RUTH

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ULTRAPRODUCTS AND HIGHER ORDER MODELS

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CONTENTS

INTRODUCTION

The programme of work for this thesis began with the somewhat general intention of paralleling in the context of higher order models the ultraproduct construction and its consequences as developed in the literature for first order models. Something of this was, of course, already available in the ultrapower construction of W.A.J. Luxemburg used in Non Standard Analysis.

It may have been considered that such a general intention was not likely to yield anything of significance over and above what was already available from viewing the higher order situation as a 'many sorted' first order one and interpreting the first order theory accordingly. In the event, however, I believe this has proved not to be so. In particular the substructure concepts developed in Chapter II of this thesis together with the various embedding theorems and their applications are not immediately available from the first order theory and seem to be of sufficient worth to warrant developing the higher order theory in its own terms. This, anyway, is the basic justification for the approach and content of the thesis.

Chapter I sets out, within a simple type theory, notation and initial concepts for higher order structures and associated languages. Because of the concern for application of the model theoretic material to algebraic situations, n-ary operations on the individuals of structures are explicitly nominated within the class of general relations. Whilst this makes the applications more natural a price is paid of making the notation and detailed development more complex. Further in Chapter I some attention is given to the two concepts of normality and fullness associated with higher order models and the manner in which these concepts find expression in the ultraproduct construction. Finally model theoretic results paralleling those of a first order theory and ultraproducts are established.

In the first section of Chapter II a substructure concept for higher order models is defined and some of its properties established. In the second section various theorems involving the embedding of a structure into an ultraproduct of a local family of its substructures are proved. Brief mention is given to the presence of 'inverse limits' in this embedding context.

In Chapter III various algebraic applications are given of the concepts and theorems of Chapter II. In Section 1 Stone's Representation Theorem for Boolean Algebras is expressed as an example of λ-embedding. In Section 2 the presence of a higher order ultraproduct construction in the mechanism of Sylow p-subgroups of locally normal groups is exposed and analysed. Perhaps the most significant and fruitful applications are those in Sections 3 and 4 concerned with the theory of local theorems in Universal Algebras.

Two appendices are included. The first discusses modification to a compactness result for a generalised first order language established by A. Shafaat. The second develops some consequences to simple variations in the first order ultraproduct construction.

Theorems, Corollaries and Lemmas are numbered consecutively within each section. Theorem II : 2.3 denotes Theorem 2.3 in Section 2 of Chapter II. Within each chapter results referred to in that chapter do not contain the chapter number.

Some of the material in the thesis has already been accepted for publication. Appendix II is to appear in Volume 13, Number 3, pp. 394-398 of the Notre Dame Journal of Formal Logic. It was submitted in October 1970. Appendix I has been accepted for publication in the Journal of the London Mathematical Society. It was submitted in April 1971 and revised in October 1971. A paper based on the work of Chapter II and the first two sections of Chapter III has been accepted for publication in the Notre Dame Journal of Formal Logic. It was submitted in February 1971. Finally, a paper, based essentially on the work of Sections 3 and 4 of Chapter III, has been submitted (January 1972) to the Editors of the Journal/Proceedings of the London Mathematical Society. (Accepted July 1972)

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CHAPTER I

HIGHER ORDER MODELS AND ULTRAPRODUCTS

Summary. Higher order (algebraic) structures and associated formal languages are described in Section 1. In Section 2 the higher order ultraproduct construction is introduced and the properties of normality and fullness discussed. In Section 3 the chief model-theoretic property of higher order ultraproducts is established, being the extension of Loš's first order result. Extensions are provided to the class of formulae to which the result is applicable.

1. Terminology and basic concepts

We first set out the notation and some standard properties of the calculus of *finite types*. (*cf*. Kreisel and Krivine [1967], page 95 ff.)

The set T of finite types is defined as follows:

(i) the symbol 0 belongs to T ;

(ii) if for any positive integer n the symbols $\sigma_1, \ldots, \sigma_n \in T$ then the symbol $(\sigma_1, \ldots, \sigma_n) \in T$;

(iii) T consists of all symbols formed by a finite number of applications of (i) and (ii).

LEMMA 1.1. *For each* $\sigma \in T$ *there exists a positive integer, called the rank of* σ *and written* $r(\sigma)$ *, such that there exists some sequence* $\sigma_1, \ldots, \sigma_{r(\sigma)} \in T$ *with* $\sigma_1 = \sigma$ *and* $\sigma_{r(\sigma)} = 0$ *and for each* $1 \leq i < r(\sigma)$, $\sigma_i = (\sigma_{i,1}, \ldots, \sigma_{i,n_i})$ *where for some* $1 \leq j \leq n_i$, $\sigma_{i,j} = \sigma_{i+1}$ *and further* $r(\sigma)$ *is maximal with respect* *to this condition.*

Proof. We will proceed by way of an induction argument on the number of applications of (i) and (ii) in the generating definition of T to form each $\sigma \in T$.

For each $\sigma \in T$ let n_σ be the least positive integer such that σ is formed in n_σ applications of (i) and (ii). Condition (iii) and the induction property of the positive integers ensures that n_σ is well defined.

If $n_\sigma = 1$ then $\sigma = 0$ and $r(\sigma) = 1$.

Now assume that, for $m > 1$, $r(\sigma)$ is defined for all $\sigma \in T$ such that $n_\sigma < m$. Take any $\delta \in T$ such that $n_\delta = m$. Thus $\delta = (\delta_1, \ldots, \delta_p)$ for some $\delta_1, \ldots, \delta_p \in T$. As $\delta_1, \ldots, \delta_p$ must be constructed in the process of constructing δ we have $n_\delta \geq 1+n_{\delta_i}$, each $1 \leq i \leq p$. Thus, for each i, $r(\delta_i)$ is defined by hypothesis and it is immediate that $r(\delta)$ can be defined such that $r(\delta) = 1 + \max\{r(\delta_i) : 1 \leq i \leq p\}$.

Hence $r(\sigma)$ with the required maximal property is defined for all $\sigma \in T$. //

$\leq$ is a binary relation defined on T by: $\sigma_1 \leq \sigma_2$ if there exists a finite sequence $\tau_1, \ldots, \tau_n \in T$ such that $\tau_1 = \sigma_2$, $\tau_n = \sigma_1$ and for each $1 \leq i < p$, $\tau_i = (\tau_{i,1}, \ldots, \tau_{i,n_i})$ where $\tau_{i+1} = \tau_{i,j}$ for some $1 \leq j \leq n_i$.

We write $\sigma_1 < \sigma_2$ for $\sigma_1 \leq \sigma_2$ and $\sigma_1 \neq \sigma_2$.

LEMMA 1.2. *$\leq$ is a partial ordering on T .*

Proof. That $\leq$ is reflexive and transitive is immediate from the definition of $\leq$. Assume that $\sigma_1 \leq \sigma_2$ and $\sigma_2 \leq \sigma_1$. Let $\tau_1, \ldots, \tau_m$ be a sequence establishing that $\sigma_1 \leq \sigma_2$ and $\delta_1, \ldots, \delta_n$ a sequence establishing that $\sigma_2 \leq \sigma_1$. Thus $\tau_1 = \delta_n = \sigma_2$ and $\tau_m = \delta_1 = \sigma_1$. If $m > 1$ then

$$r(\tau_1) \geq 1+r(\tau_2), \ldots, r(\tau_{m-1}) \geq 1+r(\tau_m) .$$

Thus $r(\sigma_2) > r(\sigma_1)$. If $n > 1$ then similarly $r(\sigma_1) > r(\sigma_2)$. Hence either $m = 1$ or $n = 1$ and so $\sigma_1 = \sigma_2$. That is $\geq$ is anti-symmetric. //

LEMMA 1.3. *If $\sigma_1, \sigma_2 \in T$ and $\sigma_1 < \sigma_2$ then $r(\sigma_1) < r(\sigma_2)$.*

Proof. Let $\tau_1, \ldots, \tau_m$ be a sequence establishing that $\sigma_1 < \sigma_2$. As $\sigma_1 \neq \sigma_2$ then $m > 1$ and so $r(\sigma_1) < r(\sigma_2)$. //

The next lemma establishes the basis for the inductive arguments used in many of the later theorems. Let P be any subset of T such that if $\sigma \in P$ and $\tau < \sigma$ then $\tau \in P$.

LEMMA 1.4. *If Q is a subset of P such that (i) $0 \in Q$ and (ii) if $\sigma \in P$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\sigma_i \in Q$, $1 \leq i \leq n$, then $\sigma \in Q$, then $Q = P$.*

Proof. Assume there exists $\sigma \in P$ and $\sigma \notin Q$. Hence, by the induction property of the positive integers, there exists a least positive integer n and a $\tau \in P$ and $\tau \notin Q$ such that $r(\tau) = n$.

$n \neq 1$ as $0 \in Q$ and so $\tau = (\tau_1, \ldots, \tau_p)$ where $\tau_1, \ldots, \tau_p \in P$. For each $1 \leq i \leq p$, $r(\tau_i) < r(\tau)$ and so $\tau_i \in Q$. But this would require $\tau \in Q$ by the condition *(ii)* in the statement of the lemma. Hence there exists no $\sigma \in P$ such that $\sigma \notin Q$. That is $Q = P$. //

The final lemma on finite types establishes that T is countable. This ensures that the formal languages associated with the higher order structures as defined below can be made countable if required.

LEMMA 1.5. *T is a countable set.*

Proof. A proof proceeds by first establishing that, for each positive integer n, the members of T of rank n form a countable set. Hence T is a countable union of countable sets and so is countable. //

Higher order structures. Let κ denote a non empty set of types such that if $\sigma \in \kappa$ and $\tau < \sigma$ then $\tau \in \kappa$. Let α, β denote two cardinals.

A $\kappa(\alpha, \beta)$ algebraic structure, (hereafter called a $\kappa(\alpha, \beta)$ structure) is a collection

$$M = \{E^\sigma : \sigma \in \kappa\} \cup \{\epsilon^\sigma : \sigma \in \kappa, \sigma \neq 0\} \cup \{f_m : m < \alpha\} \cup$$
$$\left\{R_n^{\sigma_n} : n < \beta, \sigma_n = (\sigma_{n,1}, \ldots, \sigma_{n,\phi(n)}) \text{ for some } \sigma_{n,1}, \ldots, \sigma_{n,\phi(n)} \in \kappa\right\},$$

where the E^σ's are mutually disjoint sets; for each $\sigma \in \kappa$, $\sigma \neq 0$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$, ϵ^σ is an n+1-placed relation on

$E^{\sigma_1} \times \ldots \times E^{\sigma_n} \times E^{\sigma}$; for some map θ from α to the non-negative integers and, for each $m < \alpha$, f_m is an $\theta(m)$-ary operation on E^0 ; for some map ϕ from β to the positive integers, and for each $n < \beta$, $R_n^{\sigma_n}$ is a $\phi(n)$-placed relation on $E^{\sigma_{n,1}} \times \ldots \times E^{\sigma_{n,\phi(n)}}$, where $\sigma_n = (\sigma_{n,1}, \ldots, \sigma_{n,\phi(n)})$.

For each $\sigma \in \kappa$ the members of E^{σ} are called the objects of M of type σ . The members of E^0 are also called the individuals of M .

The ϵ^{σ}'s , $\sigma \in \kappa$ are called the membership relations of M . In general if $\sigma = (\sigma_1, \ldots, \sigma_n)$ then E^{σ} will not be a subset of the power set of $E^{\sigma_1} \times \ldots \times E^{\sigma_n}$ and so ϵ^{σ} is not the ordinary set membership relation. However, as explained below (Theorem 1.6), an isomorphic structure to M can be constructed in which the corresponding relations to the ϵ^{σ}'s are set membership relations. If $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $a^{\sigma_i} \in E^{\sigma_i}$, $1 \leq i \leq n$, and $a^{\sigma} \in E^{\sigma}$, then $(a^{\sigma_1}, \ldots, a^{\sigma_n})$ is said to belong to a^{σ} , written $(a^{\sigma_1}, \ldots, a^{\sigma_n}) \epsilon^{\sigma} a^{\sigma}$, if and only if, $\epsilon^{\sigma}(a^{\sigma_1}, \ldots, a^{\sigma_n}, a^{\sigma})$, that is if, and only if, $a^{\sigma_1}, \ldots, a^{\sigma_n}, a^{\sigma}$ are related by ϵ^{σ} .

The f_m's are called the operations of M and the $R_n^{\sigma_n}$'s the constant relations of M .

If M is a $\kappa(\alpha, \beta)$ structure and N a $\kappa'(\alpha', \beta')$ structure, $N = \{F^{\sigma} : \sigma \in \kappa'\} \cup \{\epsilon'^{\sigma} : \sigma \in \kappa', \sigma \neq 0\} \cup$

$$\{g_m : m < \alpha'\} \cup \left\{S_n^{\tau_n} : n < \beta'\right\} ,$$

then M and N are *similar* structures if $\kappa = \kappa'$, $\alpha = \alpha'$, $\beta = \beta'$ and there exist permutations μ, δ of α, β respectively such that for each $m < \alpha$, $n < \beta$, f_m and $g_{\mu(m)}$ have the same arity and $\sigma_n = \tau_{\delta(n)}$ and $R_n^{\sigma_n}$ and $S_{\delta(n)}^{\tau_{\delta(n)}}$ have the same number of arguments. $\kappa(\alpha, \beta)$ is called the similarity type of M and the similarity class of M is the class of all structures similar to M .

Hereafter, if M and N are $\kappa(\alpha, \beta)$ structures we shall use the ϵ^{σ} symbols to denote the membership relations in N as well as in M . Context will prevent ambiguity. We also shall always assume that the operations and constant relations of N have been renamed so that the permutations μ, δ , arising from the similarity correspondence, can be taken as the identity permutations and thus not require explicit mention. In general, whenever we take M and N as $\kappa(\alpha, \beta)$ structures we shall understand them given in detail as above. If we take M_i, N_i as $\kappa(\alpha, \beta)$ structures, indexed by some i , we shall understand their detailed description given as for M, N respectively but carrying the index i . In all cases we shall understand the arity and argument maps, θ and ϕ, to be given without requiring explicit mention in each situation.

Let M and N be $\kappa(\alpha, \beta)$ structures. For each $\sigma \in \kappa$ let ψ^{σ} be a bijection from E^{σ} to F^{σ}. Let $\underline{\psi}$ denote the family of such maps, for all $\sigma \in \kappa$.

$\underline{\psi}$ is called an *isomorphism* between M and N (and M and N are said to be isomorphic) if

(i) for each $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, and all $a^{\sigma} \in E^{\sigma}$, $a^{\sigma_i} \in E^{\sigma_i}$, each $1 \le i \le n$, we have $(a^{\sigma_1}, \ldots, a^{\sigma_n}) \in^{\sigma} a^{\sigma}$ if, and only if, $\left(\psi^{\sigma_1}(a^{\sigma_1}), \ldots, \psi^{\sigma_n}(a^{\sigma_n})\right) \in^{\sigma} \psi^{\sigma}(a^{\sigma})$,

(ii) for each $m < \alpha$, and all $a_1^0, \ldots, a_{\theta(m)}^0 \in E^0$, we have $\psi^0\left(f_m\left(a_1^0, \ldots, a_{\theta(m)}^0\right)\right) = g_m\left(\psi^0\left(a_1^0\right), \ldots, \psi^0\left(a_{\theta(m)}^0\right)\right)$,

(iii) for each $n < \beta$, and all $a^{\sigma_{n,j}} \in E^{\sigma_{n,j}}$, each $1 \le j \le \phi(n)$, we have $R_n^{\sigma_n}(a^{\sigma_{n,1}}, \ldots, a^{\sigma_{n,\phi(n)}})$ if, and only if, $S_n^{\sigma_n}\left(\psi^{\sigma_{n,1}}(a^{\sigma_{n,1}}), \ldots, \psi^{\sigma_{n,\phi(n)}}(a^{\sigma_{n,\phi(n)}})\right)$,

where $\sigma_n = (\sigma_{n,1}, \ldots, \sigma_{n,\phi(n)})$.

We shall frequently omit the superscript σ from the map ψ^{σ} allowing context to provide the proper level of action.

A $\kappa(\alpha, \beta)$ structure M is called *normal* if for each $\sigma \in \kappa$,

$\sigma = (\sigma_1, \ldots, \sigma_n)$, and for all $a^\sigma, b^\sigma \in E^\sigma$, we have $a^\sigma = b^\sigma$ if, and only if, $\hat{a}^\sigma = \hat{b}^\sigma$, where

$$\hat{a}^\sigma = \left\{ \left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) : \left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) \epsilon^\sigma a^\sigma \right\}$$

and $\hat{b}^\sigma$ is defined likewise. $\hat{a}^\sigma$ is called the extension of a^σ .

Unless otherwise stated all given structures will be assumed normal. We shall also assume that for any given $\kappa(\alpha, \beta)$ structure M that if for any $n < \beta$, $\sigma_n \in \kappa$ then E^{σ_n} contains an element whose extension coincides with the extension of $R_n^{\sigma_n}$. We identify this member with $R_n^{\sigma_n}$.

THEOREM 1.6. *For each (normal)* $\kappa(\alpha, \beta)$ *structure*

$$M = \{E^\sigma : \sigma \in \kappa\} \cup \{\epsilon^\sigma : \sigma \in \kappa, \sigma \neq 0\} \cup \{f_m : m < \alpha\} \cup \left\{R_n^{\sigma_n} : n < \beta\right\}$$

there exists an isomorphic structure

$$N = \{F^\sigma : \sigma \in \kappa\} \cup \{\epsilon'^\sigma : \sigma \in \kappa, \sigma \neq 0\} \cup \{g_m : m < \alpha\} \cup \left\{S_n^{\sigma_n} : n < \beta\right\}$$

such that if $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ *then* F^σ *is a subset of the power set of* $F^{\sigma_1} \times \ldots \times F^{\sigma_n}$ *and the* ϵ'^σ *is the set membership relation.*

Proof. We shall proceed by induction to define N and the isomorphism $\underline{\psi}$ between M and N .

Put $F^0 = E^0$ and define $\psi^0 : E^0 \to F^0$ as the identity map.

Take $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and assume that for each $1 \le i \le n$, F^{σ_i} is defined as a subset of the power set of $F^{\sigma_{i,1}} \times \ldots \times F^{\sigma_{i,m_i}}$, where $\sigma_i = (\sigma_{i,1}, \ldots, \sigma_{i,m_i})$, and the bijection $\psi^{\sigma_i} : E^{\sigma_i} \to F^{\sigma_i}$ is defined. Now define F^σ and ψ^σ as follows. For each $a^\sigma \in E^\sigma$ put $\psi^\sigma(a^\sigma) = \left\{\left(\psi^{\sigma_1}(a^{\sigma_1}), \ldots, \psi^{\sigma_n}(a^{\sigma_n})\right) : (a^{\sigma_1}, \ldots, a^{\sigma_n}) \in^\sigma a^\sigma\right\}$

Define $F^\sigma = \{\psi^\sigma(a^\sigma) : a^\sigma \in E^\sigma\}$.

We check that ψ^σ is an injective map from E^σ to F^σ . Let $\psi^\sigma(a^\sigma) = \psi^\sigma(b^\sigma)$ for $a^\sigma, b^\sigma \in E^\sigma$. Hence if $(a^{\sigma_1}, \ldots, a^{\sigma_n}) \in^\sigma a^\sigma$ then $\left(\psi^{\sigma_1}(a^{\sigma_1}), \ldots, \psi^{\sigma_n}(a^{\sigma_n})\right) \in \psi^\sigma(b^\sigma)$ and so $(a^{\sigma_1}, \ldots, a^{\sigma_n}) \in^\sigma b^\sigma$, as $\psi^{\sigma_1}, \ldots, \psi^{\sigma_n}$ are bijective by assumption. Similarly if $(a^{\sigma_1}, \ldots, a^{\sigma_n}) \in^\sigma b^\sigma$ then $(a^{\sigma_1}, \ldots, a^{\sigma_n}) \in^\sigma a^\sigma$ and so, as M is normal, $a^\sigma = b^\sigma$.

From the definition of F^σ it is immediate that ψ^σ is surjective. Thus it is bijective. It is also plain from the definition that F^σ is a subset of the power set of $F^{\sigma_1} \times \ldots \times F^{\sigma_n}$.

We now define the relations and operations of N as follows. For each $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ we define $\in'^\sigma$ by:

$$(b^{\sigma_1}, \ldots, b^{\sigma_n}) \in'^\sigma b^\sigma \quad \text{if} \quad \left((\psi^{\sigma_1})^{-1}(b^{\sigma_2}), \ldots, (\psi^{\sigma_n})^{-1}(b^{\sigma_n})\right) \in^\sigma (\psi^\sigma)^{-1}(b^\sigma) .$$

For each $m < \alpha$ we define $g_m = f_m$. Finally, for each $n < \beta$ we define $S_n^{\sigma_n}$ by: $S_n^{\sigma_n}\left(b^{\sigma_{n,1}}, \ldots, b^{\sigma_{n,\phi(n)}}\right)$ if

$R_n^{\sigma_n}\left[\left(\psi^{\sigma_{n,1}}\right)^{-1}\left(b^{\sigma_{n,1}}\right), \ldots, \left(\psi^{\sigma_{n,\phi(n)}}\right)^{-1}\left(b^{\sigma_{n,\phi(n)}}\right)\right]$.

If $\underline{\psi}$ is as defined then it is immediate that $\underline{\psi}$ is an isomorphism between M and N . It is also apparent that for each $\sigma \in \kappa$, $\sigma \neq 0$, ϵ'^{σ} is the ordinary set membership relation. //

If M is a $\kappa(\alpha, \beta)$ structure then M is called a *full* structure if for each $\sigma \in \kappa$, $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$ and for each subset A of $E^{\sigma_1} \times \ldots \times E^{\sigma_n}$ there exists an $a^{\sigma} \in E^{\sigma}$ such that $\hat{a}^{\sigma} = A$.

Thus if M is a full structure then in the associated isomorphic structure N , as constructed in Theorem 1.6, each F^{σ} , $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$, will be the whole power set of $F^{\sigma_1} \times \ldots \times F^{\sigma_n}$.

Formal languages associated with $\kappa(\alpha, \beta)$ structures. We shall denote by $L\left(\kappa(\alpha, \beta)\right)$ a formal language associated with the similarity class of type $\kappa(\alpha, \beta)$ and described as follows.

The symbols of $L\left(\kappa(\alpha, \beta)\right)$ are

(i) for each $\sigma \in \kappa$ a countable class of variable symbols,

$$X^{\sigma} = \left\{x^{\sigma}, y^{\sigma}, \ldots, x_1^{\sigma}, y_1^{\sigma}, \ldots\right\} .$$

(ii) for each $\sigma \in \kappa$, $\sigma \neq 0$ a symbol $\underline{\epsilon}^{\sigma}$,

(iii) for each $m < \alpha$ a symbol $\underline{f}_m$,

(iv) for each $n < \beta$ a symbol $\underset{\sim}{R}_n^{\sigma_n}$ and

(v) the symbols for the logical operators, viz., $\neg$, $\wedge$, $\exists$, the identity symbol $=$, brackets ( ) , and the comma , .

With (iii) and (iv) we shall associate the maps θ from α to the non negative integers, ϕ from β to the positive integers respectively as defined for M and say $\underset{\sim}{f}_m$ has arity $\theta(m)$, $\underset{\sim}{R}_n^{\sigma_n}$ is $\phi(n)$-placed.

We put $X = \cup\{X^\sigma : \sigma \in \kappa\}$ and note that from Lemma 1.5 we can deduce X is countable. Thus if the cardinalities of α, β are finite or denumerable then the collection of symbols for $L(\kappa(\alpha, \beta))$ is countable.

The set Z of terms of $L(\kappa(\alpha, \beta))$ is $\cup\{Z^\sigma : \sigma \in \kappa\}$ where if $\sigma \in \kappa$, $\sigma \neq 0$ then Z^σ is X^σ together with the symbols $\underset{\sim}{R}_n^{\sigma_n}$, $n < \beta$, such that $\sigma_n = \sigma$, and Z^0 is defined as follows:

(i) $T^0 \subseteq Z^0$, where T^0 is X^0 together with the symbols $\underset{\sim}{f}_m$, $m < \alpha$, such that $\theta(m) = 0$, that is the arity of $\underset{\sim}{f}_m = 0$.

(ii) For any $m < \alpha$ and $t_1, \ldots, t_{\theta(m)} \in Z^0$, $\underset{\sim}{f}_m(t_1, \ldots, t_{\theta(m)}) \in Z^0$.

(iii) Z^0 consists of all strings of symbols got by a finite number of applications of (i) and (ii).

The next lemma provides the basis for inductive arguments over all members of Z^0.

LEMMA 1.7. *If* Y *is a subset of* Z^0 *such that (i)* $T^0 \subseteq Y$, *(ii) if* $m < \alpha$ *and* $t_1, \ldots, t_{\theta(m)} \in Y$ *then* $\underline{f}_m(t_1, \ldots, t_{\theta(m)}) \in Y$, *then* $Y = Z^0$.

Proof. The rank of each member of Z^0 can be defined in a manner similar to the rank of finite types. A proof of the lemma may then proceed in the manner of the proof of Lemma 1.4. //

The set of atomic formulae of $L(\kappa(\alpha, \beta))$ is the set of all strings of symbols of the following form:

(i) $t_1^\sigma = t_2^\sigma$, where $t_1^\sigma, t_2^\sigma \in Z^\sigma$, $\sigma \in \kappa$,

(ii) $\underline{R}_n^{\sigma_n}(\alpha_1, \ldots, \alpha_{\phi(n)})$, where $n < \beta$, $\sigma_n = (\sigma_{n,1}, \ldots, \sigma_{n,\phi(n)})$ and $\alpha_{n,j} \in Z^{\sigma_{n,j}}$, each $1 \leq j \leq \phi(n)$,

(iii) $(\alpha_1, \ldots, \alpha_n) \underline{\in}^\sigma \alpha$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\alpha_j \in Z^{\sigma_j}$, each $1 \leq j \leq n$, $\alpha \in Z^\sigma$.

The set of (well formed) formulae (wff's) of $L(\kappa(\alpha, \beta))$ is defined by:

(i) All atomic formulae are wff's;

(ii) If α_1, α_2 are wff's then so are $(\neg \alpha_1)$, $(\alpha_1 \wedge \alpha_2)$ and for any $\beta \in X^\sigma$, $\sigma \in \kappa$, $(\exists \beta \alpha_1)$;

(iii) The set of wff's consists of all formulae obtained by a finite number of applications of (i) and (ii).

We shall omit brackets according to the usual conventions. We shall also introduce the logical symbols $\vee$, $\Rightarrow$, $\Leftrightarrow$ and $\forall$ via the standard procedures.

The following lemma provides a basis for later induction arguments over the set of wff's.

LEMMA 1.8. *If* Q *is a subset of wff's such that (i) all atomic formulae belong to* Q *, (ii) if* $\alpha_1, \alpha_2 \in Q$ *and* $\beta \in X^{\sigma}$ *, some* $\sigma \in \kappa$ *then* $\neg\alpha_1 \in Q$ *,* $\alpha_1 \wedge \alpha_2 \in Q$ *and* $\exists\beta\alpha_1 \in Q$ *, then* Q *is the set of all wff's.*

Proof. A proof proceeds in a manner analogous to that of Lemma 1.4 but with the length of a formula replacing the notion of rank of a type symbol. //

Let M be any member of the similarity class of type $\kappa(\alpha, \beta)$. A standard interpretation ν of $L(\kappa(\alpha, \beta))$ with respect to M , (hereafter called a M-interpretation), is a map from the non-logical symbols of $L(\kappa(\alpha, \beta))$, (that is the symbols (i) to (iv) in the defining list of symbols of $L(\kappa(\alpha, \beta))$,) to the objects, operations and constant relations of M such that

(i) for each $\sigma \in \kappa$, $\nu(X^{\sigma}) \subseteq E^{\sigma}$;

(ii) for each $\sigma \in \kappa$, $\sigma \neq 0$, $\nu(\underline{\epsilon}^{\sigma})$ is the membership relation ϵ^{σ} of M ;

(iii) for each $m < \alpha$, $\nu(\underline{f}_m)$ is the operation f_m of M ;

(iv) for each $n < \beta$, $\nu\left(\underset{\rightarrow n}{R}^{\sigma_n}\right)$ is the relation $R_n^{\sigma_n}$ of M .

Each standard interpretation ν as above can be extended to map all the members of Z^0 to E^0 by the following procedure: If $m < \alpha$ and $t_1, \ldots, t_{\theta(m)} \in Z^0$ such that $\nu(t_i)$ is already defined, each $1 \leq i \leq \theta(m)$ then put

$$\nu\left(\underline{f}_m(t_1, \ldots, t_{\theta(m)})\right) = f_m\left(\nu(t_1), \ldots, \nu(t_{\theta(m)})\right) .$$

Lemma 1.7 provides the basis for an inductive definition of ν over the whole of Z^0 .

If ψ is a wff of $L(\kappa(\alpha, \beta))$ we define, in the standard manner, what it is for ψ to hold in M with respect to an M-interpretation ν , written $M \models_\nu \psi$, as follows.

(i) If ψ is an atomic wff of one of the forms,

a) $t_1^\sigma = t_2^\sigma$, $\sigma \in \kappa$, b) $(\alpha^{\sigma_1}, \ldots, \alpha^{\sigma_n}) \underline{\epsilon}^\sigma \alpha^\sigma$, or

c) $\underset{\rightarrow n}{R}^{\sigma_n}(\alpha_1, \ldots, \alpha_{\phi(n)})$ then $M \models_\nu \psi$ if

a) $\nu\left(t_1^\sigma\right) = \nu\left(t_2^\sigma\right)$, b) $\left(\nu(\alpha^{\sigma_1}), \ldots, \nu(\alpha^{\sigma_n})\right) \epsilon^\sigma \nu(\alpha^\sigma)$ or

c) $R_n^{\sigma_n}\left(\nu(\alpha_1), \ldots, \nu(\alpha_{\phi(n)})\right)$ respectively.

(ii) If ψ is of the form a) $\neg \psi_1$, b) $\psi_1 \wedge \psi_2$ or

c) $\exists\beta\psi_1$, where $\beta \in X^\sigma$, $\sigma \in \kappa$, then $M \models_\nu \psi$ if

a) it is not the case that $M \models_\nu \psi_1$, b) $M \models_\nu \psi_1$, and

$M \models_{\nu} \psi_2$ or c) there exists an interpretation ν' which agrees with ν except possibly on β and such that $M \models_{\nu'} \psi_1$, respectively.

We assume the definition is extended to include the defined logical operators, disjunction, (material) implication and equivalence, and universal quantification in the usual manner.

It is a standard consequence of the above definitions for holding in M that the decision as to whether or not $M \models_{\nu} \psi$ depends only on the values ν takes on the free terms (that is a term not occurring in the scope of a quantifier involving the term) of ψ. That is if Y is the (finite) set of free terms of ψ then $M \models_{\nu} \psi$ if, and only if, for all M interpretations ν' such that $\nu|Y = \nu'|Y$, $M \models_{\nu'} \psi$.

A wff ψ is said to hold in M, written $M \models \psi$, if for all M-interpretations ν, $M \models_{\nu} \psi$.

A wff ψ is called a sentence of $L(\kappa(\alpha, \beta))$ if it contains no free variables. It is an immediate consequence of the holding definition above that if ψ is a sentence and $M \models_{\nu} \psi$ for one M-interpretation ν then $M \models_{\nu'} \psi$ for all M-interpretations ν'. Thus either a sentence or its negation, but not both, holds in any $\kappa(\alpha, \beta)$ structure M.

2. Higher order ultraproducts

In this section we first define the notion of a higher order ultraproduct associated with a family of higher order structures of the same similarity type. Frayne, Morel and Scott [1962], page 195, give a brief history of the development of the ultraproduct (prime reduced product) construction in the first order context. Luxemburg [1969], page 23-24, extends the definition to higher order structures in the context of an ultrapower.

We assume the language and basic properties of filters and ultrafilters as set out in Bell and Slomson [1969], Chapter I.

Let $\{M_i : i \in I\}$ be a family of $\kappa(\alpha, \beta)$ systems. For each $i \in I$,

$$M_i = \{E_i^\sigma : \sigma \in \kappa\} \cup \left\{\epsilon_i^\sigma : \sigma \in \kappa, \sigma \neq 0\right\} \cup \{f_{i,m} : m < \alpha\} \cup \left\{R_{i,n}^{\sigma_n} : n < \beta\right\} .$$

If F is an ultrafilter over the index set I (that is F is an ultrafilter of the subset algebra of the power set of I) then the ultraproduct of $\{M_i : i \in I\}$ with respect to F, written

$$\pi M_i/F = \{\overline{E}^\sigma : \sigma \in \kappa\} \cup \{\overline{\epsilon}^\sigma : \sigma \in \kappa, \sigma \neq 0\} \cup \{\overline{f}_m : m < \alpha\} \cup \left\{\overline{R}_n^{\sigma_n} : n < \beta\right\}$$

is defined as follows.

For each $\sigma \in \kappa$ let

$$P^\sigma = \left\{h^\sigma : h^\sigma : I \to \cup\left\{E_i^\sigma : i \in I\right\} \text{ and } h^\sigma(i) \in E_i^\sigma \text{, each } i \in I\right\} .$$

A binary relation $\sim$ is defined on P^σ by $h^\sigma \sim k^\sigma$ if, and only if, $\{i : h^{\sigma}(i) = k^{\sigma}(i)\} \in F$.

LEMMA 2.1. *For each* $\sigma \in \kappa$, $\sim$ *as defined above is an equivalence relation on* P^{σ} .

We now define, for each $\sigma \in \kappa$, $\overline{E}^{\sigma}$ as $P^{\sigma}/\sim$, that is the set of equivalence classes of P^{σ} with respect to $\sim$.

LEMMA 2.2. *Let* $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$. *If* $h, k \in P^{\sigma}$ *and* $h \sim k$ *and if* $h_j, k_j \in P^{\sigma_j}$ *and* $h_j \sim k_j$, *each* $1 \le j \le n$, *then* $\left\{i : (h_1(i), \ldots, h_n(i)) \epsilon_i^{\sigma} h(i)\right\} \in F$ *if, and only if* $\left\{i : (k_1(i), \ldots, k_n(i)) \epsilon_i^{\sigma} k(i)\right\} \in F$.

LEMMA 2.3. *If* $m < \alpha$ *and* $h_j, k_j \in P^0$, $h_j \sim k_j$, *for* $1 \le j \le \theta(m)$, *then*

$$\{i : f_m(h_1(i), \ldots, h_{\theta(m)}(i)) = f_m(k_1(i), \ldots, k_{\theta(m)}(i))\} \in F .$$

LEMMA 2.4. *If* $n < \alpha$, $\sigma_n = (\sigma_{n,1}, \ldots, \sigma_{n,\phi(n)})$ *and* $h^{\sigma_{n,j}}, k^{\sigma_{n,j}} \in P^{\sigma_{n,j}}$, $h^{\sigma_{n,j}} \sim k^{\sigma_{n,j}}$, *for* $1 \le j \le \phi(n)$, *then* $\left\{i : R_{i,n}^{\sigma_n}(h^{\sigma_{n,1}}(i), \ldots, h^{\sigma_{n,\phi(n)}}(i))\right\} \in F$ *if, and only if* $\left\{i : R_{i,n}^{\sigma_n}(k^{\sigma_{n,1}}(i), \ldots, k^{\sigma_{n,\phi(n)}}(i))\right\} \in F$.

The proofs of these lemmas follow the pattern for the similar lemmas in Bell and Slomson [1969], pages 87 to 89. We illustrate the pattern with the proof of Lemma 2.3.

Proof (Lemma 2.3). Let $F_j = \{i : h_j(i) = k_j(i)\}$, for $1 \le j \le \theta(m)$. Put $F_0 = F_1 \cap \ldots \cap F_{\theta(m)}$ and so $F_0 \in \mathsf{F}$ as $h_j \sim k_j$, for $1 \le j \le \theta(m)$. Now

$$\{i : f_m(h_1(i), \ldots, h_{\theta(m)}(i)) = f_m(k_1(i), \ldots, k_{\theta(m)}(i))\} \supseteq F_0$$

and so belongs to F . //

We now conclude the definition of $\pi M_i/\mathsf{F}$. Let $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$. $\overline{\epsilon}^{\sigma}$ is defined by: $(\overline{h}^{\sigma_1}, \ldots, \overline{h}^{\sigma_n}) \overline{\epsilon}^{\sigma} \overline{h}^{\sigma}$ if $\left\{i : (h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)) \epsilon_i^{\sigma} h^{\sigma}(i)\right\} \in \mathsf{F}$, where $\overline{h}^{\sigma_j}$ denotes the equivalence class of $h^{\sigma_j} \in P^{\sigma_j}$, each $1 \le j \le n$.

