The Algebraic and Geometric Classification of Four Dimensional Superalgebras

by

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Abstract

The algebraic and geometric classification of k-algbras, of dimension four or less, was started by Gabriel in "Finite representation type is open" [12]. Several years later Mazzola continued in this direction with his paper "The algebraic and geometric classification of associative algebras of dimension five" [21]. The problem we attempt in this thesis, is to extend the results of Gabriel to the setting of super (or $\mathbb{Z}_2$-graded) algebras — our main efforts being devoted to the case of superalgebras of dimension four. We give an algebraic classification for superalgebras of dimension four with non-trivial $\mathbb{Z}_2$-grading. By combining these results with Gabriel's we obtain a complete algebraic classification of four dimensional superalgebras. This completes the classification of four dimensional Yetter-Drinfeld module algebras over Sweedler's Hopf algebra H_4 given by Chen and Zhang in "Four dimensional Yetter-Drinfeld module algebras over H_4" [9]. The geometric classification problem leads us to define a new variety, Salg_n — the variety of n-dimensional superalgebras — and study some of its properties. The geometry of Salg_n is influenced by the geometry of the variety Alg_n yet it is also more complicated, an important difference being that Salg_n is disconnected. While we make significant progress on the geometric classification of four dimensional superalgebras, it is not complete. We discover twenty irreducible components of Salg_4 — however there could be up to two further irreducible components.

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Chapter 1

Introduction

1.1 Background

In this thesis we consider the problems of algebraic and geometric classification of four dimensional superalgebras. The idea of algebraic classification is a very natural one, whereas the notion of geometric classification is more subtle. Before introducing the main problems and the results of this thesis, we give the reader some background to this area of research, focusing on the more interesting geometric classification problem.

The algebraic classification problem is where one is interested in determining "all possible examples of some algebraic structure up to isomorphism" (of the appropriate kind); or, more formally, the problem is to try and determine the isomorphism classes of the algebraic structure in question. As mentioned before, this question arises very naturally once one has defined some algebraic structure and a suitable notion of map between two such structures. An example of this is the problem of determining the isomorphism classes of n-dimensional k-algebras (solved for $n \leq 4$ in [12] and for $n = 5$ in [21]). Similarly, the problem of determining all isomorphism classes of modules of a given dimension over a fixed k-algebra is of this nature. The algebraic classification is very "rigid", with two structures being considered equivalent only when they really are two different views of the same structure.

The geometric classification on the other hand is not so "rigid". In this case the problem one wishes to tackle is to find all the structures which are the "most basic", in a sense which we shall try to describe in this paragraph. These structures are called generic, and the complete list of such generic structures has the following property: every possible structure is either isomorphic to one of the generic structures or "lies very close to one of the generic structures". In the latter case we can obtain this structure from a suitable degeneration of the rigid structure. Thus, from the list of all the rigid structures every other structure can be obtained in this manner. Determining the complete list of generic structures is the geometric classification problem. While we have not been very precise, we hope this description helps the reader to understand both of these points of view of classification and see the distinction between them.

We illustrate the idea of geometric classification with the example of n-dimensional k-algebras. A k-algebra structure A on an n-dimensional vector space V is a linear map $\mu : A \otimes A \to A$ called multiplication, which satisfies suitable conditions requiring that this multiplication is associative and unitary. If one chooses a basis for V, say $\{e_1, \ldots, e_n\}$, then from the algebra structure we can easily determine the structure constants $(\alpha_{ij}^k) \in k^{n^3}$, which must satisfy $e_i e_j = \sum_{k=1}^n \alpha_{ij}^k e_k$. (Where we follow the usual convention of denoting the product of two elements by their juxtaposition, so for example, $e_i e_j$ denotes $\mu(e_i \otimes e_j)$). Conversely, such structure constants (α_{ij}^k) will give rise to a k-algebra structure on V by defining multiplication of the basis vectors by the previous formula. The conditions imposed requiring μ to be associative and unitary translate into relations amongst the structure constants. These relations define a subvariety of k^{n^3}, called Alg_n.

The structure constants for an algebra depend on the choice of basis for V; in different bases the same algebra structure may be represented by different structure constants. Suppose V has a given basis, then a set of structure constants (α_{ij}^k) can be used to construct a k-algebra structure on

V. After making a change of basis in V, the k-algebra structure just constructed will now have, in general, a different set of structure constants (α'^k_{ij}). In this way, base changes in V gives rise to the transport of structure action on Alg_n. The orbits in Alg_n (under this action) can be identified with the isomorphism classes of n-dimensional k-algebras. Suppose that A is an n-dimensional algebra and one writes $O(A)$ for the orbit in Alg_n which is identified with the isomorphism class of A. Given two n-dimensional k-algebras, A and B, we say that A degenerates to B if some point in $O(B)$ also belongs to $\overline{O(A)}$ (the closure being taken in the Zariski topology). This notion extends to a well-defined partial order on the isomorphism classes of n-dimensional k-algebras, called the degeneration partial order. We call an n-dimensional algebra generic if the (Zariski-) closure of its orbit is an irreducible component of Alg_n. The geometric classification of n-dimensional k-algebras is nothing more than finding the decomposition of Alg_n into its irreducible components. Supposing that A is a generic algebra, then every algebra in the irreducible component given by $\overline{O(A)}$ is a degeneration of A.

The reader should now realise that the geometric classification problem brings into use all the tools from Algebraic Geometry. The more specialised area of actions of algebraic groups are also very useful in tackling this problem. It should be noted that the problem has a flavour of Geometric Invariant Theory: we would like to take the variety Alg_n and construct a quotient by the transport of structure action, yielding us a space in which the points would represent the orbits in Alg_n. However we are not able to do this, because the transport of structure action is not sufficiently well-behaved, and so the methods of Geometric Invariant Theory are not available to us.

In general, it is too much to hope that we can determine the orbits in the variety Alg_n. Instead of looking at such fine properties, we should first try to determine properties on a broader scale — such as determining the connected and irreducible components of the variety. Also knowing which orbits are open and which are closed is an interesting question. Thus we are naturally led to studying the geometry or perhaps "landscape" of such varieties. Flanigan in [11] coins the term "algebraic geography" for studying the variety Alg_n in this manner. Gabriel studies these important properties of Alg_n in his famous paper, "Finite Representation type is open" [12]. There have been other works written giving the same material. For example see [7, 17], both of which provide more explanation and background knowledge to the reader than [12] does.

The first efforts in the geometric classification of algebras seems to have been made by Gabriel in [12], where he gave both forms of classification for k-algebras of dimension four or less. It is, however, hard to determine exactly whom should be credited with coming up with the notion of the geometric classification of k-algebras, since the variety Alg_n was known and had been mentioned before Gabriel's paper (see for example [11]). Several years after Gabriel's paper on the classification of k-algebras of dimension four or less, Mazzola published a paper [21] giving the algebraic and geometric classification of k-algebras of dimension five. The notation alg_n is used to denote the number of irreducible components of the variety Alg_n. In this paper, Mazzola also included asymptotic bounds on alg_n as n goes to infinity. The lower asymptotic bound is exponential in n, and unsurprisingly the classification problems become increasing difficult in higher dimensions. In a later paper Mazzola [23] obtains a description of the algebras responsible for the asymptotic behaviour of the function alg_n — they are basic algebras and their quivers are of a particularly simple shape.

The geometric classification problem is a problem which can be asked in many different settings, not only that of n-dimensional k-algebras. There is also a geometric classification problem for m-dimensional modules over a fixed k-algebra. This gives rise to the module varieties Mod_m^A where A is a k-algebra and the modules in question are A-modules. From this introduction the reader may find it somewhat surprising that the module varieties have been the subject of much more research than the algebra varieties Alg_n. However our main focus is going to be on superalgebras — a notion generalising that of an algebra — explaining our motivation in focusing on the algebra varieties.

In any case, the two geometric classification problems in each setting proceed very much in parallel. The crux of the approach to the geometric classification problem is finding a variety whose points represent the type of objects you are studying, then finding the appropriate transport of structure action on this variety, and checking that the orbits under this action may be identified with the isomorphism classes of these objects. Then the properties one is interested in are: determining which orbits are open and which are closed, determining the connected components and the irreducible components, and so on. The papers [7, 12, 17] are helpful in seeing this parallel, where they give standard properties of both the algebra and module varieties.

There are many more papers which concentrate solely on the module varieties, [24] being just one which gives standard results on the module varieties. We finish our discussion of the module varieties with the following comment. One very interesting result for the module varieties is the fact that there is a characterisation of degenerations between modules purely in terms of representation theory. There is currently no such characterisation for degenerations between algebras. The work of Riedtmann [28] and Zwara [33] combined, shows that the following three statements are equivalent for m-dimensional A-modules M and N (where we denote the category of A-modules by $\mathrm{mod}\, A$):

- M degenerates to N
- There is a short exact sequence

  $$0 \to N \to M \oplus Z \to Z \to 0$$

  in $\mathrm{mod}\, A$ for some Z in $\mathrm{mod}\, A$

- There is a short exact sequence

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More recently, the work of applying geometric methods to representation theory has still been active. However, the methods used and questions asked differ from those in earlier work on this topic. The paper [13], by Goze and Makhlouf, attacks the the classification problem of rigid algebra structures on $\mathbb{C}^n$ using a new approach, based on non-standard analysis and a method of perturbing the idempotent elements. They give a full classification of such structures in $\mathbb{C}^6$ using this idea, although this does not constitute a full solution to the geometric classification problem in dimension 6 (since not all algebra structures are rigid). A year later Makhlouf published another paper [19] showing how computer algebra software can be used to study associative algebras and in particular their irreducible components. In the future, with more computing power available, such an approach may well prove useful in the attack of the higher dimensional problems, with the computations involved becoming increasingly difficult as the dimension increases. A paper of Le Bruyn and Reichstein [18] addresses questions of smoothness and the singularities that may occur in Alg_n. In particular they show that the closure of the orbit of the matrix algebra $M_r(k)$ in Alg_{r^2} is not smooth for $r \geq 3$. It should also be known that in this paper they adopt a different definition of Alg_n from the one presented in earlier works. They require that the first element of the basis for V be the identity of the algebra. We mention this, as we shall also follow this convention when we define the variety of n-dimensional superalgebras.

In recent times, we have also seen the geometric classification problem applied to different algebraic structures. This method has been used in the setting of Lie algebras, one such example of this being [6]. Another paper [20] by Makhlouf defines the varieties of bialgebras and Hopf algebras.

In this he reviews different works giving an algebraic classification of all Hopf algebras of dimension thirteen or less, and lists the irreducible components of the Hopf algebra varieties. He also discovers that all Hopf algebras with dimension thirteen or less are rigid. This highlights even more strongly the general method of attacking the geometric classification problem. It seems apparent that the geometric classification problem may be asked whenever there is an algebraic structure whose isomorphism classes may be parametrised as orbits in some variety.

1.2 Our Problem

The task which we attempt in this thesis, is to classify, both algebraically and geometrically, superalgebras of dimension four or less — thus extending the work of Gabriel. We obtain the algebraic classification theorem under the assumptions that k is an algebraically closed field with $\mathrm{ch}(k) \neq 2$. While the assumption $\mathrm{ch}(k) \neq 2$ simplifies the algebraic classification, it is vital for the geometric classification. The most basic fact about Salg_n — its disconnectedness (Proposition 3.2.12) — requires this assumption. Thus we keep the same assumptions for the geometric classification problem, and while we make significant progress towards solving the geometric classification problem, it is unfortunately not completed. While we find 20 generic superalgebras (or generic families of superalgebras) there may be up to two more generic superalgebras. Owing to time restrictions, we must leave this problem unsolved.

Since our main object of interest in this thesis is a superalgebra, we should introduce what a superalgebra is. A superalgebra (or a $\mathbb{Z}_2$-graded algebra) is an algebra A which can be written as $A = A_0 \oplus A_1$ with $A_i A_j \subseteq A_{i+j}$ for $i, j \in \mathbb{Z}_2$. A superalgebra A is equivalent to a pair (B, σ) where B is an algebra and $\sigma : B \to B$ is an algebra involution. Thus we view a superalgebra as an algebra "with some additional structure". We shall often refer to this additional structure as the $\mathbb{Z}_2$-grading.

The notion of a superalgebra is not only of interest to mathematicians, but it is of vital interest to physicists too (with one hearing quantum physicists using terms like "supersymmetry"). So extending the classification results of Gabriel and Mazzola to the setting of superalgebras should be a problem of broad appeal.

We have already mentioned that the geometric classification problem may be applied to many different situations and have given several examples of this. The case of superalgebras serves as a prototype of how one can generalise the classical approach to the geometric classification problem for n-dimensional algebras, to the analogous problem for n-dimensional module algebras. We remark that an algebra A is a superalgebra if and only if it is a $k\mathbb{Z}_2$-module algebra, where $k\mathbb{Z}_2$ denotes the sub-Hopf algebra $k1 \oplus kg$ of Sweedler's Hopf algebra H_4. The element 1 from $k\mathbb{Z}_2$ acts on the algebra A trivially, whereas the element g from $k\mathbb{Z}_2$ acts on the algebra A as an involution. Thus, our treatment of the case of superalgebras should help with setting up the geometric classification problem for this more general situation. Studying the more general problem may also shed new light on the geometric classification problem for superalgebras. One particularly interesting possible future direction we have in mind for this work, is to attempt the classification of H_4-module algebras, which are precisely the same as differential superalgebras (see [32]). We examine this idea in more detail in the following section.

Our work relies heavily on the work of Gabriel and others on the algebra variety Alg_n and the classification of n-dimensional algebras; our methods being mainly to reduce our arguments to a situation where we can apply one of their results, rather than to provide a new more general proof which gives their results as a special case. It should however be noted that while there are some similarities between our results, there are some very substantial differences too.

The algebraic classification of four dimensional algebras over an algebraically closed field k, determines the underlying algebra structure of each superalgebra. Yet it is interesting to see that all of these algebras admit at least one non-trivial $\mathbb{Z}_2$-grading, and it is also interesting to note which algebras admit multiple $\mathbb{Z}_2$-gradings. The geometry of the variety of n-dimensional superalgebras, which we call Salg_n is influenced by the geometry of Alg_n (the variety of n-dimensional algebras), yet is more complicated. It was remarked in [12], by Gabriel, that Alg_n is always connected. However we shall see that Salg_n is disconnected. We also find that from superalgebra structures (and there may be several such structures), on a given generic algebra structure at least one such structure must be generic as a superalgebra. However we also find examples in Salg_4 of generic superalgebra structures whose underlying algebra is not generic. For this reason, in a fixed dimension, there are many more generic superalgebras than generic algebras. Recall that we use alg_n to denote the number of irreducible components of Alg_n (or equivalently the number of generic algebras or generic families of algebras of dimension n). Now if we denote by salg_n the number of irreducible components of Salg_n and combine our results with those in [12, 13, 21], then we have the following table. This should help convince the reader that the varieties Salg_n are, in general, more complex than the corresponding variety Alg_n.

alg_n vs salg_n for small n					
n	2	3	4	5	6
alg_n	1	2	5	10	≥ 21
salg_n	2	5	20–22	?	?

In Chapter 2 we work on the algebraic classification of four dimensional superalgebras. Under the assumption that $\mathrm{ch}(k) \neq 2$, we classify up to isomorphism all non-trivially $\mathbb{Z}_2$-graded superalgebras of dimension four (see Proposition 2.2.12, Theorem 2.3.1 and Theorem 2.4.1). However, for the following chapter on the geometric classification, we need an algebraic classification of all superalgebras of dimension four, whether they be trivially $\mathbb{Z}_2$-graded or not. (But notice that, for the geometric classification, we must assume k is algebraically closed to apply the standard techniques of algebraic geometry). Since a trivially $\mathbb{Z}_2$-graded superalgebra is nothing more than an algebra, to complete the algebraic classification of four dimensional superalgebras over a field k with $\mathrm{ch}(k) \neq 2$ would require a classification of four dimensional algebras over a field k, with $\mathrm{ch}(k) \neq 2$. This would likely be a difficult task. Thus, we settle for a complete classification of four dimensional superalgebras in the case that k is an algebraically closed field with $\mathrm{ch}(k) \neq 2$ (see Theorem 2.5.1). We use Gabriel's results to give us the classification of four dimensional trivially $\mathbb{Z}_2$-graded superalgebras, and specialise our results on the classification of non-trivially $\mathbb{Z}_2$-graded superalgebras to the case where k is algebraically closed.

Our results in Chapter 2 serve two purposes. Firstly they equip us with the set of isomorphism classes of four dimensional superalgebras, each corresponding to an orbit in the variety Salg_4. For the geometric classification we then attempt to find those orbits whose closures give an irreducible component of Salg_4. In this way, it is not only natural to complete the algebraic classification first, but it is usually required, since the geometric classification builds upon the algebraic classification. Secondly in [9], Chen and Zhang give a classification of Hopf actions of $D(H_4)$ (the Drinfeld double of H_4) on 4-dimensional algebras. The methods they used failed to apply to the 4-dimensional $D(H_4)$-module algebras on which the action of the skew-primitive elements are trivial. After noticing that $D(H_4)$-module algebras with the skew primitive elements acting trivially are nothing other than superalgebras, we find that our classification theorem of this chapter completes the classification results of [9].

In Chapter 3, the remark that a superalgebra is simply a pair consisting of an algebra and an algebra involution, allows us to find suitable structure constants to represent superalgebra structures. We then give the definition of the variety of n-dimensional superalgebra structures, which we denote by Salg_n (see Definition 3.2.1). The appropriate transport of structure action (which arises by considering a change of basis) is determined. We show that this is well-defined and that its orbits in Salg_n correspond to isomorphism classes of n-dimensional superalgebras. There are two very useful morphisms between Alg_n and Salg_n denoted by U and i, which relate the geometry of these two varieties. The map i can be used to show that Alg_n can be identified with a closed subvariety of Salg_n — thus the algebraic and geometric classification of n-dimensional algebras should be attempted before the classications of n-dimensional superalgebras. We show that the variety Salg_n is disconnected for $n \geq 2$ (see Proposition 3.2.12). We define Salg_n^i for $i = 1, \ldots, n$ to consist of those points representing an n-dimensional superalgebra whose degree zero component has dimension i. The subsets Salg_n^i of Salg_n are clearly disjoint. In either low dimensions ($n \leq 6$) or under suitable assumptions on the characteristic of k, we show these subsets are also closed (see Lemma 3.2.10). We require this to be the case for the majority of the remainder of the chapter. We then show in this case that Salg_n has Salg_n^i for $i = 1, \ldots, n$ as its connected components (see Proposition 3.4.5). In Section 3.5 we give the partial degeneration diagrams of 4-dimensional superalgebras: these diagrams are complete apart from 1 degeneration between two superalgebras with homogeneous degree zero components having dimension 3, and 6 degenerations between superalgebras with homogeneous degree zero components having dimension 2. From these diagrams we can find twenty irreducible components in Salg_4 (giving us a partial result towards the geometric classification, see Theorem 3.5.1). However, due to some of the missing degenerations there are two other structures of which we are unsure if they give rise to irreducible components or not.

In Chapter 4, we give the algebraic and geometric classification results for superalgebras of dimensions 2 and 3. The corresponding results for 2 and 3 dimensional algebras appear in Gabriel's paper, [12]. We first present the algebraic classification of the superalgebras, which is trivial for dimension 2 and straightforward for dimension 3 (see Theorem 4.1.1 and Theorem 4.2.3). Then using these results and the general methods derived in chapter 3 we give the geometric classification of superalgebras in dimensions 2 and 3 (see Theorem 4.3.3 and Theorem 4.4.3).

Finally, in Chapter 5, we introduce the notion of a supermodule over a superalgebra. As mentioned before, the study of the module varieties $\mathrm{mod}\,{}^{A}_{m}$ shares a lot in common with the study of the algebra varieties Alg_n. The papers [7, 12, 17] are useful for background on the module varieties and their similarities with the algebra varieties Alg_n. So, naturally, one would wonder how to define the supermodule varieties, the superspace analogue of the ordinary module varieties. The main purpose of this chapter is to define and introduce the supermodule varieties. This chapter is not intended to be rigorous, but merely to indicate the similarities with the classical case of modules over an algebra, and suggest how some of the techniques used to study the superalgebra varieties in Chapter 3 may be modified to apply to the situation of supermodules over a superalgebra. Section 5.2 is dedicated to giving an example of 3-dimensional supermodules over two different superalgebra structures on the same underlying algebra. We present both algebraic and geometric classifications in this example, which enables us to see how the module varieties can change when one alters the $\mathbb{Z}_2$-grading of the algebra.

1.3 Future Research

As with most research, our work has brought up at least as many new questions as we have answered. So we will first outline the questions which have arisen in the preparation of this thesis before indicating possible directions for future research in this area.

First and foremost, it would be an interesting and satisfying result to have the geometric classification (see Theorem 3.5.1 and Remark 3.5.2) of four dimensional superalgebras completed. Once that is done, completing the degeneration diagrams may be attempted (since if the geometric classification cannot be completed, then neither can the degeneration diagram). It would be satisfying to check that the geometries of the two versions of the varieties of n-dimensional superalgebras defined by (i) simply requiring the existence of an identity, and (ii) requiring the identity to be the first element in the basis for the vector space V, do in fact coincide (see Remark 3.2.5). Another interesting question is whether the geometry of Salg_n can change in some cases where n is suitably large and the ground field k has characteristic p so that Lemma 3.2.10 doesn't apply, or whether there are methods which can be used to show that the conclusions of Lemma 3.2.10 must always hold (see Remark 3.2.11). It would be interesting to see if the criterion, namely that $H^2(A, A) = 0$, which Gabriel gives to show that an orbit of an algebra A is open in Alg_n may be generalised to a criterion to determine when the orbit of a superalgebra B is open. The natural generalisation would be that the orbit of B is open in Salg_n when $H^2(B, B) = 0$, where $H^2(B, B)$ is interpreted as the Hochschild cohomology group of the superalgebra B (see Remark 3.2.17). Does the dimension of the irreducible component of a generic family which depends on a single parameter, exceed the dimension of any given orbit in that family by exactly one? (See Remark 3.5.3). Finally, is it ever possible for a superalgebra structure to degenerate to a different superalgebra structure on the same underling algebra? (See Remark 3.5.4).

