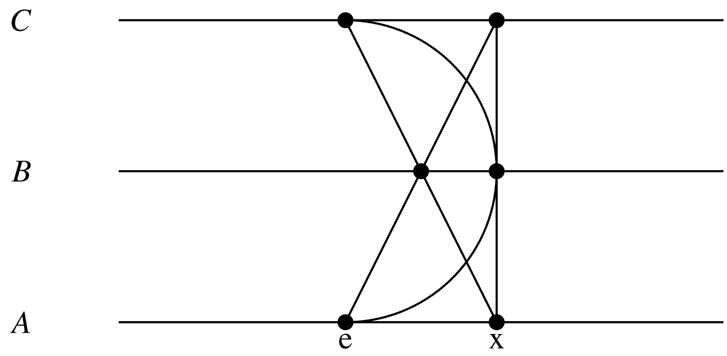


Figure 5.2: The C_2 configuration $(C_2, \circ_e)_1$. Note that for this configuration, $\circ_e$ is a strong binary operation.

The group $(C_2, \circ_e)$ consists of two elements, e and x, where $x = x^{-1}$. By inspection, we can see that $e \circ_e e = e, e \circ_e x = x \circ_e e = x$ and $x \circ_e x = e$ on every e-triangle, therefore the points of A are closed under $\circ_e$. Furthermore, $\circ_e$ is a strong binary operation. As e and x are both self-inverse, clearly the points of A are closed under inverses. Finally, $\circ_e$ is associative by Lemma 4.2.1, as at least one point of any ordered triple must be e. So $(C_2, \circ_e)_1$ is indeed a group configuration of $(C_2, \circ_e)$. There are many other e-based group configurations of $(C_2, \circ_e)$ — Figure 5.2 is only one example. Figure 5.3, as shown below, is another C_2-configuration. Note that though $\circ$ is strong in both Figures 5.2 and 5.3, there do exist group configurations of $(C_2, \circ_e)$ where $\circ_e$ is weak.

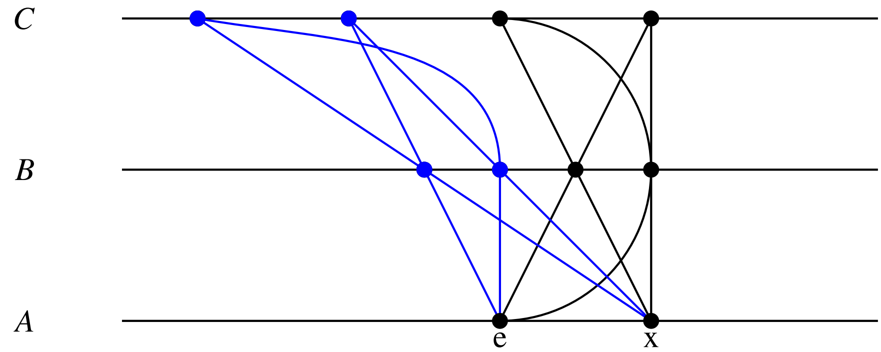


Figure 5.3:

It is not surprising that for any group, there are many corresponding group configurations. To simplify things, we will enforce some natural conditions on the e-based group configurations we consider.

5.0.3 n-replication

Let a group configuration G also be known as G^1. Given a group configuration G^1, we can always extend it to another group configuration, G^n, through a process called *n-replication*. Within a group configuration $(G^1, \{A,B,C\}, e)$, let $A = \{a_1,...,a_k\}$, $B = \{b_1,...,b_l\}$ and $C = \{c_1,...,c_m\}$. Suppose we duplicate the points of B and C to form the sets $B^2 = \{b_1^2,...,b_l^2\}$ and $C^2 = \{c_1^2,...,c_m^2\}$. Then $(G^2, \{A, B \cup B^2, C \cup C^2\}, e)$ is a 2-replication, or *duplication*, of $(G^1, \{A,B,C\}, e)$ if $a_h b_i^2 c_j^2$ is a triangle in G^2 if and only if $a_h b_i c_j$ is a triangle in G^1 for any $h \in \{1,...,k\}, i \in \{1,...,l\}$ and $j \in \{1,...,m\}$. To be clear, a configuration G^1 is itself a 1-replication. For example, we obtained Figure 5.3 from Figure 5.2 by duplication. In Figure 5.3, the blue points and lines are a duplication of the black points and lines. Performing a duplication means every basic relation configuration in our original configuration, G^1, appears twice in the duplication, G^2. We can generalize duplication to n-replication. For the configuration $(G^1, \{A,B,C\}, e)$, let $A = \{a_1, ..., a_k\}$, $B = \{b_1, ..., b_l\}$ and $C = \{c_1, ..., c_m\}$. Let $B^i = \{b^i_1, ..., b^i_l\}$ and $C^i = \{c^i_1, ..., c^i_m\}$. Then $(G^n, \{A, B \cup (\bigcup_{2 \leq i \leq n} B^i), C \cup (\bigcup_{2 \leq i \leq n} C^i\}), e)$ is an n-replication of G^1 if $a_h b^n_i c^n_j$ is a triangle in G^n if and only if $a_h b_i c_j$ is a triangle in G^1 for any $h \in \{1, ..., k\}$, $i \in \{1, ..., l\}$ and any $j \in \{1, ..., m\}$. We say a configuration G' is *prime* if it cannot be decomposed as an n-replication of some configuration G^1. For example, Figure 5.2 is a prime C_2-configuration. As for duplication, if G^n is an n-replication of a prime configuration G^1, then every basic relation configuration of G^1 is replicated $n-1$ many times, therefore appears n times in total in the n-replication G^n. However, we do not gain any new information through n-replication. So, in order to minimize configuration size and complexity, we will assume that every group configuration is prime.

5.0.4 *e*-relevance

For a point $x \in A$ within a configuration G, we say an x-triangle xbc is *used* if xbc is a triangle of a basic relation sub-configuration (based on e) of G. For the point x, if every x-triangle is used, then we say x is an *e-relevant point*. If every point of A is e-relevant, then we say the configuration G is *e-relevant*. For example, in Figure 1, as $(C_2, \circ_e)_1$ is isomorphic to the basic relation configuration R_4, all points of A are used, so the point x is trivially an e-relevant point. Recall $\circ_x$ is the relation $\circ$ based on x, *not* based on e (for example, in Figure 5.2, on both x-triangles we have $e \circ_x e = x$). Given a configuration constructed from $\circ_e$, we want to know the properties $\circ_e$ enforces on $\circ_x$ for $x \in A - \{e\}$. If we have x-triangles which are *not* e-relevant, they will interfere with the influence of $\circ_e$ on $\circ_x$. Therefore, to eliminate degenerate configurations, we will assume within any e-based group configuration, all points $x \in A - \{e\}$ are e-relevant.

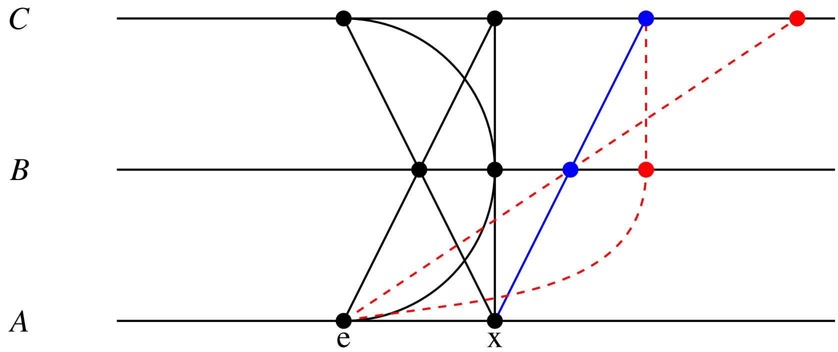


Figure 5.4: $(C_2, \circ_e)_1$ with an added blue triangle which is *not* e-relevant. Recall that a red point on a dashed line consisting of the points m, n means there exists *no* point p such that m, n, p are collinear.

To summarize, we will assume we have an e-based, e-relevant, prime group configuration. Under these conditions, what can we say about the group configurations of a given group? Do they conform to a particular geometric structure? How many group configurations exist for a given group? Given an e-based group configuration $(H_e, \circ_e)$, does $\circ_a$ define a group $(F_a, \circ_a)$ on the points of A? Furthermore, is $(F_a, \circ_a)$ isomorphic to $(H_e, \circ_e)$? We can ask these questions for both strong or weak binary operations. We will first consider the case when $\circ$ is a strong binary operation.

5.1 Strong binary operation

In this section we will assume that $\circ_e$ is a strong, full binary operation. We will eventually prove that for any group with a strong binary operation, there is a unique group configuration, given some natural constraints. First, we will motivate our understanding of group configurations by considering those of two small groups — the cyclic group C_2 and the non-cyclic Klein-4 group.

5.1.1 Group configurations of C_2

We will consider the group configurations of the group C_2, whose group presentation is $\{x \,|\, x^2 = e\}$. In any group configuration of C_2, there must be at least one e-triangle defining $x \circ_e x = e$ (and therefore trivially defining $e \circ_e x, x \circ_e e$ and $e \circ_e e$). This induces the minimal configuration of C_2, denoted $(C_2, \circ_e)_1$, as in Figure 5.2.

We say any two points lying on different distinguished lines are *non-adjacent*. Given a pair (x,y) of non-adjacent points, if there exists a triangle containing both x and y, we say the pair (x,y) is *connected*. For example, in Figure 5.2, every point of B is connected to every point of C. If all non-adjacent pairs of a configuration are connected pairs, then we say we have a *full configuration*. We now show the minimal configuration of C_2 is the *only* group configuration of C_2.

Lemma 5.1.1. *Suppose $(G, \{A,B,C\}, e)$ is an e-based, e-relevant prime C_2-configuration. Then $(G, \{A,B,C\}, e)$ is isomorphic to $(C_2, \circ_e)_1$.*

Proof. Any group configuration of C_2 must contain the minimal configuration, $(C_2, \circ_e)_1$. Note that all pairs of points in $(C_2, \circ_e)_1$ are connected so we have a full configuration. Therefore, we cannot add any triangle which contains any pair of points of $(C_2, \circ_e)_1$. In other words, the only way to extend the minimal configuration is through n-replication. As we assume our configuration is prime, our configuration must be isomorphic to $(C_2, \circ_e)_1$. □

It is clear that the points of A in $(C_2, \circ_e)_1$ are isomorphic up to labelling, resulting in the next lemma.

Lemma 5.1.2. *Given the minimal configuration of $((C_2)_e, \circ_e)$, the binary operation $\circ_x$ defines a group $(G_x, \circ_x)$ on the points of A. Furthermore, $(G_x, \circ_x)$ is isomorphic to $(C_2, \circ_e)$.*

Proof. By inspection, we see that $\circ_x$ is a full binary operation. Every point of A has an inverse and $\circ_x$ is associative, $\circ_x$ defines a group $(G_x, \circ_x)$ on the points of A. It is clear that $(G_x, \circ_x)$ is indeed isomorphic to the group $(C_2, \circ_e)$. □

Recall the main line of a configuration is the line on which $\circ$ is defined. For the configuration $(C_2, \circ_e)_1$, the main line is A, and our ordering of the distinguished lines is (A,B,C). What happens if we permute the ordering of these distinguished lines? Do we still have a C_2-group configuration? All points of $(C_2, \circ_e)_1$ are of equal degree and each of the distinguished lines has exactly two points of degree two, therefore the distinguished lines are isomorphic up to labelling. This proves the following lemma, which states that given $(C_2, \circ_e)_1$, any permutation of the distinguished lines and any labelling of points results in the unique group configuration of C_2.