If $m < \alpha$ then $\overline{f}_m$ is defined by: $\overline{f}_m(\overline{h}_1, \ldots, \overline{h}_{\theta(m)}) = \overline{h}$ if $\{i : f_{i,m}(h_1(i), \ldots, h_{\theta(m)}(i)) = h(i)\} \in \mathsf{F}$, where $\overline{h}_1, \ldots, \overline{h}_{\theta(m)}$, $\overline{h} \in \overline{E}^0$.

If $n < \alpha$ then $\overline{R}_n^{\sigma_n}$ is defined by: $\overline{R}_n^{\sigma_n}(\overline{h}^{\sigma_{n,1}}, \ldots, \overline{h}^{\sigma_{n,\phi(n)}})$ if $\left\{i : R_{i,n}^{\sigma_n}(h^{\sigma_{n,1}}(i), \ldots, h^{\sigma_{n,\phi(n)}}(i))\right\} \in \mathsf{F}$, where $\overline{h}^{\sigma_{n,j}} \in \overline{E}^{\sigma_{n,j}}$, for $1 \le j \le \phi(n)$.

Lemmas 2.2 to 2.4 ensure that the definitions are well established. We note that $\pi M_i/\mathsf{F}$ as defined is a $\kappa(\alpha, \beta)$ structure. The following theorem ensures that the property of a structure being normal is preserved under the ultraproduct construction.

THEOREM 2.5. *If each member of* $\{M_i : i \in I\}$ *is normal then so* *is* $\pi M_i/F$.

Proof. Let $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$. Take $\overline{h}^\sigma, \overline{k}^\sigma \in \overline{E}^\sigma$. Put $G = \{i : h(i) = k(i)\}$ and assume $\overline{h}^\sigma = \overline{k}^\sigma$, that is $G \in F$. Now $(\overline{h}^{\sigma_1}, \ldots, \overline{h}^{\sigma_n}) \overline{\in}^\sigma \overline{h}^\sigma$ if and only if $H \in F$, where $H = \left\{i : (h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)) \in_i^\sigma h^\sigma(i)\right\}$. But each M_i is normal and so $K \supseteq G \cap H$, where $K = \left\{i : (h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)) \in_i^\sigma k^\sigma(i)\right\}$. Hence $K \in F$ and so $(\overline{h}^{\sigma_1}, \ldots, \overline{h}^{\sigma_n}) \overline{\in}^\sigma \overline{k}^\sigma$. Similarly $(\overline{h}^{\sigma_1}, \ldots, \overline{h}^{\sigma_n}) \overline{\in}^\sigma \overline{h}^\sigma$ if $(\overline{h}^{\sigma_1}, \ldots, \overline{h}^{\sigma_n}) \overline{\in}^\sigma \overline{k}^\sigma$. Thus if $\overline{h}^\sigma = \overline{k}^\sigma$ then $\hat{\overline{h}}^\sigma = \hat{\overline{k}}^\sigma$.

Conversely, assume $\overline{h}^\sigma \neq \overline{k}^\sigma$, that is $G \notin F$ and so $CG \in F$. As each M_i is normal there exists, for each $i \in CG$, $(a_{i,1}, \ldots, a_{i,n}) \in E_i^{\sigma_1} \times \ldots \times E_i^{\sigma_n}$ such that $(a_{i,1}, \ldots, a_{i,n})$ belongs to one, and only one, of $h^\sigma(i), k^\sigma(i)$. For each $i \in CG$ define $h^{\sigma_j}(i) = a_{i,j}$, for $1 \leq j \leq n$. Thus $\overline{h}^{\sigma_j}$, $1 \leq j \leq n$, are well defined as $CG \in F$. Let

$$H_0 = \left\{i : (h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)) \in_i^\sigma h^\sigma(i)\right\}$$

and

$$K_0 = \left\{i : (h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)) \in_i^\sigma k^\sigma(i)\right\} .$$

Now $(CG \cap H_0) \cup (CG \cap K_0) = CG$ and $(CG \cap H_0) \cap (CG \cap K_0) = \emptyset$.

Therefore one, and only one, of the H_0, K_0 belongs to F . Thus $\left(\bar{h}^{\sigma_1}, \ldots, \bar{h}^{\sigma_n}\right)$ belongs to one, and only one, of $\bar{h}^{\sigma}, \bar{k}^{\sigma}$. That is $\hat{\bar{h}}^{\sigma} \neq \hat{\bar{h}}^{\sigma}$. //

In fact the above theorem is a consequence of Theorem 3.4 below, as the property of a $\kappa(\alpha, \beta)$ structure M being normal can be expressed by the collection, Σ , of sentences given by

$$\forall x^{\sigma} \forall y^{\sigma} \left(\forall x^{\sigma_1} \ldots \forall x^{\sigma_n} \left(\left(x^{\sigma_1}, \ldots, x^{\sigma_n} \right) \underline{\epsilon}^{\sigma} x^{\sigma} \Leftrightarrow \left(x^{\sigma_1}, \ldots, x^{\sigma_n} \right) \underline{\epsilon}^{\sigma} y^{\sigma} \right) \Rightarrow x^{\sigma} = y^{\sigma} \right) ,$$

for all $\sigma \in \kappa$, $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$.

Anticipating the statement of Corollary 3.5 and consequent to it we are able to assert if $M_i \models \Sigma$, (that is each member of Σ holds in M_i), each $i \in I$, then $\pi M_i/F \models \Sigma$. That is if each M_i , $i \in I$, is normal (in fact it only requires enough of the M_i's to make up a set of the filter F to be normal) then $\pi M_i/F$ is normal. It must be noted that this deduction requires that the identity relation on the M_i's transfers by the defining construction to the identity relation on the ultraproduct, $\pi M_i/F$, and not just to an equivalence relation with the substitution property. That this is so can be immediately checked from the definitions. It should also be noted that the definition given by Luxemburg, [1969], pages 23-24, for higher order ultrapowers does not in fact preserve the identity relation as such nor the property of being normal. Further discussion of these matters is contained in Appendix II.

The detail of the proof given above for Theorem 2.5 clearly illustrates the essential role of the 'ultra' property of the filter F (that is for any subset of I either it or its complement belongs to F) in the preservation of normality in the ultraproduct construction. The presence in general in a filter over I of a subset G of I such that neither G nor its complement belong to the filter enables one to construct counterexamples to the assertion that the direct product and reduced product constructions preserve normality.

Finally in this consideration of normality we comment that while in the case of first order structures it is possible to normalise a structure via a quotient substructure construction it is not apparent to the author how this can in general be achieved for higher order structures.

The final three theorems of this section discuss the fullness of the ultraproduct of a family of full structures.

For the statement of these three theorems $\{M_i : i \in I\}$ is a family of full $\kappa(\alpha, \beta)$ structures, F an ultrafilter over I and $\pi M_i/F$ the resulting ultraproduct.

THEOREM 2.6. *If* $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, *then for each subclass* K *of* $\overline{E}^{\sigma_1} \times \ldots \times \overline{E}^{\sigma_n}$ *there exists some* $\overline{h}^{\sigma} \in \overline{E}^{\sigma}$ *such that* $K \subseteq \hat{\overline{h}}^{\sigma}$.

Proof. For each $i \in I$, put

$K_i = \left\{\left(h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)\right) : \left(\bar{h}^{\sigma_1}, \ldots, \bar{h}^{\sigma_n}\right) \in K\right\}$. But each M_i is full and so there exists some object $a_i^{\sigma} \in E_i^{\sigma}$ such that $\hat{a}_i^{\sigma} = K_i$. Define $\bar{h}^{\sigma} \in \bar{E}^{\sigma}$ by: $h^{\sigma}(i) = a_i^{\sigma}$, for each $i \in I$.

Take any $\left(\bar{k}^{\sigma_1}, \ldots, \bar{k}^{\sigma_n}\right) \in K$. Hence

$\left\{i : \left(k^{\sigma_1}(i), \ldots, k^{\sigma_n}(i)\right) \epsilon^{\sigma} h^{\sigma}(i)\right\} = I$. But $I \in F$ and so

$\left(\bar{k}^{\sigma_1}, \ldots, \bar{k}^{\sigma_n}\right) \bar{\epsilon}^{\sigma} \bar{h}^{\sigma}$. That is $K \subseteq \hat{\bar{h}}^{\sigma}$. //

We recall that an ultrafilter F is defined as δ-incomplete, δ some cardinal if there exists $F_j \in F$, $j < \delta$, such that $\cap\{F_j : j < \delta\} \notin F$. (*c.f.* Bell and Slomson [1969], page 111.)

THEOREM 2.7. *If* $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, *and if* K *is a subset of* $\bar{E}^{\sigma_1} \times \ldots \times \bar{E}^{\sigma_n}$ *such that* $|K| = \delta$, *(that is the cardinality of* K *is* δ *), then there exists no* $\bar{h}^{\sigma} \in \bar{E}^{\sigma}$ *such that* $\hat{\bar{h}}^{\sigma} = K$ *only if* F *is* δ*-incomplete.*

Proof. Let the members of K be indexed by δ, that is $K = \left\{\left(\bar{g}_j^{\sigma_1}, \ldots, \bar{g}_j^{\sigma_n}\right) : j < \delta\right\}$. Further, for each $j < \delta$, let $g_j^{\sigma_1}, \ldots, g_j^{\sigma_n}$ be arbitrary but fixed representations of $\bar{g}_j^{\sigma_1}, \ldots, \bar{g}_j^{\sigma_n}$ respectively. Put $K_i = \left\{\left(g_j^{\sigma_1}(i), \ldots, g_j^{\sigma_n}(i)\right) : j < \delta\right\}$ for each $i \in I$.

Let $\overline{h}^\sigma \in \overline{E}^\sigma$ be defined, as in the proof of Theorem 2.6, such that $h^\sigma(i) = a_i^\sigma$, where $\hat{a}_i^\sigma = K_i$, each $i \in I$. Thus $K \subseteq \overline{h}^\sigma$.

Assume there exists no $\overline{g}^\sigma \in \overline{E}^\sigma$ such that $\overline{g}^\sigma = K$ and so there exists some $\left(\overline{h}^{\sigma_1}, \ldots, \overline{h}^{\sigma_n}\right) \in \overline{E}^{\sigma_1} \times \ldots \times \overline{E}^{\sigma_n}$ such that $\left(\overline{h}^{\sigma_1}, \ldots, \overline{h}^{\sigma_n}\right) \overline{\epsilon}^\sigma \overline{h}^\sigma$ but is not a member of K.

Let $F = \left\{i : \left(h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)\right) \epsilon_i^\sigma h^\sigma(i)\right\}$ and so $F \in \mathsf{F}$. Put $F_j = F \cap \left\{i : \left(h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)\right) = \left(g_j^{\sigma_1}(i), \ldots, g_j^{\sigma_n}(i)\right)\right\}$ for each $j < \delta$. Now $F_j \notin \mathsf{F}$, $j < \delta$, as $\left(\overline{h}^{\sigma_1}, \ldots, \overline{h}^{\sigma_n}\right) \notin K$. Further, $\bigcup\{F_j : j < \delta\} = F$ as, for all $i \in I$, $h^\sigma(i) = a_i^\sigma$, where $\hat{a}_i^\sigma = K_i$, and the K_i have been defined using only the fixed representations of the members of K. Therefore $\bigcap\{CF_j : j < \delta\} \cap F = \emptyset$ and thus F is δ-incomplete. //

COROLLARY 2.8. *If* K *is as in the statement of Theorem* 2.7 *and* δ *is finite then there exists some* $\overline{h}^\sigma \in \overline{E}^\sigma$ *such that* $\overline{h}^\sigma = K$.

Proof. Every ultra filter F is δ-complete (that is not δ-incomplete) for δ finite. //

The final theorem of the section is a compromise with the failure to be able to establish whether or not the incompleteness of the ultrafilter F will guarantee the non-fullness of the ultraproduct. The complicated conditions can best be understood by seeing their place in the detail of the proof.

THEOREM 2.9. *If* F *is* δ*-incomplete and if for*
$\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, *there exists some* $K \subseteq \overline{E}^{\sigma_1} \times \ldots \times \overline{E}^{\sigma_n}$, *say* $K = \left\{\left(\overline{g}_j^{\sigma_1}, \ldots, \overline{g}_j^{\sigma_n}\right) : j < \gamma\right\}$ *where* $\delta \leq \gamma$, *such that* $G \in$ F, *where* $G = \cap\{CF_{m,p} : m, p < \delta, m \neq p\}$ *and*

$$F_{m,p} = \left\{i ; \left(g_m^{\sigma_1}(i), \ldots, g_m^{\sigma_n}(i)\right) = \left(g_p^{\sigma_1}(i), \ldots, g_p^{\sigma_n}(i)\right)\right., \text{ all}$$

$m, p < \delta$, $m \neq p$, *then there exists no* $\overline{h}^{\sigma} \in \overline{E}^{\sigma}$ *such that* $\hat{\overline{h}}^{\sigma} = K$.

Proof. As F is δ-incomplete let $\{H_k : k < \delta\}$ be a family of members of F such that $\cap\{H_k : k < \delta\} = \emptyset$. Assume there exists $\overline{h}^{\sigma} \in \overline{E}^{\sigma}$ such that $\hat{\overline{h}}^{\sigma} = K$. Thus, for each $j < \gamma$,

$\left(\overline{g}^{\sigma_1}, \ldots, \overline{g}^{\sigma_n}\right) \overline{\in}^{\sigma} \overline{h}^{\sigma}$.

For each $j < \delta$ put $G_j = \left\{i : \left(g_j^{\sigma_1}(i), \ldots, g_j^{\sigma_n}(i)\right) \in_i^{\sigma} h^{\sigma}(i)\right\}$, put $G_j' = G_j \cap G$ and put $H_j' = G_j' \cap H_j$. Thus $G_j', H_j' \in$ F and $\cup\{C'H_j' : j < \delta\} = G$, where $C'H_j' = G \cap CH_j'$.

Now define $\left(\overline{h}^{\sigma_1}, \ldots, \overline{h}^{\sigma_n}\right) \in \overline{E}^{\sigma_1} \times \ldots \times \overline{E}^{\sigma_n}$ as follows. For all $i \in C'H_0'$ put $\left(h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)\right) = \left(g_0^{\sigma_1}(i), \ldots, g_0^{\sigma_n}(i)\right)$.

Assume $\left(h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)\right)$ has been defined for all $i \in \cup\{C'H_j' : j < \mu\}$, for some $\mu < \delta$, and define

$\left(h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)\right) = \left(g_\mu^{\sigma_1}(i), \ldots, g_\mu^{\sigma_n}(i)\right)$, for all $i \in \bigcap\{H'_j : j < \mu\} - H'_\mu$. By transfinite induction $\left(h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)\right)$ is defined for all $i \in G$, as $\bigcup\{C'H'_j ; j < \delta\} = G$. Hence $\left(\bar{h}^{\sigma_1}, \ldots, \bar{h}^{\sigma_n}\right)$ is well defined, as $G \in \mathcal{F}$.

But $\left(\bar{g}_j^{\sigma_1}, \ldots, \bar{g}_j^{\sigma_n}\right) \neq \left(\bar{h}^{\sigma_1}, \ldots, \bar{h}^{\sigma_n}\right)$, for any $j < \delta$, as

$$\left\{i : \left(g_j^{\sigma_1}(i), \ldots, g_j^{\sigma_n}(i)\right) = \left(h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)\right)\right\} \cap G$$
$$= \bigcap\{H'_k : k < j\} \cap CH'_j ,$$

and $CH'_j \notin \mathcal{F}$. Hence $\left(\bar{h}^{\sigma_1}, \ldots, \bar{h}^{\sigma_n}\right) \notin K$. But $\left(\bar{h}^{\sigma_1}, \ldots, \bar{h}^{\sigma_n}\right) \bar{\epsilon}^{\sigma} \bar{h}^{\sigma}$ as $\left\{i : \left(h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)\right) \epsilon_i^{\sigma} h^{\sigma}(i)\right\} \supseteq G$. This contradicts the assumption that $\hat{\bar{h}}^{\sigma} = K$ and hence establishes the theorem. //

3. Model-theoretic properties of higher order ultraproducts

We prepare the way for the main results of this section with some lemmas relating interpretations of the ultraproduct with those of its components.

Let $\{M_i : i \in I\}$ be a family of $\kappa(\alpha, \beta)$ structures and $\mathcal{F}$ any ultrafilter over I .

If for each $i \in F$, $F \in \mathcal{F}$, ν_i is a given M_i-interpretation then $\bar{\nu}$ is a $\pi M_i/\mathcal{F}$-interpretation defined as follows: For each

$\sigma \in \kappa$ and $t^\sigma \in X^\sigma$, put $\bar{\nu}(t^\sigma) = \bar{h}^\sigma$, where $h^\sigma(i) = \nu_i(t^\sigma)$, for all $i \in F$; for each $\sigma \in \kappa$, $\sigma \neq 0$, put $\bar{\nu}(\underline{\epsilon}^\sigma) = \bar{\epsilon}^\sigma$; for each $m < \alpha$, put $\bar{\nu}(\underline{f}_m) = \bar{f}_m$; for each $n < \beta$ put

$\bar{\nu}\left(\underline{R}_n^{\sigma_n}\right) = \bar{R}_n^{\sigma_n}$.

$\bar{\nu}$ is extended over all members of Z^0 by the standard inductive procedure, viz., if $t^0 = \underline{f}_m\left(t_1^0, \ldots, t_{\theta(m)}^0\right)$ then $\bar{\nu}(t^0) = \bar{f}_m\left(\bar{\nu}\left(t_1^0\right), \ldots, \bar{\nu}\left(t_{\theta(m)}^0\right)\right)$, $\bar{\nu}\left(t_j^0\right)$ being already defined for each $1 \leq j \leq \theta(m)$.

LEMMA 3.1. *If* $\bar{\nu}$ *and* $\{\nu_i : i \in F\}$ *are as defined above and if* $t^0 \in Z^0 - T^0$ *then* $\bar{\nu}(t^0) = \bar{h}^0$, *where* $h^0(i) = \nu_i(t^0)$, *each* $i \in F$.

Proof. A proof may proceed by induction on the basis of Lemma 1.7. Take $t_0 = \underline{f}_m\left(t_1^0, \ldots, t_{\theta(m)}^0\right)$ and assume the result of the lemma for $t_1^0, \ldots, t_{\theta(m)}^0$. $\bar{\nu}(t_0) = \bar{f}_m\left(\bar{\nu}\left(t_1^0\right), \ldots, \bar{\nu}\left(t_{\theta(m)}^0\right)\right)$ and hence, by the assumption and the definition of $\bar{f}_m$, we have $\bar{\nu}(t_0) = \bar{h}^0$, where $h^0(i) = f_{i,m}\left(\nu_i\left(t_1^0\right), \ldots, \nu_i\left(t_{\theta(m)}^0\right)\right)$, each $i \in F$. But $f_{i,m}\left(\nu_i\left(t_1^0\right), \ldots, \nu_i\left(t_{\theta(m)}^0\right)\right) = \nu_i\left(\underline{f}_m\left(t_1^0, \ldots, t_{\theta(m)}^0\right)\right)$, each $i \in F$. That is $h^0(i) = \nu_i(t^0)$, each $i \in F$. //

LEMMA 3.2. *If* $\bar{\nu}$ *is defined as above from*

$\{\nu_i : i \in F_1, F_1 \in \mathsf{F}\}$ and $\bar{\nu}'$ is defined similarly from $\{\nu_i : i \in F_2, F_2 \in \mathsf{F}\}$ then $\bar{\nu}|Y = \bar{\nu}'|Y$ if $\{i : \nu_i|Y = \nu_i'|Y\} \in \mathsf{F}$, for any set of variables Y.

Proof. For any $t^\sigma \in Y$, $\sigma \in \kappa$, put $\bar{\nu}(t^\sigma) = \bar{h}^\sigma$ and $\bar{\nu}'(t^\sigma) = \bar{h}'^\sigma$, where $h^\sigma(i) = \nu_i(t^\sigma)$, each $i \in F_1$, and $h'^\sigma(i) = \nu_i'(t^\sigma)$, each $i \in F_2$. Hence $\{i : h^\sigma(i) = h'^\sigma(i)\} \supseteq F_1 \cap F_2$ and so $\bar{\nu}(t^\sigma) = \bar{\nu}'(t^\sigma)$. That is $\bar{\nu}|Y = \bar{\nu}'|Y$. //

Now consider a given $\pi M_i/\mathsf{F}$ interpretation $\bar{\mu}$. For each $\sigma \in \kappa$ and $x^\sigma \in X^\sigma$ let h^σ be an arbitarily chosen representative of $\bar{\mu}(x^\sigma)$. With respect to this chosen family of representatives and for each $i \in I$ define an M_i interpretation μ_i as follows: For each $\sigma \in \kappa$ and $x^\sigma \in X^\sigma$ put $\mu_i(x^\sigma) = h^\sigma(i)$ and then extend the definition of μ_i in the standard manner so that it is a well defined M_i interpretation.

LEMMA 3.3. *Let $\bar{\mu}$ be a given $\pi M_i/\mathsf{F}$ interpretation and $\{\mu_i : i \in I\}$, $\{\mu_i' : i \in I\}$ two families of M_i interpretations, $i \in I$, defined via $\bar{\mu}$ as above but with respect to different families of representatives of the $\bar{\mu}(x^\sigma)$, $\sigma \in \kappa$, $x^\sigma \in X^\sigma$. If Y is a finite set of variable symbols of $L(\kappa(\alpha, \beta))$ then $\{i : \mu_i|Y = \mu_i'|Y\} \in \mathsf{F}$.*

Proof. Let $Y = \left\{t^{\sigma_1}, \ldots, t^{\sigma_m}\right\}$. For each $1 \le j \le m$, let h_j be the representative of $\overline{\mu}\left(t^{\sigma_j}\right)$ from which the μ_i's are constructed and h'_t the representative of $\overline{\mu}\left(t^{\sigma_j}\right)$ from which the μ'_i 's are constructed. Put $F_j = \{i : h_j(i) = h'_j(i)\}$, each $1 \le j \le m$ and $F = \cap\{F_j : 1 \le j \le m\}$. Hence $F \in \mathrm{F}$ and so $\{i : \mu_i|Y = \mu'_i|Y\} \in \mathrm{F}$. //

If for each $i \in G$, where $G \subseteq I$, ν_i is an M_i interpretation, then $\overline{\nu}$, an $\pi M_i/\mathrm{F}$ interpretation, and $\{\nu_i : i \in G\}$ are said to be compatible with respect to Y, a set of variable symbols, if $G \in \mathrm{F}$ and for each $t \in Y$, $\overline{\nu}(t) = \overline{h}$, where $h(i) = \nu_i(t)$, each $i \in G$.

We next state and prove the fundamental model-theoretic property for higher order ultraproducts. Kochen, [1961], p. 221, notes that the theorem for first order ultraproducts is implicit in Łoš [1955], (Bell and Slomson, [1969], page 90, name it Łoš's Theorem,) and proves the theorem by a method he attributes to Scott. We follow the pattern of this proof.

THEOREM 3.4. *For each wff* ψ *of* $L(\kappa(\alpha, \beta))$, $\pi M_i/\mathrm{F} \models_{\overline{\nu}} \psi$ *if, and only if,* $\left\{i : M_i \models_{\nu_i} \psi\right\} \in \mathrm{F}$, *where* $\overline{\nu}$ *and* $\{\nu_i : i \in G\}$ *are any compatible collection of interpretations with respect to the set of free variables of* ψ.

Proof. The proof proceeds by induction based on Lemma 1.8.

a) Let ψ be of the form $t_1^{\sigma} = t_2^{\sigma}$, where $t_1^{\sigma}, t_2^{\sigma} \in Z^{\sigma}$,

$\sigma \in \kappa$.

If $\pi M_i/F \models_{\bar{\nu}} \psi$ then $\bar{\nu}\left(t_1^{\sigma}\right) = \bar{\nu}\left(t_2^{\sigma}\right)$ and so $\left\{i : \nu_i\left(t_1^{\sigma}\right) = \nu_i\left(t_2^{\sigma}\right)\right\} \in F$, that is $\left\{i : M_i \models_{\nu_i} \psi\right\} \in F$. The converse is proved by reversing the argument.

b) Let ψ be of the form $\left(\alpha^{\sigma_1}, \ldots, \alpha^{\sigma_n}\right) \underline{\epsilon}^{\sigma} \alpha^{\sigma}$, where $\sigma \in \kappa$, $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$ and $\alpha^{\sigma_j} \in Z^{\sigma_j}$, each $1 \leq j \leq n$, and $\alpha^{\sigma} \in Z^{\sigma}$.

If $\pi M_i/F \models_{\bar{\nu}} \psi$ then $\left(\bar{\nu}\left(\alpha^{\sigma_1}\right), \ldots, \bar{\nu}\left(\alpha^{\sigma_n}\right)\right) \bar{\epsilon}^{\sigma} \bar{\nu}\left(\alpha^{\sigma}\right)$ and hence $\left\{i : \left(\nu_i\left(\alpha^{\sigma_1}\right), \ldots, \nu_i\left(\alpha^{\sigma_i}\right)\right) \epsilon_i^{\sigma} \nu_i\left(\alpha^{\sigma}\right)\right\} \in F$, that is $\left\{i : M_i \models_{\nu_i} \psi\right\} \in F$. Again the reversal of the argument proves the converse.

c) Let ψ be of the form $\underline{R}_n^{\sigma_n}\left(\alpha^{\sigma_{n,1}}, \ldots, \alpha^{\sigma_{n,\phi(n)}}\right)$, some $n < \beta$. If $\pi M_i/F \models_{\bar{\nu}} \psi$ then $\bar{R}_n^{\sigma_n}\left(\bar{\nu}\left(\alpha^{\sigma_{n,1}}\right), \ldots, \bar{\nu}\left(\alpha^{\sigma_{n,\phi(n)}}\right)\right)$ and so $\left\{i : R_{i,n}^{\sigma_n}\left(\nu_i\left(\alpha^{\sigma_{n,1}}\right), \ldots, \nu_i\left(\alpha^{\sigma_{n,\phi(n)}}\right)\right)\right\} \in F$, that is $\left\{i : M_i \models_{\nu_i} \psi\right\} \in F$. Again reversing the argument establishes the converse.

d) Let ψ be of the form $\neg \psi_1$ and assume the theorem is true for ψ_1 .

We have $\pi M_i/F \models_{\bar{\nu}} \psi$ if, and only if, it is not the case that $\pi M_i/ \models_{\nu_i} \psi_1$, that is, by assumption, if, and only if, $\left\{i : M_i \models_{\nu_i} \psi_1\right\} \notin F$, that is if, and only if, $\{i : M_i \models_{\nu_i} \psi\} \in F$.

e) Let ψ be of the form $\psi_1 \wedge \psi_2$ and assume the theorem is true for ψ_1 and ψ_2 .

We have $\pi M_i/F \models_{\bar{\nu}} \psi$ if, and only if, $\pi M_i/F \models_{\bar{\nu}} \psi_1$ and $\pi M_i/F \models_{\bar{\nu}} \psi_2$, that is if, and only if, $\left\{i : M_i \models_{\nu_i} \psi_1\right\} \in F$ and $\left\{i : M_i \models_{\nu_i} \psi_2\right\} \in F$, that is if, and only if, $\left\{i : M_i \models_{\nu_i} \psi\right\} \in F$.

f) Let ψ be of the form $\exists\gamma\psi_1$ where $\gamma \in X^\sigma$, $\sigma \in \kappa$, and the theorem is assumed for ψ_1 .

Assume $\pi M_i/F \models_{\bar{\nu}} \exists\gamma\psi_1$ and so there exists a $\pi M_i/F$ interpretation $\bar{\nu}'$ which agrees with $\bar{\nu}$ except possible on γ and such that $\pi M_i/F \models_{\bar{\nu}'} \psi_1$. Thus, by assumption, $\left\{i : M_i \models_{\nu_i'} \psi_1\right\} \in F$, where $\nu_i'(t) = \nu_i(t)$, each $t \in Z$, $t \neq \gamma$ and $\nu_i'(\gamma) = h^\sigma(i)$, where $\bar{\nu}'(\gamma) = \bar{h}^\sigma$, each $i \in G$. Hence $\left\{i : M_i \models_{\nu_i} \psi\right\} \in F$.

Conversely, assume $H = \left\{i : M_i \models_{\nu_i} \psi\right\} \in F$. Thus for each $i \in H$ there exists an M_i interpretation ν_i' which agrees with ν_i except possibly on γ and such that $M_i \models_{\nu_i'} \psi_1$. Define $\bar{\nu}'$ in terms of $\{\nu_i' : i \in H\}$ and so $\bar{\nu}'$ agrees with $\bar{\nu}$ except possibly on

γ . Further, $\overline{\nu}'$ and $\{\nu_i' : i \in F\}$ are compatible with respect to the free terms of ψ_1 , and so, by assumption, we have $\pi M_i/F \models_{\overline{\nu}'} \psi_1$. That is $\pi M_i/F \models_{\overline{\nu}} \psi$.

The theorem is thus established. //

The following corollary is a consequence of the theorem.

COROLLARY 3.5. *If* ψ *is a sentence of* $L(\kappa(\alpha, \beta))$ *then* $\pi M_i/F \models \psi$ *if, and only if,* $\{i : M_i \models \psi\} \in F$.

Proof. Assume $\pi M_i/F \models \psi$. Thus if $\overline{\nu}$ is any $\pi M_i/F$ interpretation then $\pi M_i/F \models_{\overline{\nu}} \psi$. Let $\{\nu_i : i \in I\}$ be a family of M_i interpretations, each $i \in I$, formed from $\overline{\nu}$ and so compatible with it. Hence from Theorem 3.4 we have $\left\{i : M_i \models_{\nu_i} \psi\right\} \in F$ and so, as ψ is a sentence, $\{i : M_i \models \psi\} \in F$. The converse follows by a similar argument. //

The completeness and compactness (finiteness principle) results for higher order model theory were first proved by Henkin [1950] using his method of construction of models via maximal consistent sets. We shall concern ourselves only with the compactness result although it is of interest to note that the method of proof of the completeness theorem for the first order case given by Rasiowa and Sikorski, [1951], and expounded in Bell and Slomson, [1969], pages 62 to 64, carries over to the higher order theory.

Morel, Scott and Tarski [1958], first used the ultraproduct construction explicitly to establish the first order compactness theorem. A. Robinson, [1966], Theorem 2.8.1, page 27, proves the higher order result by a translation of higher order formulae to related first order formulae and application of the first order compactness theorem. Young [1969], page 29, sets out a proof by direct use of higher order ultraproducts. We follow this method.

THEOREM 3.6. *Σ is a set of sentences of $L(\kappa(\alpha, \beta))$. If each finite subset of Σ has a (normal) model then so has Σ.*

Proof. Let $I = \{i : \Delta_i$ is a finite subset of $\Sigma\}$. Put $S = \{F_i : i \in I\}$, where $F_i = \{j : j \in I$ and $\Delta_j \geq \Delta_i\}$, each $i \in I$. S has the finite intersection property and thus can serve as a sub-basis for a filter over I which can then be extended (by appeal to the Prime Ideal Theorem, *c.f.* Grätzer, [1968], p. 27) to an ultrafilter F.

For each $i \in I$ let M_i be the $\kappa(\alpha, \beta)$ structure such that $M_i \models \Delta_i$. By Theorem 2.5, $\pi M_i/F$ is a normal $\kappa(\alpha, \beta)$ structure. Further, if $\psi \in \Sigma$ then $\{i : M_i \models \psi\} \supseteq F_k$, where $\Delta_k = \{\psi\}$. But $F_k \in F$ and so, by Corollary 3.5, $\pi M_i/F \models \psi$. Thus $\pi M_i/F$ is a $\kappa(\alpha, \beta)$ model of Σ. //

Finally in this section, we consider two kinds of extension to the class of well-formed formulae of $L(\kappa(\alpha, \beta))$. The first extension is somewhat unusual in that it is done not in terms of syntactical conditions but semantical ones and these with respect to a given family of $\kappa(\alpha, \beta)$ structures and an associated ultrafilter over the index set of the family. In fact the extension was developed for the purpose of later application in Chapter III, section 2, to the work on Sylow p-subgroups. A sentence expressing the fact that a group is a p-group was thought by the author to be of the kind to be defined. It proved (after the paper containing it had been accepted for publication) not to be so and the application lapsed except for a trivial vestige. The extension therefore seems to lack real purpose and so may be omitted from the reading if wished.

Let W denote the set of wff's of $L(\kappa(\alpha, \beta))$. Take $\{M_i : i \in I\}$ a family of $\kappa(\alpha, \beta)$ structures and F a given ultrafilter over I.

We first define an extended set of wff's, denoted by $W_0(\pi M_i/F)$, as follows:

(i) $W \subseteq W_0(\pi M_i/F)$;

(ii) If $\{\phi_t : t < \tau\}$, τ any non-finite cardinal, is a set of members of W such that

a) only a finite number of variables occur free in $\{\phi_t : t < \tau\}$,

b) for each $\pi M_i/F$ interpretation $\overline{\nu}$ there exists a compatible set (with respect to the free variables in the ϕ_i's) of component interpretations $\{\nu_i : i \in I\}$ such that, for all $k < \tau$, if there exists some $j \in I$ such that $M_j \models_{\nu_j} \phi_k$ then $\{i : M_i \models_{\nu_i} \phi_k\} \in F$,

then the infinite disjunction $\vee\{\phi_t : t < \tau\}$ is a member of $W_0(\pi M_i/F)$.

The set, $W(\pi M_i/F)$, is formed by the rules of formation of $L(\kappa(\alpha, \beta))$ but with the members of $W_0(\pi M_i/F)$ replacing the atomic formulae.

We extend Theorem 3.4 to all members of $W(\pi M_i/F)$.

THEOREM 3.7. *For each* $\psi \in W(\pi M_i/F)$, $\pi M_i/F \models_{\bar{\nu}} \psi$ *if, and only if,* $\left\{i : M_i \models_{\nu_i} \psi\right\} \in F$, *where* $\bar{\nu}$ *and* $\{\nu_i : i \in G\}$ *are any compatible collection of interpretations with respect to the free variables in* ψ .

Proof. In view of the inductive procedures of the proof of Theorem 3.4 it is necessary only to consider the case where ψ is of the form $\vee\{\phi_t : t < \tau\}$ as described above.

Assume that $\pi M_i/F \models_{\bar{\nu}} \vee\{\phi_t : t < \tau\}$. By the semantical rules for a disjunction there exists some $k < \tau$ such that $\pi M_i/F \models_{\bar{\nu}} \phi_k$. Hence, by Theorem 3.4, as $\phi_k \in W$, we have $\left\{i : M_i \models_{\nu_i} \phi_k\right\} \in F$ and so $\left\{i : M_i \models_{\nu_i} \vee\{\phi_t : t < \tau\}\right\} \in F$.

Conversely, assume $\left\{i : M_i \models_{\nu_i} \vee\{\phi_t : t < \tau\}\right\} \in F$. Let $\{\nu_i' : i \in I\}$ be the set of M_i interpretations, $i \in I$, compatible with $\bar{\nu}$ with respect to Y the set of free variables of the ϕ_t's , $t < \tau$, and having the property (ii), b) above establishing

$\vee\{\phi_t : t < \tau\}$ a member of $W_0(\pi M_i/F)$. Hence, by Lemma 3.3, $\{i : \nu_i'|Y = \nu_i|Y\} \in F$ and so $\left\{i : M_i \models_{\nu_i'} \vee\{\phi_t : t < \tau\}\right\} = F \in F$.

Take some $j \in F$ and so there exists some $k < \tau$ such that $M_j \models_{\nu_j'} \phi_k$. Hence $\left\{i : M_i \models_{\nu_i'} \phi_k\right\} \in F$, as $\vee\{\phi_t : t < \tau\} \in W_0(\pi M_i/F)$, and so, by Theorem 3.4, $\pi M_i/F \models_{\bar{\nu}} \phi_k$. That is $\pi M_i/F \models_{\bar{\nu}} \vee\{\phi_t : t < \tau\}$. //

COROLLARY 3.8. *If* $\psi \in W(\pi M_i/F)$ *is a sentence then* $\pi M_i/F \models \psi$ *if, and only if,* $\{i : M_i \models \psi\} \in F$.

Proof. As for Corollary 3.5. //

We further extend the class of formulae by introducing infinite conjunctions in the following manner. (For this extension application can be found in the context of Mal'cev's Interior Local Theorem as discussed in section 3 of Chapter III.)

We first form a set of formulae, W_0', by:

(i) $W \subseteq W_0'$;

(ii) If $\{\theta_t : t < \tau\}$, τ any non-finite cardinal, is a set of members of W such that only a finite number of variables occur free in $\{\theta_t : t < \tau\}$ then the infinite conjunction $\wedge\{\theta_t : t < \tau\}$ is a member of W_0'.

The set W' is now formed by the rules of formation of $L(\kappa(\alpha, \beta))$ but with the members of W_0' replacing the atomic formulae and with the restriction that the negation rule (that is, if A is a well formed formula then so is $\neg A$) may only be applied to members of W in the formulation of W' from W'_0 .