The following suggestions only skim the surface. By applying and generalising the ideas in this thesis, one can attempt the geometric classification problem for module algebras — of which there are many interesting examples.

Since Mazzola gives the algebraic and geometric classifications of five dimensional algebras, it is possible to try to generalise our work to the classication of five dimensional superalgebras. This may come up against some difficulties due to the large number of dimensions. Another problem with the geometric classification of five dimensional superalgebras, is that in the case of dimension four, our results used the degeneration diagram of Alg_4 to eliminate a large number of degenerations which could not occur. However the complete degeneration diagram in Alg_5 has not been given, although Mazzola gives the degeneration diagram for the commutative structures in Alg_5 in [22], so the degeneration diagram for Alg_5 would most likely be needed in attempting the geometric classification of five dimensional superalgebras.

One future direction is the study of the supermodule varieties. Chapter 5 introduces the basic notions and problems necessary to study these varieties. This is the natural generalisation of the classical module varieties to superspaces. In this chapter we focus on ideas and concepts rather than proving any results. It is hoped that this chapter will stimulate interest in this open area of research, and that the supermodule varieties will be studied in more detail in the future.

Another way in which this thesis could be generalised is by extending the ideas to the case of H_4-module algebras, or in other words, differential superalgebras (see [32]). The algebraic classification of 4-dimensional differential superalgebras has been done in [9] and Chaper 2, leaving the geometric classification problem open in the case of 4-dimensions. The classification of Azumaya differential superalgebras up to Morita equivalence was completed in [31], and the structures of Azumaya differential superalgebras and their invariants have been studied in [2]. We would be very interested in observing the geometric behaviour of these invariants. The variety Salg_n will appear as the closed subvariety of differential superalgebras having trivial differential (in much the same way Alg_n was identified as the closed subvariety of Salg_n with the trivial $\mathbb{Z}_2$-grading). Thus the complete geometric classification of n-dimensional H_4-module algebras requires a complete geometric classification of n-dimensional superalgebras. We also remark that since the differential is a nilpotent linear map, its trace and determinant are both zero, methods similar to those in Lemma 3.2.10 will not be able to be applied. If similar results are needed, then new methods will be required. These new methods may, however, shed more light on the variety Salg_n.

Finally, one could try applying these ideas to obtain the notion of geometric classification of super Lie algebras, combining some of our ideas on superalgebras with the literature which already exists on the geometric classification of Lie algebras, for example [6].

1.4 Notation

In this section we fix some basic notation and make a convention which shall be used throughout the thesis.

Throughout this thesis we work over a fixed ground field, k. We use $\mathrm{ch}(k)$ to denote the characteristic of k; we assume that $\mathrm{ch}(k) \neq 2$ throughout the thesis; we use k^* to denote the non-zero elements of k. This set, k^*, forms a group under multiplication and k^{*2} is used to denote the subgroup of square elements.

All vector spaces are vector spaces over k; all bases are k-bases; all linear maps are k-linear and all unadorned tensor products are implied to be taken over k (for those unfamiliar with tensor products, see Definition 2.1.13).

Chapter 2

Algebraic Classification

In this chapter we classify all non-trivially $\mathbb{Z}_2$-graded superalgebras of dimension four over a field k with $\mathrm{ch}(k) \neq 2$, up to isomorphism. Specialising to the case k is algebraically closed and utilising the results of [12] we obtain a classification of all four dimensional superalgebras, up to isomorphism. The results of this section have been made into a paper [3] which shall appear soon.

2.1 Preliminaries

It is assumed that the reader has a basic knowledge of abstract algebra. We shall give definitions of k-algebra and modules over an algebra; and present related results which we wish to use, omitting the proofs as they are standard and can easily be found in the literature. It should be noted that the definitions given are limited to those required for our purpose. It is possible to define the notion of an R-algebra and an R-module where R is a ring. In fact, in module theory, one usually defines the more general notion of an R-module, rather than a module over an algebra. Our definitions and presentation of this preliminary material is based on [1, 4, 27]. However many other texts on these subject areas would also be suitable for this purpose. We briefly introduce the notion of a G-graded algebra, where G is a group. However we quickly specialise to the case of interest, $\mathbb{Z}_2$-graded algebras or superalgebras. For a very thorough treatment of graded ring theory, the reader is referred to [26].

Definition 2.1.1 *An* **algebra over a field** k *or* k**-algebra** *is defined to be a triple* (A, m, u)*, with* A *a vector space,* $m : A \otimes A \to A$ *a linear map called multiplication,* $u : k \to A$ *a linear map called the unit map, making the following two diagrams commute:*

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id}_A \otimes m"] \arrow[d, "m \otimes \mathrm{id}_A"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd} \qquad \begin{tikzcd} A \otimes k \arrow[r, "\mathrm{id}_A \otimes u"] \arrow[dr] & A \otimes A \arrow[d, "m"] & k \otimes A \arrow[l, "u \otimes \mathrm{id}_A"'] \arrow[dl] \\ & A & \end{tikzcd}$$

where the maps $A \otimes k \to A$ *and* $k \otimes A \to A$ *are the isomorphisms induced by scalar multiplication.*

The **dimension** *of a* k*-algebra is its dimension as a vector space over* k*.*

We shall usually abbreviate this terminology and simply refer to algebra A instead of the triple (A, m, u). We shall also write multiplication of two elements of the k-algebra by juxtaposition, and instead of referring explicitly to the unit map, we shall simply identify the identity of k and the identity of A.

Definition 2.1.2 *With* k*-algebras* (A, m_A, u_A) *and* (B, m_B, u_B) *a map,* $f : A \to B$ *is a* k**-algebra map** *or* k**-algebra homomorphism** *if* f *is a linear map and the following two diagrams commute:*

$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "m_A"'] & B \otimes B \arrow[d, "m_B"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] & & B \\ & k \arrow[ul, "u_A"] \arrow[ur, "u_B"'] & \end{tikzcd}$$

With these definitions we obtain the category of k-algebras, whose objects are k-algebras and morphisms are k-algebra maps as defined above. For the reader familiar with category theory, the following definition of isomorphism is just the usual notion of an isomorphism between objects in a category.

Definition 2.1.3 *A k-algebra map $f : A \to B$ is said to be an* **isomorphism** *if there exists another k-algebra map $g : B \to A$ such that $f \circ g = \mathrm{id}_A$ and $g \circ f = \mathrm{id}_B$, in this case A and B are said to be* **isomorphic**.

Moreover, a k-algebra map $\sigma : A \to A$ having the same domain and codomain is said to be an **automorphism** *if σ is an isomorphism, and is said to be an* **algebra involution** *if $\sigma^2 = \mathrm{id}_A$.*

Remark 2.1.4 *If we were being careful, we would refer to k-algebras as unitary k-algebras since by deleting the second and fourth diagrams above we obtain the category of k-algebras (perhaps without unit). However as we shall rarely deal with non-unitary k-algberas this should not cause a problem. Since we are viewing the ground field k fixed, we shall abreviate the terminology even further, by simply talking about algebras and algebra maps.*

Examples 2.1.5 *Many examples of k-algebras abound such as:*

(a) Polynomial rings over k, $k[X_1, \ldots, X_n]$

(b) Rings of $n \times n$ square matrices with entries in k, $M_n(k)$

(c) Also, for any $\alpha \in k$ we can define a 2-dimensional k-algebra, denoted by $k(\sqrt{\alpha})$, generated by an element x, subject to $x^2 = \alpha$. The set $\{1, x\}$ is a basis for $k(\sqrt{\alpha})$ over k and $k(\sqrt{\alpha}) \cong k[X]/(X^2 - \alpha)$. When k is not algebraically closed and $\alpha \in k^* \backslash k^{*2}$, then $k(\sqrt{\alpha})$ as defined above, is a quadratic extension of k, explaining why we have used this particular notation

To each algebra, we can associate another algebra which is identical, except for the way we take products. In this new algebra, for the product of two elements, we first reverse their order and then compute the product in the original algebra. More formally, we make the following:

Definition 2.1.6 *Given an algebra (A, m_A, u_A), the* **opposite algebra** *is defined to be $(A^{op}, m_{A^{op}}, u_{a^{op}})$, with $A^{op} = A$ as vector spaces, $m_{A^{op}}(x, y) = m_A(y, x)$ and $u_{A^{op}} = u_A$.*

Definition 2.1.7 *An algebra A is said to be a* **division algebra** *if every non-zero element of A has an inverse.*

An algebra is essentially a ring with identity, which is also a vector space using the addition of the ring. Thus, many of the notions in the settings of algebras are obtained by taking the corresponding notions in the settings of rings and placing suitable vector space restrictions on them, e.g. requiring subsets to be subspaces, maps to be linear maps and so on. We have seen an example of this with Definition 2.1.2, the following definition gives us another example.

Definition 2.1.8 *A* **subalgebra** *B of an algebra A is a subspace which is closed under multiplication and contains the same identity element as A. The* **center** *of an algebra A is defined to be the subalgebra $Z(A) = \{a \in A : ab = ba \quad \forall b \in A\}$. A* **left ideal** *(respectively* **right ideal***), I, in an algebra is a vector subspace of A which satisfies $AI \subseteq I$ (respectively $IA \subseteq I$). A* **two-sided ideal** *is a subspace which is simultaneously a left and right sided ideal.*

Definition 2.1.9 *An algebra A is called* **left Artinian** *if it satisfies the descending chain condition for left ideals: that is, for any sequence $I_1 \supseteq I_2 \supseteq \ldots$ of ideals there is an integer $r \in \mathbb{N}$ such that $I_r = I_{r+1} = \ldots$. An algebra A is called* **left Noetherian** *if it satisfies the ascending chain condition for left ideals: that is, for any sequence $I_1 \subseteq I_2 \subseteq \ldots$ of ideals there is an integer $r \in \mathbb{N}$ such that $I_r = I_{r+1} = \ldots$. Similarly, we call an algebra A* **right Artinian** *(respectively* **right Noetherian***) if it satisfies the descending (respectively ascending) chain condition for right ideals. We call an algebra* **Artinian** *(respectively* **Noetherian***) in the case that it is both left and right Artinian (respectively Noetherian).*

If an algebra A is finite dimensional as a vector space over k, then it is both Artinian and Noetherian (consider the sequence of dimensions of the ideals).

Definition 2.1.10 *An ideal (or left ideal or right ideal) I is said to be* **nil** *if every element $x \in I$ is nilpotent, that is $x^n = 0$ for some integer $n \in \mathbb{N}$. An ideal I is* **nilpotent** *if $I^n = 0$ for some integer $n \in \mathbb{N}$.*

Every nilpotent ideal is nil, and in an Artinian algebra the converse holds too. To see this we first need to introduce the Jacobson radical.

Definition 2.1.11 *If A is an algebra, then the* **Jacobson radical** *of A, denoted by $J(A)$, is defined to be the intersection of all maximal left ideals of A.*

There are many equivalent ways to define the Jacobson radical. We mention just one. Definition 2.1.11 is equivalent to defining $J(A)$ to be the intersection of all maximal right ideals of A.

Lemma 2.1.12 *We have the following statements about the Jacobson radical:*

(a) If I is a nil left ideal of A, then $I \subseteq J(A)$

(b) If A is Artinian, then $J(A)$ is nilpotent

(c) If A is Artinian, then by (a) and (b), $J(A)$ is the largest nilpotent ideal of A and every nil ideal of A is nilpotent

Definition 2.1.13 *Suppose that V and W are vector spaces over k with dimensions m and n respectively and suppose that $\{e_i : 1 \leq i \leq m\}$ is a basis for V and $\{f_i : 1 \leq e \leq n\}$ is a basis for W. Then we can construct a new vector space $V \otimes W$ called the* **tensor product** *of V and W, which has dimension mn and a basis $\{e_i \otimes f_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Although we do not give the* *construction (for details, see [27] for example) the tensor product can be shown to have the following properties, for all $v, v_1, v_2 \in V, w, w_1, w_2 \in W, c \in k$:*

$$\begin{aligned} v \otimes (w_1 + w_2) &= (v \otimes w_1) + (v \otimes w_2) \\ (v_1 + v_2) \otimes w &= (v_1 \otimes w) + (v_2 \otimes w) \\ c(v \otimes w) &= (cv) \otimes w = v \otimes (cw) \\ 0 \otimes w &= v \otimes 0 = 0 \end{aligned}$$

Definition 2.1.14 *Suppose that A, B are k-algebras then the tensor product $A \otimes B$ can be given a k-algebra structure by defining multiplication in $A \otimes B$ by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$, then this has identity given by $1_{A\otimes B} = 1_A \otimes 1_B$. We call the k-algebra $A \otimes B$ the* **tensor product** *of A and B.*

Whenever A and B are k-algebras the tensor product of A and B is understood to be the *k-algebra $A \otimes B$* defined in this way.

Definition 2.1.15 *Let A be an k-algebra and M be a vector space over k. Then M is a* **left A-module** *if there exists a map $\lambda : A \otimes M \to M$ which satisfies the following four conditions, for all $a, b \in A, m, n \in M$:*

$$\begin{aligned} \lambda(a \otimes (m + n)) &= \lambda(a \otimes m) + \lambda(a \otimes n), \\ \lambda((a + b) \otimes m) &= \lambda(a \otimes m) + \lambda(b \otimes m), \\ \lambda((ab) \otimes m) &= \lambda(a \otimes \lambda(b \otimes m)), \\ \lambda(1_A \otimes m) &= m \end{aligned}$$

If we write $\lambda(a \otimes m) = a \cdot m$, then the equations become:

$$\begin{aligned} a \cdot (m + n) &= (a \cdot m) + (a \cdot n), \\ (a + b) \cdot m &= (a \cdot m) + (b \cdot m), \\ (ab) \cdot m &= (a \cdot (b \cdot m)), \\ 1_A \cdot m &= m \end{aligned}$$

Let A be an k-algebra and M be a vector space over k. Then M is a **right** A**-module** *if there exists a map $\rho : M \otimes A \to M$ which satisfies the following four conditions, for all $a, b \in A, m, n \in M$:*

$$\begin{aligned}
\rho((m+n) \otimes a) &= \rho(m \otimes a) + \rho(n \otimes a), \\
\rho(m \otimes (a+b)) &= \rho(m \otimes a) + \rho(m \otimes b), \\
\rho(m \otimes (ab)) &= \rho(\rho(m \otimes a) \otimes b), \\
\rho(m \otimes 1_A) &= m
\end{aligned}$$

If we write $\rho(m \otimes a) = m \cdot a$, then the equations become:

$$\begin{aligned}
(m+n) \cdot a &= (m \cdot a) + (n \cdot a), \\
m \cdot (a+b) &= (m \cdot a) + (m \cdot b), \\
m \cdot (ab) &= ((m \cdot a) \cdot b), \\
m \cdot 1_A &= m
\end{aligned}$$

If we do not specify on which side A acts on M, it shall always be understood to act on the left. We note that considering only the case of left modules should cause no restriction however, since a right A-module is the same thing as a left A^{op}-module.

It is useful to think about A-modules as a generalisation of vector spaces over a field. We may add any two elements in the A-module to give us another such element, and we can also mulitply by scalars from the algebra A. (Also see Example 2.1.17 (a)).

For any k-algebra A, the 0-dimensional vector space consisting of only the zero vector is always a left A-module upon setting $a \cdot 0 = 0$ for all $a \in A$. We shall call this the **zero module** and denote it by 0.

Definition 2.1.16 *Let A and B be k-algebras and M a vector space over k. Then M is an $A - B$* **bimodule** *if it is simultaneously a left A-module and right B-*

module which satisfies the following for all $a \in A, b \in B, c \in k, m \in M$:

$$(a \cdot m) \cdot b = a \cdot (m \cdot b)$$
$$(c1_A) \cdot m = m \cdot (c1_B)$$

We refer to an $A - A$ bimodule, simply as an A-bimodule.

Notice that any $A - B$ bimodule can be regarded as a left $A \otimes B^{op}$-module via $(a \otimes b) \cdot m = a \cdot m \cdot b$ (note that by the "associativity" property of bimodules we do not need to bracket the right hand side of this equation). Conversely any left $A \otimes B^{op}$-module may be regarded as an $A - B$ bimodule via $a \cdot m = (a \otimes 1_B) \cdot m, m \cdot b = (1_A \otimes b) \cdot m$.

Examples 2.1.17 *(a) Consider k itself as the 1-dimensional k-algebra, then any k-vector space V is a left k-module with the action of k on V given by scalar multiplication*

(b) Let $A = k\mathbb{Z}_2 \cong k[\sigma]/(\sigma^2 - 1)$ and let $M = k(\sqrt{\alpha})$, which is in fact a k-algebra itself (see Example 2.1.5 (c)). Then M is a left A-module with the action of A on M induced by setting:

$$1_A \cdot 1_M = 1_M, 1_A \cdot x = x, \sigma \cdot 1_M = 1_M, \sigma \cdot x = -x$$

Notice that σ is an algebra involution of M. Viewing this example in conjunction with Example 2.1.33 (b) may help the reader to realise that a superalgebra is equivalent to a pair consisting of an algebra A and an algebra involution $\sigma : A \to A$ (or equivalently a $k\mathbb{Z}_2$-module algebra).

(c) Any k-algebra A can be viewed as a left or right A-module by letting A act on itself via left or right multiplication. In fact, this gives an example of an A-bimodule.

Definition 2.1.18 *If A is an algebra and M is a left A-module then a* **left A-submodule of** *M is a subspace N of the vector space M, closed under scalar* *multiplication by A. We often abbreviate this and simply say that N is a submodule of the module M.*

Definition 2.1.19 *If $M_1, \ldots M_r$ are submodules of the left A-module M, then we define $M_1 + \ldots + M_r = \{x_1 + \ldots + x_r : x_i \in M_i, i = 1, \ldots, r\}$ which is again a submodule of M. More generally, suppose that $(M_i)_{i \in \mathcal{I}}$ is an indexed class of left A-modules, then $\sum_{i \in \mathcal{I}} M_i$ is defined to be the collection of all finite sums $\sum_{i \in \mathcal{I}'} m_i$ with $m_i \in M_i$ and $\mathcal{I}'$ finite.*

Moreover, the module M is the **internal direct sum** *of submodules $M_i, i \in \mathcal{I}$ if each $x \in M$ can be written uniquely as $x = x_{i_1} + \ldots + x_{i_n}$ where $0 \neq x_{i_j} \in M_{i_j}, j = 1, \ldots, n$ and each index i_j is distinct from the others. Note that n may depend on the particular element x. We denote this by $M = \bigoplus_{i \in \mathcal{I}} M_i$.*

For the next two results, we make the assumption that M is a left A-module, which has finite dimension as a vector space over k. (We make this assumption to avoid mentioning the notion of "finitely generated" — these results hold in a more general context)

Lemma 2.1.20 *(Nakayama's lemma, Version 1) Suppose that I is a two-sided ideal of A. If $I \subseteq J(A)$ and $IM = M$, then $M = 0$.*

As a corollary to this result, if $I \subseteq J(A)$ and M is not the zero module, then $IM \subset M$, and in particular $J(A)M \subset M$.

Lemma 2.1.21 *(Nakayama's lemma, Version 2) Suppose N is a submodule of M and I is a two-sided ideal of A, with $I \subseteq J(A)$. If $M = N + IM$, then $M = N$.*

We now explain how one can construct new A-modules out of a collection of A-modules.

Definition 2.1.22 *Suppose that $(M_i)_{i \in \mathcal{I}}$ is an indexed class of left A-modules. The catesian product $\times_{i \in \mathcal{I}} M_i$ becomes a left A-module with operations defined coordinatewise. That is, $(m_i) + (n_i) = (m_i + n_i), a \cdot (m_i) = (a \cdot m_i)$ for all*

$a \in A, (m_i), (n_i) \in \times_{i \in \mathcal{I}} M_i$. *The resulting* A*-module is called the* **direct** *(or* **cartesian***)* **product** *and is denoted by* $\prod_{i \in \mathcal{I}} M_i$.

The **support** *of an element* $m = (m_i) \in \prod_{i \in \mathcal{I}} M_i$ *is* $S(m) = \{i \in \mathcal{I} : m_i \neq 0\}$, *the element* $m = (m_i) \in \prod_{i \in \mathcal{I}} M_i$ *is said to be* **almost always zero**, *when its support* $S(m)$ *is finite. We define* $\bigoplus_{i \in \mathcal{I}} = \{m \in \prod_{i \in \mathcal{I}} :$ m *is almost always zero* $\}$ *and call it the* **external direct sum** *of* $(M_i)_{i \in \mathcal{I}}$. *This can be shown to be a submodule of* $\prod_{i \in \mathcal{I}} M_i$.

We briefly remark that direct products are the products in the category of A-modules, and external direct sums are the coproduct in this category. It should be clear from the definitions that these two notions coincide when the collection $(M_i)_{i \in \mathcal{I}}$ of A-modules is finite (i.e. $\mathcal{I}$ is a finite set).

Note that we use the same symbol for internal and external direct sums. This should not cause any confusion. The difference between the two notions is simply whether the direct summands M_i are submodules of a given module, or not.

Definition 2.1.23 *An* A*-module* M *is* **simple** *if* $M \neq 0$ *and the only submodules of* M *are* 0 *and* M.