Lemma 5.1.3. *Consider $(C_2, \circ_e)_1$, whose ordering of distinguished lines is (A,B,C). For any permutation of the distinguished lines $\{A,B,C\}$ and for any point p on the main line, the p-based operation $\circ_p$ defines a group $(F_p, \circ_p)$ on the points of the main line. Furthermore, $(F_p, \circ_p)$ is isomorphic to $(C_2, \circ_e)$.*

Therefore under the restrictions of n-replication and e-relevance, there is a unique group configuration of C_2. Furthermore, any ordering of the distinguished lines and any labelling of points of this unique configuration retains the same configuration.

Suppose we remove the restriction of n-replication. For any n-replication of $(C_2, \circ_e)_1$, we do not retain a C_2 configuration if we permute the distinguished lines so that either B or C is the main line, as $\circ_x$ will not be a full binary operation for any point x on the given main line. For example, consider Figure 5.3, which is a duplication of $(C_2, \circ_e)$. Suppose we permute the distinguished lines so the ordering is $\{C,B,A\}$. Our main line, C, has four points, so we certainly cannot have a C_2-configuration. Furthermore, we do not have a full binary operation, so this configuration cannot be a group configuration.

5.1.2 Group configurations of V_4

We will now consider the group configurations of the smallest non-cyclic group, the Klein-4 group, denoted by either V_4 or $(V_4, \circ_e)$, whose group presentation is $\{a,b \mid a^2 = b^2 = (a \circ_e b)^2 = (b \circ_e a)^2 = e\}$.

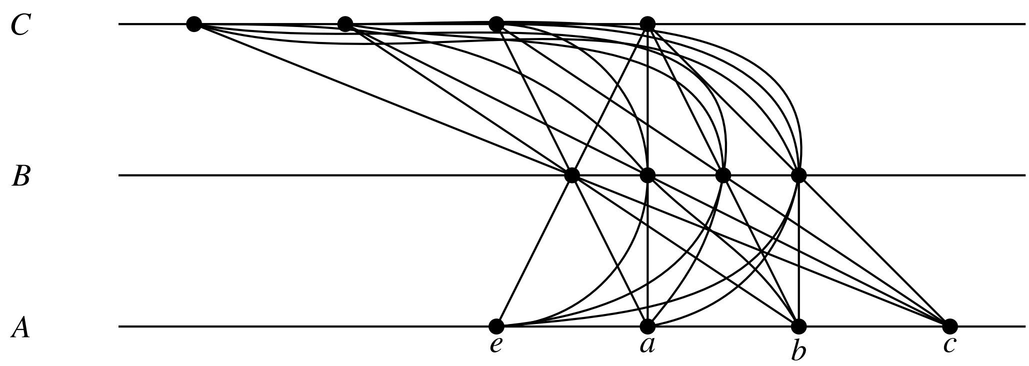
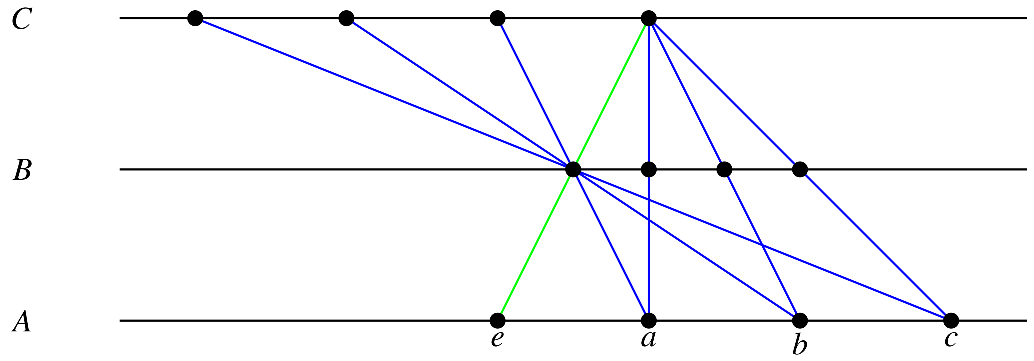


Figure 5.5: The V_4-configuration denoted $(V_4, \circ_e)_1$

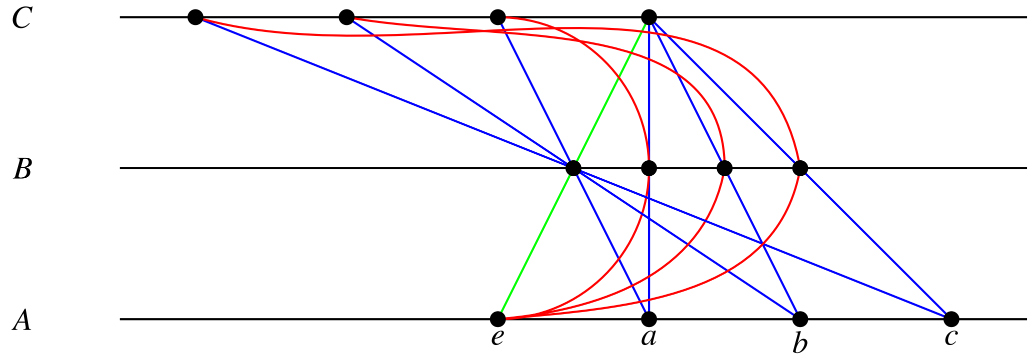
We now show that $(V_4, \circ_e)_1$ is the *only* V_4-configuration up to isomorphism.

Lemma 5.1.4. *Suppose $(G, \{A,B,C\}, e)$ is an e-based, e-relevant, prime V_4-configuration. Then $(G, \{A,B,C\}, e)$ is isomorphic to $(V_4, \circ_e)_1$.*

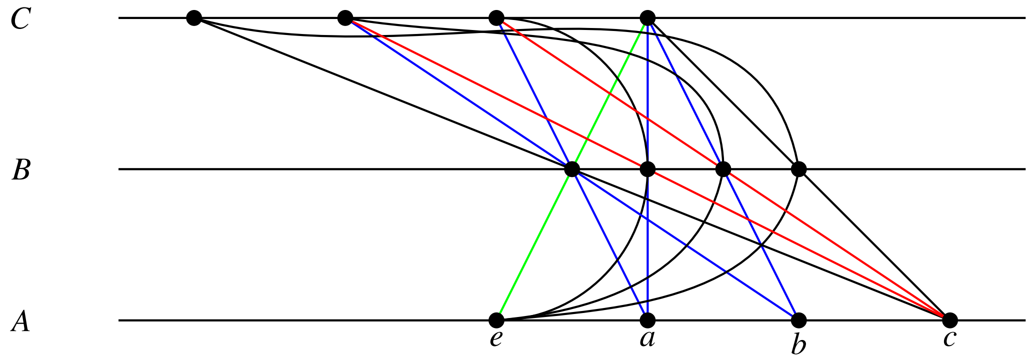
Proof. Consider any e-triangle of our V_4 configuration. As every pair is defined on this e-triangle, this forces the following necessary triangles and the following configuration:



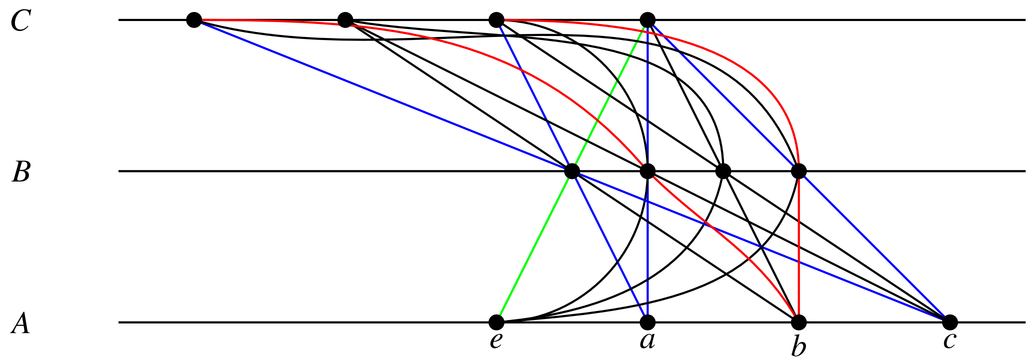
Given we have the above configuration, we are now forced to define the relation triangles of all pairs. First we define $a^2 = b^2 = c^2 = e$ on the green e-triangle, forcing the red relation triangles as in the following configuration:



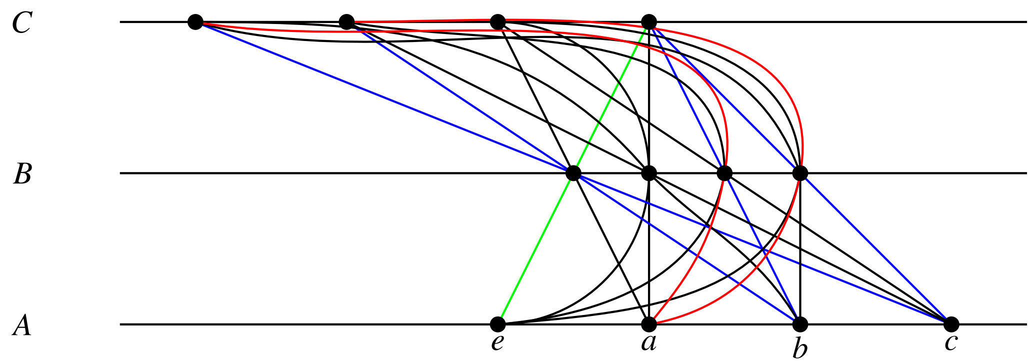
Now we will define $a \circ b = c$ and $b \circ a = c$ on the green e-triangle, forcing the red relation triangles in the following configuration:



Now will define $a \circ c = b$ and $c \circ a = b$ on the green e-triangle, forcing the red relation triangles as in the following configuration:



Finally, we will define $b \circ c = c \circ b = a$ on the green e-triangle, forcing the red relation triangles in the following configuration:



Note that the above configuration is a full configuration, so the only way to extend is through n-replication, contradicting our assumption that our configuration is prime. By inspection, we see that for all $x, y \in \{e, a, b, c\}$, $x \circ y$ is defined on every one of the four e-triangles. So this configuration is indeed a V_4-configuration and by inspection it is isomorphic to $(V_4, \circ_e)_1$. □

It is clear that the points of A in $(V_4, \circ_e)_1$ are isomorphic up to labelling, resulting in the next lemma.