We now have the following partial extension of Theorems 3.4 and 3.7.

THEOREM 3.9. *If* $\psi \in W'$ *then* $\pi M_i/F \models_{\bar{\nu}} \psi$ *if* $\left\{i : M_i \models_{\nu_i} \psi\right\} \in F$ *, where* $\bar{\nu}$ *,* $\{\nu_i : i \in I\}$ *are compatible with respect to the set of free variables of* ψ *.*

Proof. In view of the proof procedure of Theorem 3.4 the following lemma provides the necessary addition for a proof of Theorem 3.9. //

LEMMA 3.10. *If* $\wedge\{\theta_t : t < \tau\} \in W'$ *then* $\pi M_i/F \models_{\bar{\nu}} \wedge\{\theta_t : t < \tau\}$ *if* $\left\{i : M_i \models_{\nu_i} \wedge\{\theta_t : t < \tau\}\right\} \in F$.

Proof. If $\left\{i : M_i \models_{\nu_i} \wedge\{\theta_t : t < \tau\}\right\} \in F$ then for each $t < \tau$ we have $\left\{i : M_i \models_{\nu_i} \theta_t\right\} \in F$ and so, by Theorem 3.4, $\pi M_i/F \models_{\bar{\nu}} \theta_t$. Hence the result. //

COROLLARY 3.11. *If* $\psi \in W'$ *and* ψ *is a sentence then* $\pi M_i/F \models \psi$ *if* $\{i \quad M_i \models \psi\} \in F$.

CHAPTER II

SUBSTRUCTURES AND EMBEDDING THEOREMS

Summary. In section 1 we define a notion of substructure for higher order structures and develop a variety of consequential properties. In section 2 we use the concept of a local family of substructures of a given structure M to set up a number of theorems related to the embedding of M into a particular ultraproduct of the local family of substructures. At the conclusion of section 2 we briefly introduce the notion of an 'inverse limit' of the local family.

1. Substructures of higher order structures

We shall first set out a formal definition of the notion of substructure and then illustrate the definition in a simple example and comment on possible alternative definitions, one available in the literature - *c.f.* Kreisel and Krivine [1967], page 100.

Let M, N be (normal) $\kappa(\alpha, \beta)$ structures given by

$$M = \{E^{\sigma} : \sigma \in \kappa\} \cup \{\epsilon^{\sigma} : \sigma \in \kappa, \sigma \neq 0\} \cup \{f_m : m < \alpha\} \cup \left\{R_n^{\sigma_n} : n < \beta\right\} ,$$

$$N = \{F^{\sigma} : \sigma \in \kappa\} \cup \{\epsilon^{\sigma} : \sigma \in \kappa, \sigma \neq 0\} \cup \{g_m : m < \alpha\} \cup \left\{S_n^{\sigma_n} : n < \beta\right\} .$$

N is termed a *substructure* of M if

(i) $F^0 \subseteq E^0$ and for each $m < \alpha$, g_m is f_m restricted to F^0 , (which requires that F^0 is closed under the operations of M);

(ii) p^0 is a partial map from E^0 onto F^0 , with domain F^0

and such that $p^0 | F^0$ is the identity map on F^0 ;

(Note: We shall use the convention that writing $p^0(a^0)$ for any $a^0 \in E^0$ implies that a^0 is already given as a member of F^0 .)

(iii) for each $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, there exists a map p^σ from E^σ onto F^σ such that

a) for each $a^{\sigma_1}, \ldots, a^{\sigma_n}, a^{\sigma}$ $\left(p^{\sigma_1}\left(a^{\sigma_1}\right), \ldots, p^{\sigma_n}\left(a^{\sigma_n}\right)\right) \in^\sigma p^\sigma(a^\sigma)$, *iff* there exists some $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \in^\sigma a^\sigma$ such that for each $1 \leq j \leq n$, $p^{\sigma_j}\left(b^{\sigma_j}\right) = p^{\sigma_j}\left(a^{\sigma_j}\right)$,

b) for each $n < \beta$, $\sigma_n = (\sigma_{n,1}, \ldots, \sigma_{n,\phi(n)})$, $S_n^{\sigma_n}\left(p^{\sigma_{n,1}}\left(a^{\sigma_{n,1}}\right), \ldots, p^{\sigma_{n,\phi(n)}}\left(a^{\sigma_{n,\phi(n)}}\right)\right)$ *iff* there exists some $\left(b^{\sigma_{n,1}}, \ldots, b^{\sigma_{n,\phi(n)}}\right)$ such that $R_n^{\sigma_n}\left(b^{\sigma_{n,1}}, \ldots, b^{\sigma_{n,\phi(n)}}\right)$ and $p^{\sigma_{n,j}}\left(a^{\sigma_{n,j}}\right) = p^{\sigma_{n,j}}\left(b^{\sigma_{n,j}}\right)$, $1 \leq j \leq \phi(n)$.

We shall denote the family of projection maps $\{p^\sigma : \sigma \in \kappa\}$ by $\underline{p}$ and the notation $\underline{p} : M \to N$ will express the fact that N is a substructure of M where $\underline{p}$ is the family of projections. We shall often omit the superscripts σ from each p^σ and allow context to provide the appropriate type. We shall also normally denote $S_n^{\sigma_n}$ by $p\left(R_n^{\sigma_n}\right)$.

We now illustrate the definition with the following example where $\alpha = \beta = 0$, $\kappa = \left(0, (0) = 1, \left((0)\right) = 2\right)$ and the ϵ^{σ}'s , $\sigma \in \kappa$, are the set membership relations. Take

$E^0 = \{a\ b\ c\ d\}$ $\qquad F^0 = \{a\ b\}$

$E^1 = \{a^1 = \{ab\}\ b^1 = \{acd\}\ c^1 = \{ac\}\}$ $\qquad F^1 = \{a^1 = \{ab\}\ n^1 = \{a\}\}$

$E^2 = \{a^2 = \{a^1\}\ b^2 = \{b^1\}\ c^2 = \{c^1\}\}$ $\qquad F^2 = \{a^2 = \{a^1\}\ n^2 = \{n^1\}\}$.

A family of projection maps establishing $\underline{p} : M \to N$ is as follows: $p^0(a) = a$, $p^0(b) = b$; $p^1\left(a^1\right) = a^1$, $p^1\left(b^1\right) = p^1\left(c^1\right) = n^1$; $p^2\left(a^2\right) = a^2$, $p^2\left(b^2\right) = n^2 = p^2\left(c^2\right)$.

The definition of a substructure given above is a reversal of the usual first order concept. In the first order a substructure is embedded into its parent system but in the above definition the parent system M is projected downwards onto the subsystem N , except of course at the level of type 0 where the injection is retained through the partial map p^0 .

The definition differs from that given by Kreisel and Krivine [1967], page 100, who carry over the injection procedure into the higher order context. Their definition can be described as follows. If M and N are two $\kappa(\alpha, \beta)$ structures (where for technical convenience we put $\alpha = 0$) and M', N' are related to M, N respectively in the manner given by Theorem I, 1.6 then N is termed a substructure of M if for each $\sigma \in \kappa$, $F'^{\sigma} \subseteq E'^{\sigma}$ (where E'^{σ}, F'^{σ} are the sets of objects of type σ of M', N' respectively) and for each $n < \beta$ the constant relation $S'^{\sigma_n}_n$ of N' is the restriction of the constant relation $R'^{\sigma_n}_n$ of M' to $F'^{\sigma_{n,1}} \times \ldots \times F'^{\sigma_{n,\phi(n)}}$, where $\sigma_n = (\sigma_{n,1}, \ldots, \sigma_{n,\phi(n)})$.

This definition differs from the one we have taken in several important respects. For instance if M and N are as given in the illustration above then N will not be a substructure of M in terms of the Kreisel and Krivine definition. Simple illustrations (see below) are available of the reverse situation. If M and N are both full however the two definitions will coincide.

Further, and this has important consequences for the possibilities of later applications, if we take any non-empty subset F^0_0 of the individuals of a $\kappa(\alpha, \beta)$ structure M (again for technical simplicity we assume $\alpha = 0$) in terms of our definition there is a unique (within isomorphism) substructure N of M with its set of individuals equal to F^0 , (*c.f.* Theorems 1.2 and 1.4). This is not so for the Kreisel and Krivine definition. For example if M is the structure given in the illustration above and $F^0_0 = \{a\ b\ c\}$ then N_1 may be formed by taking $F^1_1 = \{a^1 = \{ab\}\}$, $F^2_1 = \{a^2 = \{a^1\}\}$ and N_2 may be formed by taking $F^1_2 = \{a^1 = \{ab\}\ c^1 = \{ac\}\}$,

$F_2^2 = \{c^2 = \{c^1\}\}$. Both N_1 and N_2 are substructures of M in accord with Kreisel and Krivine's definition and have in common the set of individuals $F_0^0 = \{a\ b\ c\}$. As it happens neither is a substructure of the given M in terms of our definition. The unique substructure (in terms of our definition) based on $F_0^0 = \{a\ b\ c\}$ is given by $F_0^1 = \{a^1 = \{ab\}\ c^1 = \{ac\}\}$, $F_0^2 = \{a^2 = \{a^1\}\ c^2 = \{c^1\}\}$ where $p^1(a^1) = a^1$. $p^1(b^1) = c^1 = p^1(c^1)$, $p^2(a^2) = a^2$, $p^2(b^2) = c^2 = p^2(c^2)$ is the family of projection maps.

We anticipate Theorem 1.2 to explain how this latter substructure was arrived at. It can be observed that $F_0^1 = \left\{F_0^0 \cap x : x \in E^1\right\}$ and p^1 is defined by $p^1(x) = F_0^0 \cap x$. Then $F_0^2 = \{\{p^1(x) : x \in y\} : y \in E^2\}$ and p^2 is defined by $p^2(y) = \{p^1(x) : x \in y\}$.

This suggests a variation to the definition of substructure as we have given it, in particular in the above example defining F_0^2 not as there done but as $F_0'^2 = \left\{F_0^1 \cap y : y \in E^2\right\}$ thus following the pattern for the construction of F_0^1 . Thus we would have $F_0'^2 = \left\{a^2 = \{a^1\}\ n_0^2 = \emptyset\ c^2 = \{c^1\}\right\}$. If M was a full structure the variation in definition would make no final difference but when M is not full as in our illustration the definitions do not agree.

Our reason for preferring the definition we have given over the latter possibility can be seen in the example. In the application of our definition $b^2 \in E^2$ induces $c^2 \in F_0^2$ (that is $p^2(b^2) = c^2$) but in the variation b^2 induces the empty relation as the projected member of $F_0'^2$. This does not seem to be what one would naturally require in algebraic situations for instance as illustrated in the context of chains of subalgebras as discussed in section 4 of Chapter III. Further, in the example of the variation in definition we have that $b^1 \in b^2$ but $p(b^1) \notin p(b^2) = n_0^2$, whereas $p(b^1) \in p(b^2) = c^2$. The failure of the projection maps to preserve the membership relation would inhibit some of the later developments.

We now proceed to state and prove a number of the basic properties of substructures as defined.

THEOREM 1.1. *If* M *and* N *are* $\kappa(\alpha, \beta)$ *structures such that* $\underline{p}_1 : M \to N$ *and* $\underline{p}_2 : M \to N$ *then* $p_1 = p_2$.

Proof. If $\sigma = 0$ then $p_1^0 = p_2^0$, as both are the identity map on F^0 .

Assume $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $p_1^{\sigma_i} = p_2^{\sigma_i}$ for $1 \le i \le n$. We shall now deduce that $p_1^\sigma = p_2^\sigma$. Take any $a^\sigma \in E^\sigma$. Consider $\left(p_1^{\sigma_1}\left(a^{\sigma_1}\right), \ldots, p_1^{\sigma_n}\left(a^{\sigma_n}\right)\right) \in^\sigma p_1^\sigma(a^\sigma)$. Therefore there exists some $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \in^\sigma a^\sigma$ such that $p_1^{\sigma_i}\left(a^{\sigma_i}\right) = p_1^{\sigma_i}\left(b^{\sigma_i}\right)$,

$1 \le i \le n$. Thus $\left(p_2^{\sigma_1}\left(b^{\sigma_1}\right), \ldots, p_2^{\sigma_n}\left(b^{\sigma_n}\right)\right) \epsilon^{\sigma} p_2^{\sigma}\left(a^{\sigma}\right)$ and so $\left(p_1^{\sigma_1}\left(a^{\sigma_1}\right), \ldots, p_1^{\sigma_n}\left(a^{\sigma_n}\right)\right) \epsilon^{\sigma} p_2^{\sigma}\left(a^{\sigma}\right)$, as $p_1^{\sigma_i} = p_2^{\sigma_i}$ all $1 \le i \le n$. Similarly if $\left(p_2^{\sigma_1}\left(a^{\sigma_1}\right), \ldots, p_2^{\sigma_n}\left(a^{\sigma_n}\right)\right) \epsilon^{\sigma} p_2^{\sigma}(a)$ then $\left(p_2^{\sigma_1}\left(a^{\sigma_1}\right), \ldots, p_2^{\sigma_n}\left(a^{\sigma_n}\right)\right) \epsilon^{\sigma} p_1^{\sigma}(a)$. Thus $\widehat{p_1^{\sigma}\left(a^{\sigma}\right)} = \widehat{p_2^{\sigma}\left(a^{\sigma}\right)}$ and so, as N is normal, we have $p_1^{\sigma}\left(a^{\sigma}\right) = p_2^{\sigma}\left(a^{\sigma}\right)$. An induction argument completes the proof. //

Let M be a $\kappa(\alpha, \beta)$ structure and F^0 a given subset of its individuals closed under the operations of M . We build, by induction, a $\kappa(\alpha, \beta)$ structure N with individuals F^0 and such that N is a substructure of M .

(i) F^0 comprises the individuals of N .

(ii) Take $\sigma \in \kappa$, $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$ and assume F^{σ_i} is defined for all $1 \le i \le n$, together with projection maps $p^{\sigma_i} : E^{\sigma_i} \to F^{\sigma_i}$, $\sigma_i \ne 0$ and $p^0 : E^0 \to F^0$ the partial identity map with domain F^0 , such that for all $1 \le i \le n$ with $\sigma_i \ne 0$,

$$p^{\sigma_i}\left(a^{\sigma_i}\right) = \left\{\left(p^{\sigma_{i,1}}\left(a^{\sigma_{i,1}}\right), \ldots, p^{\sigma_{i,m}}\left(a^{\sigma_{i,m}}\right)\right) : \left(a^{\sigma_{i,1}}, \ldots, a^{\sigma_{i,m}}\right) \epsilon^{\sigma_i} a^{\sigma_i}\right\} ,$$

$\sigma_i = \left(\sigma_{i,1}, \ldots, \sigma_{i,m}\right)$, and $a^{\sigma_i} \in E^{\sigma_i}$.

We now define

$$p^{\sigma}\left(a^{\sigma}\right) = \left\{\left(p^{\sigma_1}\left(a^{\sigma_1}\right), \ldots, p^{\sigma_n}\left(a^{\sigma_n}\right)\right) : \left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) \epsilon^{\sigma} a^{\sigma}\right\}$$

for each $a^{\sigma} \in E^{\sigma}$. We put $F^{\sigma} = \{p^{\sigma}\left(a^{\sigma}\right) : a^{\sigma} \in E^{\sigma}\}$ and thus F^{σ} and $p^{\sigma} : E^{\sigma} \to F^{\sigma}$ are defined.

(iii) For each $\sigma \in \kappa$, $\sigma \neq 0$ we define ϵ^{σ} for N as the ordinary set membership relation. That is

$\left(p^{\sigma_1}\left(a^{\sigma_1}\right), \ldots, p^{\sigma_n}\left(a^{\sigma_n}\right)\right) \epsilon^{\sigma} p^{\sigma}\left(a^{\sigma}\right)$ if

$p^{\sigma_1}\left(a^{\sigma_1}\right), \ldots, p^{\sigma_n}\left(a^{\sigma_n}\right) \in p^{\sigma}\left(a^{\sigma}\right)$.

(iv) For each $m < \alpha$ the operation g_m on F^0 is defined as f_m restricted to F^0 .

(v) For each $n < \beta$ the constant relation $S_n^{\sigma_n}\left(= p\left(R_n^{\sigma_n}\right)\right)$ of N is defined by $S_n^{\sigma_n}\left(p^{\sigma_{n,1}}\left(a^{\sigma_{n,1}}\right), \ldots, p^{\sigma_{n,\phi(n)}}\left(a^{\sigma_{n,\phi(n)}}\right)\right)$ if there exists $\left(b^{\sigma_{n,1}}, \ldots, b^{\sigma_{n,\phi(n)}}\right)$ such that $R_n^{\sigma_n}\left(b^{\sigma_{n,1}}, \ldots, b^{\sigma_{n,\phi(n)}}\right)$ and $p^{\sigma_{n,j}}\left(a^{\sigma_{n,j}}\right) = p^{\sigma_{n,j}}\left(b^{\sigma_{n,j}}\right)$, all $1 \leq j \leq \phi(n)$.

THEOREM 1.2. *N as constructed above is a substructure of M with $\underline{p}$ as the associated family of projection maps.*

Proof. We first check that N is a normal structure. Let $\sigma \in \kappa$, $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$ and take $p^{\sigma}\left(a^{\sigma}\right), p^{\sigma}\left(b^{\sigma}\right) \in F^{\sigma}$ such that $\widehat{p\left(a^{\sigma}\right)} = \widehat{p\left(b^{\sigma}\right)}$. That is $\left(p\left(a^{\sigma_1}\right), \ldots, p\left(a^{\sigma_n}\right)\right) \epsilon^{\sigma} p\left(a^{\sigma}\right)$ if, and only if, $\left\{p\left(a^{\sigma_1}\right), \ldots, p\left(a^{\sigma_n}\right)\right\} \in^{\sigma} p\left(a^{\sigma}\right)$. But $\in^{\sigma}$ is defined as the ordinary set membership relation and thus $p\left(a^{\sigma}\right) = p\left(b^{\sigma}\right)$. Hence N is normal.

It can also be checked that each p^{σ} , $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$ has the required projection properties. Hence $\underline{p} : M \to N$. //

We shall term N as constructed above the canonical substructure of M based on F^0 .

We next state and prove a lemma as a preliminary to two further theorems.

LEMMA 1.3. *If* M, N_1, N_2 *are* $\kappa(\alpha, \beta)$ *structures such that* $\underline{p}_1 : M \to N_1$, $\underline{p}_2 : M \to N_2$ *and* $F_1^0 \subseteq F_2^0$ *then for all* $\sigma \in \kappa$ *and all* $a^{\sigma}, b^{\sigma} \in E^{\sigma}$, *if* $p_2\left(a^{\sigma}\right) = p_2\left(b^{\sigma}\right)$ *then* $p_1\left(a^{\sigma}\right) = p_1\left(b^{\sigma}\right)$.

Proof. If $\sigma = 0$ then the result is immediate, as whenever $a^0, b^0 \in F_1^0$ we have $p_1\left(a^0\right) = a^0 = p_2\left(a^0\right)$ and $p_1\left(b^0\right) = b^0 = p_2\left(b^0\right)$.

Take $\sigma \in \kappa$, $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$ and assume the result is true for each σ_i , $1 \le i \le n$. Assume, further, that $p_2\left(a^{\sigma}\right) = p_2\left(b^{\sigma}\right)$. Let $\left\{p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right)\right\} \in^{\sigma} p_1\left(a^{\sigma}\right)$ and so there exists some $\left\{b^{\sigma_1}, \ldots, b^{\sigma_n}\right\} \in^{\sigma} a^{\sigma}$ such that $p_1\left(a^{\sigma_i}\right) = p_1\left(b^{\sigma_i}\right)$, $1 \le i \le n$. Thus $\left\{p_2\left(b^{\sigma_1}\right), \ldots, p_2\left(b^{\sigma_n}\right)\right\} \in^{\sigma} p_2\left(a^{\sigma}\right) = p_2\left(b^{\sigma}\right)$ and so there exists some $\left(c^{\sigma_1}, \ldots, c^{\sigma_n}\right) \in^{\sigma} b^{\sigma}$, where $p_2\left(c^{\sigma_i}\right) = p_2\left(b^{\sigma_i}\right)$, $1 \le i \le n$. By the induction hypothesis, for each $1 \le i \le n$, $p_1\left(c^{\sigma_i}\right) = p_1\left(b^{\sigma_i}\right)$. Hence $\left(p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right)\right) \in^{\sigma} p_1\left(b^{\sigma}\right)$. Similarly if $\left(p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right)\right) \in p_1\left(b^{\sigma}\right)$ then $\left(p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right)\right) \in p_1\left(a^{\sigma}\right)$. Therefore $\widehat{p_1\left(a^{\sigma}\right)} = \widehat{p_1\left(b^{\sigma}\right)}$, and so $p_1\left(a^{\sigma}\right) = p_1\left(b^{\sigma}\right)$. //

THEOREM 1.4. *If* M, N_1, N_2 *are* $\kappa(\alpha, \beta)$ *structures such that* $\underline{p}_1 : M \to N_1$, $\underline{p}_2 : M \to N_2$ *and* $F_1^0 = F_2^0$ *then* N_1 *and* N_2 *are isomorphic.*

Proof. Define $\underline{\psi} : N_1 \to N_2$ by $\psi\left[p_1\left(a^{\sigma}\right)\right] = p_2\left(a^{\sigma}\right)$, for each $\sigma \in \kappa$, and all $a^{\sigma} \in E^{\sigma}$. Lemma 1.3 ensures that ψ is well defined. We now establish that it is an isomorphism.

First we show that for each $\sigma \in \kappa$, $\psi : F_1^{\sigma} \to F_2^{\sigma}$ is bijective. If $\sigma = 0$ then $\psi : F_1^0 \to F_2^0$ is the identity map on $F_1^0 = F_2^0$ and hence is bijective. Let $\sigma \in \kappa$, $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$ and assume that $\psi : F_1^{\sigma_i} \to F_2^{\sigma_i}$ is bijective, each $1 \le i \le n$. For any $p_1\left(a^{\sigma}\right), p_1\left(b^{\sigma}\right) \in F_1^{\sigma}$, if $\psi\left[p_1\left(a^{\sigma}\right)\right] = \psi\left[p_1\left(b^{\sigma}\right)\right]$ then $p_2\left(a^{\sigma}\right) = p_2\left(b^{\sigma}\right)$

and so $p_1(a^\sigma) = p_1(b^\sigma)$ by Lemma 1.3. That is, $\psi : F_1^\sigma \to F_2^\sigma$ is injective.

If $p_2(a^\sigma) \in F_2^\sigma$ then $a^\sigma \in E^\sigma$ and so $p_1(a^\sigma) \in F_1^\sigma$ and $\psi\left(p_1(a^\sigma)\right) = p_2(a^\sigma)$. That is $\psi : F_1^\sigma \to F_2^\sigma$ is surjective. The usual inductive argument establishes that $\psi : F_1^\sigma \to F_2^\sigma$ is bijective for all $\sigma \in \kappa$.

We next show that ψ preserves the operations and relations of N_1 with respect to those of N_2. Take $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and consider any $\left(p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right)\right) \in^\sigma p_1(a^\sigma)$. Therefore there exists some $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \in^\sigma a^\sigma$ such that $p_1\left(a^{\sigma_i}\right) = p_1\left(b^{\sigma_i}\right)$, $1 \le i \le n$. Thus $\left(p_2\left(b^{\sigma_1}\right), \ldots, p_2\left(b^{\sigma_n}\right)\right) \in^\sigma p_2(a^\sigma)$. But for each $1 \le i \le n$, $p_1\left(a^{\sigma_i}\right) = p_1\left(b^{\sigma_i}\right)$ and so by Lemma 1.3, $p_2\left(a^{\sigma_i}\right) = p_2\left(b^{\sigma_i}\right)$. That is, $\left(\psi\left(p_1\left(a^{\sigma_1}\right)\right), \ldots, \psi\left(p_1\left(a^{\sigma_n}\right)\right)\right) \in^\sigma \psi\left(p_1(a^\sigma)\right)$.

Conversely, if $\left(\psi\left(p_1\left(a^{\sigma_1}\right)\right), \ldots, \psi\left(p_1\left(a^{\sigma_n}\right)\right)\right) \in^\sigma \psi\left(p_1(a^\sigma)\right)$ then $\left(p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right)\right) \in^\sigma p_1(a^\sigma)$.

By a similar argument it can be shown that for any $n < \beta$, $p_1\left(R_n^{\sigma_n}\right)\left(p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right)\right)$ if, and only if,

$p_2\left(R_n^{\sigma_n}\right)\left(\psi\left(p_1\left(a^{\sigma_1}\right)\right), \ldots, \psi\left(p_1\left(a^{\sigma_n}\right)\right)\right)$.

Finally, as ψ is the identity map on $F_1^0 = F_2^0$, ψ preserves the operations of N_1 with respect to the operations of N_2 .

Hence $\underline{\psi} : N_1 \to N_2$ is an isomorphism between N_1 and N_2 . //

COROLLARY 1.5. *If M is a $\kappa(\alpha, \beta)$ structure then every substructure of M is isomorphic to a canonical substructure of M .*

Proof. Let N_1 be any substructure of M . Let N_2 be the canonical substructure of M based on F_1^0 , the set of individuals of N_1 . Hence from Theorem 1.4, N_1 is isomorphic to N_2 . //

COROLLARY 1.6. *If M, N are $\kappa(\alpha, \beta)$ structures such that each is a substructure of the other then M and N are isomorphic.*

Proof. As each is a substructure of the other then $F^0 = E^0$ and so N is isomorphic to the canonical substructure of M based on E^0 which by the proof of Theorem I : 1.6 is isomorphic to M . //

THEOREM 1.7. *M, N_1, N_2 are $\kappa(\alpha, \beta)$ structures such that $\underline{p}_1 : M \to N_1$ and $\underline{p}_2 : M \to N_2$. If $F_1^0 \subseteq F_2^0$ then N_1 is a substructure of N_2 and $\underline{p}_3 : N_2 \to N_1$ defined by:*

$p_3\left(p_2\left(a^\sigma\right)\right) = p_1\left(a^\sigma\right)$ for all $a^\sigma \in E^\sigma$, $\sigma \in \kappa$, is the family of projection maps from N_2 to N_1 .

Proof. It is immediate from Lemma 1.3 that $\underline{p}_3$ is well defined.

If $\sigma = 0$ then $p^3 : F_2^0 \to F_1^0$ is the partial identity map on F_1^0. Further for each $\sigma \in K$, $p_3 : F_2^\sigma \to F_1^\sigma$ is surjective.

Consider $\sigma \in K$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and

$\left(p_3\left(p_2\left(a^{\sigma_1}\right)\right), \ldots, p_3\left(p_2\left(a^{\sigma_n}\right)\right)\right) \in^\sigma p_3\left(p_2(a^\sigma)\right)$. That is

$\left(p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right)\right) \in^\sigma p_1(a^\sigma)$. Hence there exists $\left(b^{\sigma_1}, \ldots, b^{\sigma n}\right) \in^\sigma a^\sigma$

such that $p_1\left(a^{\sigma_i}\right) = p_1\left(b^{\sigma_i}\right)$, for each $1 \le i \le n$. Thus

$\left(p_2\left(b^{\sigma_1}\right), \ldots, p_2\left(b^{\sigma_n}\right)\right) \in^\sigma p_2(a^\sigma)$ and for each $1 \le i \le n$,

$p_3\left(p_2\left(b^{\sigma_i}\right)\right) = p_1\left(b^{\sigma_i}\right) = p_1\left(a^{\sigma_i}\right) = p_3\left(p_2\left(a^{\sigma_i}\right)\right)$ as required.

Finally, take $n < \beta$ and consider

$p_1\left(R_n^{\sigma_n}\right)\left(p_3\left(p_2\left(a^{\sigma_{n,1}}\right)\right) \ldots, p_3\left(p_2\left(a^{\sigma_{n,\phi(n)}}\right)\right)\right)$. That is

$p_1\left(R_n^{\sigma_n}\right)\left(p_1\left(a^{\sigma_{n,1}}\right), \ldots, p_1\left(a^{\sigma_{n,\phi(n)}}\right)\right)$ and so there exists

$b^{\sigma_{n,1}}, \ldots, b^{\sigma_{n,\phi(n)}}$ such that $R_n^{\sigma_n}\left(b^{\sigma_{n,1}}, \ldots, b^{\sigma_{n,\phi(n)}}\right)$ and

$p_1\left(b^{\sigma_{n,j}}\right) = p_1\left(a^{\sigma_{n,j}}\right)$, for each $1 \le j \le \phi(n)$. Thus

$p_2\left(R_n^{\sigma_n}\right)\left(p_2\left(b^{\sigma_{n,1}}\right), \ldots, p_2\left(b^{\sigma_{n,\phi(n)}}\right)\right)$, where for each $1 \le j \le \phi(n)$,

$p_3\left(p_2\left(b^{\sigma_{n,j}}\right)\right) = p_1\left(b^{\sigma_{n,j}}\right) = p_1\left(a^{\sigma_{n,j}}\right) = p_3\left(p_2\left(a^{\sigma_{n,j}}\right)\right)$ as required.

Hence $\underset{\sim}{p}_3$ is the family of projections such that $\underset{\sim}{p}_3 : N_2 \to N_1$. //

COROLLARY 1.8. *M, N_1 are $\kappa(\alpha, \beta)$ structures such that $\underline{p}_1 : M \to N_1$. For all $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ if $\left(p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right)\right) \notin^{\sigma} p_1(a^{\sigma})$ then for any substructure N_2 of M, $\underline{p}_2 : M \to N_2$, if N_2 contains N_1 (that is if $F_2^0 \supseteq F_1^0$) then $\left(p_2\left(a^{\sigma_1}\right), \ldots, p_2\left(a^{\sigma_n}\right)\right) \notin^{\sigma} p_2(a^{\sigma})$. Similarly, for any $n < \beta$, if $p_1\left(R_n^{\sigma_n}\right)\left(p_1\left(a^{\sigma_{n,1}}\right), \ldots, p_1\left(a^{\sigma_{n,\phi(n)}}\right)\right)$ does not hold in N_1 then $p_2\left(R_n^{\sigma_n}\right)\left(p_2\left(a^{\sigma_{n,1}}\right), \ldots, p_2\left(a^{\sigma_{n,\phi(n)}}\right)\right)$ does not hold in N_2, where N_2 is as above.*

Proof. Let $\underline{p}_3 : N_2 \to N_1$ be as in Theorem 1.7. If

$\left(p_2\left(a^{\sigma_1}\right), \ldots, p_2\left(a^{\sigma_n}\right)\right) \in^{\sigma} p_2(a^{\sigma})$ then

$\left(p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right)\right) \in^{\sigma} p_1(a^{\sigma})$, as $p_3 p_2 = p_1$. Thus if

$\left(p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right)\right) \notin^{\sigma} p_1(a^{\sigma})$ then

$\left(p_2\left(a^{\sigma_1}\right), \ldots, p_2\left(a^{\sigma_n}\right)\right) \notin^{\sigma} p_2(a^{\sigma})$. The second part follows likewise. //

THEOREM 1.9. *M, N_1 are $\kappa(\alpha, \beta)$ structures such that $\underline{p}_1 : M \to N_1$. If N_2 is any substructure of N_1, $\underline{p}_3 : N_1 \to N_2$ then N_2 is a substructure of M with $\underline{p}_2 : M \to N_2$ defined by $p_2(a^{\sigma}) = p_3\left(p_1(a^{\sigma})\right)$ all $a^{\sigma} \in E^{\sigma}$, $\sigma \in \kappa$.*

Proof. If $\sigma = 0$ we agree that $p_2 : E^0 \to F_2^0$ is given by the definition as the partial identity map on F_2^0 .

Take $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and

$\left(p_2\left(a^{\sigma_1}\right), \ldots, p_2\left(a^{\sigma_n}\right)\right) \in^{\sigma} p_2(a^{\sigma})$. That is

$\left(p_3\left(p_1\left(a^{\sigma_1}\right)\right), \ldots, p_3\left(p_1\left(a^{\sigma_n}\right)\right)\right) \in^{\sigma} p_3\left(p_1(a^{\sigma})\right)$ and so there exists

$\left(p_1\left(b^{\sigma_1}\right), \ldots, p_1\left(b^{\sigma_n}\right)\right) \in^{\sigma} p_1(a^{\sigma})$, such that

$p_3\left(p_1\left(b^{\sigma_j}\right)\right) = p_3\left(p_1\left(a^{\sigma_j}\right)\right)$, each $1 \le j \le n$. Hence there exists

$\left(c^{\sigma_1}, \ldots, c^{\sigma_n}\right) \in^{\sigma} a^{\sigma}$ such that $p_1\left(c^{\sigma_j}\right) = p_1\left(b^{\sigma_j}\right)$, each $1 \le j \le n$.

Thus for each $1 \le j \le n$ we have $p_2\left(c^{\sigma_j}\right) = p_2\left(a^{\sigma_j}\right)$ as required.

A similar argument establishes the corresponding property for the constant relations. Thus N_2 is a subsystem of M with

$\underline{p}_2 : M \to N_2$. //

We observe that Theorems 1.7 and 1.9 provide a unique (within isomorphism) natural definition of the intersection of two substructures, N_1 and N_2 , of a $\kappa(\alpha, \beta)$ structure M . For if $F^0 = F_1^0 \cap F_2^0$ then $N = N_1 \cap N_2$ can be defined as the substructure of M , or N_1 , or N_2 on F^0 .

THEOREM 1.10. *Any substructure, N , of a full $\kappa(\alpha, \beta)$ structure is itself full.*

Proof. Take $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and K any subset of $F^{\sigma_1} \times \ldots \times F^{\sigma_n}$. It is required to find some $p(a^\sigma) \in F^\sigma$, where $\underline{p} : M \to N$, such that $\widehat{p(a^\sigma)} = K$.

Put $K_1 = \left\{\left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) : \left(p\left(a^{\sigma_1}\right), \ldots, \left(a^{\sigma_n}\right)\right) \in K\right\}$. M is full and so there is some $a^\sigma \in E^\sigma$ such that $\hat{a}^\sigma = K_1$. We now show $K = \widehat{p(a^\sigma)}$.

Take $\left(p\left(a^{\sigma_1}\right), \ldots, p\left(a^{\sigma_n}\right)\right) \in K$. Hence $\left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) \in K_1$ and so $\left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) \in^\sigma a^\sigma$. Thus $\left(p\left(a^{\sigma_1}\right), \ldots, p\left(a^{\sigma_n}\right)\right) \in^\sigma p(a^\sigma)$. That is $K \subseteq \widehat{p(a^\sigma)}$.

Now take $\left(p\left(a^{\sigma_1}\right), \ldots, p\left(a^{\sigma_n}\right)\right) \in^\sigma p(a^\sigma)$. Therefore there exists some $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \in^\sigma a^\sigma$, where $p\left(b^{\sigma_i}\right) = p\left(a^{\sigma_i}\right)$ each $1 \leq i \leq n$. Hence $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \in K_1$ as $\hat{a}^\sigma = K_1$, and so $\left(p\left(b^{\sigma_1}\right), \ldots, p\left(b^{\sigma_n}\right)\right) \in K$. That is $\left(p\left(a^{\sigma_1}\right), \ldots, p\left(a^{\sigma_n}\right)\right) \in K$. Thus $\widehat{p(a^\sigma)} \subseteq K$. //

We now introduce the notion of a (strong) homomorphism from one $\kappa(\alpha, \beta)$ structure to another. Let M_1, M_2 be two $\kappa(\alpha, \beta)$ structures. $\underline{\psi}$ is termed a *homomorphism* from M_1 to M_2 , written $\underline{\psi} : M_1 \to M_2$, if

(i) for each $\sigma \in \kappa$, ψ is a map from E_1^{σ} to E_2^{σ} ,

(ii) for each $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$,

$\left(\psi\left(a^{\sigma_1}\right), \ldots, \psi\left(a^{\sigma_n}\right)\right) \in^{\sigma} \psi(a^{\sigma})$, if, and only if, there exists $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \in^{\sigma} a^{\sigma}$, where $\psi\left(b^{\sigma_i}\right) = \psi\left(a^{\sigma_i}\right)$, each $1 \leq i \leq n$,

(iii) for each $m < \alpha$, $f_m^1\left(a_1^0, \ldots, a_{\theta(m)}^0\right) = a^0$ only if $f_m^2\left(\psi\left(a_1^0\right), \ldots, \psi\left(a_{\theta(m)}^0\right)\right) = \psi(a^0)$ where f_m^1, f_m^2 are corresponding operations in M_1, M_2 respectively,

(iv) for each $n < \beta$, $R_{n,2}^{\sigma_n}\left(\psi\left(a^{\sigma_{n,1}}\right), \ldots, \psi\left(a^{\sigma_{n,\phi(n)}}\right)\right)$ if, and only if, $R_{n,1}^{\sigma_n}\left(b^{\sigma_{n,1}}, \ldots, b^{\sigma_{n,\phi(n)}}\right)$, where $\psi\left(b^{\sigma_{n,j}}\right) = \psi\left(a^{\sigma_{n,j}}\right)$, each $1 \leq j \leq \phi(n)$.