Lemma 2.1.24 *Let* M *be an* A*-module, then the following conditions are equivalent and any module satisfying them is called* **semisimple***:*

(a) M *is a sum of simple modules*

(b) M *is a direct sum of simple modules*

(c) If N *is a submodule of* M, *then* N *is a direct summand of* M, *that is, there is a submodule* N' *of* M *such that* $M = N \oplus N'$

Before giving the next definition, recall from Example 2.1.17 (c) that any algebra can be regarded as a left or right module over itself, where the action is given by multiplication in the algebra.

Definition 2.1.25 *An algebra A is said to be* **semisimple** *when it is semisimple as a right module over itself.*

We remark that it can be shown (after some work) to be equivalent to requiring that A is semisimple as a left module over itself. Hence we may simply refer to an algebra as semisimple, in either case.

The following result shows that, in some sense, the Jacobson radical measures the "obstruction to semisimplicity" of an algebra.

Lemma 2.1.26 *A is semisimple if and only if A is Artinian and $J(A) = 0$.*

We have another standard fact about semisimple algebras:

Lemma 2.1.27 *Suppose that an algebra A is semisimple, then any A-module is semisimple.*

We have one final result to give before moving onto the subject of G-graded algebras.

Lemma 2.1.28 *(The Wedderburn-Artin Structure Theorem)*

Suppose that A is a semisimple algebra, then

(i) There exist natural numbers $n_1, \ldots, n_r$ and division algebras $D_1, \ldots, D_r$ such that

$$A \cong M_{n_1}(D_1) \oplus \ldots \oplus M_{n_r}(D_r)$$

When k is algebraically closed, for every $i \in \{1, \ldots, r\}$, $D_i \cong k$.

(ii) The pairs $(n_1, D_1), \ldots, (n_r, D_r)$ for which the above is satisfied are uniquely determined (up to isomorphism) by A.

(iii) If $n_1, \ldots, n_r \in \mathbb{N}$ and $D_1, \ldots, D_r$ are division algebras, then $M_{n_1}(D_1) \oplus \ldots \oplus M_{n_r}(D_r)$ is a semisimple algebra.

Most statements of the above Theorem do not include the last statement given in part (i); however, this follows at once from the fact that the only division algebra over an algebraically closed field k is k itself.

We now introduce the notion of a G-graded algebra.

Definition 2.1.29 *Let G be a group. We denote the group operation by $\circ$ and the identity of this group by e. An algebra A is a G*-**graded algebra** *if there is a family of subspaces $\{A_g : g \in G\}$ such that $A = \bigoplus_{g \in G} A_g$ and $A_g A_h \subseteq A_{g \circ h}$. The subspace A_g is said to be the* **degree** g **component of** A *and elements of A_g are said to be* **homogeneous of degree** g.

When A and B are G-graded algebras and $f : A \to B$ is a linear map, f is said to be **homogeneous of left degree** g *(respectively* **right degree** g*) if $f(A_h) \subseteq B_{g \circ h}$ (respectively $f(A_h) \subseteq B_{h \circ g}$). An algebra map which is homogeneous of left degree e must also be homogeneous of right degree e and conversely, in this case, we say that the map is a G*-**graded algebra map** .

From this we obtain the category of G-graded algebras whose objects are G-graded algebras and morphisms are G-graded algebra maps.

We note here that, from the definition, we can quickly deduce that for a G-graded algebra $k1 \subseteq A_e$, A_e is a subalgebra of A, A_g is an A_e-bimodule and that for each $g \in G$, $A_g A_{g^{-1}}$ and $A_{g^{-1}} A_g$ are ideals in A_e.

Lemma 2.1.30 *Suppose that ϕ is an invertible linear map, homogeneous of left (respectively right) degree g, then ϕ^{-1} is homogeneous of left (respectively right) degree g^{-1}.*

As an easy consequence of the above result, an algebra isomorphism is a G-graded algebra isomorphism if and only if it is homogeneous of degree e.

Examples 2.1.31 *As examples of G-graded algebras we give the following:*

(a) An algebra A can be given a "trivial" G-grading for any group G by setting $A_e = A$ and $A_g = \{0\}$ for all $e \neq g \in G$.

(b) *$A = k[X]$ is naturally graded by $\mathbb{Z}$ upon setting $A_n = \{0\}$ for $n \leq 0$ and $A_n = \{aX^n : a \in k\} = kX^n$ for $n \geq 1$. More generally, $A = k[X_1, \ldots, X_m]$ is graded by $\mathbb{Z}$ upon setting $A_n = \{0\}$ for $n \leq 0$ and $A_n = \{$ homogeneous polynomials of degree $n\}$ for $n \geq 1$. Recall that a polynomial $p(X_1, \ldots, X_m)$ is said to be* **homogeneous of degree** *n if $p(\lambda X_1, \ldots, \lambda X_m) = \lambda^n p(X_1, \ldots, X_m)$.*

(c) *Let $\mathbb{Z}_{n+1}$ be the cyclic group of order $n+1$ with generator 1, then $A = k[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^2$ is graded by $\mathbb{Z}_{n+1}$ by setting $A_0 = k1_A$ and $A_i = kX_i$ for $i \in \{1, \ldots, n\}$. This follows since $A_0A_i = A_iA_0 = A_i$ and $A_iA_j = 0 \subset A_{i+j}$ for $0 \neq i, j \in \mathbb{Z}_{n+1}$.*

(d) *The group algebra $A = kG$ is naturally graded by G by setting $A_g = kg$. Notice that the homogeneous components A_g obey $A_gA_h = A_{g \circ h}$ rather than just $A_gA_h \subseteq A_{g \circ h}$. In such cases we say that A is* **strongly G-graded**. *However, we only mention this concept in passing and will not use it further.*

In the case $G = \mathbb{Z}_2$ then we obtain the concept of a $\mathbb{Z}_2$-graded algebra, which is also called a superalgebra. Since this is our main object of interest we give the conditions defining a superalgebra (even though they are a special case of Definition 2.1.29).

Definition 2.1.32 *A* **$\mathbb{Z}_2$-graded algebra** *or a* **superalgebra** *is an algebra A with subspaces A_0, A_1 such that $A = A_0 \oplus A_1$ with $A_iA_j \subseteq A_{i+j}$ for $i, j \in \mathbb{Z}_2$ or in full: $A_0A_0 \subseteq A_0$, $A_0A_1 \subseteq A_1$, $A_1A_0 \subseteq A_1$ and $A_1A_1 \subseteq A_0$.*

Throughout the thesis we shall use the term superalgebra. We view a superalgebra A as consisting of an algebra B and "some additional structure" (which, with some thought, we discover the additional structure is an algebra involution $\sigma : B \to B$ — consider Example 2.1.17 (c) and Example 2.1.33 (b)). We shall call the algebra obtained by forgetting this additional structure the **underlying algebra** and we shall refer to the additional structure as the **$\mathbb{Z}_2$-grading** .

We note that $k1 \subseteq A_0$, A_0 is a subalgebra of A, A_1 is an A_0-bimodule and $A_1^2 = A_1 A_1$ is an ideal of A_0.

Examples 2.1.33 *As examples of superalgebras we give the following:*

(a) Any algebra A is a superalgebra endowed with the trivial $\mathbb{Z}_2$-grading $A_0 = A, A_1 = \{0\}$. This is quite an important idea which is used throughout this thesis.

(b) If $A = \mathbb{C}$ then A is a superalgbera over $\mathbb{R}$ via $A_0 = \mathbb{R}, A_1 = \mathbb{R}i$. Or more generally, if $A = k(\sqrt{\alpha})$, then A is a superalgebra over k, via $A_0 = k = k1, A_1 = kx$.

Definition 2.1.34 *For a superalgebra, A, we define* $\dim_0 A = \dim A_0$ *and* $\dim_1 A = \dim A_1$ *where these are the dimensions of A_0 and A_1 as vector spaces over k.*

We shall use $\dim = n$ substantively to refer to the set of algebras of dimension n and similarly, we shall use $\dim_0 = i$ to refer to the set of superalgebras A having $\dim_0 A = \dim A_0 = i$.

Finally, we define a useful piece of notation:

Definition 2.1.35 *Let δ_i^j be the* **Kronecker delta function** *defined by:*

$$\delta_i^j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Henceforth, δ_i^j with the sub- and superscripts shall only be used to refer to this function.

2.2 4-dimensional Algebras

As mentioned in the introduction, we assume that k is a field with $\mathrm{ch}(k) \neq 2$. We do not make the additional assumption that k is algebraically closed until Section 2.5.

Notice that any two superalgebras A and B must in particular be algebras and that a superalgebra isomorphism of A and B must in particular be an algebra isomorphism between A and B when viewed as algebras. Hence, when A and B are not isomorphic as algebras, they cannot be isomorphic as superalgebras. We shall use this fact to help us prove the classification results of non-trivially $\mathbb{Z}_2$-graded superalgebras in the later sections. This section is dedicated to proving several results which state when some 4-dimensional algebras are not isomorphic. It should be noted however that this does not give a full classification of 4-dimensional algebras.

Recall that $k\langle X_1, \ldots, X_n\rangle$ denotes the polynomial algebra in n non-commuting indeterminates.

The first result follows from the work of [12]

Proposition 2.2.1 *The following families of algebras are pairwise non-isomorphic:*

(1) $k \times k \times k \times k$,
(2) $k \times k \times k[X]/(X^2)$,
(3) $k[X]/(X^2) \times k[Y]/(Y^2)$,
(4) $k \times k[X]/(X^3)$,
(5) $k[X]/(X^4)$,
(6) $k \times k[X,Y]/(X,Y)^2$,
(7) $k[X,Y]/(X^2,Y^2)$,
(8) $k[X,Y]/(X^3,XY,Y^2)$,
(9) $k[X,Y,Z]/(X,Y,Z)^2$,
(10) M_2,

(11) $\left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} : a, b, c, d \in k \right\}$,

(12) $\wedge k^2$,

(13) $k \times \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} = \left\{ \left(a, \begin{pmatrix} b & c \\ 0 & d \end{pmatrix} \right) : a, b, c, d \in k \right\}$,

(14) $\left\{ \begin{pmatrix} a & 0 & 0 \\ c & a & 0 \\ d & 0 & b \end{pmatrix} : a, b, c, d \in k \right\}$,

(15) $\left\{ \begin{pmatrix} a & c & d \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} : a, b, c, d \in k \right\}$,

(16) $k\langle X, Y\rangle/(X^2, Y^2, YX)$,

(17) $\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ c & d & b \end{pmatrix} : a, b, c, d \in k \right\}$,

(18;λ) $k\langle X, Y\rangle/(X^2, Y^2, YX - \lambda XY)$, *where* $\lambda \in k$ *with* $\lambda \neq -1, 0, 1$,

(19) $k\langle X, Y\rangle/(Y^2, X^2 + YX, XY + YX)$

Proof:
This follows from Gabriel's results given in [12]. Suppose that two algebras on the list are isomorphic. Then take tensor products with an algebraically closed extension K of k to obtain two K-algebras. The isomorphism of the two k-algebras induces an isomorphism of the two K-algebras just constructed. However, since K is algebraically closed, this would contradict the results in [12] — impossible. Thus any two algebras on the above list are indeed isomorphic as claimed. $\square$

In the case where the families depend on some parameter, such as (18;λ) above, there may be situations in which different members of the the same family are isomorphic. For instance (18;λ) $\cong$ (18;λ') if and only if either $\lambda = \lambda'$ or $\lambda\lambda' = 1$, which again follows from the results of [12]. Since this section is only to help us with our proofs in the next few sections we do not bother listing when two such members of a given family are isomorphic. We shall however be interested in exactly this question in the next few sections when we give classification results for superalgebras.

When k is not algebraically closed, in particular when k contains non-square elements, the classification of non-trivially $\mathbb{Z}_2$-graded superalgebras over k gives rise to some algebras which are not on the list above. We treat these algebras in the next two propositions. The parameters μ and ξ which occur in the next few propositions are elements of $k^*\backslash k^{*2}$.

Proposition 2.2.2 *The following families of algebras are pairwise non-isomorphic and are all non-isomorphic with the families described in Proposition 2.2.1:*

(20;μ)	$k[X]/(X^2) \times k(\sqrt{\mu})$,
(21;μ)	$k \times k \times k(\sqrt{\mu})$,
(22;ξ,μ)	$k(\sqrt{\xi}) \times k(\sqrt{\mu}))$,
(23;μ)	$k[X,Y]/(X^2 - \mu, Y^2)$,
(24;μ)	$k[X,Y]/(\mu X^2 + Y^2, XY)$,
(25;μ)	$k\langle X,Y\rangle/(X^2 - \mu, Y^2, XY + YX)$

Proof:
We shall use a series of lemmas establishing the following:

- (20;μ)–(23;μ) are non-isomorphic (Lemma 2.2.5, Lemma 2.2.6 and Lemma 2.2.9)
- none of (20;μ)–(23;μ) is isomorphic to (1)–(9) (Lemma 2.2.8)
- (24;μ) is not isomorphic to any of (1)–(9) (Lemma 2.2.9 and Lemma 2.2.10)
- (24;μ) is not isomorphic to any of (20;μ)–(23;μ) (Lemma 2.2.8)
- (25;μ) is not isomorphic to any of (10)–(19) (Lemma 2.2.8 and Lemma 2.2.9)

To complete the proof, note the simple fact that a commutative algebra can never be isomorphic to a non-commutative algebra. $\square$

Proposition 2.2.3 *The following families of algebras are pairwise non-isomorphic and we have:*
(a) *Algebra (26;μ,θ,η) is non-isomorphic with all algebras described in Proposition 2.2.1 and Proposition 2.2.2 above, except (20;μ)–(23;μ)*
(b) *Algebra (27;μ,θ) is non-isomorphic with all algebras described in Proposition 2.2.1 and Proposition 2.2.2 above, except (10) and (25;μ)*

(26;μ,θ,η) $k[X,Y]/(X^2-\mu, Y^2-\theta-\eta X)$, *with* $\theta,\eta \in k$ *where* $\theta \neq 0$ *or* $\eta \neq 0$,
(27;μ,θ) $k\langle X,Y\rangle/(X^2-\mu, XY+YX, Y^2-\theta)$, *with* $\theta \in k$ *where* $\theta \neq 0$

Proof:
Again we use the fact that a commutative and a non-commutative algebra cannot be isomorphic. We complete the proof by using Lemma 2.2.8, which shows that :

- (26;μ,θ,η) is not isomorphic to any of (1)–(9), or (24;μ)
- (27;μ,θ) is not isomorphic to any of (11)–(19)

□

Remark 2.2.4 *There are some cases where Proposition 2.2.3 cannot be strengthened any further, for example* (26;μ,1,0) *is isomorphic to* (22;μ,μ) *and* (27;μ,1) *is isomorphic to* (10). *However for some of the other cases, for example* (27;μ,θ) *and* (25;μ)*, we are unsure if they can be isomorphic to each other or not. Determining conditions when an algebra from Proposition 2.2.3 is isomorphic to an algebra from Proposition 2.2.2 can be quite difficult. However we spend no further time on this problem since Proposition 2.2.3 is as strong as we require for its use in later sections.*

Lemma 2.2.5 (a) *Let* α, $\beta \in k$. *Then* $k(\sqrt{\alpha}) \cong k(\sqrt{\beta})$ *if and only if* $\alpha = \delta^2\beta$ *for some* $\delta \in k^*$.

(b) *Let $\mu \in k$. Then we have*

$$k[X]/(X^2 - \mu) \cong \begin{cases} k[X]/(X^2), & \mu = 0, \\ k \times k, & \mu \in k^{*2}, \\ k(\sqrt{\mu}), & \mu \in k^* \backslash k^{*2}. \end{cases}$$

Moreover, the three classes of algebras are non-isomorphic to each other.

Proof:
For part (a), by definition, there is a k-basis $\{1, x\}$ in $k(\sqrt{\alpha})$ and a k-basis $\{1, y\}$ in $k(\sqrt{\beta})$ such that $x^2 = \alpha$ and $y^2 = \beta$. Suppose $\phi : k(\sqrt{\alpha}) \to k(\sqrt{\beta})$ is an algebra isomorphism. Then $\phi(1) = 1$ and $\phi(x) = \gamma + \delta y$ for some $\gamma \in k$ and $\delta \in k^*$. From $\phi(x^2) = \phi(x)^2$, one gets $\alpha = \gamma^2 + \delta^2\beta + 2\gamma\delta y$. This implies $\gamma = 0$ and $\alpha = \delta^2\beta$ as $\delta \neq 0$. Conversely, if $\alpha = \delta^2\beta$ for some $\delta \in k^*$, then the k-linear map $k(\sqrt{\alpha}) \to k(\sqrt{\beta})$, $1 \mapsto 1$, $x \mapsto \delta y$, is an algebra isomorphism.

Part (b) follows from Part (a) and the facts that $k[X]/(X^2 - \alpha) \cong k(\sqrt{\alpha})$ and $k(\sqrt{1}) \cong k \times k$. $\square$

Lemma 2.2.6 *Suppose that B, C, E, F are finite dimensional algebras and that $B \cong E$. Then $B \times C \cong E \times F$ if and only if $C \cong F$.*

Proof:
It is easy to see that if $C \cong F$ then $B \times C \cong B \times F \cong E \times F$.

The converse follows from the Krull-Schmidt theorem for finite dimensional algebras. However, since this result is not very well-known, we give a direct proof for the sake of completeness.

Firstly, we will show the special case that $B \times C = A = E \times F$ with $B \cong E$ implies $C \cong F$.

Since A is finite dimensional it has a unique, finite complete set of primitive central idempotents. Denote this set by S. So $S = \{e_1, \ldots, e_n\}$ with each e_i non-zero and satisfying the following: $e_i e_j = \delta_i^j e_j, 1 = e_1 + \ldots + e_n$ and if $e_i = e_{i_1} + e_{i_2}$ then $e_{i_1} = 0$ or $e_{i_2} = 0$.

We can identify A with $Ae_1 \times \ldots \times Ae_n$. Order the primitive central idempotents so that $e_1, \ldots, e_t \in B$, so $B = Ae_1 \times \ldots \times Ae_t$ and hence $C = Ae_{t+1} \times \ldots \times Ae_n$.

An algebra map must preserve idempotents, and moreover an isomorphism must map distinct primitive central idempotents to distinct primitive central idempotents. Suppose that $\phi : B \to E$ is an isomorphism, such an isomorphism exists by hypothesis.

Let $e'_i = \phi(e_i)$ for $1 \leq i \leq t$. For each e'_i with $t+1 \leq i \leq n$ choose an element of $S \backslash \{e'_1, \ldots, e'_t\}$ in such a way that $e'_i = e'_j \Leftrightarrow i = j$. It is clear that $\{e'_1, \ldots, e'_n\}$ is simply a permutation of $\{e_1, \ldots, e_n\}$, $E = Ae'_1 \times \ldots \times Ae'_t$ and $F = Ae'_{t+1} \times \ldots \times Ae'_n$.

Now for $1 \leq i \leq t$,

$$\phi(Ae_i) = \phi(Ae_ie_i) = \phi(Ae_i)\phi(e_i) = \phi(Ae_i)e'_i \subseteq Ee'_i = Ae'_i$$

and since ϕ is an isomorphism

$$\phi^{-1}(Ae'_i) = \phi^{-1}(Ae'_ie'_i) = \phi^{-1}(Ae'_i)\phi^{-1}(e'_i) = \phi^{-1}(Ae'_i)e_i \subseteq Be_i = Ae_i$$

i.e. $Ae'_i \subseteq \phi(Ae_i)$. Thus $\phi(Ae_i) = Ae'_i$, so $Ae_i \cong Ae'_i$ via $\phi|_{Ae_i}$.

We wish to lift ϕ to an automorphism ψ of A. First define $T = \{e_1, \ldots, e_t\}$ and $T' = \{e'_1, \ldots, e'_t\}$. We can then construct such a $\psi : A \to A$ as follows (we also describe how to construct its inverse): if $e_j \in T$ then for $x \in Ae_j$ define $\psi(x) = \phi(x)$, if $e_j \in T'$ then for $x \in Ae_j$ define $\psi^{-1}(x) = \phi^{-1}(x)$, if $e_j \in T' \backslash T$ then for $x \in Ae_j$ define $\psi(x) = \phi^{-1}(x)$, if $e_j \in T \backslash T'$ then for $x \in Ae_j$ define $\psi^{-1}(x) = \phi(x)$ and finally, if $e_j \in S \backslash (T \cup T')$ then for $x \in Ae_j$ define $\psi(x) = \psi^{-1}(x) = x$. We then extend these maps linearly to A. One can easily check that $\psi \circ \psi^{-1} = \psi^{-1} \circ \psi = \mathrm{id}_A$.

By construction $\psi|_B = \phi$, so ψ is the required lifting. The primitive central idempotents of C are either in $T' \backslash T$ or in $S \backslash (T \cup T')$, so one quickly checks that $\psi(C) = F$. Hence $\psi|_C : C \to F$ is an isomorphism. Thus $C \cong F$ as required.

We can now prove the general case as follows. Suppose $B \times C \cong E \times F$ where $B \cong E$. Set $A = B \times C$ and suppose $\theta : E \times F \to B \times C$ is an isomorphism. The isomorphism induces a decomposition of A as $\theta(E) \times \theta(F)$ where $\theta(E) \cong E \cong B$ and $\theta(F) \cong F$. We can now apply the above special case to deduce that $\theta(F) \cong C$ thus $F \cong \theta(F) \cong C$, as required. $\square$

Now let us consider the algebras given in (22; ξ, μ). Let μ, μ_1, ξ, $\xi_1 \in k^*\backslash k^{*2}$. Then from Lemma 2.2.6 we know that $k(\sqrt{\xi}) \times k(\sqrt{\mu}) \cong k(\sqrt{\xi_1}) \times k(\sqrt{\mu_1})$ if and only if either $k(\sqrt{\xi}) \cong k(\sqrt{\xi_1})$ and $k(\sqrt{\mu}) \cong k(\sqrt{\mu_1})$, or $k(\sqrt{\xi}) \cong k(\sqrt{\mu_1})$ and $k(\sqrt{\mu}) \cong k(\sqrt{\xi_1})$, which occurs if and only if either $\xi\xi_1^{-1}$, $\mu\mu_1^{-1} \in k^{*2}$, or $\xi\mu_1^{-1}$, $\mu\xi_1^{-1} \in k^{*2}$. That is, (22; ξ, μ) $\cong$ (22; ξ_1, μ_1) if and only if either $\xi\xi_1^{-1}$, $\mu\mu_1^{-1} \in k^{*2}$, or $\xi\mu_1^{-1}$, $\mu\xi_1^{-1} \in k^{*2}$.