Lemma 5.1.5. *Consider $(V_4, \circ_e)_1$, the unique e-based, e-relevant, prime V_4-configuration. For any $x \in A$, the binary operation $\circ_x$ defines a group $(F_x, \circ_x)$ on the points of A. Furthermore, $(F_x, \circ_x)$ is isomorphic to $(V_4, \circ_e)$.*

Proof. By inspection, we see that for any $x \in A$, $\circ_x$ is a full binary operation. Every point of A has an inverse and $\circ_x$ is associative, so $\circ_x$ defines a group $(F, \circ_x)$ on the points of A. It is clear that $(F, \circ_x)$ is indeed isomorphic to the group $(V_4, \circ_e)$. □

For the configuration $(V_4, \circ_e)_1$, the main line is A, and our ordering of the distinguished lines is $\{A, B, C\}$. What happens if we permute the ordering of these distinguished lines? Do we still have a V_4-group configuration? All points of

$(V_4, \circ_e)_1$ are of equal degree and each of the distinguished lines has exactly four points of degree four, therefore the distinguished lines are isomorphic up to labelling. This proves the following lemma, which states that given $(V_4, \circ_e)_1$, any permutation of the distinguished lines and any labelling of points results in the unique group configuration of V_4.

Lemma 5.1.6. *Consider $(V_4, \circ_e)_1$, whose ordering of distinguished lines is $\{A, B, C\}$. For any permutation of the distinguished lines $\{A, B, C\}$ and for any point p on the main line, the p-based operation $\circ_p$ defines a group $(F_p, \circ_p)$ on the points of the main line. Furthermore, $(F_p, \circ_p)$ is isomorphic to $(V_4, \circ_e)$.*

Therefore under the restrictions of n-replication and e-relevance, there is a unique group configuration of V_4. Furthermore, any ordering of the distinguished lines and any labelling of points of this unique configuration retains the same configuration.

Suppose we remove the restriction of n-replication. If the ordering of the distinguished lines remains as (A, B, C), then any n-replication of $(V_4, \circ_e)_1$ remains a V_4-configuration. Given an n-replication of $(V_4, \circ_e)_1$, suppose we permute the distinguished lines so that either B or C is the main line. Our main line will have $4n$ points, so we certainly cannot have a V_4-configuration. Moreover, for any point p on the main line, $\circ_p$ will not be a full binary operation — so this configuration cannot be a group configuration.

5.1.3 The uniqueness of group configurations

We know that for two small groups — one cyclic and one non-cyclic — each has a unique prime, e-relevant group configuration. This nice property holds for groups in general, as we now prove.

Theorem 5.1.7. *Suppose $(G,\{A,B,C\},e)$ is an e-based, e-relevant, prime group configuration of the group $(G_e,\circ_e)$, where $\circ_e$ is a strong binary operation. Then $(G,\{A,B,C\},e)$ is unique up to isomorphism.*

Proof. Let $(G_e,\circ_e)$ be a group of order n, whose elements are labelled $e=x_0,x_1,x_2,\ldots,x_{n-1}$. We will prove this theorem by showing that any e-based group configuration that generates $(G_e,\circ_e)$ can be built uniquely (up to isomorphism) from $(G_e,\circ_e)$. This proof will be facilitated by a particular labelling of the points of G. We will indicate a point lies on B or C by superscripts. For example, x^b indicates x^b is a point on B. Similarly, y^c indicates y^c is a point on C. The points on the line A are exactly the elements of G_e and will be labelled without superscripts. Our construction will eventually force a triangle to be labelled xy^bz^c if $x\circ_e y=z$.

For any group configuration of G_e, there must exist at least one e-triangle. Pick any e-triangle and label it ee^be^c. By our definition of a full binary operation, $x_i\circ_e x_j$ is defined on every e-triangle for every pair $(x_i,x_j)\in A\times A$ for any $i,j\in\{0,\ldots,n-1\}$. In particular, $x_i\circ_e x_j$ must be defined on ee^be^c. So, for every element $x_i\in A$, there must exist a triangle containing x_i and e^b. We will label this triangle $x_ie^bx_i^c$. In order to satisfy Lemma 2.1.1 and remain a matroid, the points x_i^c must be distinct from one another and from e^c for all $1\le i\le n-1$. Once all triangles of the form $x_ie^bx_i^c$ are added, e^b is connected to all points on A and C, and we have a copy of the elements of the group G_e on C, which forces the configuration below in Figure 5.6.

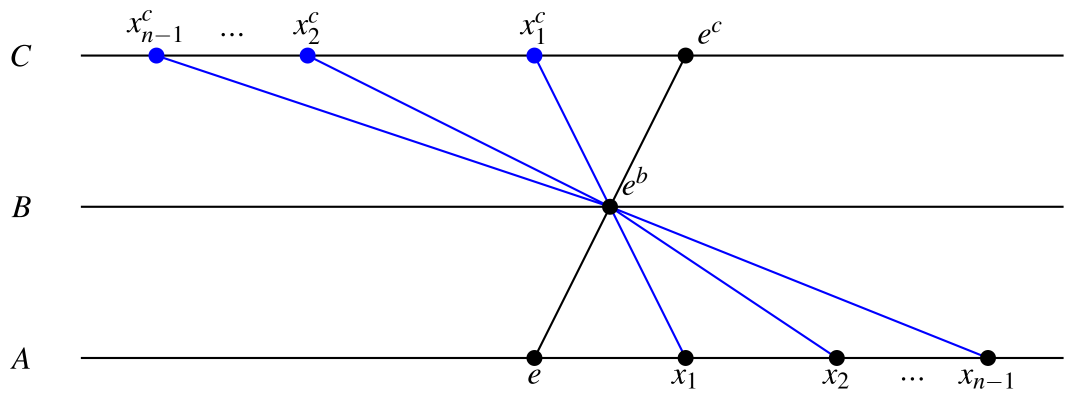


Figure 5.6:

Similarly, for every element $x_i \in A$, there must exist an e-triangle including x_i and e^c. We will label this triangle $x_i(x_i^{-1})^b e^c$. In order to satisfy Lemma 2.1.1 and remain a matroid, the points $(x_i^{-1})^b$ must be distinct from one another and from e^b for all $1 \leq i \leq n-1$. Once all triangles of the form $(x_i^{-1})^b$ are added, e^c is connected to all points on A and B and we have a copy of the elements of the group $(G_e, \circ_e)$ on B, which forces the configuration below in Figure 5.7.

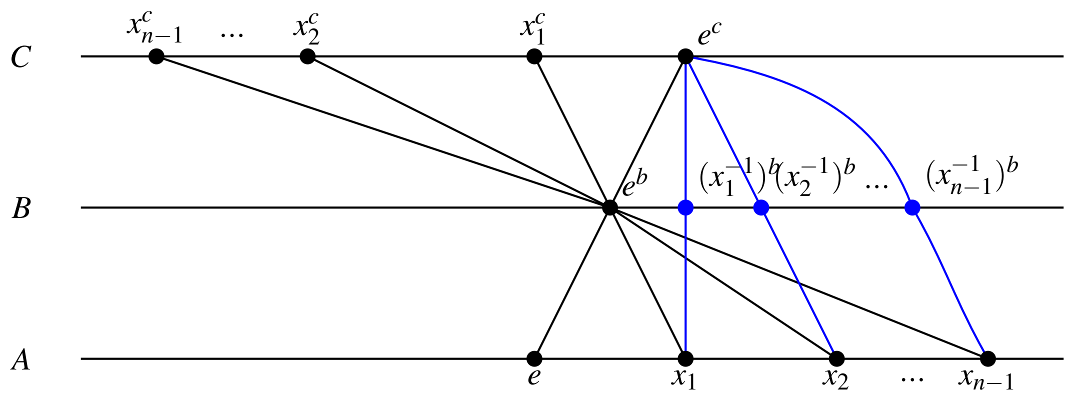


Figure 5.7:

As $x_i \circ_e x_j$ is defined on ee^be^c for all pairs $(x_i, x_j) \in A \times A$, for each such pair we must add the relation triangle $(x_i \circ_e x_j)(x_j^{-1})^b x_i^c$. We know the Latin square property holds for the points of A, that is, for each pair $(x_i, x_j) \in A \times A$, there exists unique $x_f, x_g \in A$ such that $x_f \circ_e x_i = x_j$ and $x_i \circ_e x_g = x_j$. Therefore there is no way that any two relation triangles can meet at more than one point, so Lemma 2.1.1 is satisfied and our configuration remains a matroid once the relation triangles of the form $(x_i \circ_e x_j)(x_j^{-1})^b x_i^c$ are added. However, during this process, we add new e-triangles, as the relation triangle for any pair of inverses is an e-triangle. More specifically, for any distinct pair of inverses (x_j, x_j^{-1}), we add the two new e-triangles $ex_j^b x_j^c$ and $e(x_j^{-1})^b(x_j^{-1})^c$. If an element x_j is self-inverse, we add the single e-triangle $e(x_j^{-1})^b x_j^c = ex_j^b x_j^c$. In other words, for every $x_j \in A$ we are forced to have the e-triangle $ex_j^b x_j^c$. So, on these newly added e-triangles, as $\circ$ is a strong, full binary operation, $x_i \circ_e x_k$ must be defined for all $x_i, x_k \in A$.

In order to show $x_i \circ_e x_k$ is defined on all e-triangles of the form $ex_j^b x_j^c$, first we will prove that the triangle $x_i x_j^b (x_i \circ x_j)^c$ must exist for any x_i, x_j^b. For any point $x_j \in A$, we know the triangles $ex_j^b x_j^c$ and $x_j e^b x_j^c$ exist. We also know the triangle $(x_i \circ_e x_j)e^b(x_i \circ_e x_j)^c$ must exist, which forces the configuration as in Figure 5.8.

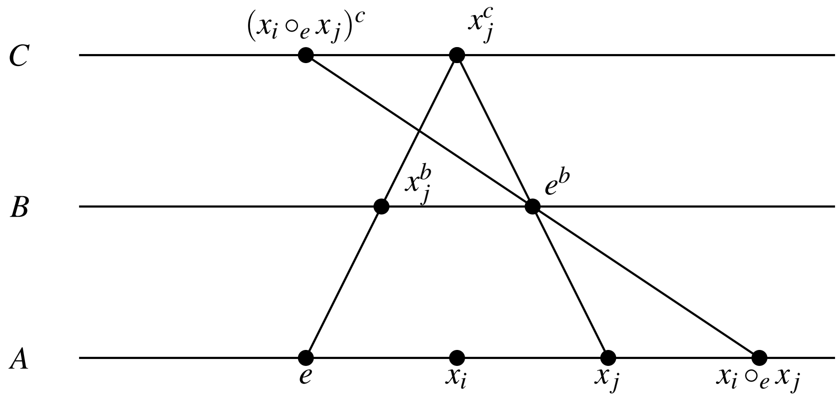


Figure 5.8:

We know there must be a triangle containing x_i and x_j^b. Suppose this triangle is *not* $x_i x_j^b (x_i \circ_e x_j)^c$, but rather $x_i x_j^b x_l^c$ for $x_l^c \neq (x_i \circ_e x_j)^c$. As $x_i \circ_e x_j$ must be defined on the e-triangle $ex_j^b x_j^c$, this forces the relation triangle $(x_i \circ_e x_j)e^b x_l^c$ and the non-matroid configuration as in Figure 5.9.