We comment that Grätzer [1968], pages 80, 81, defines several homomorphism concepts in the context of partial algebras. Similar variations are available here. We have chosen the strongest form. We also note that the projection maps defined in the substructure context are homomorphisms as defined above.

If a homomorphism $\underline{\psi}$ from M_1 to M_2 is bijective then it is an isomorphism, as defined previously, between M_1 and M_2 . An injective homomorphism is termed an *embedding*.

M_1 and M_2 are $\kappa(\alpha, \beta)$ structures and $\underline{\psi} : M_1 \to M_2$ is a homomorphism. The image of $\underline{\psi}$, written $I_{\underline{\psi}}$, is the $\kappa(\alpha, \beta)$ structure

$$I_{\underline{\psi}} = \left\{\psi\left(E_1^\sigma\right) : \sigma \in \kappa\right\} \cup \{\epsilon^\sigma : \sigma \in \kappa, \sigma \neq 0\} \cup \left\{\psi\left(f_m^1\right) : m < \alpha\right\} \cup \left\{\psi\left(R_{n,1}^{\sigma_n}\right) : n < \beta\right\}$$

where (i) for each $\sigma \in \kappa$, $\psi\left(E_1^\sigma\right)$ is the image of $\psi : E_1^\sigma \to E_2^\sigma$,

(ii) for each $\sigma \in \kappa$, $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$,

$\left(\psi\left(a^{\sigma_1}\right), \ldots, \psi\left(a^{\sigma_n}\right)\right) \epsilon^\sigma \psi\left(a^\sigma\right)$ if, and only if, there exists $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \epsilon^\sigma a^\sigma$, where $\psi\left(a^{\sigma_i}\right) = \psi\left(b^{\sigma_i}\right)$, $1 \le i \le n$,

(iii) for each $m < \alpha$, $\psi\left(f_m^1\right)$ is f_m^2 restricted to $\psi\left(E_1^0\right)$,

(iv) for each $n < \beta$, $\psi\left(R_{n,1}^{\sigma_n}\right)$ is $R_{n,2}^{\sigma_n}$ restricted to $\psi\left(E_1^{\sigma_{n,1}}\right) \times \ldots \times \psi\left(E_1^{\sigma_{n,\phi(n)}}\right)$.

We note that if $\underline{\psi}$ is an embedding of M_1 into M_2 then M_1 and $I_{\underline{\psi}}$ are isomorphic.

In general $I_{\underline{\psi}}$ as defined need not be a normal structure. The following theorem ensures that it will be normal if $\underline{\psi}$ is an embedding.

THEOREM 1.11. *If $\underline{\psi} : M_1 \to M_2$ is an embedding, where M_1, M_2 are two (normal) $\kappa(\alpha, \beta)$ structures then $I_{\underline{\psi}}$ is a normal structure.*

Proof. Take $\psi(a^{\sigma})$, $\psi(b^{\sigma})$ $\psi\left(E_1^{\sigma}\right)$, $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, such that $\left(\psi\left(a^{\sigma_1}\right), \ldots, \psi\left(a^{\sigma_n}\right)\right) \in^{\sigma} \psi(a^{\sigma})$ if, and only if, $\left(\psi\left(a^{\sigma_1}\right), \ldots, \psi\left(a^{\sigma_n}\right)\right) \in^{\sigma} \psi(b^{\sigma})$.

Consider $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \in^{\sigma} a^{\sigma}$. Thus $\left(\psi\left(b^{\sigma_1}\right), \ldots, \psi\left(b^{\sigma_n}\right)\right) \in^{\sigma} \psi(b^{\sigma})$ from above and hence there exists $\left(c^{\sigma_1}, \ldots, c^{\sigma_n}\right) \in^{\sigma} b^{\sigma}$, where $\psi\left(c^{\sigma_i}\right) = \psi\left(b^{\sigma_i}\right)$, $1 \le i \le n$. But $\underline{\psi}$ is an embedding, thus $c^{\sigma_i} = b^{\sigma_i}$, $1 \le i \le n$. That is $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \in^{\sigma} b^{\sigma}$. Conversely, if $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \in^{\sigma} a^{\sigma}$ then $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \in^{\sigma} a^{\sigma}$. Thus $\hat{a}^{\sigma} = \hat{b}^{\sigma}$ and so $a^{\sigma} = b^{\sigma}$ and therefore $\psi(a^{\sigma}) = \psi(b^{\sigma})$. //

We further note that if $\underline{\psi}$ is an embedding of M_1 into M_2 with the additional property that for each $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and each $a^{\sigma} \in E_1^{\sigma}$, if $\left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) \in^{\sigma} \psi(a^{\sigma})$ then for each $1 \le i \le n$, there exists $b^{\sigma_i} \in E_1^{\sigma_i}$ such that $\psi\left(b^{\sigma_i}\right) = a^{\sigma_i}$, then $I_{\underline{\psi}}$ is a substructure of M_2 in the sense of Kreisel and Krivine [1967], page 100.

We prepare the way for the proof of the next theorem with a preliminary lemma.

LEMMA 1.12. *$\underline{\psi} : M_1 \to M_2$ is an embedding, M_1, M_2 being $\kappa(\alpha, \beta)$ structures, and N_1 is a substructure of M_1, $\underline{p}_1 : M_1 \to N_1$. If N_2 is a substructure of $I_{\underline{\psi}}$ with $\psi\left(F_1^0\right)$ its set of individuals, $\underline{p}_2 : I_{\underline{\psi}} \to N_2$, then for all $\sigma \in \kappa$, $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$, and $a^\sigma, b^\sigma \in E_1^\sigma$, if $p_1\left(a^\sigma\right) = p_1\left(b^\sigma\right)$ then $p_2\left(\psi\left(a^\sigma\right)\right) = p_2\left(\psi\left(b^\sigma\right)\right)$.*

Proof. If $\sigma = 0$ and $a^0, b^0 \in F_1^0$, then $p_1\left(a^0\right) = a^0$ and $p_1\left(b^0\right) = b^0$. So if $p_1\left(a^0\right) = p_1\left(b^0\right)$ then $a^0 = b^0$ and $p_2\left(\psi\left(a^0\right)\right) = p_2\left(\psi\left(b^0\right)\right)$.

Take $\sigma \in \kappa$, $\sigma = \sigma_1, \ldots, \sigma_n$ and $a^\sigma, b^\sigma \in E_1^\sigma$ and assume the result is true for all σ_i, and $a^{\sigma_i}, b^{\sigma_i} \in E_1^{\sigma_i}$, $1 \leq i \leq n$. Assume, further, that $p_1\left(a^\sigma\right) = p_1\left(b^\sigma\right)$.

Take $\left(p_2\left(\psi\left(a^{\sigma_1}\right)\right), \ldots, p_2\left(\psi\left(a^{\sigma_n}\right)\right)\right) \in^\sigma p_2\left(\psi\left(a^\sigma\right)\right)$. (Note: There is ambiguity in the use of the symbol $\psi\left(a^\sigma\right)$. It denotes an object of M_2 and is also used here to denote an object of $I_{\underline{\psi}}$. The extension of $\psi\left(a^\sigma\right)$ with respect to M_2 will in general, differ from the extension of $\psi\left(a^\sigma\right)$ with respect to $I_{\underline{\psi}}$. We believe the context in which the symbol is used will remove ambiguity.) Thus there exists some $\left(\psi\left(b^{\sigma_1}\right), \ldots, \psi\left(b^{\sigma_n}\right)\right) \epsilon^{\sigma} \psi\left(a^{\sigma}\right)$, where $p_2\left(\psi\left(a^{\sigma_i}\right)\right) = p_2\left(\psi\left(b^{\sigma_i}\right)\right)$, $1 \leq i \leq n$. Hence $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \epsilon^{\sigma} a^{\sigma}$, as $\underline{\psi}$ is an embedding and so $\left(p_1\left(b^{\sigma_1}\right), \ldots, p_1\left(b^{\sigma_n}\right)\right) \epsilon^{\sigma} p_1\left(b^{\sigma}\right)$ as $p_1\left(a^{\sigma}\right) = p_1\left(b^{\sigma}\right)$. Therefore there exists some $\left(c^{\sigma_1}, \ldots, c^{\sigma_n}\right) \epsilon^{\sigma} b^{\sigma}$, where $p_1\left(c^{\sigma_i}\right) = p_1\left(b^{\sigma_i}\right)$, $1 \leq i \leq n$, and so $\left(p_2\left(\psi\left(c^{\sigma_1}\right)\right), \ldots, p_2\left(\psi\left(c^{\sigma_n}\right)\right)\right) \epsilon^{\sigma} p_2\left(\psi\left(b^{\sigma}\right)\right)$. But from the induction assumption $p_2\left(\psi\left(c^{\sigma_i}\right)\right) = p_2\left(\psi\left(a^{\sigma_i}\right)\right)$, $1 \leq i \leq n$. Hence $\left(p_2\left(\psi\left(a^{\sigma_1}\right)\right), \ldots, p_2\left(\psi\left(a^{\sigma_n}\right)\right)\right) \epsilon^{\sigma} \left(p_2\left(\psi\left(b^{\sigma}\right)\right)\right)$. The converse follows by a similar argument and so $\widehat{p_2\left(\psi\left(a^{\sigma}\right)\right)} = \widehat{p_2\left(\psi\left(b^{\sigma}\right)\right)}$. That is $p_2\left(\psi\left(a^{\sigma}\right)\right) = p_2\left(\psi\left(b^{\sigma}\right)\right)$, as $I_{\underline{\psi}}$ is normal. //

THEOREM 1.13. *If* $\underline{\psi}, M_1, M_2, N_1$ *and* N_2 *are as in the statement of Lemma* 1.12 *then* N_1 *is isomorphic to* N_2 .

Proof. For each $\sigma \in K$, define $\psi' : F_1^{\sigma} \to F_2^{\sigma}$ by: $\psi'\left(p_1\left(a^{\sigma}\right)\right) = p_2\left(\psi\left(a^{\sigma}\right)\right)$, all $a^{\sigma} \in E_1^{\sigma}$. By Lemma 1.12 we have that ψ' is well defined.

If $\sigma = 0$ then it is immediate that $\psi' : F_1^0 \to F_2^0$ is bijective. Let $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$, $\sigma \in K$, and assume $\psi' : F_1^{\sigma_i} \to F_2^{\sigma_i}$ is bijective each $1 \le i \le n$. Assume $p_2\left(\psi\left(a^\sigma\right)\right) = p_2\left(\psi\left(b^\sigma\right)\right)$ and take $p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right) \in^\sigma p_1\left(a^\sigma\right)$. Thus there exists $\left(b^{\sigma_1}, \ldots, b^{\sigma_n}\right) \in^\sigma a^\sigma$, $p_1\left(b^{\sigma_i}\right) = p_1\left(a^{\sigma_i}\right)$ each $1 \le i \le n$, and so $\left(p_2\left(\psi\left(b^{\sigma_1}\right)\right), \ldots, p_2\left(\psi\left(b^{\sigma_n}\right)\right)\right) \in^\sigma p_2\left(\psi\left(b^\sigma\right)\right)$. Hence there exists some $\left(\psi\left(c^{\sigma_1}\right), \ldots, \psi\left(c^{\sigma_n}\right)\right) \in^\sigma \psi\left(b^\sigma\right)$, where $p_2\left(\psi\left(c^{\sigma_i}\right)\right) = p_2\left(\psi\left(b^{\sigma_i}\right)\right)$, each $1 \le i \le n$, and so $\left(c^{\sigma_1}, \ldots, c^{\sigma_n}\right) \in^\sigma b^\sigma$ as ψ is an embedding.

From the induction hypothesis $p_1\left(c^{\sigma_i}\right) = p_1\left(b^{\sigma_i}\right) = p_1\left(a^{\sigma_i}\right)$ each $1 \le i \le n$. Thus $\left(p_1\left(a^{\sigma_1}\right), \ldots, p_1\left(a^{\sigma_n}\right)\right) \in^\sigma p_1\left(b^\sigma\right)$. The converse follows by a similar argument and hence $p_1\left(a^\sigma\right) = p_1\left(b^\sigma\right)$ as N_1 is normal. We have therefore that $\psi' : F_1^\sigma \to F_2^\sigma$ is injective and it is immediate that it is also surjective. That is $\underline{\psi}' : N_1 \to N_2$ is bijective.

The remainder of the morphism properties for $\underline{\psi}'$ follow from the fact that $\underline{p}_1$, $\underline{p}_2$ and $\underline{\psi}$ are homomorphisms. This establishes that ψ' is an isomorphism between N_1 and N_2 . //

Whereas the above theorem establishes that substructures are preserved under embeddings the final theorem of this section establishes that substructures are preserved under the ultraproduct construction in the following manner.

THEOREM 1.14. *Let* $\{M_i : i \in I\}$ *be a family of* $\kappa(\alpha, \beta)$ *structures and, for each* $i \in I$, N_i *is a substructure of* M_i , $\underline{p}_i : M_i \to N_i$. *If* F *is any ultrafilter over* I *then* $\pi N_i/\mathsf{F}$ *is a substructure of* $\pi M_i/\mathsf{F}$.

Proof. We define $\underline{p} : \pi M_i/\mathsf{F} \to \pi N_i/\mathsf{F}$ as follows. For $\sigma \in \kappa$ and $\overline{h}^\sigma \in \overline{E}^\sigma$ put $p\left(\overline{h}^\sigma\right) = \overline{p\left(h^\sigma\right)}$, where $p\left(h^\sigma\right) : I \to \bigcup\left\{F_i^\sigma : i \in I\right\}$ is defined by: $p\left(h^\sigma\right)(i) = p_i\left(h^\sigma(i)\right)$, each $i \in I$.

If $\sigma = 0$ we will agree the definition provides that $p : \overline{E}^0 \to \overline{F}^0$ is the partial identity map with $\overline{F}^0$ as its domain. We note that for all $\sigma \in \kappa$, and $h^\sigma, h_1^\sigma \in P^\sigma$ such that $h^\sigma \sim h_1^\sigma$ then $p\left(h^\sigma\right) \sim p\left(h_1^\sigma\right)$, as if $h^\sigma \sim h_1^\sigma$ then $\left\{i ; h^\sigma(i) = h_1^\sigma(i)\right\} \in \mathsf{F}$ and so $\left\{i : p_i\left(h^\sigma(i)\right) = p_i\left(h_1^\sigma(i)\right)\right\} \in \mathsf{F}$. Thus $p : \overline{E}^\sigma \to \overline{F}^\sigma$ is well defined.

We first check that $\overline{F}^0$ is closed under the operations of $\pi M_i/\mathsf{F}$. For any $m < \alpha$ and $\overline{h}_1^0, \ldots, \overline{h}_{\theta(m)}^0 \in \overline{F}^0$ we have $\overline{f}_m\left(\overline{h}_1^0, \ldots, \overline{h}_{\theta(m)}^0\right) = \overline{k}$ where for some $F \in \mathsf{F}$, $\left\{i : f_{i,m}\left(h_1^0(i), \ldots, h_{\theta(m)}^0(i)\right) = k(i)\right\} = F$. But $\overline{h}_1^0, \ldots, \overline{h}_{\theta(m)}^0 \in \overline{F}^0$ and so $\left\{i : k(i) \in F_i^0\right\} \in \mathsf{F}$. That is $\overline{k} \in \overline{F}^0$. Hence $\overline{F}^0$ is closed under the operations of $\pi M_i/\mathsf{F}$.

We next take $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and consider

$\left(p\left(\overline{h}^{\sigma_1}\right), \ldots, p\left(\overline{h}^{\sigma_n}\right)\right) \overline{\epsilon}^{\sigma} p(\overline{h}^{\sigma})$. Thus if

$F = \left\{i : \left(p_i\left(h^{\sigma_1}(i)\right), \ldots, p_i\left(h^{\sigma_n}(i)\right)\right) \epsilon_i^{\sigma} p_i(h^{\sigma}(i))\right\}$ then $F \in \mathrm{F}$.

Hence, for each $i \in F$, there exists some $\left(a_i^{\sigma_1}, \ldots, a_i^{\sigma_n}\right) \epsilon_i^{\sigma} h^{\sigma}(i)$,

where $p_i\left(a_j^{\sigma_j}\right) = p_i\left(h^{\sigma_j}(i)\right)$, for each $1 \le j \le n$. For each $1 \le j \le n$,

define $k^{\sigma_j} : I \to \bigcup\left\{E_i^{\sigma_j} : i \in I\right\}$ by: $k^{\sigma_j}(i) = a_i^{\sigma_j}$, each $i \in F$, and

$k^{\sigma_j}(i)$ an arbitary member of $E_i^{\sigma_j}$ for each $i \notin F$, $i \in I$. We

observe that, for each $1 \le j \le n$, $p_1\left(\overline{h}^{\sigma_j}\right) = p_1\left(\overline{k}^{\sigma_j}\right)$ and

$\left(\overline{k}^{\sigma_1}, \ldots, \overline{k}^{\sigma_n}\right) \overline{\epsilon}^{\sigma} \overline{h}^{\sigma}$.

Finally we take $n < \beta$ and consider

$p\left(\overline{R}_n^{\sigma_n}\right)\left(p\left(\overline{h}^{\sigma_{n,1}}\right), \ldots, p\left(\overline{h}^{\sigma_{n,\phi(n)}}\right)\right)$. Thus if

$G = \left\{i : p_i\left(R_{i,n}^{\sigma_n}\right)\left(p_i\left(h^{\sigma_{n,1}}(i)\right), \ldots, p_i\left(h^{\sigma_{n,\phi(n)}}(i)\right)\right)\right\}$ then $G \in \mathrm{F}$.

Hence, for each $i \in G$, there exists, for each $1 \le j \le \phi(n)$,

$b_j^{\sigma_{n,j}} \in E_i^{\sigma_{n,j}}$ such that $R_{i,n}^{\sigma_n}\left(b_i^{\sigma_{n,j}}, \ldots, b_i^{\sigma_{n,\phi(n)}}\right)$ and

$p_i\left(b_i^{\sigma_{n,j}}\right) = p_i\left(h^{\sigma_{n,j}}(i)\right)$. As in the paragraph immediately above we

can define, for each $1 \le j \le \phi(n)$, $\overline{k}^{\sigma_{n,j}}$ such that

$$p\left(\overset{\sigma}{\overline{h}}n,j\right) = p\left(\overset{\sigma}{\overline{k}}n,j\right) \text{ and } \overset{\sigma}{\overline{R}}_n n\left(\overset{\sigma}{\overline{k}}n,1, \ldots, \overset{\sigma}{\overline{k}}n,\phi(n)\right) .$$

Hence $\pi N_i/F$ is a substructure of $\pi M_i/F$ with the projection maps $\underline{p} : \pi M_i/F \to \pi N_i/F$. //

2. Embedding theorems

M is a given $\kappa(\alpha, \beta)$ structure. $L = \{M_i : i \in I\}$ is termed a *local family* of M if

(i) each M_i is a substructure of M , $\underline{p_i} : M \to M_i$,

(ii) each finite subset of individuals of M is contained in a member of L .

For example the family of finitely generated substructures of M is a local family of M . (M_i is a finitely generated substructure of M if the individuals of M_i are generated by the operations of M from a finite subset of the individuals of M .)

The above definition is in accord with that of Mal'cev [1969], page 36, McLain [1959], page 177 and D.J.S. Robinson [1968], page 126. But Kurosh [1960], page 166, and Cohn [1965], page 100, in effect add a third condition to the (i) and (ii) given above, viz, that for each $i, j \in I$ there exists $k \in I$ such that $E_i^0 \subseteq E_k^0$ and $E_j^0 \subseteq E_k^0$. In fact the example given above satisfies this latter condition as well although in general this latter condition would appear to be independent of the two conditions given. Cohn's [1965], Proposition 7.4, page 101, seems dependent on the extra condition and our own remarks at the end of this section on 'inverse limits' will also require it.

Given M and a local family, $L = \{M_i : i \in I\}$, of M we form an associated ultrafilter over I. The set $\left\{F_{a^0} : a^0 \in E^0\right\}$, where $F_{a^0} = \left\{i : a^0 \in E_i^0\right\}$, each $a^0 \in E^0$, has the finite intersection property, as L is a local family, and hence (*c.f.* Gratzer [1968], Theorem 6.7 and Corollary, page 26), can serve as a sub-basis for an ultrafilter F over I. We shall call such an F an *L-associated* ultrafilter.

A constant relation $R_n^{\sigma_n}$, $n < \beta$, of M is said to be *L-finitary* if whenever it is not the case that $R_n^{\sigma_n}\left(a^{\sigma_n,1}, \ldots, a^{\sigma_n,\phi(n)}\right)$ then $\left\{i : \text{not } p_i\left(R_n^{\sigma_n}\right)\left(p_i\left(a^{\sigma_n,1}\right), \ldots, p_i\left(a^{\sigma_n,\phi(n)}\right)\right)\right\} \in \mathsf{F}$, where $\underline{p_i} : M \to M_i$, each $i \in I$.

M is said to be *L-finitary* if

(i) each constant relation of M is L-finitary,

(ii) for each $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, if $\left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) \nvdash^{\sigma} a^{\sigma}$

then $\left\{i : \left(p_i\left(a^{\sigma_1}\right), \ldots, p_i\left(a^{\sigma_n}\right)\right) \nvdash_i^{\sigma} p_i\left(a^{\sigma}\right)\right\} \in \mathsf{F}$.

For the rest of this section we shall always assume that M is given with a local family $L = \{M_i : i \in I\}$ and an L-associated ultrafilter F.

M is termed a second order structure if the members of κ are of rank ≤ 2 . That is if $\sigma \in \kappa$ then either $\sigma = 0$ or if $\sigma = (\sigma_1, \ldots, \sigma_n)$, some n , then $\sigma_j = 0$, $1 \leq j \leq n$.

THEOREM 2.1. *If M is a second order structure with L-finitary constant relations then M is L-finitary.*

Proof. Take $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) \models^{\sigma} a^{\sigma}$. Now M is second order and so $\sigma_j = 0$, each $1 \leq j \leq n$. Hence, for each $i \in F_{a^{\sigma_1}} \cap \ldots \cap F_{a^{\sigma_n}}$, $\left(p_i\left(a^{\sigma_1}\right), \ldots, p_i\left(a^{\sigma_n}\right)\right) \models_i^{\sigma} p_i\left(a^{\sigma}\right)$. //

By the same reasoning we are able to observe that if any constant relation of M takes only individuals in its arguments then it will necessarily be L-finitary.

We note that in general higher order structures are not L-finitary with respect to any arbitarily chosen local family. A simple counter example is the following situation. Let M be a $\kappa(\alpha, \beta)$ structure where $\alpha = \beta = 0$, $\kappa = \{0\ (0) = 1\ ((0)) = 2\}$, E^0 is the counting numbers, $E^1 = 2^{E^0}$ and $E^2 = 2^{E^1}$. Let L be the local family of M given by the family of all finite substructures (that is substructures with a finite number of individuals) of L .

Consider that member a^2 of E^2 such that $a^2 = \{a^1 : a^1 \in E^1$ and a^1 is finite$\}$. Let b^1 be some non-finite subset of E^0 . Hence $b^1 \models^2 a^2$. Now take any $M_i \in L$,

$\underline{p_i} : M \to M_i$. We have that $p_i(b^1) \in^2 p_i(a^2)$, as if $c^1 = F_i^0 \cap b^1$ then c^1 is finite, as F_i^0 is finite. That is $c^1 \in^2 a^2$ and $p_i(b^1) = p_i(c^1)$. Thus no member M_i of L can be found such that $p_i(b^1) \notin^2 p_i(a^2)$.

The L-finitary condition seems necessary to the various embedding theorems developed below. We have already established that second order structures with L-finitary constant relations are L-finitary with respect to any local family L . In section 4 of Chapter III we shall introduce a significant class of third order structures which are L-finitary.

For the development of the first embedding result (Theorem 2.2) and for technical and notational convenience we shall take M as a $\kappa(\alpha, \beta)$ structure where $\alpha = 0$. We shall simply call it a $\kappa(\beta)$ structure.

Let M be a $\kappa(\beta)$ structure and M_1 a $\kappa_1(\beta_1)$ structure where $\kappa \subseteq \kappa_1$ and $\beta \leq \beta_1$. λ is an injective map from κ to κ_1 such that for each $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$,
$\lambda(\sigma) = (\lambda(\sigma_1), \ldots, \lambda(\sigma_n))$.

We shall say that M is λ-*similar* to M_1 if M, M_1 and λ are as above and if, for each constant relation $R_n^{\sigma_n}$, $n < \beta$, of M , the constant relation $R_{1,n}^{\tau_n}$ of M_1 has the same number of arguments as $R_n^{\sigma_n}$ and if $\sigma_n = (\sigma_{n,1}, \ldots, \sigma_{n,\phi(n)})$ then $\tau_n = (\lambda(\sigma_{n,1}), \ldots, \lambda(\sigma_{n,\phi(n)}))$.

M is said to be λ-*embedded* into M_1 by a λ-embedding family of maps $\underline{\psi}^\lambda$ if

(i) M and M_1 are λ-similar,

(ii) for each $\sigma \in \kappa$, ψ^λ is an injective map from E^σ to $E_1^{\lambda(\sigma)}$ such that

a) for each $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) \epsilon^\sigma a^\sigma$ if and only if, $\left(\psi^\lambda\left(a^{\sigma_1}\right), \ldots, \psi^\lambda\left(a^{\sigma_n}\right)\right) \epsilon^{\lambda(\sigma)} \psi^\lambda\left(a^\sigma\right)$,

b) for each $n < \beta$, $R_n^{\sigma_n}\left(a^{\sigma_{n,1}}, \ldots, a^{\sigma_{n,\phi(n)}}\right)$ if, and only if, $R_{1,n}^{\tau_n}\left(\psi^\lambda\left(a^{\sigma_{n,1}}\right), \ldots, \psi^\lambda\left(a^{\sigma_{n,\phi(n)}}\right)\right)$.

If λ is such that $\lambda(0) = 0$ (that is λ is the natural injection of κ into κ_1) M will be said to be embedded into M_1 .

Theorem 2.2 below, or more correctly Corollary 2.3, is a generalisation of a first order embedding theorem due to Abraham Robinson [1963], Theorem 2.4.1, page 34.

THEOREM 2.2. *M is a L-finitary $\kappa(\beta)$ structure and $\kappa_1, \beta_1, \lambda$ are as above. If for each $i \in I$, M_i can be λ-embedded into a $\kappa_1(\beta_1)$ structure N_i then M can be λ-embedded into $\pi N_i/F$,* *where* F *is the* L*-associated ultrafilter.*

Proof. For each $i \in I$ let $\underline{\psi}_i^\lambda : M_i \to N_i$ be the λ-embedding of M_i into N_i. We define $\underline{\psi}^\lambda : M \to \pi N_i/F$ as follows.

(i) If $a^0 \in E^0$ put $\psi^\lambda(a^0) = \overline{h}_{a^0}$, where $h_{a^0}(i) = \psi_i^\lambda(a^0)$, for all $i \in F_{a^0}$. Thus $\overline{h}_{a^0}$ is well defined as $F_{a^0} \in F$.

(ii) If $a^\sigma \in E^\sigma$, $\sigma \in \kappa$, $\sigma \neq 0$, put $\psi^\lambda(a^\sigma) = \overline{h}_{a^\sigma}$, where $h_{a^\sigma}(i) = \psi_i^\lambda\left(p_i(a^\sigma)\right)$ for all $i \in I$.

We now show that $\underline{\psi}^\lambda$ as defined is a λ-embedding of M into $\pi N_i/F$. We note that M is λ-similar to $\pi N_i/F$.

Take $\sigma \in \kappa$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) \in^\sigma a^\sigma$. Let J be the subset of $\{1, \ldots, n\}$ such that $j \in J$ if, and only if, $\sigma_j = 0$. Put $F = \cap\left\{F_{a^{\sigma_j}} : j \in J\right\}$ and note that if $J = \emptyset$ then $F = I$. Thus $F \in F$.

Now for each $i \in F$, we have $\left(p_i\left(a^{\sigma_1}\right), \ldots, p_i\left(a^{\sigma_n}\right)\right) \in_i^\sigma p_i(a^\sigma)$ and so $\left(\psi_i^\lambda\left(p_i\left(a^{\sigma_1}\right)\right), \ldots, \psi_i^\lambda\left(p_i\left(a^{\sigma_n}\right)\right)\right) \in_i^{\lambda(\sigma)} \psi_i^\lambda\left(p_i(a^\sigma)\right)$, as ψ_i^λ is a λ-embedding of M_i into N_i. Hence $\left(\psi^\lambda\left(a^{\sigma_1}\right), \ldots, \psi^\lambda\left(a^{\sigma_n}\right)\right) \overline{\in}^{\lambda(\sigma)} \psi^\lambda(a^\sigma)$.

Conversely, assume $\left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) \notin^\sigma a^\sigma$. M is L-finitary and so there exists $F_1 \in \mathsf{F}$, where

$F_1 = \left\{i : \left(p_i\left(a^{\sigma_1}\right), \ldots, p_i\left(a^{\sigma_n}\right)\right) \not\in_i^\sigma p_i\left(a^\sigma\right)\right\}$. For each $i \in F_1$,

$\left(\psi_i^\lambda\left(p_i\left(a^{\sigma_1}\right)\right), \ldots, \psi_i^\lambda\left(p_i\left(a^{\sigma_n}\right)\right)\right) \not\in_i^{\lambda(\sigma)} \psi_i^\lambda\left(p_i\left(a^\sigma\right)\right)$ and so

$\left(\psi^\lambda\left(a^{\sigma_1}\right), \ldots, \psi^\lambda\left(a^{\sigma_n}\right)\right) \overline{\not\in}^{\lambda(\sigma)} \psi^\lambda\left(a^\sigma\right)$.

By a similar argument it can be shown that for each $n < \beta$,

$R_n^{\sigma_n}\left(a^{\sigma_{n,1}}, \ldots, a^{\sigma_{n,\phi(n)}}\right)$ if, and only if,

$\overline{R}_n^{\tau_n}\left(\psi^\lambda\left(a^{\sigma_{n,1}}\right), \ldots, \psi^\lambda\left(a^{\sigma_{n,\phi(n)}}\right)\right)$.

We finally show that $\underline{\psi}^\lambda$ is injective. Take $a^0, b^0 \in E^0$ such that $\psi^\lambda\left(a^0\right) = \psi^\lambda\left(b^0\right)$. For each $i \in F_{a^0} \cap F_{b^0}$, $\psi_i^\lambda\left(a^0\right) = \psi_i^\lambda\left(b^0\right)$ and so $a^0 = b^0$, as ψ_i^λ is injective. That is $\psi^\lambda : E^0 \to \overline{F}^{\lambda(0)}$ is injective.

Take any $\sigma \in \kappa$, $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$ and $a^\sigma, b^\sigma \in E^\sigma$. If $a^\sigma \neq b^\sigma$ then there exists some $\left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) \in^\sigma a^\sigma$ and $\left(a^{\sigma_1}, \ldots, a^{\sigma_n}\right) \not\in^\sigma b^\sigma$, or vice-versa. Assume the former. M is L-finitary and so $G \in \mathsf{F}$, where

$G = \left\{i : \left(p_i\left(a^{\sigma_1}\right), \ldots, p_i\left(a^{\sigma_n}\right)\right) \not\in_i^\sigma p_i\left(b^\sigma\right)\right\}$. But for each $i \in G$ we have $\left(p_i\left(a^{\sigma_1}\right), \ldots, p_i\left(a^{\sigma_n}\right)\right) \in_i^\sigma p_i\left(a^\sigma\right)$. Hence, for each $i \in G$,

$\left(\psi_i^\lambda\left(p_i\left(a^{\sigma_1}\right)\right), \ldots, \psi_i^\lambda\left(p_i\left(a^{\sigma_n}\right)\right)\right)$ belongs to $\psi_i^\lambda\left(p_i\left(a^\sigma\right)\right)$ but does not belong to $\psi_i^\lambda\left(p_i\left(b^\sigma\right)\right)$. That is $\left(\psi^\lambda\left(a^{\sigma_1}\right), \ldots, \psi^\lambda\left(a^{\sigma_n}\right)\right)$ belongs to $\psi^\lambda\left(a^\sigma\right)$ but does not belong to $\psi^\lambda\left(b^\sigma\right)$. That is $\psi^\lambda\left(a^\sigma\right) \neq \psi^\lambda\left(b^\sigma\right)$. Therefore $\psi^\lambda : E^\sigma \to \overline{F}^{\lambda(\sigma)}$ is injective. //

COROLLARY 2.3. *If* $M, \kappa_1, \beta_1, \lambda$ *are as above and if each* M_i, $i \in I$, *can be* λ*-embedded into a model of* Σ, *where* Σ *is a class of sentences of* $L\left(\kappa_1\left(\beta_1\right)\right)$, *then* M *can be* λ*-embedded into a model of* Σ.

Proof. For each $i \in I$ let N_i be a model of Σ such that M_i is λ-embedded into N_i. By Theorem 2.2, M is λ-embedded into $\pi N_i/F$ which by Corollary I: 3.5, is a model of Σ. //

We observe that by Corollary I: 3.11, Corollary 2.3 would still hold if Σ were a class of sentences of W_1' where W_1 is the class of formulae of $L\left(\kappa_1\left(\beta_1\right)\right)$.

COROLLARY 2.4. *If* M *is a* L*-finitary* $\kappa(\alpha, \beta)$ *structure then* M *can be embedded into* $\pi M_i/F$.

Proof. In Theorem 2.2 take $\kappa_1 = \kappa$, $\beta_1 = \beta$, λ such that $\lambda(0) = 0$, and for each $i \in I$ take $N_i = M_i$. Hence the result. //

Strictly the statement of the above corollary, in terms of Theorem 2.2 requires $\alpha = 0$. However if $\alpha \neq 0$ it can be checked that the embedding $\underline{\psi} : M \to \pi M_i/F$ preserves the operations of M with respect to those of $\pi M_i/F$. In the sequel the embedding of M into $\pi M_i/F$ will always be denoted by $\underline{\psi} : M \to \pi M_i/F$, defined as in the proof of Theorem 2.2 but with the necessary modification that the ψ_i's are the identity maps of M_i to M_i , each $i \in I$.

COROLLARY 2.5. *If M is a L-finitary $\kappa(\alpha, \beta)$ structure and a class of sentences of a) $L(\kappa(\alpha, \beta))$ or, b) $W(\pi M_i/F)$ or c) W' such that each M_i , $i \in I$ is a model of Σ then M can be embedded in a model of Σ , viz. $\pi M_i/F$.*

Proof. By Corollary 2.4 and a) Corollary I: 3.5 or b) Corollary I: 3.8, or c) Corollary I: 3.11, respectively. //

Consider the embedding $\underline{\psi} : M \to \pi M_i/F$ of Corollary 2.4. If M is a first order structure the image of M under $\underline{\psi}$ is of course a substructure of $\pi M_i/F$, but in the higher order case this in general will not be so. The next theorem provides particular circumstances in which an isomorphic relationship can be established between M and a substructure of $\pi M_i/F$. It will prove to be the means, together with Corollary 2.5 above, of obtaining local theorems as discussed in section 3 of Chapter III. We first require a lemma and some definitions.

LEMMA 2.6. *If $\psi : M \to \pi M_i/F$ as above then $\psi(E^0)$ is closed with respect to the operations of $\pi M_i/F$.*

Proof. For $m < \alpha$ and $a_1^0, \ldots, a_{\theta(m)}^0 \in E^0$ we have

$\psi\left(f_m\left(a_1^0, \ldots, a_{\theta(m)}^0\right)\right) = \overline{f}_m\left(\psi\left(a_1^0\right), \ldots, \psi\left(a_{\theta(m)}^0\right)\right)$ and hence the result. //

We denote by

$N_\psi = \left\{\overline{F}_\psi^\sigma : \sigma \in \kappa\right\} \cup \{\overline{f}_m : m < \alpha\} \cup \left\{p\left(\overline{R}_n^{\sigma_n}\right) : n < \beta\right\}$ the substructure of $\pi M_i/F$ based on the subset of individuals of $\pi M_i/F$ given by $\psi\left(E^0\right)$, where $\underline{p} : \pi M_i/F \to N_\psi$ is the family of substructure projection maps.

A constant relation of M, $R_n^{\sigma_n}$, $n < \beta$, is said to be *L-stable* if whenever it is not the case that $R_n^{\sigma_n}\left(a^{\sigma_n,1}, \ldots, a^{\sigma_n,\theta(n)}\right)$ then it is not the case that $p\left(\overline{R}_n^{\sigma_n}\right)\left(p\left(\psi\left(a^{\sigma_n,1}\right)\right), \ldots, p\left(\psi\left(a^{\sigma_n,\phi(n)}\right)\right)\right)$.