Definition 2.2.7 *Let A be a k-algebra. A* **subring**, *B, of A is a vector subspace of A which is closed under multiplication and with the additional property that there is an element $e \in B$ which satisfies $b = eb = be$ for all $b \in B$. Then B is a k-algebra with the element e as the identity. (One can easily see that e must be an idempotent element of A).*

Lemma 2.2.8 *Algebras (1)–(9), (11)–(19) and (24;μ) have no subring isomorphic to a quadratic extension of k.*

Proof:
The general method is as follows: For any non-zero idempotent element a and an element b linearly independent from a such that $0 \neq b^2 \in ka$, we will show $b^2 = \alpha^2 a$ for some $\alpha \in k^*$ in all cases. Thus $ka \oplus kb$ cannot be isomorphic, as an algebra, to a quadratic extension of k. Since we make the minimal assumptions that $a \neq 0$ is idempotent, $\{a, b\}$ is linearly independent and that $0 \neq b^2 \in ka$, we conclude in each case that the algebra has no subring isomorphic to a quadratic extension of k.

We will illustrate this method for the algebra given in (17). The others are done similarly.

Let $a = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ \alpha_3 & \alpha_4 & \alpha_2 \end{pmatrix} \neq 0$ and $b = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ \beta_3 & \beta_4 & \beta_2 \end{pmatrix} \neq 0$ with α_i, $\beta_i \in k$ for $1 \leq i \leq 4$. Suppose $a^2 = a$. Then $\alpha_1^2 = \alpha_1$, $\alpha_2^2 = \alpha_2$, $\alpha_1\alpha_3 + \alpha_2\alpha_3 = \alpha_3$ and

$\alpha_1\alpha_4 + \alpha_2\alpha_4 = \alpha_4$. Hence

$$\alpha_1, \alpha_2 \in \{0, 1\}, \text{ and } \alpha_3 = \alpha_4 = 0 \text{ if } \alpha_1 = \alpha_2.$$

Now assume $b^2 = \gamma a$ for some $\gamma \in k^*$. Then $\beta_1^2 = \gamma\alpha_1$, $\beta_2^2 = \gamma\alpha_2$ and $(\beta_1+\beta_2)\beta_i = \gamma\alpha_i$, $i = 3, 4$. Hence $(\beta_1-\beta_2)(\beta_1+\beta_2)\beta_i = \gamma(\beta_1-\beta_2)\alpha_i$, which implies $(\alpha_1-\alpha_2)\beta_i = (\beta_1-\beta_2)\alpha_i$ as $\gamma \neq 0$, $i = 3, 4$.

If $\alpha_1 = \alpha_2$ then $\alpha_3 = \alpha_4 = 0$. Hence $\alpha_1 = \alpha_2 = 1$ as $a \neq 0$. Thus we have $\gamma = \beta_1^2 = \beta_2^2, \beta_3 = \beta_4 = 0$, and so $b^2 = \gamma a = \beta_1^2 a$. If $\alpha_1 = 1$ and $\alpha_2 = 0$, then $\gamma = \beta_1^2, \beta_2 = 0, \beta_3 = \beta_1\alpha_3, \beta_4 = \beta_1\alpha_4$, and hence $b^2 = \gamma a = \beta_1^2 a$. Similarly, if $\alpha_1 = 0, \alpha_2 = 1$, then $\gamma = \beta_2^2, \beta_1 = 0, \beta_3 = \beta_2\alpha_3, \beta_4 = \beta_2\alpha_4$ and $b^2 = \gamma a = \beta_2^2 a$. $\square$

Lemma 2.2.9 *The algebras defined by (23;μ),(24;μ) and (25;μ) contain no non-trivial idempotents.*

Proof:
We prove the lemma for the algebras given in (25; μ). The other two cases are similar.

The algebra A given in (25; μ) has two generators X and Y subject to the relations:

$$X^2 = \mu,\ Y^2 = 0, \text{ and } XY + YX = 0.$$

Hence A has a k-basis $\{1, X, Y, XY\}$. Now let $a = \alpha_1 + \alpha_2 X + \alpha_3 Y + \alpha_4 XY$ be an idempotent in A, where $\alpha_i \in k$, $1 \leq i \leq 4$. Since $a^2 = (\alpha_1 + \alpha_2 X + \alpha_3 Y + \alpha_4 XY)^2 = \alpha_1^2 + \alpha_2^2\mu + 2\alpha_1\alpha_2 X + 2a_1\alpha_3 Y + 2\alpha_1\alpha_4 XY$, we have

$$\alpha_1^2 + \alpha_2^2\mu = \alpha_1,\ 2\alpha_1\alpha_i = \alpha_i,\ i = 2,\ 3,\ 4.$$

If $\alpha_1 = \frac{1}{2}$, then $\alpha_2^2\mu = \frac{1}{2} - (\frac{1}{2})^2 = \frac{1}{2^2}$. Hence $\alpha_2 \neq 0$, and $\mu = (\frac{1}{2}\alpha_2^{-1})^2 \in k^{*2}$, a contradiction. Hence $\alpha_1 \neq \frac{1}{2}$. Thus we have $\alpha_2 = \alpha_3 = \alpha_4 = 0$ and $\alpha_1^2 = \alpha_1$. Hence $\alpha_1 = 1$ or $\alpha_1 = 0$, and so $a = 1$ or $a = 0$. $\square$

Lemma 2.2.10 *The algebra defined by (24;μ) is not isomorphic to the algebras defined by (5), and (7)–(9) respectively.*

Proof:
Let $\mu \in k^*\backslash k^{*2}$. Let $x = X + (\mu X^2 + Y^2, XY)$ and $y = Y + (\mu X^2 + Y^2, XY)$ in $k[X,Y]/(\mu X^2 + Y^2, XY)$. Let $x_1 = X + (X^4)$ in $k[X]/(X^4)$.

Assume $\phi : k[X,Y]/(\mu X^2 + Y^2, XY) \to k[X]/(X^4)$ is an algebra isomorphism. Then $\phi(1) = 1$, $\phi(x) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_1^2 + \alpha_3 x_1^3$ and $\phi(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_1^2 + \beta_3 x_1^3$ for some α_i, $\beta_i \in k$, $0 \leq i \leq 3$. From the equation $\phi(xy) = \phi(x)\phi(y)$ we obtain

$$0 = \alpha_0\beta_0 + (\alpha_0\beta_1 + \alpha_1\beta_0)x_1 + (\alpha_0\beta_2 + \alpha_1\beta_1 + \alpha_2\beta_0)x_1^2 + (\alpha_0\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_0)x_1^3.$$

This implies that $\alpha_0\beta_0 = 0$, $\alpha_0\beta_1 + \alpha_1\beta_0 = 0$, $\alpha_0\beta_2 + \alpha_1\beta_1 + \alpha_2\beta_0 = 0$ and $\alpha_0\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_0 = 0$.

If $\alpha_0 \neq 0$, then $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 0$, and so $\phi(y) = 0$. This is impossible as ϕ is an isomorphism. Hence $\alpha_0 = 0$. Similarly, one can show $\beta_0 = 0$. Thus we have $\alpha_1\beta_1 = 0$ and $\alpha_1\beta_2 + \alpha_2\beta_1 = 0$.

If $\alpha_1 \neq 0$, then $\beta_1 = \beta_2 = 0$, and so $\phi(y) = \beta_3 x_1^3$. Hence $\phi(y^2) = \phi(y)^2 = 0$, which is impossible since $y^2 \neq 0$ and ϕ is injective. Thus we have $\alpha_1 = 0$ and $\phi(x) = \alpha_2 x_1^2 + \alpha_3 x_1^3$. It follows that $\phi(x^2) = \phi(x)^2 = 0$. This is impossible since $x^2 \neq 0$ and ϕ is injective. Thus we have proven that $k[X,Y]/(\mu X^2 + Y^2, XY) \ncong k[X]/(X^4)$.

Similarly, one can show that $k[X,Y]/(\mu X^2 + Y^2, XY) \ncong k[X,Y]/(X^3, XY, Y^2)$ and $k[X,Y]/(\mu X^2 + Y^2, XY) \ncong k[X,Y,Z]/(X,Y,Z)^2$.

To show $k[X,Y]/(\mu X^2 + Y^2, XY) \ncong k[X,Y]/(X^2, Y^2)$ is slightly different. Let $x = X + (X^2, Y^2), y = Y + (X^2, Y^2)$ in $k[X,Y]/(X^2, Y^2)$ and let $x_1 = X + (\mu X^2 + Y^2, XY), y_1 = Y + (\mu X^2 + Y^2, XY)$ in $k[X,Y]/(\mu X^2 + Y^2, XY)$.

Suppose $\phi : k[X,Y]/(\mu X^2 + Y^2, XY) \to k[X,Y]/(X^2, Y^2)$ is an algebra isomorphism. $\phi(x) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_1^2 + \alpha_3 y_1, \phi(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_1^2 + \beta_3 y_1$.

From $0 = \phi(x^2) = \phi(x)\phi(x)$ we obtain: $\alpha_0^2 = 0, 2\alpha_0\alpha_1 = 0, 2\alpha_0\alpha_2 + \alpha_1^2 - \alpha_3\mu = 0, 2\alpha_0\alpha_3 = 0$ and so $\alpha_0 = 0, \alpha_1^2 - \alpha_3^2\mu = 0$. Similarly, from $0 = \phi(y^2) = \phi(y)\phi(y)$ we obtain: $\beta_0^2 = 0, 2\beta_0\beta_1 = 0, 2\beta_0\beta_2 + \beta_1^2 - \beta_3\mu = 0, 2\beta_0\beta_3 = 0$ and so $\beta_0 = 0, \beta_1^2 - \beta_3^2\mu = 0$

If $\alpha_3 \neq 0$ then $\mu = (\alpha_1\alpha_3^{-1})^2$ — impossible, since $\mu \in k^*\backslash k^{*2}$, so $\alpha_3 = 0$. If $\beta_3 \neq 0$ then $\mu = (\beta_1\beta_3^{-1})^2$ — impossible, since $\mu \in k^*\backslash k^{*2}$, so $\beta_3 = 0$. However, with $\alpha_3 = \beta_3 = 0$, ϕ cannot be surjective, so cannot be an isomorphism — a contradiction. Thus $k[X,Y]/(\mu X^2+Y^2, XY) \ncong k[X,Y]/(X^2,Y^2)$. This completes the proof. $\square$

From our remark in the introduction that a superalgebra A, must satisfy $k1 \subseteq A_0$, one can see that we can split the problem of classification of 4-dimensional superalgebras into cases: $\dim_0 A = 1, 2, 3$ or 4. That is, we may look at the cases where the degree zero component has dimensions $1, 2, 3,$ or 4 separately.

When A is a 4-dimensional superalgebra with $\dim_0 A = \dim A_0 = 4$ then we must have $A_0 = A, A_1 = \{0\}$, this is the situation where each algebra is given the trivial $\mathbb{Z}_2$-grading. In this case a superalgebra with the trivial $\mathbb{Z}_2$-grading is no more than just an algebra. Note that two trivially $\mathbb{Z}_2$-graded superalgebra are isomorphic as superalgebras if and only if they are isomorphic as algebras. Thus the results of [12] give us the classification for this case when k is algebraically closed.

Example 2.2.11 *In this example we consider superalgebra structures on algebra* (9) *from our list in Proposition 2.2.1. To this end take $A = k[X,Y,Z]/(X,Y,Z)^2$. Then one can check that any superalgebra must be isomorphic to one of the following superalgebra structures on A (where we identify X, Y, Z with their images under the natural projection $k[X,Y,Z] \to k[X,Y,Z]/(X,Y,Z)^2$):*

(a) $A_0 = A$ *and* $A_1 = \{0\}$ *(the trivial $\mathbb{Z}_2$-grading),*

(b) $A_0 = k1 \oplus kX \oplus kY$ *and* $A_1 = kZ$,

(c) $A_0 = k1 \oplus kX$ *and* $A_1 = kY \oplus kZ$,

(d) $A_0 = k1$ *and* $A_1 = kX \oplus kY \oplus kZ$

The interesting thing about Example 2.2.11 is that *all* superalgebras with $\dim A_0 = 1$, $\dim A_1 = 3$ are isomorphic to the superalgebra on (9) given the last $\mathbb{Z}_2$-grading of the example. This is a special case of the following short propostion, which shall conclude this section.

Proposition 2.2.12 *An n-dimensional superalgebra, A, with $n \geq 3$ and* $\dim A_0 =$ 1, *must have* $A_1^2 = \{0\}$.

Proof:
Suppose A is an n-dimensional superalgebra, with $n \geq 3$ and $\dim A_0 = 1$. Since $\dim A_0 = 1$, and $k1 \subseteq A_0$ we must have $A_0 = k1 = k$.

We show that the square of any element in A_1 is zero and its product with any linearly independent element from A_1 is also zero. From this, the conclusion of our proposition easily follows.

Let $0 \neq x \in A_1$ and $y \in A_1$ be such that $\{x, y\}$ are linearly independent. Thus $x^2, xy \in A_0 = k$ so $x^2 = \alpha, xy = \beta$ for some $\alpha, \beta \in k$. Now $\alpha y = (xx)y = x(xy) = \beta x$, which by linear independence of x and y implies $\alpha = \beta = 0$, which is what we wanted to show. $\square$

This leaves us with the cases $\dim_0 = 3$ and $\dim_0 = 2$, which we will deal with in the following two sections.

2.3 Case $\dim_0 = 3$

In the following, we will use (j) to denote the algebras defined as (j) of Proposition 2.2.1, Proposition 2.2.2 and Proposition 2.2.3, and use $(j|i)$ to denote the various superalgebras having the underlying algebra (j), $i = 0, 1, 2, \cdots$. We will always use $(j|0)$ to denote the superalgebra on the underlying algebra (j) with the trivial $\mathbb{Z}_2$-grading, i.e., $(j|0)_0 = (j)$ and $(j|0)_1 = 0$. For example, $(1|1)$ denotes the superalgebra structure on algebra $A = k \times k \times k \times k$ which has $A_0 = k(1,1,1,1) \oplus k(1,0,0,0) \oplus k(0,0,1,1)$ and $A_1 = k(0,0,1,-1)$.

Some algebras listed in Proposition 2.2.1, Proposition 2.2.2 and Proposition 2.2.3 admit the form of a quotient algebra A/I of an algebra A, modulo an ideal I. In this case, in order to simplify the notation, we denote by a the image $a + I$ of a under the natural projection $A \to A/I$, where $a \in A$. For example, we will write $X := X + (X^2, Y^2)$ and $Y := Y + (X^2, Y^2)$ in the algebra $(7) = k[X,Y]/(X^2, Y^2)$.

Theorem 2.3.1 *Let k be a field with $\mathrm{ch}(k) \neq 2$.*
(a) *Suppose A is a superalgebra with $\dim A_0 = 3$ and $\dim A_1 = 1$. Then A is isomorphic to one of the following pairwise non-isomorphic families of superalgebras:*

(1) $k \times k \times k \times k :$
$(1|1)_0 = k(1,1,1,1) \oplus k(1,0,0,0) \oplus k(0,0,1,1)$ *and* $(1|1)_1 = k(0,0,1,-1)$,

(2) $k \times k \times k[X]/(X^2) :$
$(2|1)_0 = k(1,1,1) \oplus k(1,0,0) \oplus k(0,1,0)$ *and* $(2|1)_1 = k(0,0,X)$,
$(2|2)_0 = k(1,1,1) \oplus k(1,1,0) \oplus k(0,0,X)$ *and* $(2|2)_1 = k(1,-1,0)$,

(3) $k[X]/(X^2) \times k[Y]/(Y^2) :$
$(3|1)_0 = k(1,1) \oplus k(1,0) \oplus k(X,0)$ *and* $(3|1)_1 = k(0,Y)$,

(4) $k \times k[X]/(X^3) :$
$(4|1)_0 = k(1,1) \oplus k(1,0) \oplus k(0,X^2)$ *and* $(4|1)_1 = k(0,X)$,

(6) $k \times k[X,Y]/(X,Y)^2 :$
$(6|1)_0 = k(1,1) \oplus k(1,0) \oplus k(0,X)$ *and* $(6|1)_1 = k(0,Y)$,

(7) $k[X,Y]/(X^2,Y^2)$:

$(7|1)_0 = k1 \oplus k(X+Y) \oplus kXY$ *and* $(7|1)_1 = k(X-Y)$,

(8) $k[X,Y]/(X^3,XY,Y^2)$:

$(8|1)_0 = k1 \oplus kX \oplus kX^2$ *and* $(8|1)_1 = kY$,

$(8|2)_0 = k1 \oplus kX^2 \oplus kY$ *and* $(8|2)_1 = kX$,

(9) $k[X,Y,Z]/(X,Y,Z)^2$:

$(9|1)_0 = k1 \oplus kX \oplus kY$ *and* $(9|1)_1 = kZ$,

(11) $\left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \middle| \, a,b,c,d \in k \right\}$:

$$(11|1)_0 = k\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \oplus k\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus k\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $(11|1)_1 = k\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

(13) $k \times \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} = \left\{ (a, \begin{pmatrix} b & c \\ 0 & d \end{pmatrix}) \middle| \, a,b,c,d \in k \right\}$:

$(13|1)_0 = k\Big(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big) \oplus k\Big(0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\Big) \oplus k\Big(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\Big)$

and $(13|1)_1 = k\Big(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Big)$,

(14) $\left\{ \begin{pmatrix} a & 0 & 0 \\ c & a & 0 \\ d & 0 & b \end{pmatrix} \middle| \, a,b,c,d \in k \right\}$:

$$(14|1)_0 = k\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus k\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus k\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $(14|1)_1 = k\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$$(14|2)_0 = k\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus k\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus k\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $(14|2)_1 = k\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,

(15) $\left\{ \begin{pmatrix} a & c & d \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a,b,c,d \in k \right\}$:

$$(15|1)_0 = k\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \oplus k\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} \oplus k\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

and $(15|1)_1 = k\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$,

$$(15|2)_0 = k\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \oplus k\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} \oplus k\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$$

and $(15|2)_1 = k\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$,

(17) $\left\{ \begin{pmatrix}a&0&0\\0&a&0\\c&d&b\end{pmatrix} \middle| \, a,b,c,d \in k \right\}$:

$$(17|1)_0 = k\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \oplus k\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} \oplus k\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$$

and $(17|1)_1 = k\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}$,

$(20;\mu)$ $k[X]/(X^2) \times k(\sqrt{\mu})$, $\mu \in k^*\backslash k^{*2}$:

$(20;\mu|1)_0 = k(1,1) \oplus k(1,0) \oplus k(X,0)$ *and* $(20;\mu|1)_1 = k(0,Y)$,

$(20;\mu|2)_0 = k(1,1) \oplus k(1,0) \oplus k(0,Y)$ *and* $(20;\mu|2)_1 = k(X,0)$,

$(21;\mu)$ $k \times k \times k(\sqrt{\mu})$, $\mu \in k^*\backslash k^{*2}$:

$(21;\mu|1)_0 = k(1,1,1) \oplus k(1,0,0) \oplus k(0,1,0)$ *and* $(21;\mu|1)_1 = k(0,0,X)$,

$(21;\mu|2)_0 = k(1,1,1) \oplus k(1,1,0) \oplus k(0,0,X)$ *and* $(21;\mu|2)_1 = k(1,-1,0)$,

$(22;\xi,\mu)$ $k(\sqrt{\xi}) \times k(\sqrt{\mu}))$, ξ, $\mu \in k^*\backslash k^{*2}$:

$(22;\xi,\mu|1)_0 = k(1,1) \oplus k(1,0) \oplus k(X,0)$ *and* $(22;\xi,\mu|1)_1 = k(0,Y)$,

$(24;\mu)$ $k[X,Y]/(\mu X^2 + Y^2, XY)$, $\mu \in k^*$:

$(24;\mu|1)_0 = k1 \oplus kX \oplus kX^2$ *and* $(24;\mu|1)_1 = kY$.

(b) *Let* μ, μ_1, ξ, $\xi_1 \in k^*\backslash k^{*2}$. *Then we have*

(b.1) $(20;\mu|1) \cong (20;\mu_1|1)$ *if and only if* $\mu\mu_1^{-1} \in k^{*2}$,

(b.2) $(20;\mu|2) \cong (20;\mu_1|2)$ *if and only if* $\mu\mu_1^{-1} \in k^{*2}$,

(b.3) $(21;\mu|1) \cong (21;\mu_1|1)$ *if and only if* $\mu\mu_1^{-1} \in k^{*2}$,

(b.4) $(21;\mu|2) \cong (21;\mu_1|2)$ *if and only if* $\mu\mu_1^{-1} \in k^{*2}$,

(b.5) $(22;\xi,\mu|1) \cong (22;\xi_1,\mu_1|1)$ *if and only if* $\xi\xi_1^{-1} \in k^{*2}$ *and* $\mu\mu_1^{-1} \in k^{*2}$, (b.6) $(24;\mu|1) \cong (24;\mu_1|1)$ *if and only if* $\mu\mu_1^{-1} \in k^{*2}$.