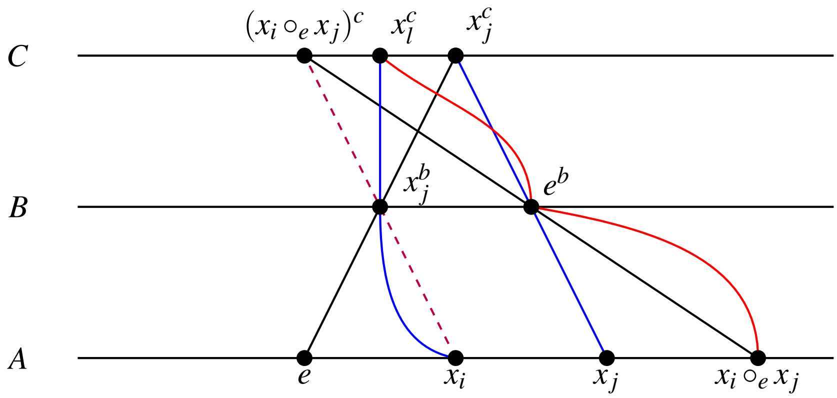


Figure 5.9:

Therefore the triangle $x_i x_j^b (x_i \circ_e x_j)^c$ must exist and we know that $x_i \circ_e x_j$ is defined on $ex_j^b x_j^c$ for any distinct $x_i, x_j \in A$. It follows that $x_i \circ_e x_k$ is defined on $ex_j^b x_j^c$ for any distinct $x_i, x_j, x_k \in A$, as the configuration in Figure 5.10, with triangles of the form $x_i x_j^b (x_i \circ_e x_j)^c$ must exist.

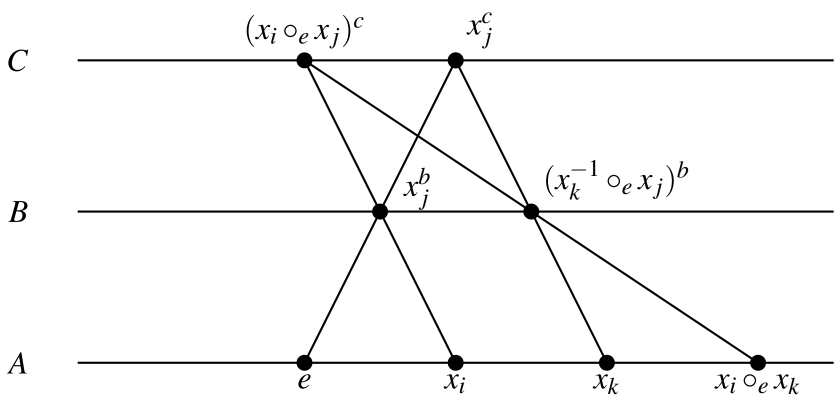


Figure 5.10:

Therefore every pair of points of A is defined on every e-triangle. As each e-triangle is of the form $ex_i x_i$ for every $x_i \in A$, and our group is of order n, there are n many e-triangles. As the points on B and C are exactly the points included in

e-triangles, each of these points has degree n. The identity point e certainly has degree n, and every other point on A has degree n. Therefore all points are of equal degree. More importantly, every pair of non-adjacent points is connected, so we have a full configuration. That is, there are no more triangles we can add to our configuration which include more than one point of the configuration. Therefore the only way we can extend our configuration is to add a new e-triangle $ee^{b'}e^{c'}$, where $e^{b'}, e^{c'}$ are distinct from all existing points. The same argument follows and we end up with a duplication of G. In other words, the only way to extend this configuration is through n-replication. But as we assume we have a prime configuration, the configuration G is the *only* group configuration of the group $(G_e, \circ_e)$, up to isomorphism. $\square$

The following corollary follows immediately from our construction in the proof of Theorem 5.1.7.

Corollary 5.1.7.1. *Consider the unique group configuration* $(G, \{A, B, C\}, e)$ *of* $(G_e, \circ_e)$. *Then xyz is a triangle of* $(G, \{A, B, C\}, e)$ *if and only if* $x \circ_e y = z$.

It also follows from Theorem 5.1.7 that any choice of identity point retains the unique group configuration.

Corollary 5.1.7.2. *Consider the unique e-based group configuration* $(G, \{A, B, C\}, e)$ *of* $(G_e, \circ_e)$. *For any* $a \in A$, $\circ_a$ *is a full binary operation which defines the group* $(F_a, \circ_a)$ *on the points of A. Furthermore,* $(F_a, \circ_a)$ *is isomorphic to* $(G_e, \circ_e)$.

Proof. For any point $a \in A$, we show that we can change the labelling of the H-configuration $(G, \{A, B, C\}, e)$ to obtain an a-based F-configuration $(G', \{A, B, C\}, a)$, where the groups $(G_e, \circ_e)$ and $(F_a, \circ_a)$ are isomorphic. Recall that $\circ_a$ is an a-based binary operation. Certainly $\circ_a$ is a full binary operation, as every pair of points in

G is connected.

For any choice of identity point $a \in A$, we will relabel the points on the lines A and C as follows. Every point $p \in A$ can be written uniquely in the form $p = aq$ for some $q \in A$. We will relabel the point $p = aq$ by $q = a^{-1}p$. This enables us to re-label a by e, as $a = ae$. As $e = aa^{-1}$, we re-label e by a^{-1}. For any $x = ay$ we re-label x by $y = a^{-1}x$. The points on B have the same labelling in both G and G'. Every triangle labelled $pq(p \circ_e q)$ in G becomes re-labelled as $(a^{-1}p)q(a^{-1}(p \circ_e q))$ in G'. Recall from Corollary 5.1.7.1 that $x \circ_e y = z$ if and only if xyz is a triangle in G, if and only if $(a^{-1}x)y(a^{-1}xy)$ is a triangle in G'. Therefore $(F_a, \circ_a)$ is isomorphic to $(G_e, \circ_e)$. $\square$

The next corollary follows from Theorem 5.1.7 and Corollary 5.1.7.2.

Corollary 5.1.7.3. *Given the unique group configuration of the group $(H_e, \circ_e)$, for any ordering of the distinguished lines $\{A, B, C\}$ and for any point p on the main line, then $\circ_p$ is a full binary operation defining the group $(F_p, \circ_p)$ on the points of the main line. Furthermore, $(F_p, \circ_p)$ is isomorphic to $(H_e, \circ_e)$.*

In conclusion, for any group under a strong binary operation, there is a unique prime, e-relevant group configuration, G. For any permutation of the main lines of G, and for any choice of identity on the main line, we retain this unique group configuration. However, this uniqueness and the symmetries which follow do not necessarily hold for group configurations under a weak binary operation, as we will see in the next section. We conclude this section by considering the repercussions of removing the constraint of n-replication.

For a group H of order h, consider $(H^1, \{A, B, C\}, e)$ — the unique prime, e-relevant, e-based H-configuration as described in Theorem 5.1.7. Note that the ordering of the distinguished lines of H^1 is (A,B,C). Consider $(H^n,\{A,B,C\},e)$, an n-replication of H^1. Certainly H^n is an H-configuration — as H^n consists of n copies of H^1, which are disjoint on the lines B and C. We will denote these copies by H^i, where $2 \leq i \leq n$. Suppose we permute the order of the distinguished lines of H^n so that either B or C is the main line. Do we get a group configuration? As the lines B and C are isomorphic in any n-replication, we can consider them as one case. We will denote H^n_p to be the configuration obtained from H^n by permuting the distinguished lines so that either B or C is the main line. In H^n, the lines B and C each have hn many points. Therefore in H^n_p, the main line has hn many points, so H^n_p cannot be an H-configuration. Furthermore, H^n_p cannot be a group configuration at all, as for any point p on the main line, $\circ_p$ is not a full binary operation. Consider any pair of points x,y on the main line such that x is contained in H^i and y is contained in H^j where $i \neq j$, where H^i and H^j are copies of H^1. As H^i and H^j are disjoint on the lines B and C, for any choice of identity p on the main line, $x \circ_p y$ is undefined. Therefore $\circ_p$ is a partial binary operation, implying that H^n_p cannot be a group configuration. We have proved the following two corollaries.

Corollary 5.1.7.4. *Let $(H^n,\{A,B,C\},e)$ be an n-replication of the unique group configuration of the group $(H_e,\circ_e)$. Then $(H^n,\{A,C,B\},e)$ is an H_e-configuration. Furthermore, $(H^n,\{A,C,B\},e)$ is isomorphic to $(H^n,\{A,B,C\},e)$.*

Corollary 5.1.7.5. *Let $(H^n,\{A,B,C\},e)$ be an n-replication of the unique group configuration of the group $(H_e,\circ_e)$. If we permute the distinguished lines of H^n so that either B or C is the main line, the resulting configuration is not an H_e-configuration. Furthermore, this resulting configuration is not a group configuration for any group.*

5.2 Weak binary operation

In this section we will assume that $\circ_e$ is a weak, full binary operation. However, we still want to refer to the unique group configurations which we described in Theorem 5.1.7. For a group H, we will refer to unique group configuration arising when $\circ$ is strong as the *full configuration of* H.

Despite having strong results for group configurations under a strong binary operation, life becomes more complicated when we consider group configurations under a weak binary operation. Luckily, we do not descend into a world of complete chaos. There are some underlying patterns to the examples on the groups C_2, and in particular C_3. However, to gain a comprehensive 'big picture' understanding of the structure of group configurations under a weak binary operation would be no mean feat.

As for the previous section, we will motivate our understanding of group configurations under a weak binary operation by considering those of two small groups — the cyclic groups C_2 and C_3. First we will introduce some terminology.

5.2.1 n-partitions

For the case when $\circ$ is a strong binary operation, we introduced the notion of n-replication. Now we are considering the case when $\circ$ is a weak binary operation, we will extend this notion, as n-replication does not fully encompass the superfluous configurations which may arise.

For a group H, we say a configuration F is a *partial H-configuration* if it is a proper sub-configuration of the full configuration of H. That is, F is not itself an H-configuration, as $\circ$ is a partial, not full, binary operation.

We say an H-configuration $(G,\{A,B,C\},e)$ is a 2-*partition* if there exist disjoint subsets B_1,B_2 of B partitioning B and disjoint subsets C_1,C_2 of C partitioning C such that $(G_1,\{A,B_1,C_1\},e)$ and $(G_2,\{A,B_2,C_2\},e)$, the *blocks* of the 2-partition, satisfy the following two conditions:

- Each block is either an H-configuration or a partial H-configuration;
- The blocks G_1 and G_2 have no triangles in common. That is, if xyz is a triangle of G_1 then $y \in B_1$ and $z \in C_1$, therefore $y \notin B_2$ and $z \notin C_2$. Similarly, if xyz is a triangle of G_2 then $y \in B_2$ and $z \in C_2$, therefore $y \notin B_1$ and $z \notin C_1$.

For example, the following Figure 5.11 is a 2-partition C_2-configuration.