LEMMA 2.7. *If* $R_n^{\sigma_n}$, $n < \beta$, *is an L-stable constant relation of* M *then it is L-finitary.*

Proof. Take $a^{\sigma_n,j} \in E^{\sigma_n,j}$, $1 \le j \le \phi(n)$, such that it is not the case that $R_n^{\sigma_n}\left(a^{\sigma_n,1}, \ldots, a^{\sigma_n,\phi(n)}\right)$. Hence it is not the case that $p\left(\overline{R}_n^{\sigma_n}\right)\left(p\left(\psi\left(a^{\sigma_n,1}\right)\right) \ldots, p\left(\psi\left(a^{\sigma_n,\phi(n)}\right)\right)\right)$ and so it is not the case that $\overline{R}_n^{\sigma_n}\left(\psi\left(a^{\sigma_n,1}\right), \ldots, \psi\left(a^{\sigma_n,\phi(n)}\right)\right)$. Hence by Theorem I: 3.4, $\left\{i : p_i\left(R_n^{\sigma_n}\right)\left(p_i\left(a^{\sigma_n,1}\right), \ldots, p_i\left(a^{\sigma_n,\phi(n)}\right)\right) \text{ does not hold}\right\} \in F$. That is $R_n^{\sigma_n}$ is L-finitary. //

THEOREM 2.8. *If the* $\kappa(\alpha, \beta)$ *structure* M *is second order, full and with each of its constant relations* L*-stable then* M *and* N_ψ *are isomorphic structures with* $\underline{p\psi}$ *a related family of isomorphisms.*

Proof. We first observe that $p\psi$ is a bijective map from E^0 to $\overline{E}^0_\psi$ which preserves the operations of M with respect to those of N_ψ.

Now take $\sigma \in \kappa$, $\sigma \neq 0$, and so $\sigma = (0, \ldots, 0)$, as M is second order. Take any $p\left(\overline{h}^\sigma\right) \in \overline{E}^\sigma_\psi$. Let

$K = \left\{\left(a_1^0, \ldots, a_n^0\right) : \left(\overline{h}_{a_1^0}, \ldots, \overline{h}_{a_n^0}\right) \overline{\epsilon}^\sigma \, \overline{h}^\sigma\right\}$, where $\psi\left(a_j^0\right) = \overline{h}_{a_j^0}$, each $1 \leq j \leq n$. M is full and so there exists some $a^\sigma \in E^\sigma$ such that $\left(a_1^0, \ldots, a_n^0\right) \epsilon^\sigma a^\sigma$ if, and only if, $\left(a_1^0, \ldots, a_n^0\right) \in K$. It can now be checked that $p\psi\left(a^\sigma\right) = p\left(\overline{h}^\sigma\right)$. That is $p\psi$ is a surjective map from E^σ onto $\overline{E}^\sigma_\psi$.

Next we observe that if $\left(a_1^0, \ldots, a_n^0\right) \epsilon^\sigma a^\sigma$ then $\left(p\psi\left(a_1^0\right), \ldots, p\psi\left(a_n^0\right)\right) \overline{\epsilon}^\sigma \, p\psi\left(a^\sigma\right)$. Conversely, take $\left(a_1^0, \ldots, a_n^0\right) \not\epsilon^\sigma a^\sigma$ and so $\left(\psi\left(a_1^0\right), \ldots, \psi\left(a_n^0\right)\right) \not\overline{\epsilon}^\sigma \psi\left(a^\sigma\right)$, as ψ is an embedding. Thus $\left(p\psi\left(a_1^0\right), \ldots, p\psi\left(a_n^0\right)\right) \not\overline{\epsilon}^\sigma \, p\psi\left(a^\sigma\right)$, as $\psi\left(a_1^0\right), \ldots, \psi\left(a_n^0\right) \in \overline{E}^0_\psi$.

From the above we can now establish that $p\psi$ is injective from E^σ to $\overline{E}^\sigma_\psi$, $\sigma = (0, \ldots, 0)$. For if $p\psi\left(a^\sigma\right) = p\psi\left(b^\sigma\right)$ then

$(a_1^0, \ldots, a_n^0) \in^\sigma a^\sigma$ if, and only if, $(a_1^0, \ldots, a_n^0) \in^\sigma b^\sigma$ and so $a^\sigma = b^\sigma$.

We finally note that each constant relation of M is L-stable and hence the result is established. //

Our use of the result of Theorem 2.8 in Chapter III, section 3, will be to assert that if a formula A of $L(\kappa(\alpha, \beta))$ holds in N_ψ then it also holds in M , where M and N_ψ are as above. The isomorphism between M and N_ψ guarantees this. For further application (to the results of Mal'cev and Kogalovskiĭ) we shall want to assert this same property for certain formulae even though we may not be able to establish a complete isomorphic relationship between M and N_ψ . The next theorem is concerned with this.

Let κ contain only types of rank ≤ 2 . Let M be a $\kappa(\alpha, \beta)$ structure that is full and with each constant relation L-stable. For some cardinal $\beta' \geq \beta$ we adjoin to M the constant relations $R_n^{\sigma_n}$, $\beta \leq n < \beta'$, anyone of which need not be L-stable with respect to M .

Let $A \in L(\kappa(\alpha, \beta'))$ be in prenex normal form with its quantifier free portion A_0 in disjunctive normal form and such that if $R_n^{\sigma_n}$ is not L-stable then it can only occur (if at all) in the negation of an atomic component of A_0 .

THEOREM 2.9. *If M and A are as above then $M \models_\nu A$ if*

and only

if $N_\psi \models_{p\bar{\nu}} A$ *, for any M-interpretation* ν *.*

Proof. A proof can proceed by an induction argument that establishes the result for A_0 and then shows that the result is preserved with the addition of each quantifier leading to the final formula A. The result for A_0 is established from the fact that if a constant relation $R_n^{\sigma_n}$ is L-stable then any atomic formula involving it holds in N_ψ if, and only if, it holds in M, (with respect to any pair of related assignments), but if $R_n^{\sigma_n}$ is not L-stable then all that can be asserted is that if the negation of an atomic formulae involving $R_n^{\sigma_n}$ holds in N_ψ then it holds in M. //

A result which extends Theorem 2.8 into a third order context is given by Theorem 4.7 of Chapter III. It could well have been included in this present section.

Finally in this section we indicate how inverse limits occur naturally in the context of a (directed) local family of substructures of a $\kappa(\alpha, \beta)$ system M. It is of interest to note that Neumann [1954], page 145 and Kurosh [1960], pages 169-170, both use an inverse limit construction on a family of subsets in proof of local theorems associated with group structures. The theorem below may indicate the common base to their methods and the method via ultraproducts set out in Chapter III.

For the rest of this section (only) we shall assume that the local family L of M is such that I can be directed by means of the partial ordering, $i \leq j$ if, and only if, $E_i^0 \subseteq E_j^0$ and that the L-associated ultrafilter F is constructed from $\{F_i : i \in I\}$ as sub-basis, where $F_i = \{j : i \leq j\}$, each $i \in I$.

For each $\sigma \in \kappa$, $\sigma \neq 0$, $\left\{E_i^\sigma : i \in I\right\}$ forms an inverse family with respect to the projection maps $p_{i,j} : E_j^\sigma \to E_i^\sigma$, where $i \leq j$ and $p_{i,j} : M_j \to M_i$. It can be checked that if $i = j$ then $p_{i,j}$ is the identity map on E_j^0 and Theorem 1.7 ensures that if $i \leq j \leq k$ then $p_{k,j}\, p_{j,i} = p_{k,i}$. Let E_∞^σ denote the inverse limit of the inverse family. That is

$$E_\infty^\sigma = \left\{h^\sigma : h^\sigma \in \pi\left\{E_i^\sigma : i \in I\right\} \text{ and for all } i, j \in I, \text{ if } i \leq j \text{ then } p_{i,j} f(j) = f(i)\right\}.$$

In the case $\sigma = 0$, $\left\{E_i^0 : i \in I\right\}$ forms a direct family of sets with respect to the p^0 maps, or more correctly, their inverses. Let E_∞^0 denote the direct limit of this family. In fact E_∞^0 can be identified with the subclass of $\pi E_i^0/\mathsf{F}$ given by the image of the embedding $\psi : E^0 \to \pi E_i^0/\mathsf{F}$ of Corollary 2.4 above, (*c.f.* Grätzer [1968], Exercise 33, page 156).

With a slight distortion of language we shall call

$M_\infty^L = \left\{E_\infty^\sigma : \sigma \in \kappa\right\} \cup \left\{\epsilon_\infty^\sigma : \sigma \in \kappa,\ \sigma \neq 0\right\} \cup \{\overline{f}_m : m < \alpha\} \cup \left\{R_{n,\infty}^{\sigma_n} : n < \beta\right\}$

the inverse limit of $L = \{M_i : i \in I\}$, where for each $\sigma \in \kappa$,

$\sigma = (\sigma_1, \ldots, \sigma_n)$, $\left(h^{\sigma_1}, \ldots, h^{\sigma_n}\right) \epsilon_\infty^\sigma h^\sigma$ if, and only if,

$\left\{i : \left(h^{\sigma_1}(i), \ldots, h^{\sigma_n}(i)\right) \epsilon_i^\sigma h^\sigma(i)\right\} \in F$, the constant relations

$R_{n,\infty}^{\sigma_n}$, $n < \beta$, are likewise defined and the $\overline{f}_m$, $m < \alpha$, are the operations as defined for $\pi M_i/F$. (Of course in the above definition if one of the $\sigma_j = 0$ then the definition must be shown independent of the particular representation of the associated member of E_∞^0 . This can be done.)

M_∞^L as defined above is a $\kappa(\alpha, \beta)$ structure. But even though M and hence each M_i , $i \in I$, are normal structures it does not seem that in general M_∞^L is a normal structure. However none the less we do have the following result.

THEOREM 2.10. *If* M *is* L*-finitary then there exist embeddings* $\underline{\psi}_1 : M \to M_\infty^L$, $\underline{\psi}_2 : M_\infty^L \to \pi M_i/F$ *and such that* $\underline{\psi}_2\underline{\psi}_1 = \underline{\psi}$ *as defined for Corollary* 2.4.

Proof. We define $\underline{\psi}_1$ by: if $a^0 \in E^0$ put $\psi_1(a^0) = \overline{h}_{a^0}$,

if $a^\sigma \in E^\sigma$, $\sigma \neq 0$, put $\psi_1(a^\sigma) = h_{a^\sigma}$.

We define $\underline{\psi}_2$ by: $\psi_2 : E_\infty^0 \to \overline{E}^0$ is the natural injection and if $h^\sigma \in E_\infty^\sigma$, $\sigma \in \kappa$, $\sigma \neq 0$, put $\psi_2(h^\sigma) = \overline{h}^\sigma$.

We check that $\underline{\psi}_1$ and $\underline{\psi}_2$ are injective. For $\sigma = 0$ this is immediate. Take $\sigma \in \kappa$, $\sigma \neq 0$ and $\psi_1(a^\sigma) = \psi_1(b^\sigma)$. Thus, for each $i \in I$, $p_i(a^\sigma) = p_i(b^\sigma)$ and so $a^\sigma = b^\sigma$, as M is L-finitary. That is $\underline{\psi}_1$ is injective. Now consider $\psi_2\left(h_1^\sigma\right) = \psi_2\left(h_2^\sigma\right)$, where $h_1^\sigma, h_2^\sigma \in E_\infty^\sigma$. Hence $F = \left\{i : h_1^\sigma(i) = h_2^\sigma(i)\right\} \in \mathsf{F}$. Take any $k \in I$. $F_k \in \mathsf{F}$ and so $F \cap F_k \neq \emptyset$. Take $j \in F \cap F_k$. Thus $h_1(k) = p_{k,j}\left(h_1^\sigma(j)\right)$ and $h_2(k) = p_{k,j}\left(h_2^\sigma(j)\right)$. But $h_1^\sigma(j) = h_2^\sigma(j)$ and so $h_1^\sigma(k) = h_2^\sigma(k)$. Hence $h_1^\sigma = h_2^\sigma$ and $\underline{\psi}_2$ is injective.

The rest of the embedding properties required for $\underline{\psi}_1$ and ψ_2 can be checked. Finally, it is immediate from the definitions of $\underline{\psi}_1$ and $\underline{\psi}_2$ that $\underline{\psi}_2\underline{\psi}_1 = \underline{\psi}$. //

CHAPTER III

SOME ALGEBRAIC APPLICATIONS

Summary. The content of this chapter deals in various ways with local properties of algebraic structures. Each of the situations taken is an already established one in its own algebraic context. What is new and of interest is the application of the model-theoretic results of the previous chapters to these situations.

In section 1 the Stone [1936] Representation Theorem is established in terms of a second order ultraproduct. In section 2 the presence of the ultraproduct construction is made explicit in the properties of Sylow (maximal) p-subgroups of locally normal groups. Section 3 develops an alternative approach to those already available in the literature to the metatheory of local theorems in universal algebras and in particular incorporates in the approach local theorems of J.P. Cleave [1969], and A.I. Mal'cev [1959]. Section 4 continues the theme of section 3 in the context of chain conditions in universal algebras with particular reference to the local theorem of D.H. McLain [1959].

1. Stone's Representation Theorem for (non-finite) Boolean algebras

THEOREM 1.1. *Any infinite boolean algebra is isomorphic to a subset subalgebra of a second order ultraproduct.*

Proof. Let M be a boolean algebra regarded as a $\kappa(\alpha, \beta)$ structure where $\kappa = \{0\}$, $\alpha = 0$, $\beta = 3$. E^0 is the individuals of the algebra. $R_0^{\sigma_0}$, $\sigma_0 = (0, 0)$, will be the two-placed relation arising from the complement operator; $R_1^{\sigma_1}$ and $R_2^{\sigma_2}$, where

$\sigma_1 = \sigma_2 = (0, 0, 0)$, will be the three-placed relations arising from the meet and join operators respectively of the boolean algebra.

$L = \{M_i : i \in I\}$ is the local family of substructures of M arising from the finitely generated (and hence finite) subalgebras of the boolean algebra. F is the L-associated ultrafilter over I .

Take $\kappa_1 = \{0 \ (0) = 1\}$, $\alpha_1 = 0$, $\beta_1 = 3$. The relation $R_0^{\tau_0}$, where $\tau_0 = (1, 1)$, we shall denote by C ; the relations $R_1^{\tau_1}, R_2^{\tau_2}$, where $\tau_1 = \tau_2 = (1, 1, 1)$, we shall denote by $\cap$ and $\cup$ respectively.

Let λ be a map from κ to κ_1 given by $\lambda(0) = 1$. Hence M is λ-similar to any $\kappa_1(\alpha_1, \beta_1)$ structure.

Σ is the set of sentences of $L(\kappa_1(\alpha_1, \beta_1))$ given as follows and where $\underline{C}x^1 = y^1$ stands for $\underline{C}(x^1, y^1)$ and likewise for $x^1 \mathbin{\underline{\cup}} y^1 = z^1$ and $x^1 \mathbin{\underline{\cap}} y^1 = z^1$.

(1) $\forall x^1 \forall y^1 \left(\underline{C}x^1 = y^1 \Leftrightarrow \forall x^0 \left(x^0 \mathbin{\underline{\epsilon}^1} x^1 \Leftrightarrow x^0 \not\epsilon^1 y^1\right)\right)$.

(2) $\forall x^1 \forall y^1 \forall z^1 \left(x^1 \mathbin{\underline{\cup}} y^1 = z^1 \Leftrightarrow \forall x^0 \left(x^0 \mathbin{\underline{\epsilon}^1} z^1 \Leftrightarrow \left(x^0 \mathbin{\underline{\epsilon}^1} x^1 \vee x^0 \mathbin{\underline{\epsilon}^1} y^1\right)\right)\right)$.

(3) $\forall x^1 \forall y^1 \forall z^1 \left(x^1 \mathbin{\underline{\cap}} y^1 = z^1 \Leftrightarrow \forall x^0 \left(x^0 \mathbin{\underline{\epsilon}^1} z^1 \Leftrightarrow \left(x^0 \mathbin{\underline{\epsilon}^1} x^1 \wedge x^0 \mathbin{\underline{\epsilon}^1} y^1\right)\right)\right)$.

We observe that the sentences of Σ have been chosen so that in any full model of Σ , $\underline{C}$, $\underline{\cap}$ and $\underline{\cup}$ will represent the set complement, set intersection and set union operations respectively.

Now for each $i \in I$ and by a standard result for finite boolean algebras M_i is isomorphic to a full subset algebra. That is each M_i can be λ-embedded in N_i a model of Σ. Thus by Theorem II: 2.2, M can be λ-embedded in $\pi N_i/F$ also a model of Σ. We note that if $\underline{\psi}^{\lambda} : M \to \pi N_i/F$ is the λ-embedding then the form of the sentences of Σ ensures that $\psi^{\lambda}(E^0)$ is closed with respect to C, $\cap$ and $\cup$. That is M is isomorphic to a subset algebra of $\pi N_i/F$. Of course, although each N_i, $i \in I$, is a full subset algebra the ultraproduct will in general not be full. //

We comment finally that Theorem I: 1.6 enables the formal membership relations of $\pi N_i/F$ to be translated into actual set membership relations thus establishing Stone's Representation Theorem in its set context.

The above discussion could of course be restricted by the elimination of the complement operator and sentence (1) of Σ to establish that every distributive lattice is isomorphic to a ring of subsets of a second order ultraproduct.

2. Sylow p-groups of locally normal groups

The following three properties from the theory of finite groups are assumed. (*c.f.* Kurosh [1960].)

I. For any two Sylow p-subgroups P and Q, of a finite group G there exists an inner automorphism of G which when restricted to P is an isomorphism from P to Q.

II. If H is a normal subgroup of a finite group G, P a Sylow p-subgroup of G, then $P \cap H$ is a Sylow p-subgroup of H.

III. If P is a p-subgroup of a finite group G, N a normal subgroup of G such that $N \supseteq P$ and Q a Sylow p-subgroup of N containing P, then there exists a Sylow p-subgroup Q', of G which contains P and such that $Q' \cap N = Q$.

We recall that a *locally normal group* is one such that each finite subset of elements of the group is contained in a finite normal subgroup.

Take a locally normal group M and regard it as a $\kappa(\alpha, \beta)$ structure where $\kappa = \{0\ (0) = 1\ \ (0, 0) = \delta\}$, $\alpha = 3$, $\beta = 0$, E^0 = the set of individuals of the group, $E^1 = 2^{E^0}$, $E^\delta = E^0 \times E^0$, $f_0 = e$, the group identity, $f_1 = {}^{-1}$, the unary inverse operator and $f_2 = \cdot$, the group binary operation. We shall use the ordinary group notation.

Let $L = \{M_i : i \in I\}$ be the family of all finite, normal subgroups of M. Hence L is a local family of M as M is a local normal group. F is a L-associated ultrafilter over I.

We describe a collection of sentences and formulae of $L(\kappa(\alpha, \beta))$. We shall also use the symbols e, ${}^{-1}$ and $\cdot$ to denote the operation symbols.

K denotes the conjunction of sentences characterising group structure.

$G_s(x^1)$ denotes the formula

$$\forall x^0 \forall y^0 \left(x^1 \neq \emptyset \wedge \left(x^0 \mathrel{\underline{\epsilon}^1} x^1 \wedge y^0 \mathrel{\underline{\epsilon}^1} x^1 \Rightarrow x^0\, y^{-1} \mathrel{\underline{\epsilon}^1} x^1\right)\right)$$

expressing that x^1 is a subgroup.

$S_1(y^0)$ denotes the formula

$$y^0 = e \ .$$

$S_n(y^0)$, (n any positive integer > 1) denotes the formula

$$\left(y^0\right)^n = e \wedge \neg\, S_{n-1}\left(y^0\right)$$

expressing that y^0 is of order n .

$x^1 \equiv y^1(\omega^\delta)$ denotes the conjunction of the following formulae.

(i) $G_s(x^1) \wedge G_x(y^1)$,

(ii) $\forall z^0 \left(z^0 \mathrel{\underline{\epsilon}^1} x^1 \Rightarrow \left(\exists^1 u^0 \left(u^0 \mathrel{\underline{\epsilon}^1} y^1 \wedge \left(z^0, u^0\right) \mathrel{\underline{\epsilon}^\delta} w^\delta\right)\right)\right)$,

(iii) $\forall x^0 \forall y^0 \forall z^0 \big(x^0 \mathrel{\underline{\epsilon}^1} x^1 \wedge y^0 \mathrel{\underline{\epsilon}^1} x^1 \wedge z^0 \mathrel{\underline{\epsilon}^1} y^1 \wedge$

$$\left(x^0, z^0\right) \mathrel{\underline{\epsilon}^\delta} w^\delta \wedge \left(y^0, z^0\right) \mathrel{\underline{\epsilon}^\delta} w^\delta \Rightarrow x^0 = y^0\big) \ ,$$

(iv) $\forall z^0 \left(z^0 \mathrel{\underline{\epsilon}^1} y^1 \Rightarrow \exists x^0 \left(x^0 \mathrel{\underline{\epsilon}^1} x^1 \wedge \left(x^0, z^0\right) \mathrel{\underline{\epsilon}^\delta} w^\delta\right)\right)$

(v) $\forall x^0 \forall y^0 \forall u^0 \forall v^0 \big(x^0 \mathrel{\underline{\epsilon}^1} x^1 \wedge y^0 \mathrel{\underline{\epsilon}^1} x^1 \wedge u^0 \mathrel{\underline{\epsilon}^1} y^1 \wedge v^0 \mathrel{\underline{\epsilon}^1} y^1 \wedge$

$$\left(x^0, u^0\right) \mathrel{\underline{\epsilon}^\delta} w^\delta \wedge \left(y^0, v^0\right) \mathrel{\underline{\epsilon}^\delta} w^\delta \Rightarrow \left(x^0 \cdot y^0, u^0 \cdot v^0\right) \mathrel{\underline{\epsilon}^\delta} w^\delta\big) \ ,$$

expressing that w^δ is an isomorphism between subgroups x^1 and y^1 .

Let $\underline{\psi} : M \to \pi M_i/F$ be the natural embedding of M into $\pi M_i/F$.
If $a^1 \in E^1$ we let $\widehat{\psi(a^1)}$ denote $\{\psi(a^0) : a^0 \in^1 a^1\}$.

THEOREM 2.1. *If M is as above then (i) $\pi M_i/F$ is a group, (ii) for each $a^1 \in E^1$, a^1 is a subgroup of M if, and only if,*

$\psi(a^1)$ *is a subgroup of* $\pi M_i/F$, *(iii) for each* $a^1 \in E^1$, a^1 *is a p-subgroup of* M *if, and only if,* $\widehat{\psi(a^1)}$ *is a p-subgroup of* $\pi M_i/F$.

Proof. (i) $\pi M_i/F \models K$, as $\{i : M_i \models K\} = I \in F$, and so $\pi M_i/F$ is a group.

(ii) Take $a^1 \in E^1$ and let ν be any M interpretation such that $\nu(x^1) = a^1$. For each $i \in I$ let ν_i be a M_i interpretation such that $\nu_i(x^1) = p_i(a^1)$, where $\underline{p}_i : M \to M_i$ is the family of substructure projections. Let $\overline{\nu}$ be the induced $\pi M_i/F$ interpretation such that $\overline{\nu}(x^1) = \overline{h}_{a^1}$, where $h_{a^1}(i) = p_i(a^1)$ each $i \in I$.

Now for any $i \in I$, a^1 is a subgroup of M if, and only if, $p_i(a^1)$ is a subgroup of M_i . Thus if a^1 is a subgroup of M then $\left\{i : M_i \models_{\nu_i} G_s(x^1)\right\} \in F$ and so by Theorem I: 3.4, $\pi M_i/ \models_{\overline{\nu}} G_s(x^1)$. That is $\psi(a^1)$ is a subgroup of $\pi M_i/F$. Conversely, if $\psi(a^1)$ is a subgroup of $\pi M_i/F$ then $\pi M_i/F \models_{\overline{\nu}} G_s(x^1)$ and so again by Theorem I: 3.4, $\left\{i : M_i \models_{\nu_i} G_s(x^1)\right\} \in F$. That is there exists some $i \in I$ such that $p_i(a^1)$ is a subgroup of M_i and hence a^1 is a subgroup of M .

(iii) A proof is established by application of Theorem II: 1.13. //

The final two theorems are results first proved by Baer [1940]. Kurosh [1960] uses the methods of projection sets, (inverse limits) to prove them. We provide alternative proofs via the ultraproduct construction.

THEOREM 2.2. *If* M *is a locally normal group as above and* a^1 *a given Sylow-p-subgroup of* M *then for each* $i \in I$, $p_i(a^1)$ *is a Sylow* p*-subgroup of* M_i *, where* $\underline{p}_i : M \to M_i$.

Proof. Assume that $p_i(a^1)$ is not a Sylow p-subgroup of M_i , for some $i \in I$. Put $F = \{j : M_j \supseteq M_i\}$ and so $F \in \mathsf{F}$ as M_i is finite. Let $p_i(b^1)$ be a Sylow p-subgroup of M_i such that $p_i(b^1) \supset p_i(a^1)$.

Now from property II of finite groups listed above we have, for each $j \in F$, that $p_j(a^1)$ is not a Sylow p-subgroup of M_j . Hence for each $j \in F$, and by property III above, there exists a Sylow p-subgroup, $p_j\left(c_j^1\right)$, of M_j such that $p_j(a^1) \subset p_j\left(c_j^1\right)$ and $p_j\left(c_j^1\right) \cap M_i = p_i(b^1)$.

Take $\overline{h}^1 \in \pi M_i/\mathsf{F}$, where $h^1(j) = p_j\left(c_j^1\right)$, each $j \in F$ and so $\overline{h}^1$ is a subgroup of $\pi M_i/\mathsf{F}$ and, further, $\overline{h}^1 \cap \psi(\hat{M})$ is a p-subgroup of $\psi(\hat{M})$, where $\psi(\hat{M}) = \{\psi(a^0) : a^0 \in E^0\}$. Also for each $j \in F$, we have $p_j(a^1) \subset p_j\left(c_j^1\right)$ and so $\psi(a^1) \subset \overline{h}^1$ and hence

$\psi\left(\widehat{a^1}\right) \subseteq \overline{h}^1 \cap \psi(\hat{M})$. But ψ is an embedding, thus $\psi\left(\hat{a}^1\right)$ is a Sylow p-subgroup of $\psi(\hat{M})$, as a^1 is a Sylow p-subgroup of M . Therefore $\psi\left(\hat{a}^1\right) = \overline{h}^1 \cap \psi(\hat{M})$.

But $p_i\left(a^1\right) \subset p_i\left(b^1\right)$ and so there exists some $a^0 \in p_i\left(b^1\right)$ but $a^0 \notin p_i\left(a^1\right)$. Further, for each $j \in F$, we have $p_j\left(c_j^1\right) \cap M_i = p_i\left(b^1\right)$ and so $a^0 \in p_j\left(c_j^1\right)$, whereas $a^0 \notin p_j\left(a^1\right)$. Therefore, if $\psi\left(a^0\right) = \overline{h}_{a^0}$ then $\overline{h}_{a^0} \in \overline{h}^1 \cap \psi(\hat{M})$ but $\overline{h}_{a^0} \notin \psi\left(\hat{a}^1\right)$ contradicting the conclusion at the end of the paragraph above.

From the contradiction it is established that $p_i\left(a^1\right)$ is a Sylow p-subgroup of M_i , each $i \in I$. //

THEOREM 2.3. *If M is a locally normal group as above then any two Sylow p-subgroups of M are isomorphic and locally conjugate.*

Proof. Let $a^1, b^1 \in E^1$ be two given Sylow p-subgroups of M . By Theorem 2.2, for each $i \in I$, we have $p_i\left(a^1\right), p_i\left(b^1\right)$ are Sylow p-subgroups of M_i . Hence, by property I of finite groups listed above, for each $i \in I$ there exists $c_i^\delta \in E^\delta$ such that $p_i\left(c_i^\delta\right)$ is an inner automorphism of M_i taking $p_i\left(a^1\right)$ to $p_i\left(b^1\right)$.

For each $i \in I$, let ν_i be a M_i interpretation such that $\nu_i\left(x^\delta\right) = p_i\left(c_i^\delta\right)$, $\nu_i\left(x^1\right) = p_i\left(a^1\right)$ and $\nu_i\left(y^1\right) = p_i\left(b^1\right)$. Let

$\overline{h}^{\delta} \in \pi M_i/\mathcal{F}$, where $h^{\delta}(i) = p_i\left(c_i^{\delta}\right)$, each $i \in I$.

Now $\left\{i : M_i \models_{\nu_i} x^1 \equiv y^1\left(x^{\delta}\right)\right\} = I$ and so, by Theorem I: 3.4, we have $\pi M_i/\mathcal{F} \models_{\bar{\nu}} x^1 \equiv y^1\left(x^{\delta}\right)$. That is $\overline{h}^{\delta}$ is an isomorphism between $\psi\left(a^1\right)$ and $\psi\left(b^1\right)$.

We first show that $\overline{h}^{\delta}$ restricted to $\psi\left(\hat{a}^1\right)$ is an isomorphism between $\psi\left(\hat{a}^1\right)$ and $\psi\left(\hat{b}^1\right)$. For this it is sufficient to show that if $\overline{h}^{\delta}\left(\overline{h}_1^0\right) = \overline{h}_2^0$, (as $\left(\overline{h}_1^0, \overline{h}_2^0\right) \in {}^{\delta}\overline{h}^{\delta}$ will be now written for notational convenience), and $\overline{h}_1^0 \in \psi\left(\hat{a}^1\right)$ then $\overline{h}_2^0 \in \psi\left(\hat{b}^1\right)$.

Take $a^0 \in a^1$ and so $\psi\left(a^0\right) = \overline{h}_{a^0}$. Put $\overline{h}^{\delta}\left(\overline{h}_{a^0}\right) = \overline{h}^0$ and we shall establish that $\overline{h}^0 \in \psi\left(b^1\right)$. Let $G = \left\{i : p_i\left(c_i^{\delta}\right)\left(a^0\right) = h^0(i)\right\}$ and so $G \in \mathcal{F}$. Take some $k \in I$ such that $a^0 \in E_k^0 = \left\{a_1^0, \ldots, a_m^0\right\}$. Put $F = \cap\left\{F_{a_j^0} : 1 \le j \le m\right\}$ and so $F \in \mathcal{F}$. Now for each $i \in F \cap G$, M_k is a normal subgroup of M_i and $p_i\left(c_i^{\delta}\right)$ is an inner automorphism of M_i and thus $p_i\left(c_i^{\delta}\right)\left(a^0\right) \in M_k$. Let

$F_j = \left\{i : i \in F \cap G \text{ and } p_i\left(c_i^{\delta}\right) = a_j^0\right\}$. Thus F_j , $1 \le j \le m$, partition $F \cap G$ and so one, and only one, of F_j , $1 \le j \le m$, $\in \mathcal{F}$. Let it be F_t . Thus, for each $i \in F_t$, $h^0(i) = a_t^0$. Hence

$\overline{h}^0 = \overline{h}_{a_t^0}$ and $\overline{h}_{a_t^0} \in \psi(\hat{b}^1)$.

We finally show that $\overline{h}^\delta$ restricted to an isomorphism between $\psi(\hat{a}^1)$ and $\psi(\hat{b}^1)$ is locally an inner autormophism. Take any $a_1^0, \ldots, a_n^0 \in a^1$. Let $b_1^0, \ldots, b_n^0 \in b^1$ be such that $\overline{h}^\delta\left(\overline{h}_{a_j^0}\right) = \overline{h}_{b_j^0}$, each $1 \le j \le n$. It is required to find some $a^0 \in E^0$ such that $\overline{h}_{a^0}^{-1} \cdot \overline{h}_{a^0} \cdot \overline{h}_{a^0} = \overline{h}_{b_j^0}$, each $1 \le j \le n$.

Let $G_j = \left\{i : h_{a_j^0}(i) = a_j^0\right\}$, $H_j = \left\{i : h_{b_j^0}(i) = b_j^0\right\}$, each $1 \le j \le n$. Put $D_j = \left\{i : p_i\left(c_i^\delta\right)\left(a_j^0\right) = b_j^0\right\}$, $1 \le j \le n$. Thus $G \in F$, where $G = \cap\{G_j \cap H_j \cap D_j : 1 \le j \le n\}$. Take some $k \in G$ and put $D = \left\{i : E_i^0 \ge E_k^0\right\}$. Thus $D \in F$.

Now $p_k\left(c_k^\delta\right)$ is an inner automorphism of M_k and so there exists some $a^0 \in E_k^0$ such that $p_k\left(c_k^\delta\right)\left(a_j^0\right) = \left(a^0\right)^{-1} \cdot a_j^0 \cdot a^0 = b_j^0$, $1 \le j \le n$. But for all $i \in D \cap G$, $p_i\left(c_i^\delta\right)\left(a_j^0\right) = b_j^0$, each $1 \le j \le n$. Thus for each $1 \le j \le n$, $\left\{i : p_i\left(c_i^\delta\right)\left(a_j^0\right) = \left(a^0\right)^{-1} \cdot a_j \cdot a^0\right\} \in F$ and so $\left(\overline{h}_{a^0}\right)^{-1} \cdot \left(\overline{h}_{a_j^0}\right) \cdot \overline{h}_{a^0} = \overline{h}^\delta\left(\overline{h}_{a_j^0}\right)$.

Hence the theorem is established. //

3. The meta-theory of some local theorems in universal algebras

Various results are known in the meta-theory of local theorems for universal algebras. Amongst these is the Interior Local Theorem of A.I. Mal'cev [1959], in which is stated syntactical conditions sufficient to ensure that a set of sentences ("quasi-universal sentences") of a second order language expresses a local property.

S.R. Kogalovskiĭ, [1965/1970], gave a generalisation of Mal'cev's result in which some of Mal'cev's syntactical conditions are replaced by a more general semantical condition. J.P. Cleave [1969], published a theorem stating that second order sentences of a particular syntactical form ("boolean universal sentences") define local properties. While Cleave mentions the results of Mal'cev he leaves as an open question the relationship between his boolean universal sentences and the quasi-universal sentences of Mal'cev. We indicate in this section some aspects of this relationship.

In [1969] Mal'cev wrote, "Most of the interesting specific local theorems (e.g. those of group theory) concern properties expressible directly in the second order rather than in the first order predicate calculus. It is therefore important to describe the broadest class of second order formulae which express predicates having the local property". It is an aim of this section to contribute to this description. We use the ultraproduct construction to obtain semantical conditions sufficient to ensure that sentences in a second order language satisfying these conditions will define local properties. We then establish some syntactical conditions sufficient to provide these semantical criteria and in particular derive the results (with extensions) of Cleave, Mal'cev and Kogalovskiĭ. In the following section we continue the discussion into third order structures, particularly discussing the local theorem of D.H. McLain [1959].

Let P denote a class of $\kappa(\alpha, \beta)$ structures such that with each member M of P there is associated a local family $L = \{M_i : i \in I\}$ and an L-associated ultrafilter, F , over I .

If A is a formula of $L(\kappa(\alpha, \beta))$ then A is called:

(i) *L-hereditary* (with respect to P) if whenever $M \models_\nu A$, for $M \in P$, ν a M-interpretation, then $\left\{i : M_i \models_{p_i\nu} A\right\} \in F$, where $\underline{p}_i : M \to M_i$, each $i \in I$;

(ii) *L-local* (with respect to P) if whenever $M \in P$, ν an M-interpretation and $\left\{i : M_i \models_{p_i\nu} A\right\} \in F$ then $M \models_\nu A$;

(iii) *Local* (with respect to P) if whenever $M \in P$, ν an M-interpretation and for each $i \in I$, $M_i \models_{p_i\nu} A$ then $M \models_\nu A$;

(iv) *Hereditary* if whenever M' is a $\kappa(\alpha, \beta)$ structure (not necessarily a member of P), ν an M'-interpretation, N a substructure of M' , $\underline{p} : M' \to N$, such that whenever x^0 is a free variable of A then $\nu(x^0)$ is an individual of N , and $M' \models_\nu A$ then $N \models_{p\nu} A$.

Some elementary relationships between the various definitions are immediately apparent. The property of being hereditary entails that of being L-hereditary and the property of being L-local entails that of being local. (Unless ambiguity requires otherwise we shall often omit explicit reference to P).