Proof:
The proof of this shall be the main goal of this section. We shall divide the proof into three lemmas, showing the following:

- Each 4-dimensional superalgebra, A, with $\dim A_0 = 3$ and $\dim A_1 = 1$ is isomorphic to one of the superalgebras listed in the theorem. (Lemma 2.3.2)
- Each pair of distinct families of superalgebras listed in the theorem are non-isomorphic. (Lemma 2.3.5)
- The conditions for superalgebra isomorphisms to exist are as stated in part (b) of the theorem. (Lemma 2.3.6)

□

Lemma 2.3.2 *Let A be a 4-dimensional superalgebra with* $\dim A_0 = 3$ *and* $\dim A_1 = 1$. *Then A is isomorphic to one of the superalgebras listed in Theorem 2.3.1* (a).

Proof:
Suppose that $A = A_0 \oplus A_1$ is a 4-dimensional superalgebra with $\dim A_0 = 3$ and $\dim A_1 = 1$. Let $\{z\}$ be a basis for A_1.

Since A_1 is an A_0-bimodule with the actions being given by multiplication in A, we can define two maps $f, g : A_0 \to k$ by $az = f(a)z$ and $za = g(a)z$ for all $a \in A_0$. With a little work, we discover that these are in fact k-algebra homomorphisms. Consider the kernels of f and g. We have two cases:

I $\mathrm{Ker}(f) = \mathrm{Ker}(g)$,

II $\mathrm{Ker}(f) \neq \mathrm{Ker}(g)$.

Suppose case I holds and $\mathrm{Ker}(f) = \mathrm{Ker}(g)$. Since $A_0 = k1 \oplus \mathrm{Ker}(f) = k1 \oplus \mathrm{Ker}(g)$, we have $f = g$. Hence $z \in Z(A)$, the center of A. Either A_0 is semisimple, or it isn't, so we get the following possibilities.

Assume A_0 is semisimple (i.e. $J(A_0) = 0$). Then $A_0 \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r)$ by the Wedderburn-Artin structure theorem, where D_i is a division algebra with dimension d_i, $1 \leq i \leq r$ and r is a positive integer with $m \leq 3$. Hence $n_1^2 d_1 + n_2^2 d_2 + n_3^2 d_3 = 3$. Since A_0 has a nontrivial ideal $\mathrm{Ker}(f)$, A_0 is not a division algebra. Thus we have the following possibilities:

(a) $A_0 \cong k \oplus D$, where D is a 2-dimensional division algebra, and hence a quadratic extension of k;

(b) $A_0 \cong k \oplus k \oplus k$.

For case (a), there is a k-basis $\{1, e, x\}$ of A_0 with 1 being the identity of A such that $e^2 = e$, $ex = xe = x$ and $x^2 = \alpha e$ for some $\alpha \in k^* \backslash k^{*2}$. For case (b), there is a k-basis $\{e_1, e_2, e_3\}$ of A_0 such that $e_i e_j = \delta_i^j e_i$, $1 \leq i, j \leq 3$ (where δ_i^j is the Kronecker delta). Let $I_i = ke_i$, $i = 1, 2, 3$. Then I_1, I_2 and I_3 are ideals of A_0, and $A_0 = I_1 \oplus I_2 \oplus I_3$.

Now assume that A_0 is not semisimple. Then $\dim J(A_0) = 1$ or 2. Let $I = \mathrm{Ker}(f) = \mathrm{Ker}(g)$.

If $\dim J(A_0) = 2$, then $I = J(A_0)$ and $I^2 \neq I$ since $J(A_0)$ is nilpotent. Thus we have two cases:

(c) I^2 is 1-dimensional;

(d) $I^2 = \{0\}$

For case (c), we may choose $0 \neq x \in I^2$ and $y \in I \backslash I^2$. Observe that $I^n = \{0\}$ for $n \geq 3$. Hence $x^2 = xy = yx = 0$, and $y^2 = \alpha x$ with $\alpha \in k^*$ since $I^2 \neq 0$. In this case, $\{x, y\}$ is a k-basis of $I = J(A_0)$.

If $\dim J(A_0) = 1$, then we get the final possibility:

(e) We discover that $J(A_0)^2 = 0$ and $J(A_0) \subset I$. Choose $0 \neq x \in J(A_0)$ and $y \in I \backslash J(A_0)$. Observe that $\{x, y\}$ is a k-basis of I. Then $x^2 = 0$, xy, $yx \in J(A_0)$ and $y^2 = \alpha_1 y + \alpha_2 x$ for some $\alpha_1, \alpha_2 \in k$. We claim that $\alpha_1 \neq 0$. In fact, if $\alpha_1 = 0$, then $I^2 \subseteq J(A_0)$, and hence I is a nilpotent ideal of A_0. This implies $I \subseteq J(A_0)$, a contradiction. Notice that $(\alpha_1^{-1} y)^2 = \alpha_1^{-1} y + \alpha_1^{-2} \alpha_2 x$; then by replacing y with $\alpha_1^{-1} y$ we may assume that $\alpha_1 = 1$.

Suppose case II holds and $\mathrm{Ker}(f) \neq \mathrm{Ker}(g)$. We first have that $\mathrm{Ker}(f) \cap$

$\mathrm{Ker}(g)$ is 1-dimensional. Let $0 \neq x \in \mathrm{Ker}(f) \cap \mathrm{Ker}(g)$. Then one can choose an element $y \in \mathrm{Ker}(f)$ such that $\{x, y\}$ is a basis for $\mathrm{Ker}(f)$ over k. Observe that $\mathrm{span}\{1, y\} \cap \mathrm{Ker}(g)$ is 1-dimensional. Let $0 \neq y' \in \mathrm{Ker}(g) \cap \mathrm{span}\{1, y\}$ such that $\{x, y'\}$ is a basis for $\mathrm{Ker}(g)$. Now by definition $xz = yz = zx = zy' = 0$ and $zy = g(y)z$ with $g(y) \neq 0$. Replacing y by $g(y)^{-1}y$, we may assume that $g(y) = 1$ and $zy = z$. Since $\mathrm{Ker}(f) \cap \mathrm{Ker}(g)$ is an ideal of A_0 and $x \in \mathrm{Ker}(f) \cap \mathrm{Ker}(g)$, we have x^2, xy, $yx \in \mathrm{Ker}(f) \cap \mathrm{Ker}(g)$, and so $x^2 = \beta_1 x$, $xy = \beta_2 x$ and $yx = \beta_3 x$ for some $\beta_i \in k$, $i = 1,\ 2,\ 3$. Now we have $y' = \gamma_1 + \gamma_2 y$ with $\gamma_1,\ \gamma_2 \in k$. Since $1 \notin \mathrm{Ker}(g)$ and $y \notin \mathrm{Ker}(g)$, we know that $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$. Replacing y' with $\gamma_1^{-1}y'$, we may assume $\gamma_1 = 1$. Hence $0 = zy' = z + \gamma_2 zy = (1+\gamma_2)z$, and so $\gamma_2 = -1$ and $y' = 1 - y$. Since $yy' \in \mathrm{Ker}(f)\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f) \cap \mathrm{Ker}(g)$, $yy' = y(1-y) = \gamma x$ for some $\gamma \in k$. It follows that $y^2 = y - \gamma x$. Since $zy = z$ and $yz = 0$, we have $z^2 = (zy)z = z(yz) = 0$.

Now we deal with each of the above cases.

I (a): A_0 has a k-basis $\{1, e, x\}$ satisfying $e^2 = e$, $ex = xe = x$ and $x^2 = \alpha e$, where $\alpha \in k^* \backslash k^{*2}$. Moreover, we have $z \in Z(A)$, $f = g$, $ez = f(e)z$ and $xz = f(x)z$. Suppose $z^2 = \eta_1 + \eta_2 e + \eta_3 x$. Since f is an algebra homomorphism, $f(e) = f(e^2) = f(e)^2$ and $f(x)^2 = f(x^2) = \alpha f(e)$. From the equation $xz^2 = (xz)z$ one gets $f(x)\eta_1 = 0$, $f(x)\eta_2 = \eta_3\alpha$ and $f(x)\eta_3 = \eta_1 + \eta_2$. Now $f(e) = f(e)^2$ implies $f(e) = 0$ or 1. If $f(e) = 1$, then $\alpha = f(x)^2$, and hence $\alpha = 0$ or $\alpha \in k^{*2}$, a contradiction. Thus $f(e) = 0$, and so $f(x) = 0$ as $f(x)^2 = \alpha f(e)$. Now from the equations $f(x)\eta_2 = \eta_3\alpha$ and $f(x)\eta_3 = \eta_1 + \eta_2$ one obtains $\eta_3 = 0$ (as $\alpha \neq 0$) and $\eta_1 + \eta_2 = 0$. It follows that $z^2 = \eta_1 - \eta_1 e$. Either $\eta_1 = 0$, $\eta_1 \in k^{*2}$ or $\eta_1 \in k^* \backslash k^{*2}$. If $\eta_1 = 0$ then $A \cong (20; \alpha|2)$, via $e \mapsto (0, 1)$, $x \mapsto (0, Y)$, $z \mapsto (X, 0)$. If $\eta_1 = \eta^2$ for some $\eta \in k^*$ then $A \cong (21; \alpha|2)$, via $e \mapsto (0, 0, 1)$, $x \mapsto (0, 0, X)$, $z \mapsto \eta(1, -1, 0)$. If $\eta_1 \in k^* \backslash k^{*2}$ then $A \cong (22; \alpha, \eta_1|1)$, via $e \mapsto (1, 0)$, $x \mapsto (X, 0)$, $z \mapsto (0, Y)$.

I (b): A_0 has a k-basis $\{1, e_1, e_2\}$ such that $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 e_2 = e_2 e_1 = 0$. Moreover, $z \in Z(A)$ and $e_i z = f(e_i)z$, $i = 1, 2$. Suppose $z^2 = \eta_0 + \eta_1 e_1 + \eta_2 e_2$. Since f is an algebra homomorphism, $f(e_i)^2 = f(e_i)$, $i = 1, 2$.

From the equation $(e_1e_2)z = e_1(e_2z)$ one gets $f(e_1)f(e_2) = 0$. Furthermore, from the equation $e_iz^2 = (e_iz)z$, $i = 1$, 2, one gets $f(e_1)\eta_0 = f(e_2)\eta_0 = 0$, $f(e_1)\eta_1 = \eta_0 + \eta_1$, $f(e_2)\eta_2 = \eta_0 + \eta_2$ and $f(e_1)\eta_2 = f(e_2)\eta_1 = 0$. Now we consider the structure of A according to whether $\eta_i \neq 0$ or $\eta_i = 0$, $i = 1$, 2, 3.

Case 1: Suppose $\eta_0 \neq 0$. In this case, we have $f(e_1) = f(e_2) = 0$ as $f(e_1)\eta_0 = f(e_2)\eta_0 = 0$. Hence $\eta_1 = \eta_2 = -\eta_0$ since $f(e_i)\eta_i = \eta_0 + \eta_i$, $i = 1$, 2. Hence we have $z^2 = \eta_0(1 - e_1 - e_2)$ and $e_iz = ze_i = 0$, $i = 1$, 2. Either $\eta_0 \in k^{*2}$ or $\eta_0 \in k^*\backslash k^{*2}$. If $\eta_0 = \eta^2$ for some $\eta \in k^*$ then $A \cong (1|1)$, via $e_1 \mapsto (1, 0, 0, 0)$, $e_2 \mapsto (0, 1, 0, 0)$, $z \mapsto \eta(0, 0, 1, -1)$. If $\eta_0 \in k^*\backslash k^{*2}$ then $A \cong (21; \eta_0|1)$, via $e_1 \mapsto (1, 0, 0)$, $e_2 \mapsto (0, 1, 0)$, $z \mapsto (0, 0, X)$.

Now suppose $\eta_0 = 0$, then we have

$$f(e_1)f(e_2) = 0, \;\; f(e_1)\eta_2 = f(e_2)\eta_1 = 0,$$

$$f(e_i)^2 = f(e_i), \;\; f(e_i)\eta_i = \eta_i, \;\; i = 1, \; 2.$$

Therefore, we get the following three cases depending on whether either of η_1 or η_2 are 0.

Case 2: $\eta_0 = 0$ and $\eta_1 \neq 0$. In this case, we have $f(e_1) = 1$, $f(e_2) = 0$ and $\eta_2 = 0$. Hence $z^2 = \eta_1e_1$, $e_1z = ze_1 = z$ and $e_2z = ze_2 = 0$. Let $e'_1 = 1 - e_1 - e_2$ and $e'_2 = e_2$. By considering the new basis $\{1, e'_1, e'_2\}$ of A_0 over k, one can see from the proof of Case 1 that $A \cong (1|1)$ if $\eta_1 \in k^{*2}$, and $A \cong (21; \eta_1|1)$ if $\eta_1 \in k^*\backslash k^{*2}$.

Case 3: $\eta_0 = 0$ and $\eta_2 \neq 0$. This is treated similarly to Case 2. (Simply interchange e_1 and e_2 in Case 2). Thus we can see that $A \cong (1|1)$ if $\eta_2 \in k^{*2}$, and $A \cong (21; \eta_2|1)$ if $\eta_2 \in k^*\backslash k^{*2}$.

Case 4: $\eta_0 = \eta_1 = \eta_2 = 0$. If $f(e_1) = f(e_2) = 0$, then we have $z^2 = 0$ and $e_iz = ze_i = 0$, $i = 1$, 2. Hence $A \cong (2|1)$, via $e_1 \mapsto (1, 0, 0)$, $e_2 \mapsto (0, 1, 0)$, $z \mapsto (0, 0, X)$. If $f(e_1) = 1$ and $f(e_2) = 0$, then by replacing e_1 with $1-e_1-e_2$ one can see that $A \cong (2|1)$ as superalgebras. Similarly, if $f(e_1) = 0$ and $f(e_2) = 1$, then we also have $A \cong (2|1)$.

I (c): A_0 has a k-basis $\{1, x, y\}$ such that $x^2 = xy = yx = 0$ and $y^2 = \alpha x$ with $\alpha \in k^*$. In this case, we have $z \in Z(A)$, $\mathrm{Ker}(f) = J(A_0)$ and hence $xz = yz = 0$. Since $J(A_0)$ is a unique maximal ideal of A_0 and A_1^2 is an ideal of A_0 with $\dim A_1^2 \leq 1$, $A_1^2 \subseteq J(A_0)$. Hence we have $z^2 = \beta_1 x + \beta_2 y$ for some β_1, $\beta_2 \in k$. Then from the equation $yz^2 = (yz)z$ one gets $\beta_2 = 0$ (as $\alpha \neq 0$). Hence, $z^2 = \beta_1 x$. Either $\beta_1 = 0$, $-\beta_1\alpha^{-1} \in k^{*2}$ or $-\beta_1\alpha^{-1} \in k^*\backslash k^{*2}$. If $\beta_1 = 0$ then $A \cong (8|1)$, via $x \mapsto \alpha^{-1}X^2$, $y \mapsto X$, $z \mapsto Y$. If $\beta_1 \neq 0$ and $-\beta_1\alpha^{-1} = \gamma^2$ for some $\gamma \in k^*$, then $A \cong (7|1)$, via $x \mapsto 2\alpha^{-1}XY$, $y \mapsto X+Y$, $z \mapsto \gamma(X - Y)$. If $\beta_1 \neq 0$ and $-\beta_1\alpha^{-1} \in k^*\backslash k^{*2}$ then $A \cong (24; -\beta_1\alpha^{-1}|1)$, via $x \mapsto \alpha^{-1}X^2$, $y \mapsto X$, $z \mapsto Y$.

I (d): A_0 has a k-basis $\{1, x, y\}$ such that $x^2 = y^2 = xy = yx = 0$. In this case, we have $z \in Z(A)$ and $J(A_0) = kx + ky = \mathrm{Ker}(f)$ and so $xz = yz = 0$. By the same reason as in 1 (c) we have $z^2 = \beta_1 x + \beta_2 y$ for some β_1, $\beta_2 \in k$. Either $\beta_1 = \beta_2 = 0$, $\beta_1 \neq 0$ or $\beta_2 \neq 0$. If $\beta_1 = \beta_2 = 0$ then $A \cong (9|1)$, via $x \mapsto X$, $y \mapsto Y$, $z \mapsto Z$. If $\beta_1 \neq 0$, then $A \cong (8|2)$, via $x \mapsto \beta_1^{-1}(X^2 - \beta_2 Y)$, $y \mapsto Y$, $z \mapsto X$. Similarly, if $\beta_2 \neq 0$ then $A \cong (8|2)$, too.

I (e): A_0 has a k-basis $\{1, x, y\}$ such that $x^2 = 0$, $y^2 = y + \alpha x$, $xy = \beta x$ and $yx = \gamma x$ for some α, β, $\gamma \in k$. In this case, we have $z \in Z(A)$ and $xz = yz = 0$. Suppose $z^2 = \delta_0 + \delta_1 x + \delta_2 y$. Then from the equations $xy^2 = (xy)y$ and $y^2 x = y(yx)$ one gets $\beta^2 = \beta$ and $\gamma^2 = \gamma$. From the equation $y^2 y = yy^2$ one gets $\alpha\beta = \alpha\gamma$. Similarly, from the equations $z^2 x = xz^2, z^2 y = yz^2, xz^2 = (xz)z, yz^2 = (yz)z$ one obtains $\delta_2\gamma = \delta_2\beta$, $\delta_1\beta = \delta_1\gamma$, $\delta_0 + \delta_2\beta = 0$, $\delta_0 + \delta_2 = 0$ and $\delta_2\alpha + \delta_1\gamma = 0$. Hence

$$\delta_2 = -\delta_0,\ \delta_0\alpha = \delta_1\gamma,\ \alpha\beta = \alpha\gamma,$$

and so

$$\beta^2 = \beta,\ \gamma^2 = \gamma,\ \alpha\beta = \alpha\gamma,\ \delta_2 = -\delta_1,\ \delta_0\beta = \delta_0\gamma,\ \delta_1\beta = \delta_1\gamma,\ \delta_0 = \delta_0\beta,\ \delta_0\alpha = \delta_1\gamma.$$

Thus we get 4 cases from this, listed 1–4 in the following, depending on whether β is 0 or 1 and whether γ is 0 or 1.

Case 1: $\beta = \gamma = 0$. In this case, we have $\delta_0 = \delta_2 = 0$. Hence $y^2 = y + \alpha x$, $xy = yx = 0$ and $z^2 = \delta_1 x$. Notice $(y + \alpha x)^2 = y + \alpha x$. Then by replacing

y with $y+\alpha x$, we may assume that $\alpha = 0$, or equivalently, $y^2 = y$. Either $\delta_1 = 0$ or $\delta_1 \neq 0$. If $\delta_1 = 0$ then $A \cong (6|1)$, via $x \mapsto (0, X)$, $y \mapsto (1, 0)$, $z \mapsto (0, Y)$. If $\delta_1 \neq 0$ then $A \cong (4|1)$, via $x \mapsto \delta_1^{-1}(0, X^2)$, $y \mapsto (1, 0)$, $z \mapsto (0, X)$.

Case 2: $\beta = 0$ and $\gamma = 1$. In this case, we have $\delta_0 = \delta_1 = \delta_2 = \alpha = 0$. Hence $y^2 = y$, $xy = 0$, $yx = x$ and $z^2 = 0$. It follows that $A \cong (14|1)$, via $x \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $z \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Case 3: $\beta = 1$ and $\gamma = 0$. In this case, we have $\delta_0 = \delta_1 = \delta_2 = \alpha = 0$. Hence $y^2 = y$, $xy = x$, $yx = 0$ and $z^2 = 0$. It follows that $A \cong (15|1)$, via $x \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $z \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Case 4: $\beta = \gamma = 1$. In this case, we have $\delta_1 = \delta_0\alpha$ and $\delta_2 = -\delta_0$. Hence $xy = yx = x$, $y^2 = \alpha x + y$ and $z^2 = \delta_0 + \delta_0\alpha x - \delta_0 y = \delta_0(1 - (y - \alpha x))$. Let $y_1 = y - \alpha x$. Then $y_1^2 = y_1$, $xy_1 = y_1 x = x$ and $z^2 = \delta_0(1 - y_1)$. Note that $\{1, x, y_1\}$ is a k-basis of A_0. Either $\delta_0 = 0$, $\delta_0 \in k^{*2}$ or $\delta_0 \in k^*\backslash k^{*2}$. If $\delta_0 = 0$ then $A \cong (3|1)$, via $x \mapsto (X, 0)$, $y_1 \mapsto (1, 0)$, $z \mapsto (0, Y)$. If $\delta_0 = \delta^2$ for some $\delta \in k^*$, then $A \cong (2|2)$, via $x \mapsto (0, 0, X)$, $y_1 \mapsto (0, 0, 1)$, $z \mapsto \delta(1, -1, 0)$. If $\delta_0 \in k^*\backslash k^{*2}$, then $A \cong (20; \delta_0|1)$, via $x \mapsto (X, 0)$, $y_1 \mapsto (1, 0)$, $z \mapsto (0, Y)$.