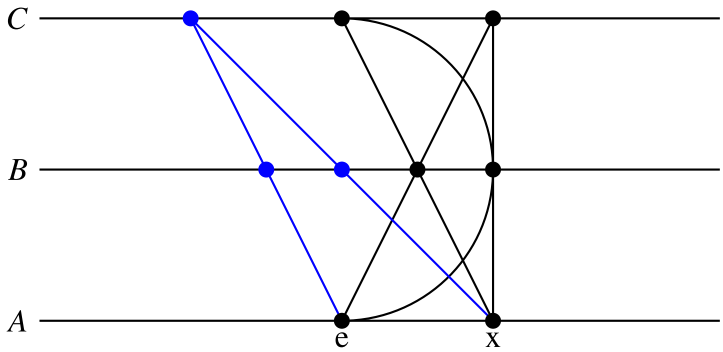


Figure 5.11: A 2-partition group configuration of C_2.

Note that in Figure 5.11, the blue lines make up a partial C_2-configuration, as this block is isomorphic to the basic relation configuration R_6. Similarly, the black lines make up a partial C_2-configuration, as this block is isomorphic to the basic relation configuration R_4.

We can extend the notion of a 2-partition to an n-partition. A H-configuration is an *n-partition* if there exist disjoint subsets $B_1, B_2, ..., B_n$ whose union is B and disjoint subsets $C_1, C_2, ..., C_n$ whose union is C such that $(G_1, \{A, B_1, C_1\}, e)$, $(G_2, \{A, B_2, C_2\}, e)$,...,$(G_n, \{A, B_n, C_n\}, e)$, the *blocks* of the n-partition, satisfy the following two properties:

- Each block is either an H-configuration or a partial H-configuration;
- No pair of blocks share a triangle in common. That is, for any $i \in \{1, ..., n\}$, if xyz is a triangle of G_i then $y \in B_i$ and $z \in C_i$ and therefore $y \notin \bigcup_{j \neq i} B_j$ and $z \notin \bigcup_{k \neq i} C_k$.

We say a configuration which is not an n-partition, i.e. a configuration in which there is a walk between every pair of points, is a 1-*block configuration*. Within a 1-block configuration, we say an e-triangle eb_1c_1 is *redundant* if the only pair defined on eb_1c_1 is $e \circ x$ or $x \circ e$ for some $x \in A$. For example, in Figure 5.11, the blue e-triangle is redundant. As this condition strengthens the notion of e-relevance, if a configuration has no redundant e-triangles, we say it is *strongly e-relevant*.

To summarize, for the rest of this section, unless otherwise stated, we will assume our group configurations are strongly e-relevant, 1-block configurations. Given these conditions, we now prove that under a weak binary operation, there is a unique C_2-configuration.

5.2.2 Group configurations of C_2

Lemma 5.2.1. *Suppose $(G, \{A, B, C\}, e)$ is an e-based, strongly e-relevant,* 1-*block C_2-configuration where $\circ_e$ is a weak binary operation. Then $(G, \{A, B, C\}, e)$ is isomorphic to $(C_2, \circ_e)_1$.*

Proof. Any group configuration of C_2 must contain the configuration $(C_2, \circ_e)_1$. As all pair of points are connected, the only way to extend this configuration is to add a new e-triangle. As we assume we have a 1-block configuration, $(C_2, \circ_e)_1$ is the only group configuration up to isomorphism. $\square$

5.2.3 Group configurations of C_3

We will now consider the strongly e-relevant, 1-block group configurations of C_3 under a weak binary operation.

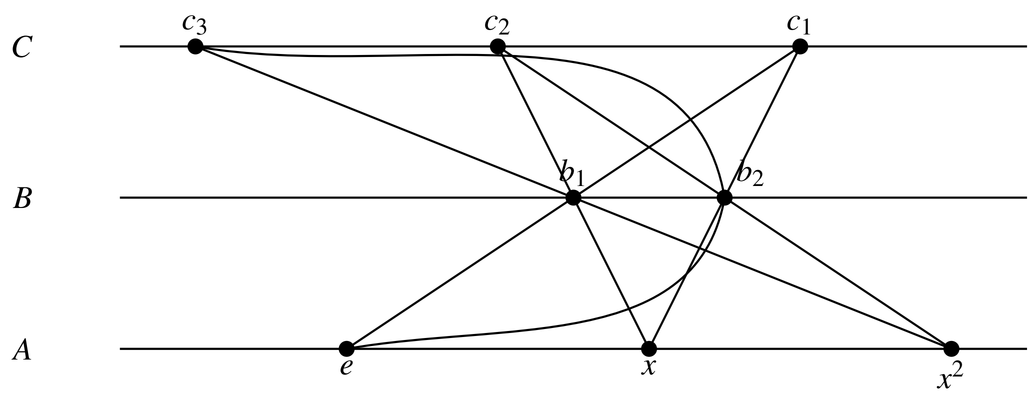


Figure 5.12: The C_3-configuration denoted $(C_3)_1$

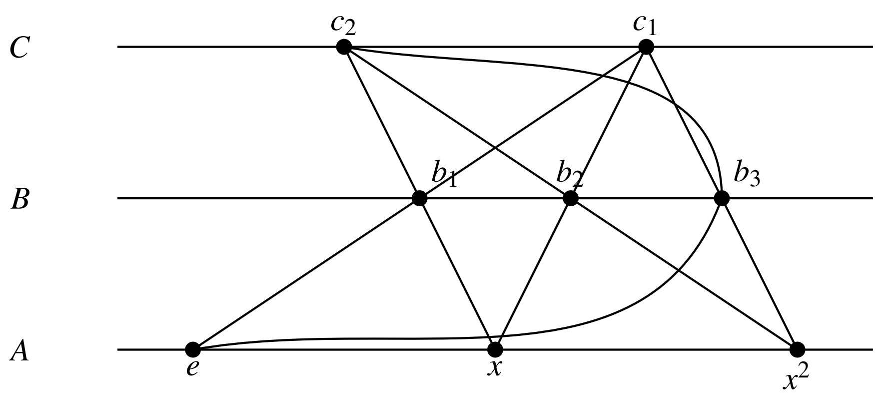


Figure 5.13: The C_3-configuration denoted $(C_3)_2$

We will prove later that the above two configurations are the only strongly e-relevant, 1-block C_3-configurations. First, we will prove some necessary lemmas. Given a configuration G, we say the configuration G' is an *extension* of G if G is a proper sub-configuration of G'.

Lemma 5.2.2. *Any extension of the C_3-configuration $(C_3)_1$ results in a configuration isomorphic to the full configuration of C_3.*

Proof. We prove this by considering all possible extensions of $(C_3)_1$. If we extend, we must have one of the four following cases.

Case (i). Suppose the triangle xb_3c_3 exists, where $b_3 \notin \{b_1, b_2\}$. This forces $x \circ x$ to be defined on eb_2c_3, which forces the relation triangle $x^2b_3c_1$ to exist. To be well-defined, we must have $x \circ x^2$ defined on eb_1c_1, resulting in a configuration isomorphic to the full configuration of C_3.

Case (ii). Suppose the triangle $x^2b_3c_1$ exists, where $b_3 \notin \{b_1, b_2\}$, then we must have $x \circ x^2$ defined on eb_1c_1. This forces the relation triangle eb_3c_2 to exist, which forces $x^2 \circ x^2$ to be defined on eb_1c_1 and the triangle xb_3c_3 to exist — resulting in a configuration isomorphic to the full configuration of C_3.

Case (iii). Suppose the e-triangle eb_3c_1 exists, where $b_3 \notin \{b_1, b_2\}$. If there is a triangle containing xb_3, this must be the triangle xb_3c_3, in order for our configuration to remain well-defined. If the triangle xb_3c_3 exists, this forces $x \circ x$ to be defined on eb_2c_3, which forces the relation triangle $x^2b_3c_1$ to exist — resulting in a configuration isomorphic to the full configuration of C_3. If there is a triangle containing x^2b_3, this must be the triangle xb_3c_1 in order for our configuration to remain well-defined. If the triangle xb_3c_1 exists, this forces $x^2 \circ x^2$ to be defined on eb_1c_1. This forces the relation triangle xb_3c_3

to exist and we have a configuration isomorphic to the full configuration of C_3.

Case (iv). Suppose the e-triangle eb_3c_4 exists, where $b_3 \notin \{b_1, b_2\}$ and $c_4 \notin \{c_1, c_2, c_3\}$. It follows from above that we cannot have the triangles xb_3c_3 or $x^2b_3c_1$, as this forces a configuration isomorphic to the full configuration of C_3 — which being a full configuration, cannot also have the triangle eb_3c_4. So if we define any pair on eb_3c_4, the necessary triangles and relation triangle must be disjoint from lines B and C of the current configuration — that is, we have a 2-partition.

□

Lemma 5.2.3. *Any extension of the C_3-configuration $(C_3)_2$ results in a configuration isomorphic to the full configuration of C_3.*

Proof. We prove this by considering all possible extensions of $(C_3)_2$. If we extend, we must have one of the four following cases.

Case (i). The triangle xb_3c_3 exists (where $c_3 \notin \{c_1, c_2\}$) if and only if $x \circ x$ is define on eb_3c_2 if and only if the triangle $x^2b_1c_3$ exists if and only if $x^2 \circ x$ is defined on eb_1c_1 if and only if the triangle eb_2c_3 exists if and only if we have a configuration isomorphic to the full configuration of C_3.

Case (ii). Suppose the triangle xb_1c_3 exists. Then we must have $x^2 \circ x$ defined on eb_1c_1, forcing the relation triangle eb_2c_3. To be well-defined, this forces $x^2 \circ x^2$ to be defined on eb_1c_1. Then we must have the relation triangle xb_3c_3 — and we have a configuration isomorphic to the full configuration of C_3.

Case (iii). Suppose the e-triangle eb_2c_3 exists. If there is a triangle containing xc_3, it must be the triangle xb_3c_3 in order for our configuration to remain well-defined. If the triangle xb_3c_3 exists then $x \circ x$ is defined on eb_3c_2, forcing the relation triangle $x^2b_1c_3$ to exist, resulting in a configuration isomorphic to the full configuration of C_3. If there is a triangle containing x^2c_3, it must be the triangle $x^2b_1c_3$ in order for our configuration to remain well-defined. If the triangle $x^2b_1c_3$ exists, this forces $x^2 \circ x^2$ to be defined on eb_1c_1. Then we must have the relation triangle xb_3c_3 and we have a configuration isomorphic to the full configuration of C_3.

Case (iv). Suppose the e-triangle eb_4c_3 exists where $b_4 \notin \{b_1, b_2, b_3\}$ and $c_3 \notin \{c_1, c_2\}$. It follows from above that we cannot have the triangles $x^2b_1c_3$ or xb_3c_3, as this forces a configuration isomorphic to the full configuration of C_3 — which being a full configuration cannot also have the triangle eb_4c_3. So if we define any pair on eb_4c_3, the necessary triangles and relation triangle must be disjoint from the current configuration on the lines B and C — i.e. we have a 2-partition.

□

We now use the previous lemmas to prove there are exactly two strongly e-relevant, 1-block C_3-configurations, up to isomorphism.