THEOREM 3.1. *Let A be any formula of $L(\kappa(\alpha, \beta))$. A is L-local if, and only if, $\neg A$ is L-hereditary.*

Proof. Take $M \in P$ and assume that A is L-local and $M \models_{\nu} \neg A$. If $\{i : M_i \models_{p_i\nu} \neg A\} \notin F$ then $\{i : M_i \models_{p_i\nu} A\} \in F$. But A is L-local and so if $\{i : M_i \models_{p_i\nu} A\} \in F$ then $M \models_{\nu} A$. Hence $\{i : M_i \models_{p_i\nu} \neg A\} \in F$ and $\neg A$ is L-hereditary. The converse follows by a similar argument. //

THEOREM 3.2. *Let A be a formula of $L(\kappa(\alpha, \beta))$. If A is L-local then $\forall x^{\sigma_1} \ldots \forall x^{\sigma_n} A$ is L-local, for any $\sigma_1, \ldots, \sigma_n \in \kappa$.*

Proof. Take $M \in P$ and assume that A is L-local and $\{i : M_i \models_{p_i\nu} \forall x^{\sigma_1} \ldots \forall x^{\sigma_n} A\} \in F$. If $M \not\models_{\nu} \forall x^{\sigma_1} \ldots \forall x^{\sigma_n} A$ then for some M interpretation ν' such that ν, ν' agree except possibly on $x^{\sigma_1}, \ldots, x^{\sigma_n}$, $M \models_{\nu'} \neg A$. Thus, from Theorem 3.1, we have $\{i : M_i \not\models_{p_i\nu'} \neg A\} \in F$. But this requires that

$\{i : M_i \not\models_{p_i\nu} \forall x^{\sigma_1} \ldots \forall x^{\sigma_n} A\} \in F$ which is not so. Hence

$M \models_{\nu} \forall x^{\sigma_1} \ldots \forall x^{\sigma_n} A$. //

Theorem 3.1 allows us to state the dual result to Theorem 3.2, viz:

THEOREM 3.3. *Let* A *be a formula of* $L(\kappa(\alpha, \beta))$. *If* A *is* *L-hereditary then* $\exists x^{\sigma_1} \ldots \exists x^{\sigma_n} A$ *is* *L-hereditary, for all* $\sigma_1, \ldots, \sigma_n \in \kappa$.

The next result requires that the class P be restricted to full second order systems such that for each member M of the class the constant relations are L-stable. For the rest of this section P_1 will denote such a class with similarity type $\kappa_1(\alpha, \beta)$ where κ_1 is such that $0 \in \kappa_1$ and for all $\sigma \in \kappa_1$ if $\sigma = (\sigma_1, \ldots, \sigma_n)$ then $\sigma_j = 0$, each $1 \leq j \leq n$.

THEOREM 3.5. *If* A *, a formula of* $L(\kappa_1(\alpha, \beta))$ *, is hereditary then* A *is* *L-local with respect to* P_1 .

Proof. Take $M \in P_1$, ν an M interpretation and assume $\{i : M_i \models_{p_i \nu} A\} \in F$. Thus $\pi M_i / F \models_{\bar{\nu}} A$, where for each $i \in I$, $\nu_i = p_i \nu$. Let N_ψ be the subsystem of $\pi M_i / F$, $\underline{p} : \pi M_i / F \to N_\psi$, as defined for Theorem II: 2.8. If x^0 is a free individual variable of A then $\bar{\nu}(x^0) = \bar{h}_{\nu(x^0)}$ is an individual of N_ψ . Hence $N_\psi \models_{p\bar{\nu}} A$, as A is hereditary, and so by Theorem II: 2.8, we have $M \models_{\nu} A$ and the result is established. //

In fact we can obtain a wider result than Theorem 3.5 by extending our context in a similar manner to that leading from Theorem II: 2.8 to Theorem II: 2.9. Let constant relations $\left\{R_n^{\sigma_n} : \beta \leq n < \beta_1\right\}$ be adjoined to each member M of P_1, and such relations need not be L-stable with respect to each member of P_1. Again for the rest of this section P_1' will denote the class P_1 but where each member has adjoined to it the extra relations.

Take $A \in L(\kappa_1(\alpha, \beta_1))$ in prenex normal form with its quantifier free portion A_0 in disjunctive normal form and such that if $R_n^{\sigma_n}$, $n < \beta_1$, is not L-stable with respect to each member of P_1' then it can only occur (if at all) in the negation of an atomic component of A_0.

THEOREM 3.6. *If* P_1' *and* $A \in L(\kappa_1(\alpha, \beta_1))$ *are as above and if* A *is hereditary then* A *is* L*-local with respect to* P_1'.

A proof of this result follows the pattern of the proof given for Theorem 3.5 but appealing to Theorem II: 2.9, rather than to Theorem II: 2.8.

Extending the terminology of Cleave [1969], page 122/3, we denote a formula A of $L(\kappa_1(\alpha, \beta_1))$ as **crypto-universal* if it is a member of the following class H:

(i) Every quantifier free formula of $L(\kappa_1(\alpha, \beta_1))$ that contains no constant relation symbol $\underset{\sim}{R}_n^{\sigma_n}$, where $\sigma_n = (\sigma_{n,1}, \ldots, \sigma_{n,\phi(n)})$

and some $\sigma_{n,j} \neq 0$, is a member of H ;

(ii) if $A \in H$ then so do a) $\forall x^{0}A$, b) $\forall x^{\sigma}A$ and c) $\exists x^{\sigma}A$, $\sigma \in \kappa_1$ and $\sigma \neq 0$;

(iii) if $A_1, A_2 \in H$ then so do $A_1 \vee A_2$ and $A_1 \wedge A_2$;

(iv) if $A \in H$ then so do a) $\forall x^{\sigma}\left(\underline{R}^{(\sigma)}\left(x^{\sigma}\right) \Rightarrow A\right)$ and b) $\exists x^{\sigma}\left(\underline{R}^{(\sigma)}\left(x^{\sigma}\right) \wedge A\right)$, where $\sigma = (0, \ldots, 0)$ and $\underline{R}^{(\sigma)}$ is a constant relation symbol of $L\left(\kappa_1\left(\alpha, \beta_1\right)\right)$;

(v) H consists of all formulae gained by a finite number of application of steps (i) to (iv).

If A is a *crypto-universal formula gained without application of step (iv) then A is a crypto-universal formula (or more correctly the prenex normal form of A is such) as defined by Cleave [1969]. As we shall observe below, step (iv) a, above enables *crypto-universal formulae to have a close relationship to the quasi-universal systems of Mal'cev [1969], and the generalised quasi-universal systems of formulae of Kogalovskiĭ [1970].

Cleave [1969], page 123, appeals to results of Tarski [1954/55] to establish that crypto-universal sentences are hereditary. Kogalovskiĭ [1970], page 116, sketches a proof for an almost similar result. We shall obtain the result in a different manner and for the wider class of *crypto-universal formulae by means of the following lemmas.

LEMMA 3.7. *If* $A \in L\left(\kappa_1\left(\alpha, \beta_1\right)\right)$ *denotes an atomic formula of* *the form* $\left(\alpha_1^0, \ldots, \alpha_n^0\right) \underline{\epsilon}^{\sigma}\ \alpha^{\sigma}$ *or* $\underline{R}^{\sigma}\left(\alpha_1^0, \ldots, \alpha_n^0\right)$ *, or their negations, where* $\sigma = (0, \ldots, 0)$ *and the* α*'s are terms, then* A *is hereditary.*

The proof of the lemma is immediate.

LEMMA 3.8. *If* $A, B \in L\left(\kappa_1(\alpha, \beta_1)\right)$ *are hereditary then so are* $A \vee B$ *and* $A \wedge B$.

Again the proof is immediate.

LEMMA 3.9. *If* $A \in L\left(\kappa_1(\alpha, \beta_1)\right)$ *is hereditary then so are* *(i)* $\forall x^0 A$ *, (ii)* $\forall x^{\sigma} A$ *, (iii)* $\exists x^{\sigma} A$ *, (iv)* $\forall x^{\sigma}\left(\underline{R}^{(\sigma)}\left(x^{\sigma}\right) \Rightarrow A\right)$ *, (v)* $\exists x^{\sigma}\left(\underline{R}^{(\sigma)}\left(x^{\sigma}\right) \wedge A\right)$ *, where* $\sigma = (0, \ldots, 0)$ *and* $\underline{R}^{(\sigma)}$ *is a constant relation symbol.*

Proof. Assume A contains $x_1^0, \ldots, x_n^0$ as free variables of type 0 . (If A does not contain any free variable of type 0 then (i) is immediately established.) Let M be a structure of type $\kappa_1(\alpha, \beta_1)$ and N a given substructure, $\underline{p} : M \to N$.

(i) If $N \not\models_{p\nu} \forall x_1^0 A$, where ν is an M interpretation such that $\nu\left(x_1^0\right), \ldots, \nu\left(x_n^0\right) \in N$, then for some M interpretation ν' such that ν and ν' agree except possibly on x_1^0 we have $N \not\models_{p\nu'} A$. Hence $M \not\models_{\nu'} A$ as A is hereditary and so $M \not\models_{\nu} \forall x_1^0 A$. Thus $\forall x_1^0 A$ is hereditary.

(ii) follows by an argument similar to that used in (i).

Proofs of (iii) and (v) are immediate.

To establish (iv) assume that $N \not\models_{p\nu} \forall x^{\sigma}\left(\underline{R}^{(\sigma)}\left(x^{\sigma}\right) \Rightarrow A\right)$ and so for some ν' such that ν, ν' agree except possibly on x^{σ} we have $N \models_{p\nu'} \left(\underline{R}^{(\sigma)}\left(x^{\sigma}\right) \wedge \neg A\right)$. Thus for some ν'' such that ν', ν'' agree except possibly on x^{σ} and such that $p\nu'\left(x^{\sigma}\right) = p\nu''\left(x^{\sigma}\right)$ we have $M \models_{\nu''} \left(\underline{R}^{(\sigma)}\left(x^{\sigma}\right) \wedge \neg A\right)$. Hence $M \not\models_{\nu} x^{\sigma}\left(\underline{R}^{(\sigma)}\left(x^{\sigma}\right) \Rightarrow A\right)$ and (iv) is established. //

Lemmas 3.7, 3.8 and 3.9 provide a basis for an inductive proof of the following theorem.

THEOREM 3.10. *If* $A \in L\left(\kappa_1(\alpha, \beta_1)\right)$ *is *crypto-universal then* A *is hereditary.*

In fact the definition of *crypto-universal formulae could have been extended a little and the hereditary property retained. In (i) of the definition formulae containing constant relation symbols $\underset{\rightarrow n}{R}^{\sigma_n}$, $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$ and some $\sigma_i \neq 0$, could have been included as long as they did not occur in the negation of an atomic formula of a disjunctive normal form of the formula concerned. Lemma 3.7 could be immediately extended to include this case. However the extension can not be maintained for later results, *c.f.* Theorem 3.11, and so has not been included.

Further, step (iv) could be extended by allowing relation symbols $\underline{R}^{\sigma}$, where $\sigma = \left(\sigma_1, \ldots, \sigma_n\right)$ with some of the $\sigma_i \neq 0$. (iv) a would then read

$$\forall x^{\sigma_{i,1}} \ldots \forall x^{\sigma_{i,k}}\left(\underline{R}^{\sigma}\left(x^{\sigma_1}, \ldots, x^{\sigma_n}\right) \Rightarrow A\right) ,$$

where $\sigma_{i,1}, \ldots, \sigma_{i,k}$ is the complete list of all σ_i , $1 \leq i \leq n$, such that $\sigma_i \neq 0$. Step (iv) b would read $\exists x^{\sigma_{i,1}} \ldots \exists x^{\sigma_{i,k}}\left(\underline{R}^{\sigma}\left(x^{\sigma_1}, \ldots, x^{\sigma_n}\right) \wedge A\right)$. While this extension can be maintained for Theorem 3.11 it is not required for the particular applications discussed and so again has not been included.

As a direct consequence of Theorems 3.6 and 3.10 we can state the following theorem.

THEOREM 3.11. *If* $A \in L(\kappa_1(\alpha, \beta_1))$ *is *crypto-universal then* A *is* L*-local with respect to* P_1' .

Again following the example of Cleave [1969], page 123, we term a formula $A \in L(\kappa_1(\alpha, \beta_1))$ as **boolean-universal* if $A = \forall x^{\sigma_1} \ldots \forall x^{\sigma_n} B$, where B is formed from *crypto-universal formulae by sentential connectives alone.

THEOREM 3.12. *If* $A \in L(\kappa_1(\alpha, \beta_1))$ *is *boolean-universal then* A *is* L*-local with respect to* P_1' .

Proof. If $A_1, A_2 \in L(\kappa_1(\alpha, \beta_1))$ are L-local then so are $A_1 \vee A_2$ and $A_1 \wedge A_2$. Further, if A_1 is hereditary then it is L-hereditary and so, by Theorem 3.1, $\neg A_1$ is L-local. Hence by an inductive argument and using Theorems 3.11 and 3.2 we have that A is

L-local. //

We observe that the class of boolean-universal sentences as defined by Cleave [1969], page 123, forms a sub-class (in form at least a proper sub-class) of the *boolean-universal formulae and so Cleave's result that boolean-universal sentences define local properties is a special case of Theorem 3.12.

We now turn to the results of Mal'cev and in particular to his Interior Local Theorem [1959]. In [1959] Mal'cev describes his local theorem as follows. He considers a set of formulae

(1) $A_i\left[y_i^{\sigma_i}\right] = \forall x_1^0 \ldots \forall x_{n_i}^0 A_i'\left[x_1^0, \ldots, x_{n_i}^0; R_1^{\tau_1}, \ldots, R_s^{\tau_s}; y_i^{\sigma_i}\right]$, each $1 \leq i \leq k$,

and the formula

(2) $Q_k y_k^{\sigma_k} \forall x_1^0 \ldots \forall x_{m_k}^0 \ldots Q_1 y_1^{\sigma_1} \forall x_{m_2+1}^0 \ldots \forall x_{m_1}^0 B\left[x_1^0, \ldots, x_{m_1}^0; R_1^{\tau_1}, \ldots, R_s^{\tau_s}; y_1^{\sigma_1}, \ldots, y_k^{\sigma_k}\right]$,

where $\tau_j = (0, \ldots, 0)$, $1 \leq j \leq s$, $\sigma_j = (0, \ldots, 0)$, $1 \leq j \leq k$; $Q_\lambda = \exists$ or $\forall$, $1 \leq \lambda \leq k$; A_i', each $1 \leq i \leq k$, are quantifier free formulae consisting of the variables and constant relation symbols appearing in the parenthesis; B is a formula without quantifiers.

A system of formulae given by (1) and (2) Mal'cev denotes as quasi-universal. Such a quasi-universal set is fulfilled on a system M if the formula (2) holds in M under the conditions that its quantifiers associated with the $y_i^{\sigma_i}$, $1 \le i \le k$, are interpreted as bounded quantifiers over the collections of all the objects of M which satisfy the corresponding axioms (1) - that is the set of all objects $\nu\left(y_i^{\sigma_i}\right)$ such that $M \models_\nu A_i\left[y_i^{\sigma_i}\right]$, for all M interpretations ν . If conditions (1) are tautologous or lacking then the associated quantifiers are interpreted as ordinary second order quantifiers.

The Interior Local Theorem states that all properties expressible by means of quasi-universal sets of formulae have the local property.

Mal'cev further notes that in general fulfillment on some system M of the quasi-universal set (1) and (2) is equivalent to the truth in M of a second order sentence, having existence quantifiers over individuals in its prenex normal form. We shall denote by C (and call it the associated Mal'cev formula of the quasi-universal set (1) and (2)) the last member of the following sequence of k formulae.

The first member C_0 of the sequence is the formula B of (2). We assume that t-th member of the sequence, C_t , is defined. We denote by alternative

$$\text{a)} \quad C_{t+1} = \forall y_t^{\sigma_t}\left(A_t\left[y_t^{\sigma_t}\right] \Rightarrow \forall x^0_{m_{t+1}+1} \ldots \forall x^0_{m_t} C_t\right)$$

and by alternative

$$\text{b)} \quad C_{t+1} = \exists y_t^{\sigma_t}\left(A_t\left[y_t^{\sigma_t}\right] \wedge \forall x^0_{m_{t+1}+1} \ldots \forall x^0_{m_t} C_t\right) ,$$

where if Q_{t+1} in (2) is $\forall$ then alternative (a) is taken, otherwise alternative (b), and $A_t\left[y_t^{\sigma_t}\right]$ is the t-th member of (1).

It is apparent that C as defined is equivalent to the quasi-universal set of formulae (1) and (2). We shall now proceed to show how C can be associated with a *crypto-universal formula and via this formula be shown to express a local property.

We take a class P_1 as above and assume the constant relations denoted in (1) by $R_1^{\tau_1}, \ldots, R_s^{\tau_s}$ are included in the constant relations of the members of P_1. With each $A_i\left[y_i^{\sigma_i}\right]$ of (1) that is associated with a universal quantifier, $\forall y_i^{\sigma_i}$, in (2) we define, for each $M \in P_1$, a constant relation $R_i^{(\sigma_i)}$ by: $\nu\left(y_i^{\sigma_i}\right)$ satisfies $R_i^{(\sigma_i)}$ if, and only if, $M \models_\nu A_i\left[y_i^{\sigma_i}\right]$, for each M interpretation ν. As we shall see below (Theorem 3.15) it is possible to establish that each such $R^{(\sigma_i)}$ is L-finitary and L-stable. The derivation of Mal'cev's results do not however require such a property. Let P_1' be the resulting class of systems M with the new relations adjoined.

The formula C^* is formed by the same sequence of steps as formed C except that if C_t and $C_t{}^*$ have been defined and C_{t+1} is got from C_t by alternative (a) then

$$C^*_{t+1} = \forall y_t^{\sigma_t}\left(R_t^{(\sigma_t)}\left(y_t^{\sigma_t}\right) \Rightarrow \forall x^0_{m_{t+1}+1} \cdots \forall x^0_{m_t} C_t{}^*\right),$$

otherwise

$$C^*_{t+1} = \exists y_t^{\sigma_t}\left(A_t\left[y_t^{\sigma_t}\right] \wedge \forall x^0_{m_{t+1}+1} \cdots \forall x^0_{m_t} C_t^*\right) .$$

It is immediate that C^* as defined is *crypto-universal and so hereditary and L-local with respect to P'_1.

It should be noted that C and C^* are not equivalent formulae. This arises from the fact that although for $M \in P'_1$, $R_i^{(\sigma_i)}$ is defined by the formula $A_i\left[y_i^{\sigma_i}\right]$ being satisfied in M yet if M_j is a substructure of M, $\underline{p}_j : M \to M_j$, then while we can assert that if $p_j\left(R_i^{(\sigma_i)}\right)\left(p_j\nu\left(y_i^{\sigma_i}\right)\right)$ then $M_j \models_{p_j\nu} A_i\left[y_i^{\sigma_i}\right]$, ν an M interpretation, (because $A_i\left[y_i^{\sigma_i}\right]$ is hereditary) we cannot assert the converse. Thus while $M \models_\nu C$ if, and only if, $M \models_\nu C^*$ this need not be the case if M is replaced by one of its subsystems. We do however have the following result.

THEOREM 3.13. *If $M \in P'_1$ and N is a substructure of M, $\underline{p} : M \to N$, such that $N \models_\nu C$ then $N \models_\nu C^*$, ν an N interpretation.*

Proof. The proof proceeds by induction on the steps of formation of C and C^*. Assume that if $N \models_\nu C_t$ then $N \models_\nu C_t^*$. For $t = 0$ the assumption is true. Let

$$C^*_{t+1} = \forall y_t^{\sigma_t}\left(\underline{R}_t^{(\sigma_t)}\left(y_t^{\sigma_t}\right) \Rightarrow \forall x^0_{m_{t+1}+1} \cdots \forall x^0_{m_t} C_t^*\right)$$

and assume $N \not\models_{\nu} C^{*}_{t+1}$. Thus for some ν' which agrees with ν except possibly on $y_t^{\sigma_t}$, $x^0_{m_{t+1}+1}, \ldots, x^0_{m_t}$ we have $N \models_{\nu'} \left(\underline{R}_t^{(\sigma_t)}\left(y_t^{\sigma_t}\right) \wedge \neg C_t{}^*\right)$ and so $N \models_{\nu'} \left(A_t\left[y_t^{\sigma_t}\right] \wedge \neg C_t\right)$. Hence $N \not\models_{\nu} C_{t+1}$. That is if $N \models_{\nu} C_{t+1}$ then $N \models_{\nu} C^{*}_{t+1}$.

If C_{t+1} comes from C_t by alternative (b) then it is immediate that if $N \models_{\nu} C_{t+1}$ then $N \models_{\nu} C^{*}_{t+1}$. The result is thus established. //

We are now able to derive (a special form of) the Interior Local Theorem of Mal'cev.

THEOREM 3.14. *All properties of models (in a class* P_1 *) expressible by means of a quasi-universal system of formulae have the local property.*

Proof. Take the quasi-universal system (1) and (2) and C the associated Mal'cev formula. Take $M \in \mathrm{P}_1$ such that $\{i : M_i \models C\} \in \mathrm{F}$. By Theorem 3.13 we have that $\{i : M_i \models C^*\} \in \mathrm{F}$ and hence, by Theorem 3.11, $M \models C^*$ and so $M \models C$. //

For ease of exposition we have taken the simplified version of Mal'cev's quasi-universal set of formulae. More generally (1) can consist of a possibly infinite set of formulae in which each $A_i\left[y_i^{\sigma_i}\right]$ may be replaced by a possibly infinite conjunction of formulae of the form $A_i\left[y_i^{\sigma_i}\right]$. Further, (2) may contain an infinite collection of formulae each interpreted with bounded quantifiers in relation to (1).

All of these modifications can be incorporated into the above seuqence of results by appropriate changes to the detail of the arguments. In particular the infinite conjunctions included in (1) can be incorporated as well formed formulae of the associated languages and the necessary ultraproduct results extended to include them, as set out at the end of section 3 in Chapter I.

Indeed, and apart from Kogalouskiĭ's [1970] extension which we shall discuss below, further generalisations can be made to the form Mal'cev's Theorem by allowing the $A_i\left[y_i^{\sigma_i}\right]$ of (1) to be *crypto-universal, as all that is required of them is that they be hereditary.

This leads us to the nature of Kogalovskiĭ's generalisation of Mal'cev's result. In terms of a quasi-universal system (1) and (2) Kogalovskiĭ allows a universal quantifier, say $\forall y_i^{\sigma_i}$, of (2) to be bound not necessarily by satisfaction of an appropriate formula $A_i\left[y_i^{\sigma_i}\right]$, but by some semantical condition not necessarily statable in a formula of the language, but which transfers from a structure to its substructures. Specifically, and in terms of this section, for $M \in P_1$, a constant relation $R_i^{(\sigma_i)}$ is defined by: $R_i^{(\sigma_i)}\left(\nu\left(y_i^{\sigma_i}\right)\right)$ if, and only if, M together with $\nu\left(y_i^{\sigma_i}\right)$ as an adjoined constant relation satisfies a condition T which is such that if N is a substructure of M , $\underline{p} : M \to N$, and $\left(M, \nu\left(y_i^{\sigma_i}\right)\right)$ satisfies T then $\left(N, p\nu\left(y_i^{\sigma_i}\right)\right)$ also satisfies T , ν being any M interpretation. This latter condition allows us to associate with a generalised quasi-universal system of formulae as defined by Kogalovskiĭ a *crypto-universal formula which plays the corresponding role to the generalised quasi-universal system as the previously defined C^* played to the quasi-universal system and hence establishes the local property.

Although it plays no part in the above results it is of interest that the constant relation $R_i^{(\sigma_i)}$ defined for the construction of C^* in the context of the Mal'cev quasi-universal system of formulae has properties that cannot be asserted for the corresponding relation defined by means of Kogalovskiĭ's semantical condition. This is exhibited in the following theorem.

THEOREM 3.15. *If* M *belongs to some* P_1' *and* $A[\underline{x}^{\sigma}] \in L(\kappa_1(\alpha, \beta_1))$ *, is *crypto-universal, where* $\sigma = (0, \ldots, 0)$ *and* x^{σ} *is the only free variable in* $A[x^{\sigma}]$ *, then the relation* $R^{(\sigma)}$ *defined by* $R^{(\sigma)}(\nu(x^{\sigma}))$ *if, and only if,* $M \models_{\nu} A[x^{\sigma}]$, ν *any* M *interpretation, is L-finitary and L-stable.*

Proof. Let $\nu(x^{\sigma})$ not satisfy $R^{(\sigma)}$ for some M interpretation ν . If $\left\{i : p_i(R^{(\sigma)})\left(p_i\nu(x^{\sigma})\right)\right\} \in F$ then $\left\{i : M_i \models_{p_i\nu} A[x^{\sigma}]\right\} \in F$

and so $M \models_{\nu} A[\underline{x}^{\sigma}]$, as $A[\underline{x}^{\sigma}]$ is *crypto-universal. That is $R^{(\sigma)}(\nu(x^{\sigma}))$. But this contradicts that $\nu(x^{\sigma})$ does not satisfy $R^{(\sigma)}$. Hence $\left\{i : p_i\nu(x^{\sigma}) \text{ does not satisfy } p_i(R^{(\sigma)})\right\} \in F$. That is $R^{(\sigma)}$ is L-finitary.

By Theorem II: 2.9, we have $N_{\psi} \not\models_{p\bar{\nu}} A[\underline{x}^{\sigma}]$, where $\bar{\nu}$ is got from the $p_i\nu_i$'s , $i \in I$, as it is not the case that $R^{(\sigma)}(\nu(x^{\sigma}))$ and so $M \not\models_{\nu} A[\underline{x}^{\sigma}]$. If $p(\bar{R}^{(\sigma)})(p\bar{\nu}(x^{\sigma}))$, $(\underline{p} : \pi M_i/F \to N_{\psi})$, then for some $\pi M_i/F$ interpretation $\bar{\mu}$ which agrees with $\bar{\nu}$ except possibly on x^{σ} and such that $p\bar{\nu}(x^{\sigma}) = p\bar{\mu}(x^{\sigma})$ we have $\bar{R}^{(\sigma)}(\bar{\mu}(x^{\sigma}))$. Let $\bar{\mu}(x^{\sigma}) = \bar{h}^{\sigma}$ and for each $i \in I$ define an M_i interpretation μ_i which agrees with $p_i\nu$ except possibly on x^{σ} and such that $\mu_i(x^{\sigma}) = h^{\sigma}(i)$. From $\bar{R}^{(\sigma)}(\bar{\mu}(x^{\sigma}))$ we have $\left\{i : p_i(R^{(\sigma)})\left[\mu_i(x^{\sigma})\right]\right\} \in F$. But for any $i \in I$, if $p_i(R^{(\sigma)})\left[\mu_i(x^{\sigma})\right]$ then for some M interpretation μ' such that $p_i\mu'(x^{\sigma}) = \mu_i(x^{\sigma})$ we have $R^{(\sigma)}(\mu'(x^{\sigma}))$ and so $M \models_{\mu'} A[\underline{x}^{\sigma}]$. But $A[\underline{x}^{\sigma}]$ is *crypto-universal and so hereditary. Hence $M_i \models_{\mu_i} A[\underline{x}^{\sigma}]$ if $p_i(R^{(\sigma)})\left[\mu_i(x^{\sigma})\right]$.

Therefore if $\left\{i : p_i(R^{(\sigma)})\left[\mu_i(x^{\sigma})\right]\right\} \in F$ then

$\left\{i : M_i \models_{\mu_i} A[x^\sigma]\right\} \in F$ and so $\pi M_i/F \models_{\bar{\mu}} A[\bar{x}^\sigma]$. Thus $N_\psi \models_{p\bar{\nu}} A[x^\sigma]$ which is not so. Thus $p\bar{\nu}(x^\sigma)$ does not satisfy $p(R^{(\sigma)})$ and the result is established. //

The next theorem enables a relationship to be established between *crypto-universal formulae and certain kinds of quasi-universal systems of formulae.

THEOREM 3.16. *If* $A \in L(\kappa_1(\alpha, \beta_1))$ *is *crypto-universal and contains no quantifiers associated with individual variables then* $\neg A$ *is *crypto-universal.*

Proof. A can be gained by a finite number of applications of steps (i) to (iv), but excluding (ii), a), as set out for the definition of *crypto-universal formulae. Any formulae established as *crypto-universal by step (i) is immediately such that its negation is *crypto-universal again by step (i). It can now be checked that if any of the steps (ii) to (iv), but excluding (ii), a), are applied to formulae such that each one and its negation are *crypto-universal then the newly gained formula and its negation will be *crypto-universal. An inductive argument establishes the theorem. //

We can now observe that if we take a quasi-universal system as given in (1) and (2) above but in which the $A_i\left[y_i^{\sigma_i}\right]$ formulae in (1) are *crypto-universal and those that are associated with universal quantifiers in the formula of (2) contain no quantifiers associated with individual variables then the associated Mal'cev formula C is *crypto-universal. The converse result that any *crypto-universal formula is equivalent to a quasi-universal system is immediate by noting that (1) can be taken as vacuous and (2) the given formula itself.

The author has sought unsuccessfully for analogous relationships between *boolean-universal formulae and (generalised) quasi-universal systems of formulae.

4. Chain conditions in third order algebraic structures

The initiative for the work of this section comes from the paper of D.H. McLain [1959], and particularly his Theorem 1 stated on p. 178 of that paper. While it is true, as asserted by Mal'cev [1969], page 39, that this theorem can be translated into a form that enables it to be derived within the scope of the Interior Local Theorem of Mal'cev yet in its natural setting it involves quantification over third order objects and so can be the means of illustrating some interesting extensions of the results of the previous section and application of the notions of the first two chapters in third order structures.

Let $\mathcal{Q}$ denote a class of $\kappa(\alpha, \beta)$ algebraic structures, where throughout this section $\kappa = \{0\ (0) = 1\ \{(0)\} = 2\}$. Further if $M \in \mathcal{Q}$ then the members of E^1 are the sets of individuals of all the sub-algebras of M and E^2 consists of all chains of elements of E^1 complete with respect to union, $\cup$, and intersection, $\cap$. The membership relations of M are those of ordinary set membership. We shall use $\underset{\bullet}{\in}$ in place of $\in^0$, $\in^1$ and $\in^2$.

With each $M \in \mathcal{Q}$ is associated a local family $L = \{M_i : i \in I\}$ and L-associated ultra filter F over I. The constant relations of each $M \in \mathcal{Q}$ are assumed to be L-stable, where $\underline{\psi} : M \to \pi M_i/F$ and N_ψ with $\underline{p} : \pi M_i/F \to N_\psi$ are defined as previously.

We let $C[x^2]$ stand for the formula

$$\forall x^1 \forall y^1 (x^1 \underline{\in} x^2 \wedge y^1 \underline{\in} x^2 \Rightarrow x^1 \subseteq y^1 \vee y^1 \subseteq x^1) ,$$

where $x^1 \subseteq y^1$ stands for the formula $\forall x^0 (x^0 \underline{\in} x^1 \Rightarrow x^0 \underline{\in} y^1)$. It is apparent that $C[x^2]$ is hereditary and expresses the property of being a chain.

THEOREM 4.1. *If $M \in \mathcal{Q}$ then for each $i \in I$ the members of E_i^2 are chains of members of E_i^1 and complete with respect to $\cup$ and $\cap$.*

Proof. That the members of E_i^2 are chains of members of E_i^1 follows from the observation that the sentence $\forall x^2 C[x^2]$ is hereditary.

Let V_i be a non-empty subset of members of $p_i(a^2) \in E_i^2$ and put $V = \left\{a^1 : a^1 \in a^2 \wedge p_i(a^1) \in V_i\right\}$. Now a^2 is complete with respect to $\cup$ and $\cap$ and so $\cup V$, $\cap V$ (being $\cup\{a^1 : a^1 \in V\}$ and $\cap\{a^1 : a^1 \in V\}$ respectively) belong to a^2. It can now be checked that $p_i(\cup V) = \cup V_i$ and $p_i(\cap V) = \cap V_i$ and hence $p_i(a^2)$ is complete with respect to $\cup$ and $\cap$. //

THEOREM 4.2. *If* $M \in \mathcal{Q}$ *then the members of* $\pi E_i^2/\mathsf{F} = \overline{E}^2$ *and* $p(\overline{E}^2)$ *are chains.*

Proof. From Theorem 4.1, $\left\{i : M_i \models \forall x^2 C[x^2]\right\} = I$ and so $\pi M_i/\mathsf{F} \models \forall x^2 C[x^2]$. Further, as $\forall x^2 C[x^2]$ is hereditary, we have $N_\psi \models \forall x^2 C[x^2]$. The result is thus established. //

It is not possible in general to establish that the chains in $\pi M_i/\mathsf{F}$ or N_ψ are complete with respect to $\cup$ and $\cap$. However we do have the following result which provides a significant property of these chains.

THEOREM 4.3. *If* $M \in \mathcal{Q}$ *then each chain* $\overline{h}^2$ *of* $\pi M_i/\mathsf{F}$ *has the property, denote it by* $S[\overline{h}^2]$, *that for each individual* $\overline{h}^0$ *of* $\pi M_i/\mathsf{F}$ *if there exists at least one member of* $\overline{h}^2$ *that contains* $\overline{h}^0$ *then there exists a least member of* $\overline{h}^2$ *that contains it and if there exists at least one member of* $\overline{h}^2$ *that does not contain* $\overline{h}^0$ *then there exists a greatest member of* $\overline{h}^2$ *that does not contain it.*

Proof. Let $S_1[x^2, x^0]$ be the formula

$$\forall x^1 \exists y^1 \forall z^1 \Big(x^1 \underline{\in} x^2 \wedge x^0 \underline{\in} x^1 \Rightarrow$$
$$\big(y^1 \underline{\in} x^2 \wedge x^0 \underline{\in} y^1 \wedge (z^1 \underline{\in} x^2 \wedge x^0 \underline{\in} z^1 \Rightarrow y^1 \subseteq z^1)\big)\Big) .$$

Let $S_2[x^2, x^0]$ be the formula

$$\forall x^1 \exists y^1 \forall z^1 \Big(x^1 \underline{\in} x^2 \wedge x^0 \underline{\notin} x^1 \Rightarrow$$

$$\big(y^1 \underline{\in} x^2 \wedge x^0 \underline{\notin} y^1 \wedge \big(z^1 \underline{\in} x^2 \wedge x^0 \underline{\notin} z^1 \Rightarrow z^1 \underline{\subseteq} y^1\big)\big)\Big) .$$

Finally, let $S[x^2]$ stand for

$$\forall x^0 \Big(S_1[x^2, x^0] \wedge S_2[x^2, x^0] \Big) .$$

We observe that for each $i \in I$, $M_i \models \forall x^2 S[x^2]$, as each chain of M_i is complete with respect to $\cup$ and $\cap$. Hence $\pi M_i/F \models \forall x^2 S[x^2]$ and the theorem is established.

COROLLARY 4.4. *If* $M \in Q$ *then for each* $\overline{h}^2 \in \pi M_i/F$, $p(\overline{h}^2)$ *has the property* S *, where* $\underline{p} : \pi M_i/F \to N_\psi$.

Proof. It can be seen by examination that the formula $S[x^2]$ is hereditary and hence the corollary is established. //

THEOREM 4.5. *If* $M \in Q$ *then for each* $a^2 \in E^2$ *, and each* $a^1 \in E^1$ *, if* $a^1 \notin a^2$ *then* $\{i : p_i(a^1) \notin p_i(a^2)\} \in F$ *and* $p\psi(a^1) \notin p\psi(a^2)$.

Proof. We first consider the case when $a^1 = \emptyset$ and $a^1 \notin a^2$. Let b^1 be the smallest member of a^2 and so $b^1 \neq \emptyset$. For each $i \in I$, $p_i(b^1)$ is the smallest member of $p_i(a^2)$. Take $b^0 \in b^1$ and so for all $i \in F_{b^0}$, $p_i(b^1) \neq \emptyset$. Thus $p_i(a^1) = \emptyset \notin p_i(a^2)$. That is $\{i : p_i(a^1) \notin p_i(a^2)\} \in F$. Further, $\overline{h}_{b^1}$ is the smallest member of $\bar{h}_{a^1}$ and $p\left(\bar{h}_{b^1}\right)$ is the smallest member of $p\left(\bar{h}_{a^2}\right)$ and $p\left(\bar{h}_{b^1}\right) \neq \emptyset$ as $\bar{h}_{b^0} \in p\left(\bar{h}_{b^1}\right)$. Thus $p\left(\bar{h}_{a^1}\right) = \emptyset \not\in p\left(\bar{h}_{a^2}\right)$.

Now assume $a^1 \neq \emptyset$ and $a^1 \not\in a^2$. Let $a^2 = \left\{a_i^1 : i < \rho\right\}$, ρ some ordinal, and let $H_1 = \left\{a_i^1 : i < \rho \text{ and } a_i^1 \subseteq a^1\right\}$. Now $\cup H_1 \in a^2$ and is the greatest member of H_1. $\cup H_1 \subseteq a^1$ but $a^1 \not\in a^2$ and so $\cup H_1 \subset a^1$. Take some $a^0 \not\in \cup H_1$ but $a^0 \in a^1$. If $H_1 = \emptyset$ then a^0 can be taken as any member of a^1.