II: $\mathrm{Ker}(f) \neq \mathrm{Ker}(g)$. In this case, A_0 has a k-basis $\{1, x, y\}$ such that $x^2 = \beta_1 x$, $xy = \beta_2 x$, $yx = \beta_3 x$, $y^2 = y - \gamma x$, $xz = zx = yz = 0$, $zy = z$ and $z^2 = 0$, where $\beta_i, \gamma \in k$, $i = 1,\ 2,\ 3$. From the equations $y^2 y = yy^2$ and $x(yx) = (xy)x$ one gets $\gamma\beta_2 = \gamma\beta_3$ and $\beta_1\beta_2 = \beta_1\beta_3$. Similarly, from the equations $y(yx) = y^2 x$ and $(xy)y = xy^2$ one gets $\beta_3^2 = \beta_3 - \gamma\beta_1$ and $\beta_2^2 = \beta_2 - \gamma\beta_1$. Either $\gamma = 0$ or $\gamma \neq 0$. If $\gamma = 0$, then $\beta_1(\beta_2 - \beta_3) = 0$, $\beta_2^2 = \beta_2$ and $\beta_3^2 = \beta_3$ and so $\beta_2, \beta_3 \in \{0, 1\}$. If $\gamma \neq 0$, then $\beta_2 = \beta_3$ and $\beta_1 = \gamma^{-1}\beta_3(1 - \beta_3)$. We get 5 cases from this, listed 1–5 in the following:

Case 1: $\gamma = 0$, $\beta_2 = 0$ and $\beta_3 = 1$. In this case, we have $\beta_1 = 0$. Hence, $x^2 = 0$, $xy = 0$, $yx = x$, $y^2 = y$, $xz = zx = yz = 0$, $zy = z$ and $z^2 = 0$. Thus $A \cong (11|1)$, via $x \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $z \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Case 2: $\gamma = 0$, $\beta_2 = 1$ and $\beta_3 = 0$. In this case, we have $\beta_1 = 0$. Hence, $x^2 = 0$, $xy = x$, $yx = 0$, $y^2 = y$, $xz = zx = yz = 0$, $zy = z$ and $z^2 = 0$. Thus $A \cong (17|1)$, via $x \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $z \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Case 3: $\gamma = 0$ and $\beta_2 = \beta_3 = 0$. In this case, we have $x^2 = \beta_1 x$, $xy = yx = 0$, $y^2 = y$, $xz = zx = yz = 0$, $zy = z$ and $z^2 = 0$. Either $\beta_1 = 0$ or $\beta_1 \neq 0$. If $\beta_1 = 0$ then $x^2 = 0$, and $A \cong (15|2)$, via $x \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $z \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. If $\beta_1 \neq 0$ then $A \cong (13|1)$, via $x \mapsto \left(\beta_1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \right)$, $y \mapsto \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \right)$, $z \mapsto \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$.

Case 4: $\gamma = 0$ and $\beta_2 = \beta_3 = 1$. In this case, we have $x^2 = \beta_1 x$, $xy = yx = x$, $y^2 = y$, $xz = zx = yz = 0$, $zy = z$ and $z^2 = 0$. Either $\beta_1 = 0$ or $\beta_1 \neq 0$. If $\beta_1 = 0$ then $A \cong (14|2)$, via $x \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $z \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. If $\beta_1 \neq 0$, then $(y - \beta_1^{-1}x)^2 = y - \beta_1^{-1}x$, $x(y - \beta_1^{-1}x) = (y - \beta_1^{-1}x)x = 0$, $(y - \beta_1^{-1}x)z = 0$ and $z(y - \beta_1^{-1}x) = z$. Replacing y with $y - \beta_1^{-1}x$, one can see from Case 3 that $A \cong (13|1)$.

Case 5: $\gamma \neq 0$. In this case, we have $x^2 = \gamma^{-1}\beta_3(1-\beta_3)x$, $xy = yx = \beta_3 x$, $y^2 = y - \gamma x$, $xz = zx = yz = 0$, $zy = z$ and $z^2 = 0$. Either $\beta_3 = 0$, $\beta_3 = 1$ or $\beta_3 \neq 0, 1$. If $\beta_3 = 0$, then $x^2 = 0$, $x(y - \gamma x) = (y - \gamma x)x = 0$, $(y - \gamma x)^2 = y - \gamma x$, $(y - \gamma x)z = 0$ and $z(y - \gamma x) = z$. Replacing y with $y - \gamma x$, it follows from Case 3 that $A \cong (15|2)$. If $\beta_3 = 1$, then $x^2 = 0$ and $x(y + \gamma x) = (y + \gamma x) = x$, $(y + \gamma x)^2 = y + \gamma x$, $(y + \gamma x)z = 0$, $z(y + \gamma x) = z$. Replacing y with $y + \gamma x$, it follows from Case 4 that $A \cong (14|2)$. Now assume $\beta_3 \neq 0$ and $\beta_3 \neq 1$. Let $y_1 = y - \gamma(1-\beta_3)^{-1}x$. Then a straightforward verification shows that $y_1^2 = y_1$, $xy_1 = y_1 x = 0$, $y_1 z = 0$ and $zy_1 = z$. It follows from Case 3 that $A \cong (13|1)$.

This completes the proof. □

For the next two short results, which will be used in the proof of

Lemma 2.3.5, suppose that A and B are superalgebras.

Lemma 2.3.3 *If $A_1^2 \neq \{0\}$ and $B_1^2 = \{0\}$, then $A \ncong B$.*

Proof:
Suppose that $\phi : A \to B$ is a superalgebra map. Take $y, z \in A_1$ such that $yz \neq 0$. Then $\phi(yz) = \phi(y)\phi(z) = 0$ since $\phi(y), \phi(z) \in B_1 \Rightarrow 0 \neq yz \in \mathrm{Ker}\phi$. Hence ϕ cannot be one-to-one, hence cannot be an isomorphism. □

Lemma 2.3.4 *Consider a superalgebra map $\phi : A \to B$. If there exists $x \in A_0$ such that either:*

(a) $xA_1 \neq \{0\}$, but $\phi(x)B_1 = \{0\}$, or

(b) $A_1x \neq \{0\}$, but $B_1\phi(x) = \{0\}$

then ϕ isn't one-to-one, and in particular can't be an isomorphism.

Proof:
We prove (a) here. (b) is proved similarly.

Take $y \in A_1$ such that $xy \neq 0$ then $\phi(xy) = \phi(x)\phi(y) = 0$ as $\phi(y) \in B_1$ $\Rightarrow 0 \neq xy \in \mathrm{Ker}\phi$. Hence ϕ cannot be one-to-one, hence cannot be an isomorphism. □

Lemma 2.3.5 *Each pair of distinct families of superalgebras listed in Theorem 2.3.1 are non-isomorphic.*

Proof:
By Proposition 2.2.1, Proposition 2.2.2 and Proposition 2.2.3 we simply need to show that different $\mathbb{Z}_2$-gradings on the same underlying algebra are non-isomorphic.

From Lemma 2.3.3 we have that $(2|1) \ncong (2|2)$, $(8|1) \ncong (8|2)$ and $(20|1) \ncong (20|2)$.

Now we consider the superalgebras $(14|1)$ and $(14|2)$. Observe that $(14|2)$ has a k-basis $\{1, x, y, z\}$ with $(14|2)_0 = k1 \oplus kx \oplus ky$ and $(14|2)_1 = kz$

such that $x^2 = 0$, $xy = yx = x$, $y^2 = y$, $xz = zx = yz = 0$, $zy = z$ and $z^2 = 0$. $(14|1)$ has a k-basis $\{1, x_1, y_1, z_1\}$ with $(14|1)_0 = k1 \oplus kx_1 \oplus ky_1$ and $(14|1)_1 = kz_1$ such that $x_1^2 = 0$, $x_1y_1 = 0$, $y_1x_1 = x_1$, $y_1^2 = y_1$, $x_1z_1 = z_1x_1 = y_1z_1 = z_1y_1 = 0$ and $z_1^2 = 0$.

Suppose, contrary to what we wish to show, that $(14|2) \cong (14|1)$ and $\phi : (14|2) \to (14|1)$ is an isomorphism. Then $\phi(1) = 1$, $\phi(x) = \alpha_0 + \alpha_1x_1 + \alpha_2y_1$, $\phi(y) = \beta_0 + \beta_1x_1 + \beta_2y_1$ and $\phi(z) = \gamma z_1$ for some α_i, β_i, $\gamma \in k$ with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ and $\gamma \neq 0$. Thus we have $0 = \phi(yz) = \phi(y)\phi(z) = (\beta_0 + \beta_1x_1 + \beta_2y_1)\gamma z_1 = \beta_0\gamma z_1$, which implies $\beta_0 = 0$ as $\gamma \neq 0$. Thus $zy = z \neq 0$ yet $z_1y_1 = z_1x_1 = 0$, hence $(14|1)_1\phi(y) = \{0\}$, which would contradict Lemma 2.3.4 — impossible. Thus $(14|2) \ncong (14|1)$.

The proof of $(15|1) \ncong (15|2)$ is similar to the above argument showing $(14|2) \ncong (14|1)$.

Next, we consider $(21; \mu|2)$ and $(21; \mu_1|1)$ with μ, $\mu_1 \in k^*\backslash k^{*2}$. Let $x = (0, 0, 1)$, $y = (0, 0, X)$ and $z = (1, -1, 0)$ in $(21; \mu|2)$. Then $\{1, x, y, z\}$ is a basis for $(21; \mu|2)$ over k such that $(21; \mu|2)_0 = k1 \oplus kx \oplus ky$ and $(21; \mu|2)_1 = kz$. Let $x_1 = (1, 0, 0)$, $y_1 = (0, 1, 0)$ and $z_1 = (0, 0, X)$ in $(21; \mu_1|1)$. Then $\{1, x_1, y_1, z_1\}$ a basis for $(21; \mu_1|1)$ over k such that $(21; \mu|1)_0 = k1 \oplus kx_1 \oplus ky_1$ and $(21; \mu|1)_1 = kz_1$.

Suppose, contrary to what we wish to show, that $(21; \mu|2) \cong (21; \mu_1|1)$ and $\phi : (21; \mu|2) \to (21; \mu_1|1)$ is an isomorphism. Then $\phi(1) = 1$, $\phi(x) = \alpha_0 + \alpha_1x_1 + \alpha_2y_1$, $\phi(y) = \beta_0 + \beta_1x_1 + \beta_2y_1$ and $\phi(z) = \gamma z_1$ for some α_i, β_i, $\gamma \in k$ with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ and $\gamma \neq 0$. From the equations $\phi(xz) = \phi(x)\phi(z)$ and $\phi(yz) = \phi(y)\phi(z)$ one gets $\alpha_0 = \beta_0 = 0$. Hence $\phi(x) = \alpha_1x_1 + \alpha_2y_1$ and $\phi(y) = \beta_1x_1 + \beta_2y_1$. Now since $\alpha_1x_1 + \alpha_2y_1 = \phi(x) = \phi(x^2) = \phi(x)^2 = (\alpha_1x_1 + \alpha_2y_1)^2 = \alpha_1^2x_1 + \alpha_2^2y_1$, we have $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$, and hence $\alpha_i = 0$ or 1, $i = 1, 2$. Similarly, from the equation $\phi(y)^2 = \phi(y^2) = \phi(\mu x)$ one gets $\mu\alpha_1 = \beta_1^2$ and $\mu\alpha_2 = \beta_2^2$. Since ϕ is an isomorphism, at least one of α_1, α_2 is non-zero. If $\alpha_1 \neq 0$ then $\alpha_1 = 1$, and so $\mu = \beta_1^2$, which is impossible as $\mu \in k^*\backslash k^{*2}$. Similarly, if $\alpha_2 \neq 0$ then $\alpha_2 = 1$, and so $\mu = \beta_2^2$, which is impossible too. Thus we have proved $(21; \mu|2) \ncong (21; \mu_1|1)$.

This completes the proof. □

Lemma 2.3.6 *The conditions given in part* (b) *of Theorem 2.3.1 for two superalgebras from the same family to be isomorphic are as stated there.*

Proof:
Observe that $(20; \mu|1)$ has a k-basis $\{1, e, x, y\}$ with $(20; \mu|1)_0 = k1 \oplus ke \oplus kx$ and $(20; \mu|1)_1 = ky$ such that $e^2 = e$, $ex = xe = x$, $x^2 = 0$, $ey = ye = xy = yx = 0$ and $y^2 = \mu(1 - e)$. Similarly, $(20; \mu_1|1)$ has a k-basis $\{1, e_1, x_1, y_1\}$ with $(20; \mu_1|1)_0 = k1 \oplus ke_1 \oplus kx_1$ and $(20; \mu_1|1)_1 = ky_1$ such that $e_1^2 = e_1$, $e_1x_1 = x_1e_1 = x_1$, $x_1^2 = 0$, $e_1y_1 = y_1e_1 = x_1y_1 = y_1x_1 = 0$ and $y_1^2 = \mu_1(1 - e_1)$. If $\mu\mu_1^{-1} \in k^{*2}$ then $\mu = \delta^2\mu_1$ for some $\delta \in k^*$. Define a k-linear isomorphism $f : (20; \mu|1) \to (20; \mu_1|1)$ by $f(1) = 1$, $f(e) = e_1$, $f(x) = x_1$ and $f(y) = \delta y_1$. Then it is straightforward to check that f is a superalgebra isomorphism. Conversely, if $(20; \mu|1) \cong (20; \mu_1|1)$ as superalgebras, then $(20; \mu|1) \cong (20; \mu_1|1)$ as ungraded algebras, i.e., $(20; \mu) \cong (20; \mu_1)$. Now it follows from Lemma 2.2.5 and Lemma 2.2.6 that $\mu\mu_1^{-1} \in k^{*2}$. Thus we have proved Part (b.1).

Similarly, one can show Parts (b.2), (b.3) and (b.4).

Now we show Part (b.5). Clearly, $(22; \xi, \mu|1)$ has a k-basis $\{1, e, x, y\}$ with $(22; \xi, \mu|1)_0 = k1 \oplus ke \oplus kx$ and $(22; \xi, \mu|1)_1 = ky$ such that $e^2 = e$, $ex = xe = x$, $x^2 = \xi e$, $ey = ye = xy = yx = 0$ and $y^2 = \mu(1 - e)$. Similarly, $(22; \xi_1, \mu_1|1)$ has a k-basis $\{1, e_1, x_1, y_1\}$ with $(22; \xi_1, \mu_1|1)_0 = k1 \oplus ke_1 \oplus kx_1$ and $(22; \xi_1, \mu_1|1)_1 = ky_1$ such that $e_1^2 = e_1$, $e_1x_1 = x_1e_1 = x_1$, $x_1^2 = \xi_1e_1$, $e_1y_1 = y_1e_1 = x_1y_1 = y_1x_1 = 0$ and $y_1^2 = \mu_1(1 - e_1)$. If $\xi = \delta^2\xi_1$ and $\mu = \gamma^2\mu_1$ for some δ, $\gamma \in k^*$, then there is a superalgebra isomorphism $f : (22; \xi, \mu|1) \to (22; \xi_1, \mu_1|1)$ given by $f(e) = e_1$, $f(x) = \delta x_1$ and $f(y) = \gamma y_1$. Conversely, if f is a superalgebra isomorphism from $(22; \xi, \mu|1)$ to $(22; \xi_1, \mu_1|1)$, then $f((22; \xi, \mu|1)_1) = (22; \xi_1, \mu_1|1)_1$. Hence there is a $\gamma \in k^*$ such that $f(y) = \gamma y_1$. Since $y^2 = \mu(1 - e)$ and $y_1^2 = \mu_1(1 - e_1)$, we have $\mu(1 - f(e)) = f(y^2) = f(y)^2 = \gamma^2 y_1^2 = \gamma^2\mu_1(1 - e_1)$. Since both $1 - f(e)$ and $1 - e_1$ are non-trivial idempotents in $(22; \xi_1, \mu_1|1)_0$, we have $\mu = \gamma^2\mu_1$ and

$1 - f(e) = 1 - e_1$, and hence $f(e) = e_1$. Since $ex = x$ and $x \in (22; \xi, \mu|1)_0$, $f(x) \in (22; \xi_1, \mu_1|1)_0$ and $e_1 f(x) = f(x)$. It follows that $f(x) = \beta e_1 + \delta x_1$ for some β, $\delta \in k$ with $\delta \neq 0$ since f is an isomorphism and $f(e) = e_1$. Since $x^2 = \xi e$, $\xi e_1 = f(x^2) = f(x)^2 = (\beta e_1 + \delta x_1)^2 = (\beta^2 + \delta^2 \xi_1) e_1 + 2\beta\delta x_1$. This implies $\beta\delta = 0$ and $\xi = \beta^2 + \delta^2\xi_1$, and hence $\beta = 0$ and $\xi = \delta^2 \xi_1$ as $\delta \neq 0$.

Finally, we show Part (b.6). We use X and Y to denote the generators of $(24; \mu|1)$, and use X_1 and Y_1 to denote the generators of $(24; \mu_1|1)$. Note that $X_1^3 = X_1 X_1^2 = -\mu_1^{-1} X_1 Y_1^2 = 0$. If $\mu = \delta^2 \mu_1$ for some $\delta \in k^*$, then there is a superalgebra isomorphism $f : (24; \mu|1) \to (24; \mu_1|1)$ given by $f(X) = X_1$ and $f(Y) = \delta Y_1$. Conversely, if $(24; \mu|1) \cong (24; \mu_1|1)$, suppose $f : (24; \mu|1) \to (24; \mu_1|1)$ is a superalgebra isomorphism, then we have $f(X) = \alpha + \beta X_1 + \gamma X_1^2$ and $f(Y) = \delta Y_1$ for some $\alpha,\ \beta,\ \gamma \in k$ and $\delta \in k^*$ with $\beta \neq 0$ or $\gamma \neq 0$. From the equations $XY = 0$ and $X_1 Y_1 = 0$ one gets $\alpha = 0$ as $\delta \neq 0$. Since $Y^2 = -\mu X^2$ and $Y_1^2 = -\mu_1 X_1^2$, we have $-\delta^2 \mu_1 X_1^2 = \delta^2 Y_1^2 = f(Y)^2 = f(Y^2) = f(-\mu X^2) = -\mu f(X)^2 = -\mu(\beta X_1 + \gamma X_1^2)^2 = -\mu\beta^2 X_1^2$. This implies $\delta^2 \mu_1 = \beta^2 \mu$, and hence $\beta \neq 0$ as $\delta, \mu, \mu_1 \neq 0$. It follows that $\mu\mu_1^{-1} \in k^{*2}$. □

This completes the proof of Theorem 2.3.1. To conclude this section we mention that, when k is algebraically closed, the superalgebras $(20; \mu|1)$–$(24; \mu|1)$ listed in Theorem 2.3.1 can never arise.

2.4 Case $\dim_0 = 2$

In this section we complete the classification of the last case of non-trivially $\mathbb{Z}_2$-graded superalgebras of dimension 4.

Theorem 2.4.1 *Let k be a field with $\mathrm{ch}(k) \neq 2$.*
(a) *Suppose A is a superalgebra with $\dim A_0 = \dim A_1 = 2$. Then A is isomorphic to one of the following pairwise non-isomorphic families of superalgebras:*

(1) $k \times k \times k \times k$:
$(1|2)_0 = k(1,1,1,1) \oplus k(1,1,0,0)$ *and* $(1|2)_1 = k(1,-1,0,0) \oplus k(0,0,1,-1)$,

(2) $k \times k \times k[X]/(X^2)$:
$(2|3)_0 = k(1,1,1) \oplus k(1,1,0)$ *and* $(2|3)_1 = k(1,-1,0) \oplus k(0,0,X)$,

(3) $k[X]/(X^2) \times k[Y]/(Y^2)$:
$(3|2)_0 = k(1,1) \oplus k(1,0)$ *and* $(3|2)_1 = k(X,0) \oplus k(0,Y)$,
$(3|3)_0 = k(1,1) \oplus k(X,Y)$ *and* $(3|3)_1 = k(1,-1) \oplus k(X,-Y)$,

(5) $k[X]/(X^4)$:
$(5|1)_0 = k1 \oplus kX^2$ *and* $(5|1)_1 = kX \oplus kX^3$,

(6) $k \times k[X,Y]/(X,Y)^2$:
$(6|2)_0 = k(1,1) \oplus k(1,0)$ *and* $(6|2)_1 = k(0,X) \oplus k(0,Y)$,

(7) $k[X,Y]/(X^2,Y^2)$:
$(7|2)_0 = k1 \oplus kX$ *and* $(7|2)_1 = kY \oplus kXY$,

(9) $k[X,Y,Z]/(X,Y,Z)^2$,
$(9|2)_0 = k1 \oplus kX$ *and* $(9|2)_1 = kY \oplus kZ$,

(10) M_2 :
$(10|1)_0 = k\begin{pmatrix}1&0\\0&1\end{pmatrix} \oplus k\begin{pmatrix}1&0\\0&0\end{pmatrix}$ *and* $(10|1)_1 = k\begin{pmatrix}0&1\\0&0\end{pmatrix} \oplus k\begin{pmatrix}0&0\\1&0\end{pmatrix}$,

(11) $\left\{ \begin{pmatrix} a&0&0&0\\0&a&0&d\\c&0&b&0\\0&0&0&b \end{pmatrix} \middle| \, a,b,c,d \in k \right\}$:

$$(11|2)_0 = k\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \oplus k\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\textit{and } (11|2)_1 = k\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix} \oplus k\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},$$

$$(11|3)_0 = k\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \oplus k\begin{pmatrix}0&0&0&0\\0&0&0&1\\1&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\textit{and } (11|3)_1 = k\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} \oplus k\begin{pmatrix}0&0&0&0\\0&0&0&-1\\1&0&0&0\\0&0&0&0\end{pmatrix},$$

(12) $\wedge k^2 \cong k\langle X, Y\rangle/(X^2, Y^2, XY + YX)$:

$(12|1)_0 = k1 \oplus kX$ *and* $(12|1)_1 = kY \oplus kXY$,

(14) $$\left\{ \begin{pmatrix}a&0&0\\c&a&0\\d&0&b\end{pmatrix} \middle| \, a, b, c, d \in k \right\} :$$

$$(14|3)_0 = k\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \oplus k\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

$$\textit{and } (14|3)_1 = k\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} \oplus k\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},$$

(15) $$\left\{ \begin{pmatrix}a&c&d\\0&a&0\\0&0&b\end{pmatrix} \middle| \, a, b, c, d \in k \right\} :$$

$$(15|3)_0 = k\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \oplus k\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

$$\textit{and } (15|3)_1 = k\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} \oplus k\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},$$