Lemma 5.2.4. *Suppose $(G, \{A, B, C\}, e)$ is an e-based, strongly e-relevant, 1-block C_3-configuration where $\circ_e$ is a weak binary operation. Then $(G, \{A, B, C\}, e)$ is isomorphic to either $(C_3, \circ_e)_1$ or $(C_3, \circ_e)_2$, where the isomorphism is the identity for the point e.*

Proof. Any C_3 configuration must contain the following basic relation configuration R_3, defining $x \circ x = x^2$.

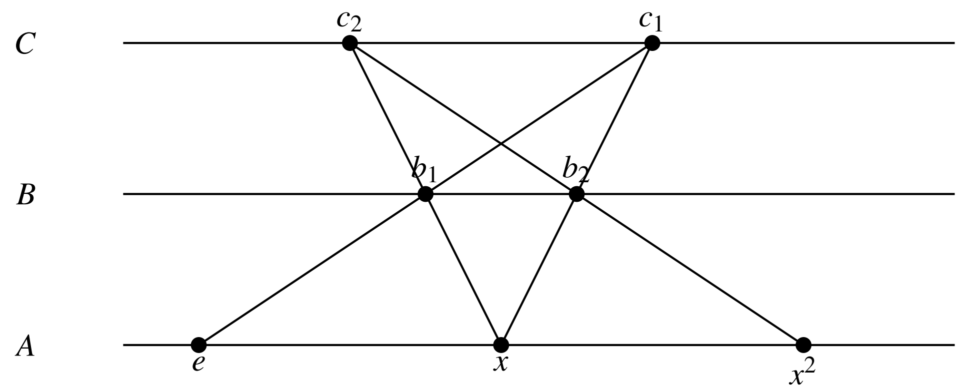


Figure 5.14:

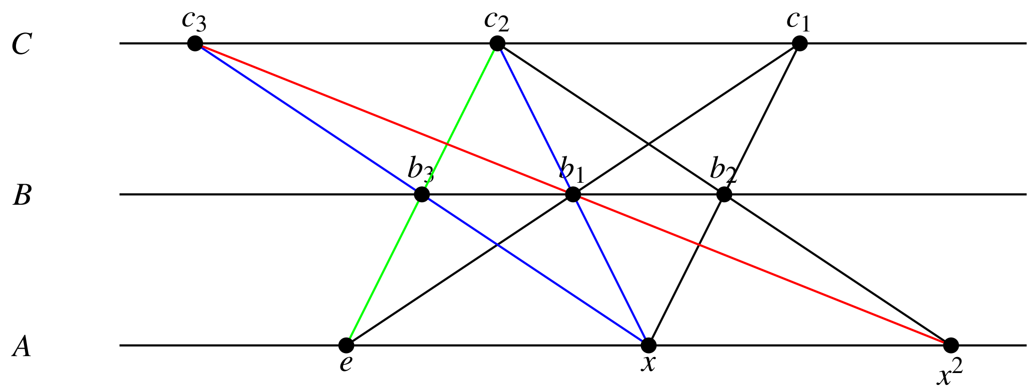
Case (i). First, we will consider extensions on the e-triangle eb_1c_1. We must have one of the two following sub-cases.

Sub-case (a). We can extend this by defining $x \circ x^2 = e$ on the e-triangle eb_1c_1, resulting in a configuration isomorphic to $(C_3)_2$. We know from Lemma 5.2.3 that any extension results in the full configuration of C_3.

Sub-case (b). We can also extend by defining $x^2 \circ x^2$ on eb_1c_1, resulting in a configuration isomorphic to $(C_3)_1$. We know from Lemma 5.2.2 that any extension results in the full configuration of C_3.

Case (ii). Now suppose we extend Figure 5.14 by adding a new e-triangle. We must have one of the three following sub-cases.

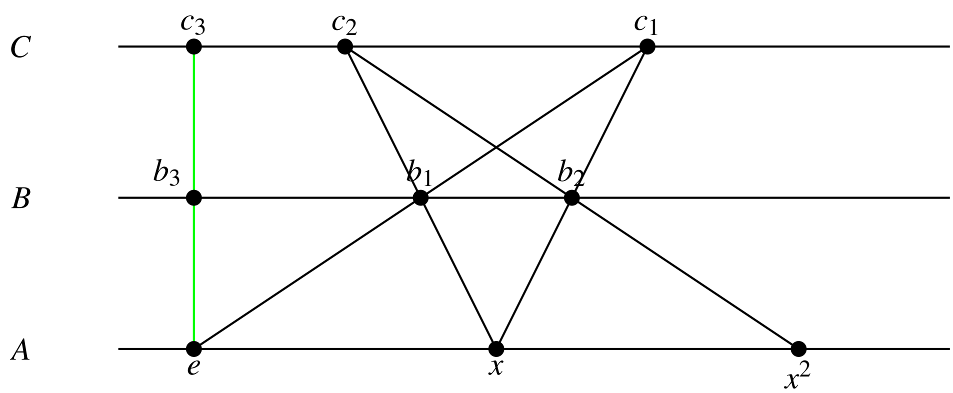
Sub-case (a). Suppose we add the e-triangle eb_3c_2 where $b_3 \notin \{b_1, b_2\}$. If we define $x \circ x$ on eb_3c_2, this forces the triangle xb_3c_3 where $c_3 \notin \{c_1, c_2\}$ and the following configuration:



In order to be well-defined, the triangles eb_2c_3 (defining $x \circ x^2$ on the e-triangle eb_3c_2) and $x^2b_3c_1$ (defining $x^2 \circ x$ on the e-triangle eb_1c_1) are forced, and we have a configuration isomorphic to the full configuration of C_3.

Sub-case (b). Suppose we add the e-triangle eb_2c_3 where $b_3 \notin \{b_1, b_2\}$. We cannot have a triangle containing x^2 and c_3, as this forces $x^2 \circ x^2$ to be defined on eb_2c_3, forcing a non-matroid configuration. If we add the triangle xb_3c_3, where $b_3 \notin \{b_1, b_2\}$, then we have the same configuration as in Figure 5.2.3, and we are forced to extend to the full configuration of C_3.

Sub-case (c). Suppose we add a new e-triangle, eb_3c_3, as in Figure 5.2.3 below:



We cannot have the triangle $x^2b_3c_1$, as then we cannot define $x \circ x^2$ or $x^2 \circ x^2$ on eb_3c_3 and remain a matroid. Similarly, we cannot have the triangle $x^2b_1c_3$, as we cannot define $x^2 \circ x$ on eb_1c_1 and remain a matroid. So if we define any pair on eb_3c_3, the necessary triangles and relation triangle must be disjoint from the lines B and C of existing configuration as in Figure 5.14, i.e. we have a 2-partition.

Note that if we start with the configuration defining $x^2 \circ x^2 = x$, we get an analysis isomorphic to above up to labelling — simply swap the labels for x and x^2. Similarly, if we start with the configuration defining $x \circ x^2 = x^2 \circ x = e$, we get an analysis isomorphic to above up to labelling — simply swap x and e. Therefore any strongly e-relevant, 1-block C_3-configuration must be isomorphic to either $(C_3)_1$ or $(C_3)_2$.

□

Note that $(C_3)_1$ and $(C_3)_2$ are isomorphic up to swapping the distinguished lines B and C.

For both $(C_3)_1$ and $(C_3)_2$, we show that for any choice of identity point on the main line A, we retain the configuration $(C_3)_1$ or $(C_3)_2$ respectively.

Lemma 5.2.5. *For the C_3-configurations $(C_3)_1$ and $(C_3)_2$, for any $a \in A$, the binary operation $\circ_a$ defines a group $(G_a, \circ_a)$ on the points of A. Furthermore, $(G_a, \circ_a)$ is isomorphic to $(C_3, \circ_e)$.*

Proof. Suppose x becomes our new identity, with the full binary operation $\circ_x$, where $e \circ_x e = x^2$, $e \circ_x x^2 = x^2 \circ_x e = x^2$, $x^2 \circ_x x^2 = e$. Then $\circ_x$ defines the group G_x on A, where x is the identity. Moreover, $(G_x, \circ_x)$ is isomorphic to $(C_3, \circ_e)$. Suppose x^2 becomes our new identity, with full binary operation $\circ_{x^2}$, where $e \circ_{x^2} e = x^2$,

$e \circ_{x^2} x = x \circ_{x^2} e = x^2$ and $x \circ_{x^2} x = e$. Then $\circ_{x^2}$ defines the group G_{x^2} on A, where x^2 is the identity. Moreover, $(G_x, \circ_x)$ is isomorphic to $(C_3, \circ_e)$. □

The next lemma shows that for certain permutations of the distinguished lines of $(C_3)_1$, we retain a configuration isomorphic to either $(C_3)_1$ or $(C_3)_2$.

Lemma 5.2.6. *Given the C_3-configuration $(C_3)_1$, for any permutation of the distinguished lines such that A or C is the main line, and for any point p on the main line, the binary operation $\circ_p$ defines a group $(G_p, \circ_p)$ on the points of the main line. Furthermore, $(G_p, \circ_p)$ is isomorphic to $(C_3, \circ_e)$.*

Proof. Clearly the permutation (A,B,C) satisfies the lemma.

Suppose we permute the distinguished lines so that the ordering is (C,B,A). As lines A and C each have exactly three points of degree four, they are isomorphic up to labelling, therefore $\circ_x$ defines a C_3-configuration for any $x \in A$.

Suppose we permute the distinguished lines so that the ordering is either (C,A,B) or (A,C,B). For both cases, observe that we have a configuration isomorphic to $(C_3)_2$ — which is indeed a C_3-configuration. □

The next lemma shows that for certain permutations of the distinguished lines of $(C_3)_2$, we retain a configuration isomorphic to either $(C_3)_2$ or $(C_3)_1$.

Lemma 5.2.7. *For the C_3-configuration $(C_3)_2$, for any permutation of the distinguished lines such that A or B is the main line, and for any point p on the main line, the binary operation $\circ_p$ defines a group $(G_p, \circ_p)$ on the points of the main line. Furthermore, $(G_p, \circ_p)$ is isomorphic to $(C_3, \circ_e)$.*

Proof. Recall that $(C_3)_2$ is isomorphic to $(C_3)_1$ up to swapping the distinguished lines B and C. Therefore it follows from Lemma 5.2.6 that the lemma holds for $(C_3)_2$. □

Given either $(C_3)_1$ or $(C_3)_2$, we now consider the permutations of the distinguished lines which *do not* retain a group structure on the main line.

Lemma 5.2.8. *Consider the C_3-configuration $(C_3)_1$ whose ordering of the distinguished lines is $\{A,B,C\}$. For any permutation of the distinguished lines such that B is the main line, the resulting configuration is not a C_3-configuration. Furthermore, the resulting configuration is not a group configuration.*

Proof. Note that the distinguished lines A and C are isomorphic, so the orderings $\{B,A,C\}$ and $\{B,C,A\}$ are isomorphic. Consider $(C_3)_1$ with the ordering $\{B,A,C\}$ of the distinguished lines, as shown below in Figure 5.15.