Let $H_2 = \left\{a_i^1 : i < \rho \text{ and } a^0 \in a_i^1\right\}$. If $H_2 = \emptyset$ then a^0 lies outside all members of a^2 and so for all $i \in F_{a^0}$,

$p_i(a^1) \not\in p_i(a^2)$. That is $\left\{i : p_i(a^1) \not\in p_i(a^2)\right\} \in \mathsf{F}$. Further, $\bar{h}_{a^1} \not\in \bar{h}_{a^2}$ and $\bar{h}_{a^0} \in \bar{h}_{a^1}$. But $\bar{h}_{a^0}$ does not belong to any members of $\bar{h}_{a^2}$ and so does not belong to any member of $p\left(\bar{h}_{a^2}\right)$. That is $p\left(\bar{h}_{a^1}\right) \not\in p\left(\bar{h}_{a^2}\right)$.

Consider the case when $H_2 \neq \emptyset$. $\cap H_2$ is the smallest member of H_2 and is not a member of H_1. Thus take some $b^0 \in \cap H_2$ and $b^0 \not\in a^1$. Let $F = F_{a^0} \cap F_{b^0}$ and so $F \in \mathsf{F}$. For any $a_i^1 \in a^2$ such that $a_i^1 \in H_2$ we have $b^0 \in a_i^1$ and for any $a_i^1 \in a^2$, such that

$a_i^1 \notin H_2$ we have $a^0 \notin a_i^1$. Hence $\{i : p_i(a^1) \notin p_i(a^2)\} \in F$.

Finally, if we assume $p(\overline{h}_{a^1}) \in p(\overline{h}_{a^2})$ then there exists some $\overline{h}^1 \in \overline{h}_{a^2}$ such that $p(\overline{h}^1) = p(\overline{h}_{a^1})$. Let $F_1 = \{i : h^1(i) \in p_i(a^2)\}$ and $F_2 = F \cap F_1$. Now $F_2 = F_3 \cap F_4$ where

$F_3 = \{i : i \in F_2$ and $b^0 \in h^1(i)\}$ and

$F_4 = \{i : i \in F_2$ and $a^0 \notin h^1(i)\}$. Thus $F_3 \in F$ or $F_4 \in F$.

But if $F_3 \in F$ then $\overline{h}_{b^0} \in \overline{h}^1$ and $\overline{h}_{b^0} \in p(\overline{h}_{a^1})$ which is not so.

Also if $F_4 \in F$ then $\overline{h}_{a^0} \in \overline{h}^1$ and $\overline{h}_{a^0} \in p(\overline{h}_{a^1})$ which again is not so. Hence $p(\overline{h}_{a^1}) \notin p(\overline{h}_{a^2})$. //

COROLLARY 4.6. *If $M \in \mathcal{Q}$ then $\psi : M \to \pi M_i/F$ is an embedding.*

Proof. Theorem 4.5 establishes that M is L-finitary and so the result follows by Corollary II: 2.4. //

We now come to what is perhaps the chief result of this section. For $M \in \mathcal{Q}$ it is concerned with the relationship between $p\psi(M)$ and N_ψ and is a generalisation of Theorem II: 2.8.

THEOREM 4.7. *If $M \in \mathcal{Q}$ and N'_ψ is the $\kappa(\alpha, \beta)$ structure formed from N_ψ by deleting those chains of N_ψ that are not complete with respect to $\cup$ and $\cap$ then M is isomorphic to N'_ψ with $p\psi$ a family of related isomorphisms.*

Proof. F^0, F^1 and F^2 denote the sets of objects of appropriate types of N_ψ and F^0, F^1 and F'^2 denote the corresponding sets of objects of N'_ψ. Thus $F'^2 \subseteq F^2$.

That $p\psi$ is bijective from E^0 to F^0 and E^1 to F^1 follows from the argument in the proof of Theorem II: 2.8.

Take any $a^2 \in E^2$ then $p\left(\overline{h}_{a^2}\right)$ is complete with respect to $\cup$ and $\cap$. For let V_p be a non-empty subset of members of $p\left(\overline{h}_{a^2}\right)$ and put $V = \left\{a^1 : a^1 \in a^2 \text{ and } p\psi\left(a^1\right) \in V_p\right\}$. Thus $\cup V$ and $\cap V$ belong to a^2 and so $p\psi(\cup V) \in p\psi\left(a^2\right)$ and $p\psi(\cap V) \in p\psi\left(a^2\right)$. Further, it can be checked that $\cup V_p = p\psi(\cup V)$ and $\cap V_p = p\psi(\cap V)$. Hence $p\left(\overline{h}_{a^2}\right)$ is complete. That is $p\psi$ is a map from E^2 into F'^2.

Take any $a^2, b^2 \in E^2$ such that $p\psi\left(a^2\right) = p\psi\left(b^2\right)$. From Theorem 4.5 we have $a^1 \in a^2$ if, and only if, $p\psi\left(a^1\right) \in p\psi\left(a^2\right)$. Hence $a^1 \in a^2$ if, and only if, $a^1 \in b^2$. That is $a^2 = b^2$ and $p\psi : E^2 \to F'^2$ is injective.

Take any $p\left(\overline{h}^2\right) \in F'^2$. $p\left(\overline{h}^2\right)$ is complete with respect to $\cup$ and $\cap$. Let $K = \{a^1 : p\psi\left(a^1\right) \in p\left(\overline{h}^2\right)\}$. It can be immediately checked that K is a chain. Let V be a non-empty subset of K and put $V_p = \{p\psi\left(a^1\right) : a^1 \in V\}$. Thus V_p is a subset of members of

$p(\bar{h}^2)$ and so $\cup V_p$ and $\cap V_p$ belong to $p(\bar{h}^2)$. But $p\,(\cup V) = \cup V_p$ and $p\psi(\cap V) = \cap V_p$ and so $\cup V$ and $\cap V$ belong to K. Thus there exists some $a^2 \in E^2$ such that $a^2 = K$ and for which it can be checked that $p\psi(a^2) = p(\bar{h}^2)$. That is $p\psi : E^2 \to F'^2$ is surjective.

Finally we note that $p\psi$ preserves the operations of M with respect to those of N'_ψ and recall that all the constant relations are L-stable by assumption. Hence the theorem is established. //

We are now in a position to describe and derive the local theorem of McLain [1959]. We let $J[x^1, y^1, z^2]$ be the formula

$$x^1 \underline{\in} z^2 \wedge y^1 \underline{\in} z^2 \wedge x^1 \subseteq y^1 \wedge \forall w^1 \big(w^1 \underline{\in} z^2 \wedge x^1 \subseteq w^1 \wedge w^1 \subseteq y^1 \Rightarrow x^1 = w^1 \vee w^1 = y^1\big) .$$

That is the formula expresses the property of two elements of a chain forming a jump in the chain. Further, we let $Z[x^2]$ stand for the formula

$$\forall x_1^0 \ldots \forall x_n^0 \forall x_1^1 \ldots \forall x_m^1 \Big(J\big[x_1^1, x_2^1, x^2\big] \wedge \ldots \wedge J\big[x_{2p-1}^1, x_{2p}^1, x^2\big] \wedge x_{2p+1}^1 \underline{\in} x^2 \wedge \ldots \wedge x_m^1 \underline{\in} x^2 \Rightarrow T\big[x_1^0, \ldots, x_m^1\big] \vee x_1^1 = x_2^1 \vee \ldots \vee x_{2p-1}^1 = x_{2p}^1 ,$$

where T is a quantifier free first order formula containing the variables as shown.

We proceed to the theorem by way of several lemmas.

LEMMA 4.8. *If* $M \in Q$ *and* $\bar{h}^2$ *is a chain of* $\pi M_i/\mathcal{F}$ *then there* *exists an* $a^2 \in E^2$ *such that* $p(\overline{h}^2) \sqsubseteq p\left(\overline{h}_{a^2}\right)$ *and for any* $a^1, b^1 \in E^1$, $p\psi(a^1)$, $p\psi(b^1)$ *form a jump in* $p\psi(a^2)$ *if, and only if, they form a jump in* $p(\overline{h}^2)$.

Proof. As in the proof of Theorem 4.7 let $K = \{a^1 : p\psi(a^1) \in p(\overline{h}^2)\}$. K is a chain but may not be complete with respect to $\cup$ and $\cap$ as $p(\overline{h}^2)$ need not be so. Let K' be formed of the members of K together with the union and intersection of all non empty subsets of K. Thus K' forms a chain complete with respect to $\cup$ and $\cap$ and so there exists some $a^2 \in E^2$ whose members are those of K'. It can now be established that this a^2 fulfills the requirements of the lemma. //

LEMMA 4.9. *The formula* $J[x^1, y^1, z^2]$ *is hereditary.*

Proof. By direct examination of the syntactical form of the formula. //

LEMMA 4.10. *Let* M *be as any member of* $\underline{Q}$ *except that the chains of* M *need not be complete but only satisfy the property* S *of Theorem* 4.3. *If* N *is a substructure of* M, $\underline{p} : M \to N$, *and* ν *any* M *interpretation such that* $N \models_{p\nu} J[x^1, y^1, z^2]$ *then for some* *M-interpretation* ν' *such that* ν' *agrees with* ν *except possibly on* x^1, y^1, *but* $p\nu'(x^1) = p\nu(x^1)$ *and* $p\nu'(y^1) = p\nu(y^1)$ *we have* $M \models_{\nu'} J[x^1, y^1, z^2]$.

Proof. Put $\nu(x^1) = a^1$, $\nu(y^1) = b^1$ and $\nu(z^2) = c^2$ and assume $N \models_{p\nu} J[x^1, y^1, z^2]$, that is $p(a^1)$, $p(b^1)$ form a jump in $p(c^2)$. If $p(a^1) = p(b^1)$ then take any $c^1 \in c^2$ such that $p(c^1) = p(a^1)$ and take ν' such that $\nu'(x^1) = \nu'(y^1) = c^1$. Hence $M \models_{\nu'} J[x^1, y^1, z^2]$.

But if $p(a^1) \subset p(b^1)$ then there exists some $a^0 \in E^0$ such that $a^0 \notin p(a^1)$ but $a^0 \in p(b^1)$. Take c^1 as the largest member of c^2 not containing a^0 and d^1 as the smallest member of c^2 containing a^0 . (Both c^1 and d^1 are available by property S .) c^1 and d^1 form a jump in c^2 and $p(c^1) = p(a^1)$ and $p(d^1) = p(b^1)$. Thus take ν' such that $\nu'(x^1) = c^1$ and $\nu'(y^1) = d^1$. //

LEMMA 4.11. *If M and N are as in Lemma 4.10 then $N \models_{p\nu} Z[x^2]$ if $M \models_{\nu} Z[x^2]$, ν any M interpretation.*

Proof. Assume $M \models_{\nu} Z[x^2]$ but $N \not\models_{p\nu} Z[x^2]$. Hence for some M interpretation ν' such that $\nu(x^2) = \nu'(x^2)$ we have

$$N \models_{p\nu'} J\left[x_1^1, x_2^1, x^2\right] \wedge \ldots \wedge J\left[x_{2p-1}^1, x_{2p}^1, x^2\right] \wedge x_{2p+1}^1 \underline{\in} x^2 \wedge \ldots$$
$$\wedge\ x_m^1 \underline{\in} x^2 \wedge \neg \left(T\left[x_1^0, \ldots, x_m^1\right] \vee x_1^1 = x_2^1 \vee \ldots \vee x_{2p-1}^1 = x_{2p}^1\right) .$$

Hence, by Lemma 4.10 and the fact that

$T\left[x_1^0, \ldots, x_m^1\right] \vee x_1^1 = x_2^1 \vee \ldots \vee x_{2p-1}^1 = x_{2p}^1$ is hereditary we have $M \not\models_{\nu} Z[x^2]$. Thus if $M \models_{\nu} Z[x^2]$ then $N \models_{p\nu} Z[x^2]$. //

LEMMA 4.12. *If $M \in \mathcal{Q}$ and $\overline{h}^2$ a chain of $\pi M_i/F$ such that*

$N_\psi \models_{p\nu} Z[x^2]$ *then* $N'_\psi \models_{p\nu'} Z[x^2]$ *, where* ν *is a* $\pi M_i/F$ *interpretation such that* $\nu(x^2) = \overline{h}^2$ *and* ν' *agrees with* ν *except that* $\nu'(x^2) = \overline{h}_{a^2}$ *, where* a^2 *is as constructed in Lemma* 4.8.

Proof. Assume there exists some N'_ψ interpretation μ such that $\mu(x^2) = \nu'(x^2)$ and such that

$$N'_\psi \models_\mu J\left[x_1^1, x_2^1, x^2\right] \wedge \ldots \wedge x_m^1 \underline{\in} x^2$$

but $N'_\psi \not\models_\mu T\left[x_1^0, \ldots, x_m^1\right] \vee x_1^1 = x_2^1 \vee \ldots \vee x_{2p-1}^1 = x_{2p}^1$. By Lemma 4.8, $\mu\left(x_j^1\right), \mu\left(x_{j+1}^1\right)$ form a jump in $p\nu(x^2)$, each $1 \le j \le 2p-1$. Recalling from Lemma 4.8 that $p(\overline{h}^2) \subseteq p\left(\overline{h}_{a^2}\right)$ assume that for some k , $2p < k \le m$, $\mu\left(x_k^1\right) \in p\left(\overline{h}_{a^2}\right)$ but $\mu\left(x_k^1\right) \notin p(\overline{h}^2)$. Hence for some t , $1 \le t \le n$, we have $N_\psi \not\models_\mu x_t^0 \in x_k^1$ or $N_\psi \not\models_\mu x_t^0 \notin x_k^1$ where $x_t^0 \in x_k^1$ or $x_t^0 \notin x_k^1$ is a component in the conjunctive normal form of $T\left[x_1^0, \ldots, x_m^1\right]$ respectively. But $\mu\left(x_k^1\right) \in p\left(\overline{h}_{a^2}\right)$ and so will be the union or intersection of a non empty set of members of $p(\overline{h}^2)$. Thus it is possible to select a member of this union or intersection (say $p\left(a_k^1\right) \in p(\overline{h}^2)$) such that if $\mu\left(x_t^0\right) \notin \mu\left(x_k^1\right)$ then $\mu\left(x_t^0\right) \notin p\left(a_k^1\right)$ or if $\mu\left(x_t^0\right) \in \mu\left(x_k^1\right)$ then $\mu\left(x_t^0\right) \in p\left(a_k^1\right)$.

Therefore it is possible to select a N_ψ interpretation μ'

which agrees with μ except that $\mu'(x^2) = p(\overline{h}^2)$, $\mu'\left(x_k^1\right) = a_k^1$, as given above, and $\mu'\left(x_j^1\right)$, each $2p < j \le m$, but $j \ne k$, are chosen arbitary members of $p(\overline{h}^2)$, and such that

$N_\psi \models_\mu J\left[x_1^1, x_2^1, x^2\right] \wedge \ldots \wedge x_m^1 \underline{\in} x^2$ but

$N_\psi \not\models_{\mu'} T\left[x_1^0, \ldots, x_m^1\right] \vee x_1^1 = x_2^1 \vee \ldots \vee x_{2p-1}^1 = x_{2p}^1$. Hence

$N_\psi \not\models_{p\nu} Z[x^2]$. That is if $N_\psi \models_{p\nu} Z[x^2]$ then $N'_\psi \models_{p\nu'} Z[x^2]$. //

We now state McLain's Local Theorem [1959], (Theorem 1, page 178), although worded in accord with the context developed here. We let λ stand for the sentence

$$\forall x^2 \exists y^2 \left(x^2 \subseteq y^2 \wedge Z[y^2]\right) .$$

THEOREM 4.13. *If* $M \in \mathcal{Q}$ *and* $\{i : M_i \models \lambda\} \in \mathrm{F}$ *then* $M \models \lambda$.

Proof. Assume that $M \in \mathcal{Q}$ and $\{i : M_i \models \lambda\} \in \mathrm{F}$. Hence $\pi M_i/\mathrm{F} \models \lambda$ and so by Lemma 4.11 and direct examination of $x^2 \underline{\subseteq} y^2$ we have $N_\psi \models \lambda$. From Lemma 4.12 we conclude $N'_\psi \models \lambda$ and hence the result by Theorem 4.7. //

Finally, we comment on the relationship between Mal'cev's use of the compactness theorem from formal logic to establish his local theorem and McLain's appeal to Steenrod's theorem on inverse limits of compact spaces. The result that underlies and supports each of these alternative procedures is the Prime Ideal Theorem (*c.f.* Grätzer [1968], Theorem 6.7 and Corollary, page 26/27). By means of this result and via a suitably chosen ultrafilter and associated ultraproduct the compactness theorem of first (and higher) order logic can be established, (*c.f.* Morel, Scott and Tarski [1958] and Theorem I: 3.4).

Again by means of this same result and via another suitably chosen ultrafilter the result used by McLain - viz. that the inverse limit of a family of finite (non-empty) sets is non-empty, can also be established (*c.f.* Grätzer [1968], Theorem 21.1, page 132).

Indeed the use of the inverse limit result in McLain's proof of his local theorem can be avoided by noting that the result of the basic lemma (*c.f.* McLain [1959], page 179) which he establishes by means of it is an immediate consequence of the properties of the L-associated ultrafilter F as constructed in this paper.

As an aside we mention in this connection that the results of the two basic lemmas set out by Derek Robinson (D. Robinson [1968], Lemmas 5.12 and 5.13, pages 128 and 131), in support of his exposition of a method for obtaining local theorems are also an immediate consequence of the properties of this same ultrafilter.

The above remarks together with the detailed exposition of this section and section 3 suggest that the concepts of ultrafilters and associated ultraproducts provide the natural and most effective means for the derivation of the full spread of local theorems in algebraic structures.

APPENDIX I

ON A COMPACTNESS THEOREM OF A. SHAFAAT

Summary. In a paper published in 1967 A. Shafaat [1967] established a compactness property (principle of localization) for certain languages of a general relational calculus. His proof was based on a topological theorem of Steenrod. Shafaat gave the details of proof only for a special case of his result, but claimed these details could be modified to obtain the general result.

In fact, however, it would seem that this general result cannot be established without an additional restriction on the class of sentences involved, viz, that only a finite number of existential quantifiers are associated with any sentence involved in the compactness result.

Section 1 summarises Shafaat's description of his general calculus with its special notation and then gives counter-examples establishing the necessity of the modification to Shafaat's results.

Section 2 develops an ultraproduct construction and associated 'Łoš' theorem for the kind of languages set out in Shafaat's paper and so gives an alternative proof of his compactness and embedding theorems, modified as above. The basic idea and procedure of using ultraproducts to establish compactness results is due to Morel, Scott and Tarski [1958]. The interest in this appendix is that the same ideas and method produce the compactness result for Shafaat's more abstract language formulation and dependent only on his structural principles $A1$, $A2$ and $A3$ as set out below.

1. Counter-examples

Shafaat [1967], page 630, describes a language L of the general relational calculus as an ordered 6-tuple (A, X, R, P, Φ, S), where

(i) A is a set of symbols called 'individuals';

(ii) X is the set of symbols $x_1, x_2, \ldots$ called 'variables';

(iii) R is a set of symbols written as $r(-, \ldots, -, \ldots)$ and called 'relative symbols', where the sequence of dashes $(-)$ may be infinite. The order of a relative symbol is the ordinal number of the sequence of dashes occurring in it.

(iv) From A, X, R are formed 'atomic formulae' in the usual manner.

(v) P is a set of symbols containing the set Q of all atomic formulae.

(vi) Φ is a mapping from P into the power set of Q such that if p is an atomic formula then $\Phi(p) = p$.

(vii) From P, Φ are formed the ordered pairs $(p, \Phi(p))$, $p \in P$, called formulae. Given $\mu : A \to M$, $\nu : X \to M_1$, $p_{\mu,\nu}$ is written for the pair $(p, \Phi(p)_{\mu,\nu})$, where $\Phi(p)_{\mu,\nu}$ is the set of symbols obtained from the atomic formulae in $\Phi(p)$ by replacing individuals and variables by their images under μ, ν respectively.

(viii) S is the set of formulae not involving any variables together with all symbols of the form $s \equiv ((q_1\, x_{j1}) \ldots (q_n\, x_{jn}) \ldots)p$, where p, called the matrix of s, is a formula involving $x_{j1}, \ldots, x_{jn}, \ldots$, and every q is either the universal quantifier $\forall$ or the existential quantifier $\exists$.

An R-relational system, (M, f) , is an ordered pair where M is any set of symbols and f is a mapping from $\cup\left\{R_n \times M^n : n < \alpha\right\}$ to $\{0, 1, \ldots, N\}$, the set of truth values, where α is some given ordinal number.

Shafaat characterises a language L of a general relational calculus as *meaningful* if given any R-relational system (M, f) , there exists a uniquely determined mapping

$$t(f) : \{p_{\mu,\nu} : p \in P,\ \nu : X \to M,\ \mu : A \to M,\ \mu|A_p \text{ injective}\} \to \{0, 1, \ldots, N\} ,$$

where A_p is the set of individual symbols in p , and such that the two following statements hold.

A.1. If f, f_1 coincide on $(\Phi(p))_{\mu,\nu}$ then $t(f)(p_{\mu,\nu}) = t(f_1)(p_{\mu,\nu})$.

A.2. If (M, f) is an R-relational system, let $\varepsilon : M^* \to M$ be bijective. Then there is an obvious bijective mapping $e : \cup\left\{R_n \times M^{*n} : n < \alpha\right\} \to \cup\left\{R_n \times M^n : n < \alpha\right\}$ where (M^*, ef) is an R-relational system. It is postulated that $t(f)(p_{\mu,\nu}) = t(ef)(p_{\varepsilon\mu,\varepsilon\nu})$, for all p, μ, ν, ε and f .

Shaffaat adds a third definition. He calls a formula p *finitary* if

A.3. $\Phi(p)_{\mu,\nu}$ is finite whenever $|\nu(x)|$ is finite.

Finally we describe the semantics Shafaat provides for his language L . Let N_1 be any fixed integer satisfying $0 \leq N_1 < N$.

$p \in P$ holds in (M, f) if, and only if, $t(f)\left(p_{\mu,\nu}\right) > N_1$.

Let $s = \left(\left(q_1\ x_{j1}\right)\ \ldots\ \left(q_n\ x_{jn}\right)\ \ldots\right)p$. If $\mu : A \to M$ is such that $\mu | A_p$ is injective and if s only involves a finite number of quantifiers then s holds in (M, f) under the normal definitions. When s involves infinitely many variables the following example explains the method of interpretation. Put

$$s = \left(\forall x_{i_1}\ \ldots\ \forall x_{i_n}\right)\left(\exists x_{j1}\ \ldots\ \exists x_{jn}\ \ldots\right)\left(\forall x_{l_1}\ \ldots\ \forall x_{l_n}\ \ldots\right)\ldots\ p\ ,$$

which is written as $s = \forall X_i \exists X_j \forall X_l\ \ldots\ p$, where

$X_i = \left\{x_{i_1},\ \ldots,\ x_{i_n},\ \ldots\right\}$, $X_j = \{x_{j1},\ \ldots,\ x_{jn}\ \ldots\}\ldots$ are disjoint. Then s holds in (M, f) under μ if, and only if, for all $\nu_i : X_i \to M$ there exists $\nu_j : X_j \to M$ such that for all $\nu_l : X_l \to M,\ \ldots,\ p_{\mu,\nu}$ holds in (M, f) , where $\nu_i\left(X_i\right)$, $\nu_j\left(X_j\right)$, ... and $\nu_i\left(X_i\right) \cup \nu_j\left(X_j\right) \cup \ldots$ are finite and ν is such that its restriction to X_i, X_j, ... is ν_i, ν_j, ... respectively.

Shafaat's claimed main result now states:

If Σ *is a set of finitary sentences in a meaningful language* L *of the general relational calculus and if every finite subset of* Σ *possesses a model then* Σ *possesses a model.*

We now set out two counter-examples to the full generality of this result.

Let $L = (A, X, R, P, \Phi, S)$ be a particular language of the general relational calculus given as follows:

$A = \emptyset$. $X = \{x_n : n < \omega\} \cup \{y_n : n < \omega\}$, ω being the first non-finite ordinal. $R = \{r(-), e(-, -), u(-, \ldots)\}$, where $u(-, \ldots)$ is of order ω . P is the set of symbols made up from the atomic formulae, Q , together with the symbols of negation, disjunction, conjunction and implication, viz, $\neg$, $\vee$, $\wedge$, and $\Rightarrow$, by the standard formation rules.

$\Phi : P \to 2^Q$ is defined by $\Phi(p) = \{p\}$, if $p \in Q$, $\Phi(\neg p) = \Phi(p)$, $\Phi(\vee\{p_i : i \in I\}) = \bigcup\{\Phi(p_i) : i \in I\}$, $\Phi(\wedge\{p_i : i \in I\}) = \bigcup\{\Phi(p_i) : i \in I\}$ and $\Phi(p \Rightarrow q) = \Phi(p) \cup \Phi(q)$, for all formulae p and index set I .

If (M, f) is an R-relational system, where f takes values 0 or 1 , then $t(f)$ is defined by: $t(f)(e(x, y)_\nu) = 1$ if, $\nu(x) = \nu(y)$, otherwise $t(f)(e(x, y)_\nu) = 0$; $t(f)(r(x)_\nu) = f(r(\nu(x)))$; $t(f)(u(x_1, \ldots)_\nu) = f(u(\nu(x_1), \ldots,))$; $t(f)(\wedge\{p_i : i \in I\}_\nu) = 1$ if, for each $i \in I$, $t(f)(p_{i_\nu}) = 1$, otherwise $t(f)(\wedge\{p_i : i \in I\}_\nu) = 0$; $t(f)(\vee\{p_i : i \in I\}_\nu) = 1$ if, for some $i \in I$, $t(f)(p_{i_\nu}) = 1$ otherwise $t(f)(\vee\{p_i : i \in I\}_\nu) = 0$; $t(f)(\neg p_\nu) = 1 - t(f)(p_\nu)$; $t(f)(p \Rightarrow q_\nu) = 0$ if $t(f)(p_\nu) = 1$ and $t(f)(q_\nu) = 0$, otherwise $t(f)(p \Rightarrow q_\nu) = 1$; for all $\nu : X \to M$.

Counter-example 1. Let $K = \{T_n : n < \omega\}$ be a class of sentences of L defined as follows:

$$T_1 = \exists(X_\omega)\forall(Y_\omega)p_1 \ ,$$

where $\exists(X_\omega)$ stands for $\exists x_1 \exists x_2 \ldots \exists x_n \ldots$, $\forall(Y_\omega)$ stands for $\forall y_1 \forall y_2 \ldots \forall y_m \ldots$, and

$$p_1 = (\wedge\{r(x_i) : i < x\}) \wedge (\wedge\{r(y_i) : i < \omega\} \Rightarrow \wedge\{\vee\{e(y_i, x_j) : j < \omega\} : i < \omega\}) .$$

For each $n > 1$, $T_n = \exists(X_\omega)p_n$, where

$$p_n = (\wedge\{r(x_i) : i < \omega\}) \wedge (\wedge\{\neg e(x_i, x_j) : 1 \le i, j \le n, i \ne j\}) .$$

It is evident, from inspection, that for each $k < \omega$, $|\Phi(p_k)_\nu|$ is finite, for each $\nu : X \to M$ such that $|\nu(X)|$ is finite. Thus Shafaat's A.3 condition is satisfied, as also are his A1 and A2 .

Let K_0 be any finite subset of K . Now K_0 has a model. For let m be the largest member of ω such that $T_m \in K_0$. Take $M = \{a_i : i < \omega\}$ and define $f : \{r\} \times M \to \{0\ 1\}$ by: $f(r(a_i)) = 1$ if $1 \le i \le m$, otherwise $f(r(a_i)) = 0$. Also put $f(e(a_i, a_j)) = 1$ if, and only if, $i = j$. It can be immediately checked that (M, f) is a model of K_0 .

But K itself has no model. For, assume some (M, f) is a model of K . Thus (M, f) is a model of T_1 . Let $\nu : X \to M$ be such that $(M, f) \models_\nu \forall(Y_\omega)p_1$, where $|\nu(X)|$ is finite and $\{\nu(x_i) : i < \omega\} = \{a_1, \ldots, a_k\}$, for some $a_1, \ldots, a_k \in M$, k some positive integer. But (M, f) is also a model of T_{k+1} . Let $\nu' : X \to M$ be such that $(M, f) \models_{\nu'} p_{k+1}$, where $|\nu'(X)|$ is finite and contains $\nu'(x_1), \ldots, \nu'(x_{k+1})$ as distinct members. Now take $\nu'' : X \to M$ such that $\nu''(x_i) = \nu(x_i)$, all $i < w$, $\nu''(y_i) = \nu'(x_i)$, all $i < w$. By inspection $(M, f) \not\models_{\nu''} p_1$, contrary to the requirement that (M, f) is a model of T_1. Hence K has no model.

Counter-example 2. Let $K = \{T_n : n < w\}$ be a class of sentences of L defined as follows:

$T_1 = \exists(X_w)u(x_1, x_2, \ldots)$,

$T_n = \forall(X_w)\left(u(x_1, x_2, \ldots) \Rightarrow \wedge\{\neg\, e(x_i, x_j) : 1 \leq i, j \leq n, i \neq j\}\right)$,

for each $n > 1$.

All of Shafaat's conditions are satisfied, but, by an analogous argument to that used in the first counter-example, it can be shown that while every finite subset of K has a model, K itself has no model.

2. Ammended compactness result

Let $\{(M_i, f_i) : i \in I\}$ be a family of R-relational systems. Let F be an ultrafilter over I. $(\pi M_i/F, f)$, called the ultra-product of the family with respect to F, is an R-relational system defined as follows: $\pi M_i/F$ is the quotient set of the cartesian product $\pi\{M_i : i \in I\}$ under the relation, $m_1 \sim m_2$ if, and only if, $\{i : m_1(i) = m_2(i)\} \in F$, where $m_1, m_2 \in \pi\{M_i : i \in I\}$. The necessary lemmas to justify these assertions are assumed. If $m \in \pi\{M_i : i \in I\}$ then $\overline{m}$ will denote the equivalence class of m.

$f : \cup\{R_n \times (\pi M_i/F)^n : n < \alpha\} \to \{0, \ldots, N\}$ is defined by $f(r(\overline{m}_1, \ldots,)) = k$, where $F_k \in F$ and $F_k = \{i : f_i(r(m_1(i), \ldots))\} = k$. The following lemma justifies this definition of f.

LEMMA 2.1. *Let* $F_j = \{i : f_i(r(m_1(i), \ldots)) = j\}$, *for all* $0 \le j \le N$. *Then for one, and only one,* $j \in \{0, \ldots, N\}$, $F_j \in F$.

Proof. The collection $F_0, \ldots, F_N$ partitions I. F is an ultrafilter over I and hence one, and only one, of $F_0, \ldots, F_N$ belongs to F. //

The next lemma establishes that when $r(\overline{m}_1, \ldots)$ involves only a finite number of distinct members of $\pi M_i/F$ then $f(r(\overline{m}_1, \ldots))$ is independent of the representation chosen for those members.

LEMMA 2.2. *Let* $\overline{m}_1, \ldots, \overline{m}_t$ *be the finite number of distinct members of* $\pi M_i/F$ *involved in* $r(\overline{m}_1, \ldots)$ *and let* $m_j \sim n_j$, *each* $1 \le j \le t$. *Then* $f(r(\overline{m}_1, \ldots)) = f(r(\overline{n}_1, \ldots))$.

Proof. Put $H_j = \{i : m_j(i) = n_j(i)\}$, and so $H_j \in F$, each $1 \le j \le t$. Let $F_j = \{i : f_i(r(m_1(i), \ldots)) = j\}$, $G_j = \{i : f_i(r(n_1(i), \ldots)) = j\}$, each $1 \le j \le N$. For each such j, $F_j \supseteq G_j \cap H_1 \cap \ldots \cap H_t$ and $G_j \supseteq F_j \cap H_1 \cap \ldots \cap H_t$. Hence $F_j \in F$ if, and only if, $G_j \in F$, $1 \le j \le n$. That is $f(r(\overline{m}_1, \ldots)) = f(r(\overline{n}_1, \ldots))$. //

The ambiguity in definition of $f(r(\overline{m}_1, \ldots))$ when $r(\overline{m}_1, \ldots)$ involves a non-finite number of distinct members of $\pi M_i/F$ will cause no later embarassment as, in such cases, a fixed representation for those members will be given. (*c.f.* the μ and μ_i's representation as later defined.)

If A_0 is a subset of A, the constant symbols of L, and if for some $F \in F$ and for each $i \in F$, $\mu_i : A_0 \to M_i$, then $\mu : A_0 \to \pi M_i/F$ is defined by $\mu(a) = \overline{m}$, where $m(i) = \mu_i(a)$, for all $i \in F$, all $a \in A_0$.

LEMMA 2.3. *μ, as above, is well defined and if, for each $i \in F$, μ_i is injective then μ is injective.*

Proof. Immediate from the definition. //

If $\nu : X \to \pi M_i/F$ then, for each $i \in I$, $\nu_i : X \to M_i$ can be defined by $\nu_i(x) = m(i)$, where $\nu(x) = \overline{m}$, for all $x \in X$.

Conversely, if for some $F \in F$ and, for each $i \in F$, $\nu_i : X \to M_i$, then $\nu : X \to \pi M_i/F$ is defined by, $\nu(x) = \overline{m}$, where $m(i) = \nu_i(x)$, all $i \in F$, all $x \in X$.

LEMMA 2.4. *Let ν and $\{\nu_i : i \in F_1, F_1 \in F\}$, ν' and $\{\nu'_i : i \in F_2, F_2 \in F\}$ be two collections of associated maps as described above. Then (i) $|\nu(X)|$ is finite if, and only if, for some positive integer n, $\{i : |\nu_i(X)| < n\} \in F$. (Similarly for ν' and the ν'_i's.) Also (ii), if $|\nu(X)|$ and $|\nu'(X)|$ are* *finite then* $\nu = \nu'$ *if, and only if,* $\{i : \nu_i = \nu_i'\} \in \mathcal{F}$.

Proof. Assume $|\nu(X)|$ is finite. Hence there is some positive integer t such that $|\nu(X)| = t$. Put $\nu(X) = \{\overline{m}_1, \ldots, \overline{m}_t\}$ where for each $i \in F_1$, and all $x \in X$, $\nu_i(x) = m_k(i)$, where $\nu(x) = \overline{m}_k$. Hence for each $i \in F_1$, $\nu_i(X) = \{m_1(i), \ldots, m_t(i)\}$. That is $\{i : |\nu_i(X)| \le t\} \supseteq F_1$ and so belongs to $\mathcal{F}$.

Conversely, assume $F \in \mathcal{F}$, where $F = \{i : |\nu_i(X)| \le n\}$, some positive integer n . Hence $|\nu(X)| \le n$. For if not let $\overline{m}_1, \ldots, \overline{m}_{n+1}$ be $n + 1$ distinct members of $\nu(X)$ and such that if $\nu(x) = \overline{m}_t$, $1 \le t \le n+1$ then, for all $i \in F_1$, $\nu_i(x) = m_t(i)$. Put $F_{k,j} = \{i : m_k(i) \ne m_j(i)\}$ for $1 \le k$, $j \le n+1$ and $k \ne j$. Thus each such $F_{k,j} \in \mathcal{F}$ as $\overline{m}_1, \ldots, \overline{m}_{n+1}$ are distinct. Hence $\{i : |\nu_i(X)| \ge n+1\} \supseteq \cap\{F_{k,j} : 1 \le k, j \le n+1, k \ne j\} \cap F_1$ and so belongs to $\mathcal{F}$, which cannot be, as $F \in \mathcal{F}$. Thus $|\nu(X)|$ is finite.

(2) Assume $\nu(X)$ and $\nu'(X)$ are both finite. Take $\nu = \nu'$ and $|\nu(X)| = t$, t some positive integer. Let $\nu(X) = \{\overline{m}_1, \ldots, \overline{m}_t\}$ where for each $i \in F_1$ if $\nu(x) = \overline{m}_k$ then $\nu_i(x) = m_k(i)$, and let $\nu'(X) = \{\overline{n}_1, \ldots, \overline{n}_t\}$ where for each $i \in F_2$ if $\nu'(x) = \overline{n}_k$ then $\nu_i'(x) = n_k(i)$. As $\nu(X) = \nu'(X)$, then for all $x \in X$, $\nu(x) = \nu'(x)$, and so $\overline{m}_k = \overline{n}_k$, $1 \le k \le t$. Let $G_k = \{i : m_k(i) = n_k(i)\}$, hence $G_k \in \mathcal{F}$, each $1 \le k \le t$. Thus

$\{i : \nu_i = \nu'_i\} \supseteq F_1 \cap F_2 \cap G_1 \cap \ldots \cap G_t$ and so belongs to F.
Conversely, if $\{i : \nu_i = \nu'_i\} \in \mathrm{F}$ then $\nu = \nu'$. For take any $x \in X$ and let $\nu(x) = \overline{m}$, $\nu(x) = \overline{n}$, where for each $i \in F_1$, $\nu_i(x) = m(i)$, and each $i \in F_2$, $\nu'_i(i) = n(i)$. Thus for each $i \in F_1 \cap F_2 \cap \{i : \nu_i = \nu'_i\}$, $m(i) = n(i)$. That is $\overline{m} = \overline{n}$. //

LEMMA 2.5. *Let p be a finitary formula of a meaningful language L. Let $\mu_i : A \to M_i$ such that $\mu_i|A_p$ is injective, for each $i \in I$. Let $\nu : X \to \pi M_i/\mathrm{F}$ such that $|\nu(X)|$ is finite. Then $t(f)(p_{\mu,\nu}) = k$, where $E_k \in \mathrm{F}$ and $E_k = \{i : t(f_i)(p_{\mu_i,\nu_i}) = k\}$, and μ, $\{\mu_i : i \in I\}$ and ν, $\{\nu_i : i \in I\}$ are associated families of maps as set out above in Lemmas 2.3 and 2.4 respectively.*

Proof. Put $E_j = \{i : t(f_i)(p_{\mu_i,\nu_i}) = j\}$, each $0 \le j \le N$. Hence one, and only one, of $E_0, \ldots, E_N$ belongs to F. Let it be E_k.