(16) $k\langle X, Y\rangle/(X^2, Y^2, YX)$:

$(16|1)_0 = k1 \oplus kX$ *and* $(16|1)_1 = kY \oplus kXY$,

$(16|2)_0 = k1 \oplus kY$ *and* $(16|2)_1 = kX \oplus kXY$,

(17) $$\left\{ \begin{pmatrix}a&0&0\\0&a&0\\c&d&b\end{pmatrix} \middle| \, a, b, c, d \in k \right\} :$$

$$(17|2)_0 = k\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \oplus k\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

$$\textit{and } (17|2)_1 = k\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix} \oplus k\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix},$$

$(18;\lambda)$ $\quad k\langle X,Y\rangle/(X^2,Y^2,YX-\lambda XY)$, *where* $\lambda\in k$ *with* $\lambda\neq -1,0,1$:
$(18;\lambda|1)_0 = k1\oplus kX$ *and* $(18;\lambda|1)_1 = kY\oplus kXY$,

$(20;\mu)$ $\quad k[X]/(X^2)\times k(\sqrt{\mu})$, *where* $\mu\in k^*\backslash k^{*2}$,
$(20;\mu|3)_0 = k(1,1)\oplus k(1,0)$ *and* $(20;\mu|3)_1 = k(X,0)\oplus k(0,Y)$,

$(21;\mu)$ $\quad k\times k\times k(\sqrt{\mu})$, *where* $\mu\in k^*\backslash k^{*2}$:
$(21;\mu|3)_0 = k(1,1,1)\oplus k(1,1,0)$ *and* $(21;\mu|3)_1 = k(1,-1,0)\oplus k(0,0,X)$,

$(22;\xi,\mu)$ $\quad k(\sqrt{\xi})\times k(\sqrt{\mu}))$, *where* $\xi,\ \mu\in k^*\backslash k^{*2}$:
$(22;\xi,\mu|2)_0 = k(1,1)\oplus k(1,0)$ *and* $(22;\xi,\mu|2)_1 = k(X,0)\oplus k(0,Y)$,

$(23;\mu)$ $\quad k[X,Y]/(X^2-\mu,Y^2)$, *where* $\mu\in k^*\backslash k^{*2}$:
$(23;\mu|1)_0 = k1\oplus kX$ *and* $(23;\mu|1)_1 = kY\oplus kXY$,
$(23;\mu|2)_0 = k1\oplus kY$ *and* $(23;\mu|2)_1 = kX\oplus kXY$,

$(25;\mu)$ $\quad k\langle X,Y\rangle/(X^2-\mu,Y^2,XY+YX)$, *where* $\mu\in k^*\backslash k^{*2}$:
$(25;\mu|2)_0 = k1\oplus kX$ *and* $(25;\mu|2)_1 = kY\oplus kXY$,
$(25;\mu|3)_0 = k1\oplus kY$ *and* $(25;\mu|3)_1 = kX\oplus kXY$,

$(26;\mu,\theta,\eta)$ $k[X,Y]/(X^2-\mu,Y^2-\theta-\eta X)$, *where* $\mu\in k^*\backslash k^{*2}$ *and* $\theta,\eta\in k$ *with* $\theta\neq 0$ *or* $\eta\neq 0$:
$(26;\mu,\theta,\eta|1)_0 = k1\oplus kX$ *and* $(26;\mu,\theta,\eta|1)_1 = kY\oplus kXY$,

$(27;\mu,\theta)$ $\quad k\langle X,Y\rangle/(X^2-\mu,XY+YX,Y^2-\theta)$, *where* $\mu\in k^*\backslash k^{*2}$ *and* $\theta\in k^*$:
$(27;\mu,\theta|1)_0 = k1\oplus kX$ *and* $(27;\mu,\theta|1)_1 = kY\oplus kXY$,

$(28;\theta,\eta,\lambda,\kappa)$
$k\langle X,Y,Z\rangle/(X^2,XY,XZ,YX,ZX,Y^2-\theta X,YZ-\eta X,ZY-\lambda X,Z^2-\kappa X)$,
where $\theta,\eta,\lambda,\kappa\in k$ *with at least one of them* $\neq 0$:
$(28;\theta,\eta,\lambda,\kappa|1)_0 = k1\oplus kX$ *and* $(28;\theta,\eta,\lambda,\kappa|1)_1 = kY\oplus kZ$.

(b) *Moreover, we have*

(b.1) $(18;\lambda|1)\cong(18;\lambda_1|1)$ *if and only if* $\lambda=\lambda_1$,

(b.2) $(20;\mu|3)\cong(20;\mu_1|3)$ *if and only if* $\mu\mu_1^{-1}\in k^{*2}$,

(b.3) $(21;\mu|3)\cong(21;\mu_1|3)$ *if and only if* $\mu\mu_1^{-1}\in k^{*2}$,

(b.4) $(22;\xi,\mu|2)\cong(22;\xi',\mu'|2)$ *if and only if* $\xi\xi_1^{-1}$, $\mu\mu_1^{-1}\in k^{*2}$, *or* $\mu\xi_1^{-1}$, $\xi\mu_1^{-1}\in k^{*2}$,

(b.5) $(23;\mu|1)\cong(23;\mu_1|1)$ *if and only if* $\mu\mu_1^{-1}\in k^{*2}$,

(b.6) $(23;\mu|2) \cong (23;\mu_1|2)$ *if and only if* $\mu\mu_1^{-1} \in k^{*2}$,
(b.7) $(25;\mu|2) \cong (25;\mu_1|2)$ *if and only if* $\mu\mu_1^{-1} \in k^{*2}$,
(b.8) $(25;\mu|3) \cong (25;\mu_1|3)$ *if and only if* $\mu\mu_1^{-1} \in k^{*2}$,
(b.9) $(26;\mu,\theta,\eta|2) \cong (26;\mu_1,\theta_1,\eta_1|2)$ *if and only if there exist* $\beta \in k^*$ *and* $\gamma,\delta \in k$ *with* $\gamma \neq 0$ *or* $\delta \neq 0$ *such that*

$$\begin{aligned} \mu &= \beta^2\mu_1 \\ \theta &= \gamma^2\theta_1 + 2\gamma\delta\mu_1\eta_1 + \delta^2\mu_1\theta_1 \\ \beta\eta &= \gamma^2\eta_1 + 2\gamma\delta\theta_1 + \delta^2\mu_1\eta_1 \end{aligned}$$

(b.10) $(27;\mu,\theta|1) \cong (27;\mu_1,\theta_1|1)$ *if and only if there exist* $\beta \in k^*$ *and* $\gamma,\delta \in k$ *with* $\gamma \neq 0$ *or* $\delta \neq 0$ *such that*

$$\begin{aligned} \mu &= \beta^2\mu_1 \\ \theta &= \gamma^2\theta_1 - \delta^2\mu_1\theta_1 \end{aligned}$$

(b.11) $(28;\theta,\eta,\lambda,\kappa|1) \cong (28;\theta_1,\eta_1,\lambda_1,\kappa_1|1)$ *if and only if there exist* $\beta \in k^*$ *and* $\gamma,\delta,\epsilon,\rho \in k$ *with* $\gamma\rho - \delta\epsilon \neq 0$ *such that*

$$\begin{aligned} \beta\theta &= \gamma^2\theta_1 + \gamma\delta\eta_1 + \gamma\delta\lambda_1 + \delta^2\kappa_1 \\ \beta\eta &= \gamma\epsilon\theta_1 + \gamma\rho\eta_1 + \delta\epsilon\lambda_1 + \delta\rho\kappa_1 \\ \beta\lambda &= \gamma\epsilon\theta_1 + \delta\epsilon\eta_1 + \gamma\rho\lambda_1 + \delta\rho\kappa_1 \\ \beta\kappa &= \epsilon^2\theta_1 + \epsilon\rho\eta_1 + \epsilon\rho\lambda_1 + \rho^2\kappa_1 \end{aligned}$$

Proof:
The proof of this shall be the main goal of this section. We shall divide the proof into three lemmas, showing the following:

- Each 4-dimensional superalgebra, A, with $\dim A_0 = \dim A_1 = 2$ is isomorphic to one of the superalgebras listed in the theorem. (Lemma 2.4.3)

- Each pair of distinct families of superalgebras listed in the theorem are non-isomorphic. (Lemma 2.4.5)
- The conditions for superalgebra isomorphisms to exist are as stated in part (b) of the theorem. (Lemma 2.4.6)

□

Remark 2.4.2 *This list is not entirely satisfactory. While the superalgebras listed are non-isomorphic as superalgebras, there are some cases where the underlying algebras may be isomorphic. In this way there can be different non-isomorphic $\mathbb{Z}_2$-gradings on an algebra listed as $\mathbb{Z}_2$-gradings on some other algebra. As an example* $(27; \mu, 1) \cong (10) = M_2$ *via* $1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X \mapsto \begin{pmatrix} 0 & \mu \\ 1 & 0 \end{pmatrix}, Y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *So, in the case that k isn't algebraically closed, there are, in fact, more $\mathbb{Z}_2$-gradings on M_2 than immediately obvious. This is because they are isomorphic to some of the superalgebras of the form $(27; \mu, \theta|1)$, hence do not get listed as $\mathbb{Z}_2$-gradings on M_2.*

We list them as has been done in Theorem 2.4.1, because it is too difficult to determine conditions where some of the different underlying algebras will be isomorphic as algebras, for instance $(27; \mu, 1)$ and M_2 as in our example.

Lemma 2.4.3 *Let A be a 4-dimensional superalgebra with $\dim_0 A = 2$ and $\dim_1 A = 2$. Then A is isomorphic to one of the superalgebras listed in Theorem 2.4.1* (a).

Proof:
Since A_0 is a 2-dimensional algebra, we may choose an element $x \in A_0$ such that $\{1, x\}$ is a k-basis of A_0 and $x^2 = \alpha \in k$. (If $x^2 = \alpha + \beta x$, notice that $(x - \frac{\beta}{2})^2 = \alpha + (\frac{\beta}{2})^2$, so then replace x with $x - \frac{\beta}{2}$). We then have the following three cases labelled I, II and III respectively to consider, depending on whether $\alpha = 0$, $\alpha \in k^{*2}$ or $\alpha \in k^* \backslash k^{*2}$:

I. If $\alpha = 0$, then $x^2 = 0$ and so $J(A_0) = kx$. Nakayama's lemma implies $J(A_0)A_1 \subset A_1$. Thus $\dim(J(A_0)A_1) = 0$ or 1. Similarly, we have $\dim(A_1 J(A_0)) = 0$ or 1. So we get three cases:

(a) $\dim(J(A_0)A_1) = 1$. In this case, choose an element $y \in A_1 \backslash J(A_0)A_1$. Then $A_1 = ky \oplus J(A_0)A_1 \subseteq A_0 y + J(A_0)A_1 \subseteq A_0A_1 = A_1$, and hence $A_1 = A_0 y + J(A_0)A_1$. Now from Nakayama's lemma (Version 2) we conclude that $A_1 = A_0 y = ky + kxy$ and $\{y, xy\}$ is a basis for A_1 over k.

(b) $\dim(J(A_0)A_1) = 0$ and $\dim(A_1J(A_0)) = 0$. In this case, $xA_1 = 0$ and $A_1 x = 0$.

(c) $\dim(J(A_0)A_1) = 0$ and $\dim(A_1J(A_0)) = 1$. In this case, a similar argument to (a) shows that there is an element $z \in A_1 \backslash A_1 J(A_0)$ such that $A_1 = kz + kzx$. We also have $xA_1 = 0$, and hence $xz = 0$.

II. If $\alpha = \delta^2$ for some $\delta \in k^*$, then $(\delta^{-1}x)^2 = 1$. Replacing x with $\delta^{-1}x$, we may assume $\alpha = 1$, and hence A_0 has a basis $\{1, x\}$ with $x^2 = 1$. Let $e_1 = \frac{1}{2}(1 + x)$ and $e_2 = \frac{1}{2}(1 - x)$. Then $e_1^2 = e_1$ $e_2^2 = e_2$ and $e_1e_2 = e_2e_1 = 0$. Since A_0 is a commutative algebra, the opposite algebra $A_0^{op} = A_0$. Hence $A_0 \otimes A_0^{op} = A_0 \otimes A_0 = \mathrm{span}\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\} \cong k \times k \times k \times k$, and $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ is a set of orthogonal idempotents with the sum being equal to 1. Thus $A_0 \otimes A_0$ is semisimple. In this case, any $A_0 \otimes A_0$-module is semisimple and any simple $A_0 \otimes A_0$-module is of dimension 1. Since A_1 is an A_0-bimodule, A_1 is a left $A_0 \otimes A_0$-module with the action given by $(a \otimes b)x = axb$, $a,\ b \in A_0$, $x \in A_1$. Thus we may choose a k-basis $\{x_1, x_2\}$ for A_1 such that kx_i is a simple $A_0 \otimes A_0$-submodule of A_1, $1 \leq i \leq 2$. Now by the Wedderburn-Artin Theorem, one gets the following six cases for which one of the four idempotents does not annihilate each x_i:

(a) $(e_1 \otimes e_1)x_i = x_i$, $1 \leq i \leq 2$.

(b) $(e_1 \otimes e_2)x_i = x_i$, $1 \leq i \leq 2$.

(c) $(e_1 \otimes e_1)x_1 = x_1$ and $(e_1 \otimes e_2)x_2 = x_2$.

(d) $(e_1 \otimes e_1)x_1 = x_1$ and $(e_2 \otimes e_1)x_2 = x_2$.

(e) $(e_1 \otimes e_1)x_1 = x_1$ and $(e_2 \otimes e_2)x_2 = x_2$.

(f) $(e_1 \otimes e_2)x_1 = x_1$ and $(e_2 \otimes e_1)x_2 = x_2$.

Note that we actually have ten other cases:

(g) $(e_2 \otimes e_1)x_i = x_i$, $1 \leq i \leq 2$.

(h) $(e_2 \otimes e_2)x_i = x_i$, $1 \leq i \leq 2$.

(i) $(e_1 \otimes e_2)x_1 = x_1$ and $(e_1 \otimes e_1)x_2 = x_2$.

(j) $(e_1 \otimes e_2)x_1 = x_1$ and $(e_2 \otimes e_2)x_2 = x_2$.

(k) $(e_2 \otimes e_1)x_1 = x_1$ and $(e_1 \otimes e_1)x_2 = x_2$.

(l) $(e_2 \otimes e_1)x_1 = x_1$ and $(e_1 \otimes e_2)x_2 = x_2$.

(m) $(e_2 \otimes e_1)x_1 = x_1$ and $(e_2 \otimes e_2)x_2 = x_2$.

(n) $(e_2 \otimes e_2)x_1 = x_1$ and $(e_1 \otimes e_1)x_2 = x_2$.

(o) $(e_2 \otimes e_2)x_1 = x_1$ and $(e_1 \otimes e_2)x_2 = x_2$.

(p) $(e_2 \otimes e_2)x_1 = x_1$ and $(e_2 \otimes e_1)x_2 = x_2$.

By relabelling e_i, $1 \leq i \leq 2$, or relabelling x_i, $1 \leq i \leq 2$, or relabelling both e_i and x_i, $1 \leq i \leq 2$, each of these cases can be reduced to one of the cases (a), (b), (c), (d), (e) and (f). For example, case (m) is reduced to case (c) by relabelling e_1 and e_2 and relabelling x_1 and x_2.

III. If $\alpha \in k^*\backslash k^{*2}$, then $A_0 \cong k(\sqrt{\alpha})$ is an extension field of k. In this case, A_1 is a free A_0-module of rank 1, and hence $A_1 = A_0 y = ky + kxy$ for some $0 \neq y \in A_1$.

Now we deal with each of the above cases.

Cases I (a) and III: $A_0 = k1 + kx$ and $A_1 = A_0 y = ky + kxy$ with $x^2 = \alpha \in k$, where $\alpha = 0$ or $\alpha \in k^*\backslash k^{*2}$. Since $A_1 A_0 = A_1$ and $A_1^2 \subseteq A_0$, we may suppose that $yx = \beta y + \gamma xy = (\beta + \gamma x)y$ and $y^2 = \delta + \epsilon x$ for some β, γ, δ, $\epsilon \in k$. From the equations $y^2 y = yy^2$, $yx^2 = (yx)x$ and $y^2 x = y(yx)$, one obtains $\beta\epsilon = 0,\ \gamma\epsilon = \epsilon,\ \beta^2 + \alpha\gamma^2 = \alpha,\ 2\beta\gamma = 0, \alpha\epsilon = \beta\delta + \beta\gamma\delta + \alpha\gamma^2\epsilon,\ \delta = \beta\epsilon + \beta\gamma\epsilon + \gamma^2\delta$. In the case $\alpha = 0$ we see that $\beta = 0$ straightaway. In the case $\alpha \in k^{*2}$, if $\beta \neq 0$, then $\gamma = 0$ and hence $\alpha = \beta^2 \in k^{*2}$, a contradiction. This contradiction shows that $\beta = 0$. Hence in both cases we must have $\beta = 0$. Thus we have $yx = \gamma xy$ and

$$\gamma\epsilon = \epsilon,\ \alpha\gamma^2 = \alpha,\ \delta\gamma^2 = \delta.$$

Note that $\gamma = \pm 1$ if $\alpha \in k^*\backslash k^{*2}$. From this we get five cases labelled I (a) 1–I (a) 3, III 1 and III 2 in the following.

Case I (a) 1: $\alpha = 0$ and $\gamma = 1$. In this case, we have $x^2 = 0$, $yx = xy$ and $y^2 = \delta + \epsilon x$. Either $\delta = \epsilon = 0$, $\delta = 0, \epsilon \neq 0$, $\delta \in k^{*2}$ or $\delta \in k^*\backslash k^{*2}$. If

$\delta = \epsilon = 0$ then $A \cong (7|2)$, via $x \mapsto X$, $y \mapsto Y$. If $\delta = 0$ and $\epsilon \neq 0$ then $A \cong (5|1)$, via $x \mapsto \epsilon^{-1}X^2$, $y \mapsto X$. If $\delta = \theta^2$ for some $\theta \in k^*$, then $A \cong (3|3)$, via $x \mapsto 2\theta(X, Y)$, $y \mapsto (\theta + \epsilon X, -\theta - \epsilon Y)$. If $\delta \in k^*\backslash k^{*2}$, then $A \cong (23; \delta|2)$, via $x \mapsto 2\delta Y$, $y \mapsto X + \epsilon XY$.

Case I (a) 2: $\alpha = 0$ and $\gamma = -1$. In this case, we have $\epsilon = 0$, and hence $x^2 = 0$, $yx = -xy$ and $y^2 = \delta$. Either $\delta = 0$, $\delta \in k^{*2}$ or $\delta \in k^*\backslash k^{*2}$. If $\delta = 0$ then $A \cong (12|1)$, via $x \mapsto X$, $y \mapsto Y$. If $\delta = \theta^2$ for some $\theta \in k^*$, then $A \cong (11|3)$, via

$$x \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; y \mapsto \theta \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

If $\delta \in k^*\backslash k^{*2}$, then $A \cong (25; \delta|3)$, via $x \mapsto Y$, $y \mapsto X$.

Case I (a) 3: $\alpha = 0$ and $\gamma \neq \pm 1$. In this case, we have $\delta = \epsilon = 0$, and hence $x^2 = 0$, $yx = \gamma xy$ and $y^2 = 0$. Either $\gamma = 0$ or $\gamma \neq 0$. If $\gamma = 0$ then $A \cong (16|1)$, via $x \mapsto X$ and $y \mapsto Y$. If $\gamma \neq 0$, then $A \cong (18; \gamma|1)$, via $x \mapsto X$ and $y \mapsto Y$.

This completes our treatment of case I(a).

Case III 1: $\alpha \in k^*\backslash k^{*2}$ and $\gamma = 1$. In this case, we have $x^2 = \alpha$, $yx = xy$ and $y^2 = \delta + \epsilon x$. Either $\delta = \epsilon = 0$ or at least one of δ and ϵ is non-zero. If $\delta = \epsilon = 0$, then $A \cong (23; \alpha|1)$, via $x \mapsto X$ and $y \mapsto Y$. If $\delta \neq 0$ or $\epsilon \neq 0$, then $A \cong (26; \alpha, \delta, \epsilon|1)$, via $x \mapsto X$ and $y \mapsto Y$.

Case III 2: $\alpha \in k^*\backslash k^{*2}$ and $\gamma = -1$. In this case, we have $\epsilon = 0$, and hence $x^2 = \alpha$, $yx = -xy$ and $y^2 = \delta$. Either $\delta = 0$ or $\delta \neq 0$. If $\delta = 0$ then $A \cong (25; \alpha|2)$, via $x \mapsto X$ and $y \mapsto Y$. If $\delta \neq 0$ then $A \cong (27; \alpha, \delta|1)$, via $x \mapsto X$ and $y \mapsto Y$.

This completes our treatment of case III.

Case I (b): $A_0 = k1 + kx$ and $xA_1 = A_1x = 0$ with $x^2 = 0$. In this case, we have $A_1^2 \subseteq A_0$, and A_1^2 is an ideal of A_0. Since $xA_1^2 = 0$, $A_1^2 \neq A_0$. It follows that $A_1^2 = 0$ or $A_1^2 = kx$. Either $A_1^2 = 0$ or $A_1^2 \neq 0$. If $A_1^2 = 0$, then clearly $A \cong (9|2)$. If $A_1^2 = kx$ then there is a k-basis $\{y, z\}$ of A_1 with $y^2 = \alpha_1 x$, $yz = \alpha_2 x$, $zy = \alpha_3 x$ and $z^2 = \alpha_4 x$ for some $\alpha_i \in k$, $1 \leq i \leq 4$, such that at least one of α_i is nonzero. Hence $A \cong (28; \alpha_1, \alpha_2, \alpha_3, \alpha_4|1)$, via $x \mapsto X$, $y \mapsto Y$, $z \mapsto Z$.