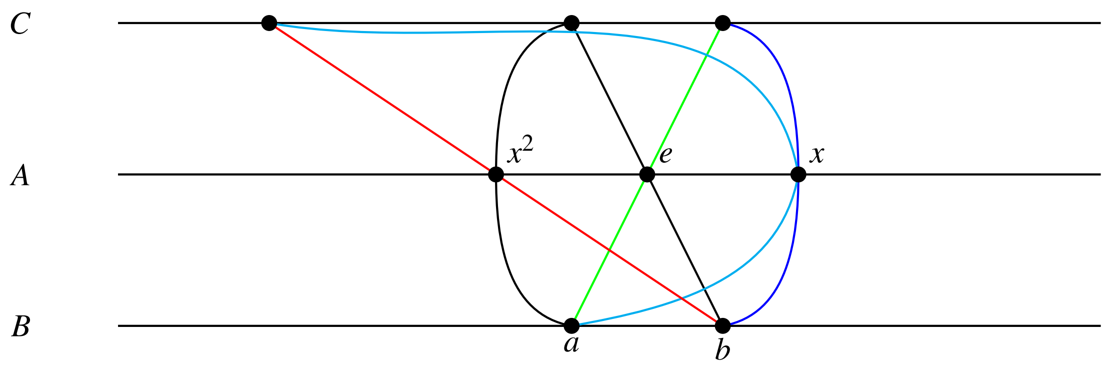


Figure 5.15: The configuration $(C_3)_1$ with permuted distinguished lines with the ordering $\{B,A,C\}$.

Firstly, consider a as the identity under the a-based operation $\circ_a$. By inspection we see that $\circ_a$ is not a well-defined operation, as in Figure 5.15 both $b \circ_a b = a$ and $b \circ_a b \neq a$. Therefore $\circ_a$ does not form a group with identity a on the main line B.

Secondly, consider b as the identity under the b-based operation $\circ_b$. By inspection we see that $\circ_b$ is not a full binary operation, as in Figure 5.15, $a \circ_b a$ is undefined. Therefore $\circ_b$ does not form a group with identity b on the main line B. $\square$

We now prove an equivalent lemma for the C_3-configuration $(C_3)_2$.

Lemma 5.2.9. *Consider the C_3-configurations $(C_3)_2$, whose ordering of the distinguished line is $\{A,B,C\}$. For any permutation of the distinguished lines such that C is the main line, the resulting configuration is not a C_3-configuration. Furthermore, the resulting configuration is not a group configuration.*

Proof. Recall that $(C_3)_2$ is isomorphic to $(C_3)_1$ up to swapping the distinguished lines B and C. Therefore, it follows by the same argument as in Lemma 5.2.9 that given $(C_3)_2$, the permutations $\{C,B,A\}$ and $\{C,A,B\}$ of the distinguished lines do *not* result in group configurations. $\square$

If we remove the restrictions of n-partitions and strong e-relevance, this results in more complicated C_3-configurations. These configurations are compressions of the full configuration of C_3, and the blocks of these configurations are isomorphic to either one of $(C_3)_1$ or $(C_3)_2$.

5.2.4 Conjectures

As evident from Chapter 4, the difficulty with a weak binary operation is that the case analysis escalates extremely quickly. Even for groups of orders four and five, the possibilities snowball. This is clear by the stark contrast in both the length and simplicity of the arguments for the unique, prime, e-relevant group configuration of V_4 under a strong binary operation, compared with the proof that there are exactly two strongly e-relevant, 1-block C_3-configurations.

For a group H of order n, the full configuration of H will have $3n$ points, each of degree n, and a total of n^2 triangles. Consequently, such configurations become very difficult to digest. There is surely an underlying structure to group configurations under a weak binary operation, despite my many unsuccessful hours spent attempting to realize one!

It is interesting that the two strongly e-relevant 1-block C_3-configurations are isomorphic up to swapping two of the distinguished lines. It is possible that similar symmetries exist between the group configurations of other cyclic groups — or even for groups in general.

The analysis of the group configurations of C_3, as well as the analysis of the group configurations of other small groups, lead to the following conjectures.

Conjecture 5.2.10. Suppose $(G, \{A, B, C\}, e)$ is an e-based, strongly e-relevant 1-block group configuration of the group $(H, \circ_e)$. For any $p \in A$, $\circ_p$ is a full binary operation which defines the group $(F, \circ_p)$ on the points of A. Furthermore, $(F, \circ_p)$ is isomorphic to $(H, \circ_e)$.

Conjecture 5.2.11. Let $(H, \circ_e)$ be a group where $\circ_e$ is a weak binary operation, and let $G_1, \ldots, G_n$ be the list of all strongly e-relevant, 1-block H-configurations. Then for any H-configuration without the constraints of n-partitions and strong e-relevance, each block of the configuration is isomorphic to one of G_i, where $1 \leq i \leq n$.

There are many other lines of inquiry concerning group configurations, some of which are briefly discussed in Chapter 7.

Chapter 6

Biased graphs and their matroids

In the previous chapter, we proved that for any group H with a strong binary operation, there is a unique prime, e-relevant, e-based H-configuration. We will now explore the connection between these group configurations and the jointless Dowling matroids obtained from group-labelled biased graphs.

Before defining group-labelled biased graphs, we will define biased graphs. First we will define some terminology. A *θ-graph* consists of two vertices with three edge-disjoint paths, which we denote P_1, P_2 and P_3, between them. In other words, any two cycles intersecting in exactly one non-empty path form a θ-graph. A *biased graph* is a pair $(G, \mathscr{B})$, where G is a graph and $\mathscr{B}$ is a set of cycles of G, called the *balanced cycles*, which satisfy the "θ-property". The *θ-property* states that for any pair of balanced cycles $C_1 = P_1 \cup P_2$ and $C_2 = P_2 \cup P_3$ which form a θ-graph, the third cycle, $C_3 = P_1 \cup P_3$, is also in $\mathscr{B}$. That is, the θ-property says we cannot have exactly two balanced cycles in a θ-graph. Any cycle which is not in $\mathscr{B}$ is said to be *unbalanced*. A *θ-graph* whose cycles are all unbalanced is called an *unbalanced θ-graph*. We will prove later that the θ-property ensures an associated matroid of the biased graph.

There are many important examples of biased graphs. These include biased graphs with only balanced cycles, biased graphs with only unbalanced cycles, and signed graphs, whose balanced cycles are exactly those with an even number of edges. We are interested in group-labelled biased graphs. Given a graph G and group H, we obtain a *group-labelled graph* by assigning a direction and a group element of H to each edge of G. Before describing our choice of balanced cycles to obtain a group-labelled biased graph from a group-labelled graph, we will introduce some more terminology. Let $C = v_1, e_1, v_2, e_2, ..., v_n, e_n, v_1$ be a cycle of the group-labelled graph G, which we will traverse in the direction from e_1 to e_n. We say an edge $e_i \in C$ is a *forward edge* if e_i is directed from v_i to v_{i+1}. We say an edge $e_i \in C$ is a *reverse edge* if e_i is directed from v_{i+1} to v_i. Given a group-labelled graph G, we obtain a *group-labelled biased graph* by defining the set $\mathscr{B}$ of balanced cycles as in the following theorem, in which we show this choice of $\mathscr{B}$ satisfies the θ-property, thus ensuring $(G, \mathscr{B})$ is indeed a biased graph.

Theorem 6.0.12. *Suppose G is a group-labelled graph. Let $h(e_i)$ denote the group label on the edge e_i. We say a cycle C is balanced if and only if*

$$\prod_{forward\,edges} h(e_i) \prod_{reverse\,edges} h(e_i)^{-1} = 1.$$

Let $\mathscr{B}$ be the set of all balanced cycles. Then $(G, \mathscr{B})$ is a biased graph.

Proof. We need only check the θ-property is satisfied. Suppose C_1 and C_2 are cycles intersecting in a non-empty path with endpoints v_1 and v_n. We must show that if C_1 and C_2 are balanced, then the third cycle, C_3, must also be balanced. As C_1 and C_2 form a θ-graph, there will be three paths joining v_1 and v_n. Call these paths P_1, P_2, P_3 and assume the edges are directed from v_1 to v_2 along P_1 and directed from v_2 to v_1 along P_2 and P_3. Note that if the edges are not directed in this way, we can reverse their direction and relabel the reversed edge e_i with group label $h(e_i)^{-1}$. Let ρ_i be the product of the new labels along P_i for each $i \in \{1,2,3\}$. Let $C_1 = P_1 \cup P_2$, $C_2 = P_2 \cup P_3$ and $C_3 = P_1 \cup P_3$. As C_1 and C_2 are balanced, we have $\rho_1 \circ \rho_2 = 1$ and $\rho_2 \circ \rho_3^{-1} = 1$. Therefore the product of the edge labels of C_3 is $\rho_1 \circ \rho_3 = \rho_1 \circ \rho_2 = 1$, therefore C_3 is balanced, so the θ-property holds and the theorem follows. $\square$

We are interested in the associated matroids associated of group-labelled biased graphs. For biased graphs in general, the two most commonly associated matroids are the *biased matroids* and *lift matroids*.

6.1 Biased matroids

Before defining the biased matroids associated with biased graphs, we will introduce some more terminology. Two cycles with exactly one vertex in common is called a *tight handcuff*. Two vertex-disjoint cycles with a minimal path joining them is called a *loose handcuff*. A *handcuff* is either a tight handcuff or a loose handcuff. We call two disjoint cycles a *bicycle*. A handcuff or bicycle is *unbalanced* if all cycles contained in the handcuff or bicycle respectively are unbalanced.

We will use the following lemma from [1] when proving the existence of biased matroids.

Lemma 6.1.1. *A connected graph G with at least two cycles has a θ-graph, a loose handcuff or a tight handcuff as a sub-graph.*

Proof. Any two distinct cycles C_1, C_2 of G are either vertex-disjoint, have a unique common vertex or have more than one common vertex. If C_1, C_2 are vertex disjoint, as G is connected there must be a path connecting C_1 and C_2 and we have a loose handcuff. If C_1, C_2 have a unique common vertex, then we have tight handcuff. If C_1, C_2 have multiple vertices in common, these vertices must form a path and we have a θ-graph. □

We will now prove the existence of biased matroids, the associated matroids of biased graphs, by outlining the following theorem which can be found in [1].

Theorem 6.1.2. *Let $(G, \mathscr{B})$ be a biased graph with edge set $E(G)$. Let $\mathscr{C}$ be the collection of subsets of $E(G)$ that are contained in $\mathscr{B}$, the edges of unbalanced handcuffs and the edges of unbalanced θ-graphs. Let $B(G)$ be the pair $(E(G), \mathscr{C})$. Then $B(G)$ is a matroid with ground set $E(G)$ and $\mathscr{C}$ as its collection of circuits.*

Proof. Clearly $\emptyset \notin \mathscr{C}$. As every member of $\mathscr{C}$ contains a cycle, no member of $\mathscr{C}$ is a proper subset of another. So we are left to show that the third circuit axiom holds. That is, for any distinct $C_1, C_2 \in \mathscr{C}$ such that $e \in C_1 \cap C_2$, there exists $C_3 \in \mathscr{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$. We will prove this by contradiction. As C_1 and C_2 share an edge in common, $G[C_1 \cup C_2]$ is connected. Let $H = G[C_1 \cup C_2] \backslash e$. By our assumption, H contains no member of $\mathscr{C}$, therefore no balanced cycles.