Let $p_{\mu,\nu} = \{r_1(\alpha_{1,1}, \alpha_{1,2}, \ldots) \ldots, r_t(\alpha_{t,1}, \ldots)\}$, t some positive integer, and each α stands for some $\mu(a)$, $a \in A_p$, or $\nu(x)$, $x \in X_p$. For each $1 \le j \le t$, let $f(r_j(\alpha_{j,1}, \ldots)) = k_j$, and so $F_{k_j} \in \mathrm{F}$, where $F_{k_j} = \{i : f_i(r_j(\alpha_{j,1}(i), \ldots)) = k_j\}$, where $\alpha_{j,1}(i) = \mu_i(a)$, if $\alpha_{j,1} = \mu(a)$, or $\alpha_{j,1}(i) = \nu_i(x)$, if $\alpha_{j,1} = \nu(x)$. Hence $E' \in \mathrm{F}$, where $E' = E_k \cap F_{k_1} \cap \ldots \cap F_{k_t}$.

For any $i \in E'$, it is desired to set up an embedding of $\pi M_i/\mathcal{F}$ into M_1 or vice-versa. However, E' may be too large for this and so is restricted in the following way. Let

$E'' = \{i : \text{if } \nu x_1 \neq \nu x_2 \text{ then } \nu_i(x_1) \neq \nu_i(x_2) \text{, all } x_1, x_2 \in X\}$.

$E'' \in \mathcal{F}$ as $|\nu(X)|$ is finite. Let $E = E'' \cap E'$.

Now take any $i \in E$. Either $|\pi M_i/\mathcal{F}| \leq |M_i|$ or vice-versa. Assume $|\pi M_i/\mathcal{F}| \leq |M_i|$ and define a bijective map $\varepsilon : \pi M_i/\mathcal{F} \to M_i^*$, where M_i^* is a subset of M_i, such that if $\overline{m} \in \mu(A_p)$ then $\varepsilon(\overline{m}) = \mu_i(a)$, where $\mu(a) = \overline{m}$, and if $\overline{m} \in \nu(X) - \mu(A_p)$ then $\varepsilon(\overline{m}) = \nu_i(x)$, where $\nu(x) = \overline{m}$. For each $a \in A_p$, $\varepsilon(\mu(a)) = \mu_i(a)$ and for each $x \in X_p$, $\varepsilon(\nu(x)) = \nu_i(x)$. It should be noted that this latter statement requires that if $\overline{m} \in \mu(A_p) \cap \nu(X_p)$, that is, if for some $a \in A_p$, $x \in X_p$, $\mu(a) = \nu(x) = \overline{m}$, then the representation of $\overline{m}$ is so chosen that $m(i) = \mu_i(a)$, each $i \in I$, and $\nu_i(x)$ is defined, for each $i \in I$, in relation to this representative m. Lemma 2.4 ensures that ν is unaffected by this choice of ν_i, $i \in I$.

Now from condition A2 we have that (M_i^*, f_ie) is an R-relational system and moreover $t(f)(p_{\mu,\nu}) = t(fe)(p_{\varepsilon\mu,\varepsilon\nu})$. But $\varepsilon\mu = \mu_i$ when restricted to A_p and $\varepsilon\nu = \nu_i$ when restricted to X_p. Hence $t(f)(p_{\mu,\nu}) = t(fe)(p_{\mu_i,\nu_i})$. But $i \in F_{k_1} \cap \ldots \cap F_{k_t}$ and so

fe, f_i coincide on $\Phi(p)_{\mu_i,\nu_i}$. Thus from condition A1, we have $t(f_i)(p_{\mu_i,\nu_i}) = t(fe)(p_{\mu_i,\nu_i})$. That is $t(f)(p_{\mu,\nu}) = t(f_i)(p_{\mu_i,\nu_i})$.

If $|M_i| < |\pi M_i/F|$ then M_i is embedded in $\pi M_i/F$ and by a similar argument $t(f)(p_{\mu,\nu}) = t(f_i)(p_{\mu_i,\nu_i})$. Hence for all $i \in E$, $t(f)(p_{\mu,\nu}) = t(f_i)(p_{\mu_i,\nu_i})$. But $E_k \supseteq E$ and so $t(f)(p_{\mu,\nu}) = k$. //

Let s_k stand for a formula of the form $q_k(x_k) \ldots q_1(x_1)p$, where each q_i stands either for the universal quantifier or the existential quantifier and p is a formula involving $x_1, \ldots, x_k$ as some, but not necessarily all, of its variable symbols. For some given assignment $\nu : X \to M$, $|\nu(X)|$ finite, $\forall(x_1)p$ holds in (M, f, μ), that is $(M, f, \mu) \models_\nu \forall(x)p$, if, and only if, for all $\nu' : X \to M$, $|\nu'(X)|$ finite, such that $\nu(x) = \nu'(x)$, for all $x \in X - \{x_1\}$, $t(f)(p_{\mu,\nu'}) > N_1$, (N_1 some chosen member of $\{0, \ldots, N\}$). Similarly, $\exists(x_1)p$ holds in (M, f, μ) if, and only if, for some $\nu' : X \to M$, $|\nu'(X)|$ finite, such that $\nu(x) = \nu'(x)$, for all $x \in X - \{x_1\}$, $t(f)(p_{\mu,\nu'}) > N_1$. $(M, f, \mu) \models_\nu s_k$ is then defined by the natural induction.

THEOREM 2.6. *If* p *is a finitary formula of a meaningful language* L *then* $(\pi M_i/F, f, \mu) \models_\nu s_k$ *if, and only if,* $\{i : (M_i, f_i, \mu_i) \models_{\nu_i} s_k\} \in F$, *where* $\nu : X \to \pi M_i/F$, $|\nu(X)|$ *finite, is associated with* ν_i, $i \in I$, *as set out above in Lemma*

2.4 *and* μ *and the* μ_i*'s as above in Lemma* 2.3.

Proof. We proceed by induction on the number of quantifiers in s_k. Assume $(\pi M_i/\mathsf{F}, f, \mu) \models_\nu q_j(x_j) \ldots q_1(x_1)p$ if, and only if, $\{i : (M_i, f_i, \mu_i) \models_{\nu_i} q_j(x_j) \ldots q_1(x_1)p\} \in \mathsf{F}$, for $0 \le j < k$.

(Put $s_0 = p$).

First let $q_{j+1}(x_{j+1})$ be $\forall(x_{j+1})$. Assume that $F \in \mathsf{F}$, where $F = \{i : (M_i, f_i, \mu_i) \models_{\nu_i} \forall(x_{j+1})s_j\}$. Take any $\nu' : X \to \pi M_i/\mathsf{F}$, such that $\nu(x) = \nu'(x)$, for all $x \in X - \{x_{j+1}\}$. Let ν'_i, $i \in I$ be associated with ν', and so $G = \{i : (M_i, f_i, \mu_i) \models_{\nu'_i} s_j\} \in \mathsf{F}$ as $G \supseteq F$. Hence $(\pi M_i/\mathsf{F}, f, \mu) \models_{\nu'} s_j$, by the induction hypothesis. Thus $(\pi M_i/\mathsf{F}, f, \mu) \models_\nu \forall(x_{j+1})s_j$.

Conversely, assume that $(\pi M_i/\mathsf{F}, f, \mu) \models_\nu \forall(x_{j+1})s_j$. Let $F = \{i : (M_i, f_i, \mu_i) \models_{\nu_i} \forall(x_{j+1})s_j\}$. If $F \notin \mathsf{F}$ then CF, the complement of F, belongs to F. Assume $CF \in \mathsf{F}$. Thus, for each $i \in CF$, there exists $\nu'_i : X \to M_i$, $\nu_i(x) = \nu'_i(x)$, for all $x \in X - \{x_{j+1}\}$, such that $(M_i, f_i, \mu_i) \not\models_{\nu'} s_j$, that is $(\pi M_i/\mathsf{F}, f, \mu) \not\models_\nu \forall(x_{j+1})s_j$, contrary to the initial assumption. Hence $F \in \mathsf{F}$.

Secondly, if $q_{j+1}(x_{j+1}) = \exists(x_{j+1})$ then a similar type of argument establishes that $(\pi M_i/\mathsf{F}, f, \mu) \models_\nu \exists(x_{j+1})s_j$ if, and only if,

$\left\{(M_i, f_i, \mu_i) \models_{\nu_i} \exists(x_{j+1})s_j\right\} \in F$. Hence the result of the theorem for all finite k . //

We introduce the following notation. If β is some ordinal number then $\forall(X_\beta)$ stands for the sequence $\forall(x_1)\forall(x_2) \ldots \forall(x_n) \ldots _{n<\beta}$. Similarly for the symbol $\exists(X_\beta)$.

We recall the special semantical interpretation that Shafaat gives to quantification involving a non-finite string of variables. For instance if β is non-finite then the formula $\forall(X_\beta)p$, where p contains the variables $X_\beta = \{x_i : i < \beta\}$, holds in a model (M, f, μ) if, and only if, for all $\nu : X_\beta \to M$ such that $|\nu(X_\beta)|$ is finite, $p_{\mu,\nu}$ holds in (M, f) . A similar finitary restriction applies to formulae such as $\exists(X_\beta)p$ and to those involving combinations of both universal and existential quantifiers.

THEOREM 2.7. *If* p *is a finitary formula of a meaningful language* L *containing variables* $\{x_i : i < \beta\}$, β *some non-finite ordinal then*

a) $(\pi M_i/F, f, \mu) \models_\nu \forall(X_\beta)p$ *if*

$$\left\{i : (M_i, f_i, \mu_i) \models_{\nu_i} \forall(X_\beta)p\right\} \in F ,$$

b) $(\pi M_i/F, f, \mu) \models_\nu \exists(X_\beta)p$ *only if*

$$\left\{i : (M_i, f_i, \mu_i) \models_{\nu_i} \exists(X_\beta)p\right\} \in F ,$$

where $\nu : X \to \pi M_i/F$ *such that* $|\nu(X)|$ *is finite and the* ν_i*'s are* *associated with* ν .

Proof. *a)* Assume that $(\pi M_i/F, f, \mu) \not\models_\nu \forall(X_\beta)p$. Thus there exists some $\nu' : X \to \pi M_i/F$, $|\nu'(X)|$ finite, and such that $\nu x = \nu' x$, for all $x \in X - \{x_j : j < \beta\}$, such that $(\pi M_i/F, f, \mu) \not\models_{\nu'} p$. Thus, from Theorem 2.6 with $k = 0$, $\left\{i : (M_i, f_i, \mu_i) \not\models_{\nu'_i} p\right\} \in F$, where the ν_i's are associated with ν' . Hence,

$\left\{i : (M_i, f_i, \mu_i) \models_{\nu_i} \forall(X_\beta)p\right\} \notin F$. That is, if

$\left\{(M_i, f_i, \mu_i) \models_{\nu_i} \forall(X_\beta)p\right\} \in F$ then $(\pi M_i/F, f, \mu) \models_\nu \forall(X_\beta)p$.

b) Assume that $(\pi M_i/F, f, \mu) \models_\nu \exists(X_\beta)p$. Hence, there exists some $\nu' : X \to \pi M_i/F$, $|\nu'(X)|$ finite, and $\nu x = \nu' x$, for all $x \in X - \{x_j : j < \beta\}$, such that $(\pi M_i/F, f, \mu) \models_{\nu'} p$. Thus, again from Theorem 2.6, with $k = 0$, $\left\{i : (M_i, f_i, \mu_i) \models_{\nu'_i} p\right\} \in F$, where the ν_i's are associated with ν' . Hence,

$\left\{i : (M_i, f_i, \mu_i) \models_{\nu_i} \exists(X_\beta)p\right\} \in F$. //

We observe that the sentence $\forall(X_\omega)\exists y(y \neq x_1 \wedge \ldots \wedge y \neq x_i \ldots)_{i<\omega}$ interpreted with respect to Shafaat's special conditions, holds in any infinite model and does not hold in any finite model. Hence it cannot be expected that either *a)* or *b)* in Theorem 2.7 above hold with 'if and only if' in place of 'if' or 'only if'. It is this consideration that underlies the counter-examples of section one.

THEOREM 2.8. *Let* $s = Qp$ *, where* p *is a finitary formula of a meaningful language* L *and* Q *is a quantifier prefix:*

a) if Q *contains only a finite number of existential quantifiers then*

$$(\pi M_i/F, f, \mu) \models s \quad \textit{if} \quad \{i : (M_i, f_i, \mu_i) \models s\} \in F \; ;$$

b) if Q *contains only a finite number of universal quantifiers then*

$$(\pi M_i/F, f, \mu) \models s \quad \textit{only if} \quad \{i : (M_i, f_i, \mu_i) \models s\} \in F \; .$$

Proof. *a)* If Q contains only a finite number of quantifiers then the result follows immediately from Theorem 2.6. Let $Q = q_1(x_1)q_2(x_2) \ldots$ and assume Q contains an infinite string of quantifiers. As s contains only a finite number of existential quantifiers there exists some positive integer k such that q_n is a universal quantifier for all $n > k$. The result now follows by using Theorem 2.7 and an induction argument similar to that used in a proof of Theorem 2.6.

b) The proof follows by a similar argument to part *a)*. //

The compactness and embedding results of Shafaat, modified by the restriction as explained in the summary, now follow by standard procedures from Theorem 2.8. We indicate the proofs in outline.

THEOREM 2.9. Σ *is a class of finitary sentences of a meaningful language* L *, such that each member of* Σ *contains only a finite number of existential quantifiers. If every finite subset of*

Σ *has a model then so has* Σ .

Proof. Let I be an index set for all finite subsets of Σ . For each subset Σ_i of Σ , $i \in I$, let (M_i, f_i, μ_i) be a model of Σ_i . For each $s \in \Sigma$ let $F_s = \{i : s \in \Sigma_i\}$. Let F be the ultrafilter on the sub-base $\{F_s : s \in \Sigma\}$. Then $(\pi M_i/F, f, \mu)$ is a model of Σ . //

It was Shafaat's intention, as stated in his paper, to develop as general a situation as possible for the statement of a compactness result. Comparison with other relevant results in the literature suggests that the extent of generalisation beyond the classical first-order compactness theorem (*c.f.* Frayne, Morel and Scott, [1962]), is limited. Hatcher, [1968], has already noted, in a review of Shafaat's paper, that Chang and Keisler, [1966] have obtained a compactness theorem in a theory of models with truth values in a (possibly infinite) compact Hausdorff space. Fuhrken, Keisler and Slomson (*c.f.* Bell and Slomson [1969], pages 260ff), have developed compactness results for languages involving special quantifiers that lie outside the range of Shafaat's general relational calculus. Further, Bell and Slomson [1969], pages 286ff, discuss a number of results in which the notion of compactness itself is extended in the context of infinitary languages $L_{\alpha,\beta}$, whose rules allow quantification over strings of variables $< \beta$, and conjunctions and disjunctions of length $< \alpha$, where α, β are non-finite cardinals. Again these results lie outside the range of Shafaat's considerations.

THEOREM 2.10. *Σ is a class of finitary sentences of a meaningful language* L *, such that each member of Σ contains only a finite number of existential quantifiers. Then an R-relational system, (M, f), is embeddable in a model of Σ, if every finite sub-relational system of (M, f) is so embeddable.*

Proof. Let I be the index set for all finite sub-relational systems, (M_i, f_i), of (M, f). For each $i \in I$, let (N_i, g_i, μ_i) be a model of Σ such that (M_i, f_i) is embeddable in (N_i, g_i), by an embedding map ϕ_i. For each $m \in M$, let $F_m = \{i : m \in M\}$ and let F be the ultrafilter on the sub-base $\{F_m : m \in M\}$. It can now be checked that the mapping $\phi : M \to \pi N_i/\mathrm{F}$, given by: $\phi(m) = \overline{n}$, where $n(i) = \phi_i(m)$, $i \in I$, is an embedding of (M, f) into $(\pi N_i/\mathrm{F}, g)$. Moreover, by Theorem 2.8, part *a)*, $(\pi N_i/\mathrm{F}, g, \mu)$ is a model of Σ. //

APPENDIX II

VARIATION IN DEFINITION OF FIRST ORDER ULTRAPRODUCTS

Summary. Two variations are made in the standard definitions (*c.f.* Bell and Slomson [1969], pages 87 to 89) of an ultraproduct of a family of first order relational structures with respect to a chosen ultrafilter F over the index set I . The first variation, following the method used by Luxemburg [1969] in the construction of higher order ultrapowers, relaxes the requirement of similarity on the members of the family. The second variation used subfilters of F to define the individuals and relations of the ultraproduct.

In §1 the construction of the ultraproduct with these variations is set out and some consequences developed, particularly those relating to the identity relation. In §2 a family of similar structures is taken and a necessary and sufficient condition is established under which the first variation produces more relations, from an extensional view-point, than the standard definition.

1. Variations in ultraproduct construction

Let $\{M_i : i \in I\}$ be a collection of first order relational structures. For each $i \in I$ let $M_i = \left\{R_i^0; R_i^1, R_i^2, \ldots\right\}$ where R_i^0 is the class (non empty) of individuals in the ith structure and for each positive integer k , R_i^k is the class of k-placed relations of the structure. Each R_i^k contains at least the empty relation and each R_i^2 contains the identity relation denoted by e_i . It is further assumed that the distinct members of each R_i^k are distinct from a set theoretic and extensional point of view.

Let F be an ultrafilter defined over I . For each $k \geq 0$, F^k is a subfilter of F ; that is F^k is a subclass of F and is a filter. For each $k \geq 0$, let R_I^k be the class $\left\{f^k : f^k : I \to \cup\left\{R_i^k : i \in I\right\} \text{ and for all } i \in I, f^k(i) \in R_i^k\right\}$. Let $\sim_k$ denote the relation defined on R_I^k by: $f^k \sim_k g^k$ if $\{i : f^k(i) = g^k(i)\} \in F^k$.

LEMMA 1.1. *For each integer* $k \geq 0$, $\sim_k$ *is an equivalence relation.*

For each integer $k \geq 0$, let $R^k_{F^k}$ denote the quotient class of R_I^k with respect to $\sim_k$. If $f^k \in R_I^k$ then $\overline{f}^k$ denotes its equivalence class. The next lemma prepares the way for the definition of the individuals and relations of the ultraproduct.

LEMMA 1.2. *For each* $k \geq 0$, $\left\{i : f^k(i)\left[f_1^0(i), \ldots, f_k^0(i)\right]\right\} \in F$ *if, and only if,* $\left\{i : g^k(i)\left[g_1^0(i), \ldots, g_k^0(i)\right]\right\} \in F$, *where* $f_j \sim_0 g_j$, $1 \leq j \leq k$, *and* $f^k \sim_k g^k$.

Proof. Put $F_j^0 = \left\{i : f_j^0(i) = g_j^0(i)\right\}$ each $1 \leq j \leq k$. Put $F^k = \{i : f^k(i) = g^k(i)\}$. Now $F^k \cap F_1^0 \cap \ldots \cap F_k^0 \cap F_1 \subseteq F_2$ and

$F^k \cap F_1^0 \cap \ldots \cap F_k^0 \cap F_2 \subseteq F_1$ where

$F_1 = \left\{i : f^k(i)\left[f_1^0(i), \ldots, f_k^0(i)\right]\right\}$ and

$F_2 = \left\{i : g^k(i)\left[g_1^0(i), \ldots, g_k^0(i)\right]\right\}$. But F^0 and F^k are subfilters of F . Hence $F_1 \in \mathsf{F}$ if, and only if, $F_2 \in \mathsf{F}$. //

The ultraproduct, denoted by $\pi M_i/\left(\mathsf{F};\ \mathsf{F}^0, \ldots\right)$ can now be defined. The class of individuals is $\mathsf{R}^0_{\mathsf{F}^0}$. For each integer $k > 0$, and for each $\bar{f}^k \in \mathsf{R}^k_{\mathsf{F}^k}$, a k-placed relation of the ultraproduct, denoted by the same symbol $\bar{f}^k$, is defined by: $\bar{f}^k\left[\bar{f}_1^0, \ldots, \bar{f}_k^0\right]$ if $\left\{i : f^k(i)\left[f_1^0(i), \ldots, f_k^0(i)\right]\right\} \in \mathsf{F}$. The symbol $\mathsf{R}^k_{\mathsf{F}^k}$ is also used to denote the class of k-placed relations of the ultraproduct.

Lemma 1.2, which justifies the definitions as given, has not required the 'ultra' property of F . If this requirement is dropped the definition provides a variation to the standard construction of reduced products. Further it is noted that Łoś's theorem as stated for an ultraproduct in relation to a suitable first order language still holds for an ultraproduct defined as above.

The theorem below establishes that from a set theoretic and extensional view point the use of subfilters F^k , for $k > 0$, adds no extra relations to those gained by taking $\mathsf{F}^k = \mathsf{F}$.

THEOREM 1.3. *For each* $k > 0$ *, if* $f^k, g^k \in R_I^k$ *such that* $f^k \neq g^k$ *but* $\{i : f^k(i) = g^k(i)\} \in F$ *then for all* $\bar{f}_1^0, \ldots, \bar{f}_k^0 \in R_{F^0}^0$ *,* $\bar{f}^k\left(\bar{f}_1^0, \ldots, \bar{f}_k^0\right)$ *if, and only if,* $\bar{g}^k\left(\bar{f}_1^0, \ldots, \bar{f}_k^0\right)$.

Proof. $\bar{f}^k\left(\bar{f}_1^0, \ldots, \bar{f}_k^0\right)$ if, and only if, $\left\{i : f^k(i)\left(f_1^0(i), \ldots, f_k^0(i)\right)\right\} \in F$, that is if, and only if, $\left\{i : g^k(i)\left(f_1^0(i), \ldots, f_k^0(i)\right)\right\} \in F$, as $\{i : f^k(i) = g^k(i)\} \in F$; that is if, and only if, $\bar{g}^k\left(\bar{f}_1^0, \ldots, \bar{f}_k^0\right)$. //

From now on and for all $k > 0$ we put $F^k = F$. The next theorem establishes that for all $k > 0$, the distinct members of R_F^k provide distinct k-placed relations on an extensional basis.

THEOREM 1.4. *For each* $k > 0$ *and* $f^k, g^k \in R_I^k$ *,* $\bar{f}^k \neq \bar{g}^k$ *if, and only if, there exist* $\bar{f}_1^0, \ldots, \bar{f}_k^0 \in R_{F^0}^0$ *satisfying one, and only one, of the relations* $\bar{f}^k, \bar{g}^k$.

Proof. Assume $\bar{f}^k \neq \bar{g}^k$ and let $G = \{i : f^k(i) \neq g^k(i)\}$. Hence $G \in F$. For each $i \in G$, there exists $a_1^i, \ldots, a_k^i \in R_i^0$ which satisfy one, and only one, of the relations $f^k(i), g^k(i)$, as $f^k(i) \neq g^k(i)$. Let $G_0 = \left\{i : i \in G \text{ and } f^k(i)\left(a_1^i, \ldots, a_k^i\right)\right\}$ and

$G_1 = \left\{i : i \in G \text{ and } g^k(i)\left(a_1^i, \ldots, a_k^i\right)\right\}$.

Now $G = G_0 \cup G_1$ and $G_0 \cap G_1 = \emptyset$, so one, and only one, of G_0, G_1 belongs to F . Define, for each $1 \le j \le k$, $\overline{f}_j^0$ as follows: for all $i \in G$, put $f_j^0(i) = a_j^i$; for $i \notin G$ choose $f_j^0(i)$ some arbitrary member of R_i^0 . Hence $\overline{f}_j^0$ is uniquely defined as $G \in \mathsf{F}$. Further $\overline{f}_j^0$ satisfies one, and only one, of $\overline{f}^k, \overline{g}^k$, as one, and only one, of G_0, G_1 belongs to F .

Conversely, if $\overline{f}^k = \overline{g}^k$ then for all $\overline{f}_j^0 \in R^0_{\mathsf{F}^0}$, $1 \le j \le k$, $\overline{f}^k\left(\overline{f}_1^0, \ldots, \overline{f}_k^0\right)$ if, and only if, $\overline{g}^k\left(\overline{f}_1^0, \ldots, \overline{f}_k^0\right)$. //

The next results are concerned with the manner in which the identity relations in the component structures transfer to the ultraproduct. For technical reasons a short lemma is set out.

LEMMA 1.5. *Let* $G = \left\{i : \left|R_I^0\right| = 1\right\}$. $R^0_{\mathsf{F}^0} \ne R^0_{\mathsf{F}}$ *if, and only if, there is an* $F \in \mathsf{F}$ *such that* $F \supseteq G$ *and* $F \notin \mathsf{F}^0$.

Proof. Assume that $R^0_{\mathsf{F}^0} \ne R^0_{\mathsf{F}}$ and so there exist $f^0, g^0 \in R_I^0$ such that $f^0 \sim g^0$ but $f^0 \not\sim_0 g^0$. Let $F = \{i : f^0(i) = g^0(i)\}$ and so $F \supseteq G$, $F \in \mathsf{F}$ but $F \notin \mathsf{F}^0$.

equal to some arbitrary member of R_i^0 ; for all $i \notin F$ take $f(i) \neq g(i)$ but otherwise arbitrary. Thus $f^0 \sim g^0$ but $f^0 \not\sim_0 g^0$ and so $R^0_{F^0} \neq R^0_F$. //

F^0 will be called a *distinct* subfilter of F if $R^0_{F^0} \neq R^0_F$, otherwise it will be called *indistinct*.

THEOREM 1.6. *If $f^2 \in R^2_I$ is defined by $f^2(i) = e_i$, for all $i \in I$, then $\overline{f}^2$ is the identity relation of $\pi M_i/(F;\ F^0)$ if, and only if, F^0 is an indistinct subfilter of F .*

Proof. Assume F^0 is an indistinct subfilter of F . For all $\overline{f}^0, \overline{g}^0 \in R^0_{F^0}$, $\overline{f}^2(\overline{f}^0, \overline{g}^0)$ if, and only if,

$\left\{i : e_i\left[f^0(i),\ g^0(i)\right]\right\} \in F$; that is if, and only if, $f^0 \sim g^0$; that is if, and only if, $\overline{f}^0 = \overline{g}^0$, as $R^0_{F^0} = R^0_F$. Hence $\overline{f}^2$ is the identity relation.

Conversely, assume F^0 is a distinct subfilter of F . Hence, as in lemma 1.5, there exists $\overline{f}^0, \overline{g}^0 \in R^0_{F^0}$ such that $\overline{f}^0 \neq \overline{g}^0$ but $f^0 \sim g^0$. Thus $\{i : f^0(i) = g^0(i)\} \in F$ and so $\overline{f}^2(\overline{f}^0, \overline{g}^0)$. Hence $\overline{f}^2$ is not the identity relation. //

We note that $\overline{f}^2$ as defined in the above theorem is always an equivalence relation and moreover one with the general substitution property. Thus the theorem has given that a distinct subfilter gives rise to a non-normal structure. The next theorem sets out the usual relationship between such a non-normal structure and the normal ultraproduct got by putting F^0 equal to F.

THEOREM 1.7. $\pi M_i/(F;\ F)$ *is isomorphic to a quotient structure of* $\pi M_i/(F;\ F^0)$.

Proof. Define a map $\beta : R^0_{F^0} \to R^0_F$ by: for each $\bar{f}^0 \in R^0_{F^0}$, put $\beta(\bar{f}^0) = [\bar{f}^0]$, where $[\bar{f}^0]$ is the equivalence class of f^0 with respect to $\sim$. β is well defined and surjective. Further, for all $\bar{f}^k \in R^k_F$, and for all $\bar{f}^0_1, \ldots, \bar{f}^0_k \in R^0_{F^0}$, $\bar{f}^k(\bar{f}^0_1, \ldots, \bar{f}^0_k)$ if, and only if, $\bar{f}^k(\beta(\bar{f}^0_1), \ldots, \beta(\bar{f}^0_k))$. Let $\sim_\beta$ be the binary relation defined on $R^0_{F^0}$ by: $\bar{f}^0 \sim_\beta \bar{g}^0$ if $\beta(\bar{f}^0) = \beta(\bar{g}^0)$. Now $\sim_\beta$ as defined is a congruence of $\pi M_i/(F;\ F^0)$ and hence the quotient structure with respect to this congruence is isomorphic to $\pi M_i/(F;\ F)$. //

2. Relations of ultraproducts of similar structures

Let $\{M_i : i \in I\}$ now be a family of similar structures. For each $k > 0$ let $\{R^k_{i,j} : j < \alpha_k\}$ be the set of k-placed relations of M_i, $i \in I$, where α_k is a given ordinal, the same for all

$i \in I$.

In Robinson [1963] page 241 and [1966] page 10 the individuals of the ultraproduct are defined as in §1 but with $F^0 = \{I\}$. In Bell and Slomson [1969], page 87, the individuals are defined by taking $F^0 = F$. It is this we call the standard definition.

Further, the k-placed relations of the ultraproduct in this standard definition, viz. R_j^k , all $j < \alpha_k$, are defined by:

$R_j^k\left(\overline{f}_1^0, \ldots, \overline{f}_k^0\right)$ if $\left\{i : R_{i,j}^k\left(f_1^0(i), \ldots, f_k^0(i)\right)\right\} \in F$.

Now in terms of §1 this definition has selected from R_I^k the subclass $S_I^k = \left\{h_j^k : h_j^k : I \to \cup\left\{R_I^k : i \in I\right\}\right.$ such that for all $i \in I$, $\left.h_j^k(i) = R_{i,j}^k , \ j < \alpha_k\right\}$ and associated with each member of this subclass a k-placed relation of the ultraproduct. The next theorem establishes a necessary and sufficient condition under which the construction of §1 applied to this family of similar structures reproduces only the standard relations. Of course at least the standard relations will always be produced for if $h_m^k \neq h_n^k$ then $\overline{h}_m^k \neq \overline{h}_n^k$.

THEOREM 2.1. *For all* $k > 0$ *, there exists an* $f^k \in R_I^k$ *such that for all* h_j^k *,* $j < \alpha_k$ *,* $\overline{f}^k \neq \overline{h}^k$ *if, and only if,* F *is* α_k*-incomplete.*

Proof. Assume that F is α_k-incomplete and so let β_k be the first cardinal, $\beta_k \le \alpha_k$, such that $\mathcal{F}$ is β_k-incomplete. Thus there exists, for each $j < \beta_k$ a $F_j \in \mathcal{F}$ such that $\cap\{F_j : j < \beta_k\} = \emptyset$.

Construct f^k inductively as follows: for all $i \notin F_0$ put $f^k(i) = R^k_{i,0}$; for all $i \in F_1 - F_2$, put $f^k(i) = R^k_{i,1}$; assume that $f^k(i)$ has been defined for all $i \in \cup\{CF_t : t < \delta\}$ for some ordinal $\delta < \beta_k$ and define $f^k(i) = R^k_{i,\delta}$ for all $i \in \cap\{F_t : t < \delta\} - F_\delta$. By induction f^k is well defined and domain $f^k = I$, as $\cap\{F_j : j < \beta_k\} = \emptyset$.

Now $\left\{i : f^k(i) = R^k_{i,0}\right\} = CF_0$ and so $\overline{f}^k \ne \overline{h}^k_0$ as $F_0 \in \mathcal{F}$. For $0 < j < \beta_k$, $\left\{i : f^k(i) = R^k_{i,j}\right\} = \cap\{F_t : t < j\} - F_j$ and so $\overline{f}^k \ne \overline{h}^k_j$, as $F_j \in \mathcal{F}$. Finally if $\beta_k \le j < \alpha_k$ then $\left\{i : f^k(i) = R^k_{i,j}\right\} = \emptyset$ and so $\overline{f}^k \ne \overline{h}^k_j$.

Conversely, assume there is an $f^k \in R^k_I$ such that for all $h^k_j \in S^k_I$, $\overline{f}^k \ne \overline{h}^k_j$. For each $j < \alpha_k$, define $G_j = \left\{i : f^k(i) = R^k_{i,j}\right\}$. Now $\cup\{G_j : j < \alpha_k\} = I$ and so $\cap\{CG_j : j < \alpha_k\} = \emptyset$. But for all $j < \alpha_k$, $CG_j \in \mathcal{F}$ and so $\mathcal{F}$ is α_k-incomplete. //

While the above theorem establishes the distinctness of $\overline{f}^k$ in terms of an equivalence class of maps theorem 1.4 ensures that the distinctness is carried over to the relations of the ultraproduct on an extensional basis.

COROLLARY 2.2. *For each* $k > 0$, *if* α_k *is finite then for each* $f^k \in R_I^k$, *there exists some* $h^k \in S_I^k$ *such that* $\overline{f}^k = \overline{h}^k$.

Proof. If α_k is finite then F is α_k-complete. //

COROLLARY 2.3. *If* F *is a principal ultrafilter then for all integers* $k > 0$, *and for all* $f^k \in R_I^k$, *there exists some* $h^k \in S_I^k$ *such that* $\overline{f}^k = \overline{h}^k$.

Proof. A principal ultrafilter is α_k-complete for all α_k . //

The final theorem concerns the relationship between two ultraproducts, each formed by the standard definition from the same family of similar structures with respect to the same ultrafilter, but where in the case of the second ultraproduct the similarity correspondence may, for each $k > 0$, link different k-placed relations from each structure from those linked in the first case.

Let $\pi M_i/F$ be the standard ultraproduct formed as noted at the beginning of §2. Let $\pi' M_i/F$ be a second ultraproduct formed by the standard definition but following possible rearrangements of the relations connected under the similarity correspondence; that is, for each $i \in I$, and for each $k > 0$, if θ_i^k is a permutation of the set $\{j : j < \alpha_k\}$ then the k-placed relations of $\pi' M_i/F$ are given by R'^k_j , $j < \alpha_k$, where $R'^k_j\left(\overline{f}^0_1, \ldots, \overline{f}^0_k\right)$ if, and only if,

$\left\{i \cdot R^k_{i,\theta^k_i(j)}\left(f^0_1(i), \ldots, f^0_k(i)\right)\right\} \in F$.

THEOREM 2.4. *There exists such a* $\pi' M_i/F$ *as above distinct from* $\pi M_i/F$ *if, and only if, there exists some* $k > 0$ *such that* F *is* α_k*-incomplete.*

Proof. Assume that for each $k > 0$, F is α_k-complete. Associate each standard relation R^k_j , $j < \alpha_k$ of $\pi M_i/F$ with $\overline{h}^k_j$, where for all $i \in I$, $h^k_j(i) = R^k_{i,j}$. Associate R'^k_j , $j < \alpha_k$, in $\pi' M_i/F$ with $\overline{h}'^k_j$, where for all $i \in I$, $h'^k_j(i) = R^k_{i,\theta^k_i(j)}$. From theorem 2.1 it follows that $\left\{\overline{h}^k_j : j < \alpha_k\right\} = \left\{\overline{h}'^k_j . j < \alpha_k\right\}$. Hence $\pi M_i/F$ is the same structure as $\pi' M_i/F$.

Conversely, assume that for some $k > 0$, F is α_k-incomplete. Hence from theorem 2.1 there exists $\overline{f}^k \in R^k_F$ such that $\overline{f}^k$ is distinct from each of the standard k-placed relations of $\pi M_i/F$. For each $j < \alpha_k$, let $G_j = \left\{i : f^k(i) = R^k_{i,j}\right\}$. Thus $\{G_j : j < \alpha_k\}$ partitions I and for each $j < \alpha_k$, $G_j \notin F$. For each $i \in I$, and each $m \neq k$, $m > 0$, take θ^m_i as the identity permutation of α_m . For each $i \in I$, take θ^k_i as one of the permutations of α_k such that $\theta_i^k(0) = j$, where j is such that $i \in G_j$. Hence the relation $R_0'^k$ of $\pi' M_i/F$ is associated with $\bar{f}^k$ and so is distinct from all the k-placed relations of $\pi M_i/F$. Thus $\pi' M_i/F$ is distinct from $\pi M_i/F$. //

In the above theorem even when $\pi' M_i/F$ is distinct from $\pi M_i/F$ it is not established if it is also non-isomorphic to $\pi M_i/F$. Whether this is so or not does not seem to have an immediate answer.

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