We will now prove by contradiction that H is connected. Suppose H is not connected. Then e cannot be in a cycle of $G[C_1]$ or $G[C_2]$, otherwise H would be connected by the endpoints of e. So $G[C_1]$ and $G[C_2]$ cannot be θ-graphs or tight handcuffs. Therefore they must be loose handcuffs where e is a member of the path connecting the two cycles. When we delete e from H, this divides the two cycles of $G[C_1]$ and $G[C_2]$, so H has a component H_1 whose edge set is not contained in C_1. The edges of C_1 in H_1 induce an unbalanced cycle D_1 together with a path P_1 from a vertex of $G[D_1]$ to an end vertex v of e. Similarly, the edges of C_2 in H_1 induce an unbalanced cycle D_2 together with a path P_2 from a vertex of $G[D_2]$ to an end vertex v of e. By our assumption, H does not contain any balanced cycles, unbalanced handcuffs or unbalanced θ-graphs. As H doesn't contain any balanced cycles, it cannot contain any balanced handcuffs or balanced θ-graphs. So by the converse of the previous lemma, H_1 contains a unique cycle and it must be $D_1 = D_2$. We know $E(H_1) - C_1$ is non-empty, therefore P_2 must contain an edge f that is not in C_1. It is clear that $G[C_2 \cap E(H_1)] \backslash f$ contains paths from f to v and from f to $V[D_1]$. Connecting these paths to P_1, we see that $H_1 \backslash f$ is connected. Therefore f is contained in a cycle which is not D_1, contradicting the fact that H_1 has a unique cycle. Therefore H is connected.

We will now prove that H contains a cycle. If H contains no cycles, then both C_1 and C_2 must be cycles both containing e. By the third circuit axiom applied to $M(G)$, it follows that $(C_1 \cup C_2) - e = H$ contains a cycle, contradicting our assumption that H contains no cycles. We can now assume that H is connected and contains a cycle. Again, by the converse of the previous lemma, H must have a unique cycle. This cycle must be unbalanced as we assume H contains no balanced cycles.

Neither $G[C_1]$ or $G[C_2]$ has any vertices of degree one, so every vertex of degree one in H must be an end of e. Therefore H consists of one the following four cases, where each path described has non-zero length:

Case (i). a single cycle having e as a diagonal, i.e. a θ-graph;

For the remaining three cases, e is connected to at least one vertex of a path off a cycle:

Case (ii). A cycle C together with two paths P_1 and P_2 that are attached to distinct vertices u_1 and u_2 of the cycle such that the only vertices common to any two of P_1, P_2, C are u_1 and u_2. In this case, e joins the ends of P_1 and P_2 that are not in C;

Case (iii). A cycle C, a (u,v)-path P that has one end u on C, but which is otherwise vertex disjoint from both P and C. In this case, possibly $u = w$, and e must join v to the end of Q that differs from w; or

Case (iv). A cycle C and a path P that has one end u on C, but which is otherwise vertex disjoint from C. In this case, e joins the other end of P to either (a) a vertex of C other than u; or (b) a vertex of P, possibly u.

For Case (i), C_1, C_2 must both be balanced cycles and $C_1 \cup C_2$ is a θ-graph, the third cycle of which is in H. Since $(G, \mathscr{B})$ is a biased graph, it satisfies the θ-property and H is a balanced cycle; a contradiction.

For Case (ii), $G[C_1 \cup C_2]$ is a θ-graph, and C_1, C_2 must both contain P_1, P_2 and e. As neither C_1 nor C_2 contains the other (by the second circuit axiom), each of C_1 and C_2 is a cycle and hence is balanced. This implies the cycle in H is also balanced; a contradiction.

Case (iii) cannot occur, as this forces one of C_1, C_2 to be a loose handcuff and one to be a cycle contained in the other; a contradiction.

Similarly, Case (iv)(b) cannot occur as this forces one of C_1, C_2 to be a handcuff and one to be a cycle contained in the other; a contradiction. Finally, in case (iv)(a), C_1 and C_2 must both contain P and e. As neither circuit can contain the other, each of C_1 and C_2 are cycles and must be balanced. This implies the cycle in H is also balanced; a contradiction.

Therefore, the third circuit axiom holds and $B(G)$ is indeed a matroid with ground set $E(G)$ and circuit set $\mathscr{C}$. $\square$

We call $B(G)$ the *biased matroid* of the biased graph $(G, \mathscr{B})$. There are various examples of biased matroids. For example, given a graph G whose cycles are all balanced, the corresponding biased matroid is the graphic matroid of G.

6.1.1 The connection between jointless Dowling geometries and group configurations

We are interested in the *jointless Dowling geometries*, which are the biased matroid of a particular group-labelled graph. In order to define the more general *Dowling geometries*, we will first define the biased graphs from which the Dowling geometries — and consequently the jointless Dowling geometries — arise.

Take n vertices and between each pair of vertices take a set of $|H|$ parallel edges, labelled by each of the elements of the group H. We can assume that within any parallel class, all edges are directed the same way. Add to each vertex a loop, whose label is any non-identity element of H. This gives a biased graph, which we denote HK_n°, where $\circ$ represents the existence of loops. The *Dowling geometry*, denoted $Q_n(H)^\circ$, is the biased matroid of the graph HK_n°. The loops of HK_n° correspond to the corner points, or *joints*, of $Q_n(H)^\circ$. If we delete the loops of HK_n°, we obtain the loopless graph HK_n. The *jointless Dowling geometry*, denoted $Q_n(H)$, is the biased matroid of the loopless graph HK_n. We are interested in $Q_3(H)$, the rank-3 jointless Dowling geometry of HK_3. The following lemma will be useful when we discuss the connection between $Q_3(H)$ and the unique group configuration of H.

Lemma 6.1.3. *Each parallel class of edges in HK_3 corresponds to an $|H|$-point line of $Q_3(H)$.*

Proof. The graph HK_3 consists of three vertices, and between each pair of vertices there are $|H|$ parallel edges, labelled by each of the elements of the group H. We will label these three sets of parallel edges by A, B, C and assume the edges of A and B are oriented in the same direction, with the edges of C oriented in the reverse direction. We say a 2-*cycle* is a cycle consisting of two edges. Any 2-cycle contained in a parallel class of HK_3 is unbalanced. Therefore, by the θ-property, any θ-graph contained in a parallel class is also unbalanced. So every triple of edges within a parallel class of HK_3 correspond to a triangle in $Q_3(H)$. Therefore each parallel class of edges in HK_3 corresponds to an $|H|$-point line (containing a point for every group element of H) of $Q_3(H)$. □

We call the $|H|$-point lines described in Lemma 6.1.3 the *distinguished lines* of $Q_3(H)$. Note that for the case when $|H| = 3$, there will be other triangles of $Q_3(H)$ which are not distinguished lines. Otherwise, the distinguished lines will be the only $|H|$-point lines.

The following two theorems reveal the beautiful bijection between the rank-3 jointless Dowling geometries and the unique group configurations from Chapter 5.

Theorem 6.1.4. *Let $(H, \circ_e)$ be a finite group where $\circ_e$ is a strong binary oper-*

ation. Let $Q_3(H)$ be the rank-3 jointless Dowling geometry whose distinguished lines, partitioning the points of $Q_3(H)$, are labelled A,B,C and let $e \in A$. Then $(Q_3(H),\{A,B,C\},e)$ is a prime, e-relevant, e-based group configuration $(H',\circ_e)$. Moreover, $H' \cong H$.

Proof. By Lemma 6.1.3, each parallel class of edges of HK_n corresponds to a distinguished line of $Q_3(H)$. As HK_3 is loopless, it contains no loose handcuffs. The tight handcuffs of HK_3 are all sets of size four. The only θ-graphs of size three are those consisting of triples of parallel edges, but the corresponding circuit in $Q_3(H)$ already exists as a subset of one of the distinguished lines. Finally, for $a \in A, b \in B, c \in C$, a cycle abc is balanced in HK_3 if and only if $a \circ_e b = c$ if and only if abc is a triangle of $Q_3(H)$. Recall abc is a triangle of $(G,\{A,B,C\},e)$ if and only if $a \circ_e b = c$, and the isomorphism follows. □

Theorem 6.1.5. *Let $(G,\{A,B,C\},e)$ be the prime, e-relevant, e-based group configuration of the group $(H,\circ_e)$, where $\circ_e$ is a strong binary operation. Then $G \cong Q_3(H)$, the rank-3 jointless Dowling geometry of H. Moreover, the distinguished lines of $Q_3(H)$ are $\{A,B,C\}$.*

Proof. As in the proof of the previous theorem, this follows from the fact that for $a \in A, b \in B, c \in C$, abc is a triangle of $(G,\{A,B,C\},e)$ if and only if $a \circ_e b = c$ if and only if abc is a balanced cycle of $Q_3(H)$. □

Therefore prime, e-relevant, e-based group configurations under a strong binary operation are exactly the rank-3 jointless Dowling geometries.

Chapter 7

Open questions

To conclude this thesis, we briefly touch on some open questions — some of which have been considered, while others are yet to be explored.

Multiple 3-line configurations

Within rank-3 matroids we only consider the algebra of throws locally to 3-line configurations. Suppose we expand our outlook and consider matroids with numerous 3-line configurations. Given we can check in polynomial time whether a single 3-line configuration is well-defined, can we also check in polynomial time whether $\circ$ is well-defined across multiple 3-line configurations simultaneously?

Group configurations

There are various unanswered questions regarding group configurations. Given a group configuration, can we check in polynomial time which group it represents? It follows from Chapter 4 that we can check in polynomial time whether a group is abelian. Can we also check in polynomial time whether a group configuration satisfies other properties? For example, can we easily check whether a group configuration represents a cyclic group?

n-partitions

It is possible that applying the constraint of n-partitions to the forbidden configurations for commutativity and associativity would reduce their number, particularly for associativity.

For group configurations under a weak binary operation, we can also consider removing the restriction of every group configuration being a 1-block. That is, if we allow n-partitions, how would this affect the structure of group configurations? What can we say about the structure of the blocks of any n-partition?

For group configurations under a weak binary operation, it may be that the requirement for a configuration to be a 1-block does not provide as much structural insight as other natural constraints.

Properties of $\circ$

Within a configuration G with main line A, given an e-based binary operation — which may be full or partial, and either strong or weak — what does this imply about the potential for binary operations on other non-identity points of A? For example, given a strong, full e-based binary operation, this implies at least a weak, partial a-based binary operation on any other non-identity point $a \in A$. There are many other similar questions we can pose. For another example, suppose we have a associative, weak, partial e-based binary operation defined on the main line of a configuration. What does this imply about the binary operations — if they exist — based on other the other points of the main line?

Conclusion

It is evident there are many avenues for future exploration. Whichever path is taken, it is highly likely that for any conjecture, the case concerning a weak binary operation will yield more obstacles than the corresponding case concerning a strong binary operation. Having said this, it is possible that given the right perspective, there is less disparity between weak and strong binary operations than one would believe.

Chapter 8

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