Case I (c): $A_0 = k1 + kx$ and $A_1 = kz + kzx$ with $x^2 = 0$ and $xz = 0$. Since $z^2 \in A_0$ and $xz^2 = (xz)z = 0$, one can see $z^2 = \beta x$ for some $\beta \in k$. Now we have $z^3 = z^2 z = \beta xz = 0$ and $z^3 = zz^2 = \beta zx$. Hence $\beta = 0$ as $zx \neq 0$, and so $z^2 = 0$. Thus $A \cong (16|2)$, via $x \mapsto Y$, $z \mapsto X$.

Case II (a): $A_0 = ke_1 + ke_2$ and $A_1 = kx_1 + kx_2$ with $e_i e_j = \delta_i^j e_i$ and $e_1 x_i e_1 = x_i$, $1 \leq i, j \leq 2$, recall δ_i^j is the Kronecker delta. In this case, $e_2 A_1 = A_1 e_2 = 0$. Since $A_1^2 \subseteq A_0$ and $e_2 A_1^2 = 0$, we have $A_1^2 \subseteq ke_1$, and hence $x_1^2 = \alpha e_1$, $x_1 x_2 = \beta e_1$, $x_2 x_1 = \gamma e_1$ and $x_2^2 = \delta e_1$ for some $\alpha, \beta, \gamma, \delta \in k$. From the equation $x_1^2 x_2 = x_1(x_1 x_2)$ one gets $\alpha x_2 = \beta x_1$. This implies $\alpha = \beta = 0$ since $\{x_1, x_2\}$ is linearly independent over k. Similarly, from the equation $x_2^2 x_1 = x_2(x_2 x_1)$, one obtains $\gamma = \delta = 0$. Thus $A_1^2 = 0$, and hence $A \cong (6|2)$, via $e_1 \mapsto (0, 1)$, $e_2 \mapsto (1, 0)$, $x_1 \mapsto (0, X)$, $x_2 \mapsto (0, Y)$.

Case II (b): $A_0 = ke_1 + ke_2$ and $A_1 = kx_1 + kx_2$ with $e_i e_j = \delta_i^j e_i$ and $e_1 x_i e_2 = x_i$, $1 \leq i, j \leq 2$. In this case, $x_i x_j = (x_i e_2)(e_1 x_j) = 0$, and hence $A_1^2 = 0$. Thus $A \cong (17|2)$, via $e_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $x_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $x_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Case II (c): $A_0 = ke_1 + ke_2$ and $A_1 = kx_1 + kx_2$ with $e_i e_j = \delta_i^j e_i$, $1 \leq i, j \leq 2$, $e_1 x_1 e_1 = x_1$ and $e_1 x_2 e_2 = x_2$. An argument similar to Case II(b) shows that $x_2 x_1 = 0$ and $x_2^2 = 0$. Now we have $A_1^2 \subseteq A_0$, $e_2(x_1 x_2) = (x_1 x_2)e_1 = 0$. It follows $x_1 x_2 = 0$. Since $e_1 x_1^2 = x_1^2 e_1 = x_1^2$, we have $x_1^2 = \alpha e_1$ for some $\alpha \in k$. Then from the equation $x_1^2 x_2 = x_1(x_1 x_2)$, one gets $\alpha = 0$, and hence $x_1^2 = 0$. Thus $A_1^2 = 0$, and $A \cong (15|3)$, via $e_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $x_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $x_2 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Case II (d): $A_0 = ke_1 + ke_2$ and $A_1 = kx_1 + kx_2$ with $e_i e_j = \delta_i^j e_i$, $1 \leq i, j \leq 2$, $e_1 x_1 e_1 = x_1$ and $e_2 x_2 e_1 = x_2$. An argument similar to Case II(c) shows that $x_ix_j = 0$, and hence $A_1^2 = 0$. Thus $A \cong (14|3)$, via $e_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $x_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $x_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Case II (e): $A_0 = ke_1 + ke_2$ and $A_1 = kx_1 + kx_2$ with $e_ie_j = \delta_i^j e_i$, $1 \leq i, j \leq 2$, $e_1x_1e_1 = x_1$ and $e_2x_2e_2 = x_2$. An argument similar to Case II(c) shows that $x_1x_2 = x_2x_1 = 0$, $x_1^2 = \alpha e_1$ and $x_2^2 = \beta e_2$. We shall consider the different cases which arise depending on whether α and β belong to $\{0\}$, k^{*2} or $k^*\backslash k^{*2}$. If $\alpha = \beta = 0$, then $A \cong (3|2)$, via $e_1 \mapsto (1, 0)$, $e_2 \mapsto (0, 1)$, $x_1 \mapsto (X, 0)$ and $x_2 \mapsto (0, Y)$. If $\alpha = 0$ and $\beta = \gamma^2$ for some $\gamma \in k^*$, then $A \cong (2|3)$, via $e_1 \mapsto (0, 0, 1)$, $e_2 \mapsto (1, 1, 0)$, $x_1 \mapsto (0, 0, X)$ and $x_2 \mapsto \gamma(1, -1, 0)$. Similarly, if $\beta = 0$ and $\alpha = \gamma^2$ for some $\gamma \in k^*$, then $A \cong (2|3)$. If $\alpha = 0$ and $\beta \in k^*\backslash k^{*2}$, then $A \cong (20; \beta|3)$, via $e_1 \mapsto (1, 0)$, $e_2 \mapsto (0, 1)$, $x_1 \mapsto (X, 0)$ and $x_2 \mapsto (0, Y)$. Similarly, if $\beta = 0$ and $\alpha \in k^*\backslash k^{*2}$, then $A \cong (20; \alpha|3)$. If $\alpha = \gamma^2$ and $\beta = \delta^2$ for some γ, $\delta \in k^*$, then $A \cong (1|2)$, via $e_1 \mapsto (1, 1, 0, 0)$, $e_2 \mapsto (0, 0, 1, 1)$, $x_1 \mapsto \gamma(1, -1, 0, 0)$, $x_2 \mapsto \delta(0, 0, 1, -1)$. If $\alpha = \gamma^2$ for some $\gamma \in k^*$ and $\beta \in k^*\backslash k^{*2}$, then $A \cong (21; \beta|3)$, via $e_1 \mapsto (1, 1, 0)$, $e_2 \mapsto (0, 0, 1)$, $x_1 \mapsto \gamma(1, -1, 0)$, $x_2 \mapsto (0, 0, X)$. Similarly, if $\alpha \in k^*\backslash k^{*2}$ and $\beta = \gamma^2$ for some $\gamma \in k^*$, then $A \cong (21; \alpha|3)$. If $\alpha \in k^*\backslash k^{*2}$ and $\beta \in k^*\backslash k^{*2}$, then $A \cong (22; \alpha, \beta|2)$, via $e_1 \mapsto (1, 0)$, $e_2 \mapsto (0, 1)$, $x_1 \mapsto (X, 0)$ and $x_2 \mapsto (0, Y)$.

Case II (f): $A_0 = ke_1 + ke_2$ and $A_1 = kx_1 + kx_2$ with $e_ie_j = \delta_i^j e_i$, $1 \leq i, j \leq 2$, $e_1x_1e_2 = x_1$ and $e_2x_2e_1 = x_2$. An argument similar to Case II(c) shows that $x_1^2 = x_2^2 = 0$, $x_1x_2 = \alpha e_1$ and $x_2x_1 = \beta e_2$ for some α, $\beta \in k$. From the equation $(x_1x_2)x_1 = x_1(x_2x_1)$ one gets $\alpha = \beta$. Either $\alpha = 0$ or $\alpha \neq 0$. If $\alpha = 0$ then $A \cong (11|2)$, via $e_1 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $x_1 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $x_2 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. If $\alpha \neq 0$ then $A \cong (10|1)$, via $e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $x_1 \mapsto \alpha\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $x_2 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

This completes the proof. $\square$

The following helpful result will be used in the proof of Lemma 2.4.5.

Lemma 2.4.4 *Consider superalgebras A and B with $\dim_0 A = \dim_0 B = 2$. Suppose that $\{1, x\}, \{1, y\}$ are bases of A_0 and B_0 respectively with $x^2 = \omega \in k, y^2 = \nu \in k$. If $A \cong B$ then $\omega = \beta^2 \nu$ for some $\beta \in k^*$.*

Proof:
After noticing that, if $A \cong B$ as superalgebras then we must have $A_0 \cong B_0$ as algebras, the result immediately follows from Lemma 2.2.5 (a). □

Lemma 2.4.5 *Each pair of distinct families of superalgebras listed in Theorem 2.4.1 are non-isomorphic.*

Proof:
From Proposition 2.2.1, Proposition 2.2.2 and Proposition 2.2.3 we have to show the following:

- different superalgebras defined on the same underlying algebra are non-isomorphic;
- the superalgebra $(26; \mu, \theta, \eta|1)$ is not isomorphic to any of $(20; \mu|3)$, $(21; \mu|3)$, $(22; \xi, \mu|2)$, $(23; \mu|1)$ and $(23; \mu|2)$;
- the superalgebra $(27; \mu, \theta|1)$ is not isomorphic to $(10|1)$, $(25; \mu|2)$ or $(25; \mu|3)$;
- the superalgebra $(28; \theta, \eta, \lambda, \kappa|1)$ is not isomorphic to any other superalgebra on the list.

In Theorem 2.4.1 we gave A_0 and A_1 in the forms $k1 \oplus kx$, $ky \oplus kz$ respectively. Hence we may give A_0 a basis of $\{1, x\}$ and A_1 a basis of $\{y, z\}$. For the remainder of the proof we endow each superalgebra with the basis obtained from Theorem 2.4.1 in this manner, except for $(6|2)$, instead giving $(6|2)_0$ and $(6|2)_1$ bases $\{(1, 1), (0, 1)\}$ and $\{(0, X), (0, Y)\}$ respectively.

In this paragraph we describe the arguments used in the rest of the proof, and also show that different superalgebras defined on the same underlying algebra are non-isomorphic. To see $(3|2) \ncong (3|3)$, $(23;\mu|1) \ncong (23;\mu|2)$ and $(25;\mu|2) \ncong (25;\mu|3)$ assume that they are isomorphic, apply Lemma 2.4.4 and find the contradiction that no such non-zero β as described in Lemma 2.4.4 can exist. We call this *Approach 1*. Apply Lemma 2.3.3 to discover that $(11|2) \ncong (11|3)$. We call this *Approach 2*. Let A and A' be two superalgebras with $\dim A_0 = \dim A_1 = \dim A'_0 = \dim A'_1 = 2$. Then A_0 has a k-basis $\{1, x\}$ such that $x^2 \in k1$ and x is uniquely determined up to a nonzero scalar multiple. Similarly, A'_0 has a k-basis $\{1, y\}$ such that $y^2 \in k1$ and y is uniquely determined up to a nonzero scalar multiple. If $f : A \to A'$ is a superalgebra isomorphism, then f restricts to an isomorphism from A_0 to A'_0. It follows from the proof of Lemma 2.2.5 that f must satisfy $f(x) = \alpha y$ for some $\alpha \in k^*$. Hence if $A_1 x = 0$ and $A'_1 y \neq 0$, then $A \ncong A'$, by Lemma 2.3.4. Similarly, if $xA_1 = 0$ and $yA'_1 \neq 0$, then $A \ncong A'$, by Lemma 2.3.4. We call this method *Approach 3*. For instance, assume $f : (16|2) \to (16|1)$ is an isomorphism. Then by the above discussion we have $f(Y) = \alpha X$ for some $\alpha \in k^*$. Now $(16|1)_1 f(Y) = (16|1)_1 X = 0$, however $(16|2)_1 Y \neq 0$. This is impossible. Hence $(16|2) \ncong (16|1)$.

Using Approach 1, one can see that $(26;\mu,\theta,\eta|1)$ is not isomorphic to any of $(20;\mu'|3)$, $(21;\mu'|3)$, $(22;\xi,\mu'|2)$ and $(23;\mu'|2)$, where μ, μ', $\xi \in k^*\backslash k^{*2}$ and θ, $\eta \in k$ with $\theta \neq 0$ or $\eta \neq 0$. Using Approach 2, one discovers that $(26;\mu,\theta,\eta|1)$ is not isomorphic to $(23;\mu'|1)$, where μ, μ', θ and η are given as above.

Using Approach 1, one gets that $(27;\mu,\theta|1)$ is not isomorphic to $(10|1)$ or $(25;\mu'|3)$; using Approach 2, one gets that $(27;\mu,\theta|1)$ is not isomorphic to $(25;\mu'|2)$, where μ, $\mu' \in k^*\backslash k^{*2}$ and $\theta \in k^*$.

Using Approach 2, one can see that $(28;\theta,\eta,\lambda,\kappa|1)$ is not isomorphic to $(9|2)$, where θ, η, λ, $\kappa \in k$ with at least one of them $\neq 0$. Finally, using Approach 3, one knows that $(28;\theta,\eta,\lambda,\kappa|1)$ is not isomorphic to any of the remaining superalgebras on the list in Theorem 2.4.1 (a). Thus

$(28;\theta,\eta,\lambda,\kappa|1)$ is not isomorphic to any other superalgebra on the list in Theorem 2.4.1 (a).

This completes the proof. □

Lemma 2.4.6 *The conditions given in part* (b) *of Theorem 2.4.1 for two superalgebras from the same family to be isomorphic are as stated there.*

Proof:
(b.1). Let X and Y be the generators of $(18;\lambda|1)$ as given in Theorem 2.4.1(a), and let X_1 and Y_1 denote the corresponding generators of $(18;\lambda_1|1)$. Obviously, if $\lambda = \lambda_1$ then $(18;\lambda|1) \cong (18;\lambda_1|1)$. Conversely, assume $f : (18;\lambda|1) \to (18;\lambda_1|1)$ is a superalgebra isomorphism. Then as pointed out in the proof of Lemma 2.4.5, f must be of the form $f(X) = \alpha X_1$, $f(Y) = \gamma Y_1 + \delta X_1 Y_1$, where α, γ, $\delta \in k$ with $\alpha \neq 0$ and $\gamma \neq 0$. Now $0 = f(YX - \lambda XY) = f(Y)f(X) - \lambda f(X)f(Y) = (\gamma Y_1 + \delta X_1 Y_1)\alpha X_1 - \lambda\alpha X_1(\gamma Y_1 + \delta X_1 Y_1) = \gamma\alpha Y_1 X_1 - \lambda\gamma\alpha X_1 Y_1 = \gamma\alpha(\lambda_1 - \lambda)X_1 Y_1$. This implies $\lambda = \lambda_1$ as $\gamma\alpha \neq 0$.

(b.2). Let $(X,0)$, $(0,Y)$ and $(X_1,0)$, $(0,Y_1)$ be the generators of $(20;\mu|3)$ and $(20;\mu_1|3)$, respectively, as given in Theorem 2.4.1(a). If $\mu = \delta^2\mu_1$ for some $\delta \in k^*$, then there is a superalgebra isomorphism $f : (20;\mu|3) \to (20;\mu_1|3)$ given by $f((X,0)) = (X_1,0)$ and $f((0,Y)) = \delta(0,Y_1)$. Conversely, assume $(20;\mu|3) \cong (20;\mu_1|3)$. Then $(20;\mu|3) \cong (20;\mu_1|3)$ as ungraded algebras, i.e., $(20;\mu) \cong (20;\mu_1)$. By Lemma 2.2.5 and Lemma 2.2.6, one knows that $\mu\mu_1^{-1} \in k^{*2}$.

(b.3). Is proved similarly to (b.2).

(b.4). Let $(X,0)$ and $(0,Y)$ be the generators of $(22;\xi,\mu)$, and $(X_1,0)$ and $(0,Y_1)$ be the generators of $(22;\xi_1,\mu_1)$, as described in Theorem 2.4.1(a). Suppose that either $\xi\xi_1^{-1}$, $\mu\mu_1^{-1} \in k^{*2}$ or $\mu\xi_1^{-1}$, $\xi\mu_1^{-1} \in k^{*2}$. Observe that $(22;\xi,\mu|2) \cong (22;\mu,\xi|2)$ as superalgebras. Hence we may assume that $\mu = \gamma^2\mu_1$ and $\xi = \delta^2\xi_1$ for some γ, $\delta \in k^*$. Then there is a superalgebra isomorphism f from $(22;\xi,\mu|2)$ to $(22;\xi_1,\mu_1|2)$ given by $f((X,0)) = \gamma(X_1,0)$ and $f((0,Y)) = \delta(0,Y_1)$. Conversely assume $(22;\xi,\mu|2) \cong (22;\xi_1,\mu_1|2)$

as superalgebras, then $(22; \xi, \mu) \cong (22; \xi_1, \mu_1)$ as algebras. By the comments following Lemma 2.2.6 we must have either $\xi\xi_1^{-1}$, $\mu\mu_1^{-1} \in k^{*2}$ or $\mu\xi_1^{-1}$, $\xi\mu_1^{-1} \in k^{*2}$.

(b.5). Let X and Y be the generators of $(23; \mu|1)$, and X_1 and Y_1 be the generators of $(23; \mu_1|1)$, as described in Theorem 2.4.1(a). If $\mu = \gamma^2\mu_1$ for some $\gamma \in k^*$, then $(23; \mu|1) \cong (23; \mu_1|1)$ via $X \mapsto \gamma X_1$ and $Y \mapsto Y_1$. Conversely, if $(23; \mu|1) \cong (23; \mu_1|1)$ then $(23; \mu|1)_0 \cong (23; \mu_1|1)_0$. However, we have $(23; \mu|1)_0 \cong k(\sqrt{\mu})$ and $(23; \mu_1|1)_0 \cong k(\sqrt{\mu_1})$. Thus by Lemma 2.2.5 one gets $\mu\mu_1^{-1} \in k^{*2}$.

(b.6). If $\mu = \gamma^2\mu_1$ for some $\gamma \in k^*$, then the algebra isomorphism from $(23; \mu|1)$ to $(23; \mu_1|1)$ given in (b.5) is also a superalgebra isomorphism from $(23; \mu|2)$ to $(23; \mu_1|2)$. Conversely, suppose $f : (23; \mu|2) \to (23; \mu_1|2)$ is an isomorphism. We use the notation given in the proof of (b.5). Then from the proofs of Lemma 2.2.5 and Lemma 2.4.5, one can see that f is given by $f(Y) = \beta Y_1$ and $f(X) = \gamma X_1 + \delta X_1Y_1$, where $\beta, \gamma \in k^*$ and $\delta \in k$. Now we have $\mu = f(X^2) = f(X)^2 = (\gamma X_1 + \delta X_1Y_1)^2 = \gamma^2\mu_1 + 2\gamma\delta\mu_1Y_1$, which implies $\mu = \gamma^2\mu_1$ and $\delta = 0$. Thus $\mu\mu_1^{-1} \in k^{*2}$.

(b.7). Is proved similarly to (b.5).

(b.8). Is proved similarly to (b.6).

(b.9). Let X and Y be the generators of $(26; \mu, \theta, \eta)$, and X_1 and Y_1 be the generators of $(26; \mu_1, \theta_1, \eta_1)$, as described in Theorem 2.4.1(a). If $f : (26; \mu, \theta, \eta|1) \cong (26; \mu_1, \theta_1, \eta_1|1)$, then f must be given by $f(X) = \beta X_1$ and $f(Y) = \gamma Y_1 + \delta X_1Y_1$ for some $\beta \in k^*$ and $\gamma, \delta \in k$ with $\gamma \neq 0$ or $\delta \neq 0$. From the equations $f(X^2) = f(X)^2$ and $f(Y^2) = f(Y)^2$ one gets

$$\begin{aligned}
\mu &= \beta^2\mu_1, \\
\theta &= \gamma^2\theta_1 + 2\gamma\delta\mu_1\eta_1 + \delta^2\mu_1\theta_1, \\
\beta\eta &= \gamma^2\eta_1 + 2\gamma\delta\theta_1 + \delta^2\mu_1\eta_1.
\end{aligned}$$

Conversely, if there exist $\beta \in k^*$ and $\gamma, \delta \in k$ with $\gamma \neq 0$ or $\delta \neq 0$ such that the above three equations are satisfied, then there is a superalgebra isomorphism $f : (26; \mu, \theta, \eta|1) \cong (26; \mu_1, \theta_1, \eta_1|1)$ given by $f(X) = \beta X_1$ and $f(Y) = \gamma Y_1 + \delta X_1Y_1$.

(b.10) and (b.11). Are proved similarly to (b.9).

This completes the proof. □

This completes the proof of Theorem 2.4.1. To conclude this section we give a corollary of Theorem 2.4.1 which shall be used later.

Corollary 2.4.7 *Assume that k is an algebraically closed field, then*
(a) *superalgebras* $(20;\mu|3)$–$(27;\mu,\theta|1)$ *listed in Theorem 2.4.1 can never arise, and*
(b) *the superalgebra* $(28;\theta,\eta,\lambda,\kappa|1)$ *can be simplified to the following non-isomorphic superalgebras:*

(7) $k[X,Y]/(X^2,Y^2)$:
$(7|3)_0 = k1 \oplus kXY$ *and* $(7|3)_1 = kX \oplus kY$.

(8) $k[X,Y]/(X^3,XY,Y^2)$:
$(8|3)_0 = k1 \oplus kX^2$ *and* $(8|3)_1 = kX \oplus kY$.

(12) $\wedge k^2 \cong k\langle X,Y\rangle/(X^2,Y^2,XY+YX)$:
$(12|2)_0 = k1 \oplus kXY$ *and* $(12|2)_1 = kX \oplus kY$.

(16) $k\langle X,Y\rangle/(X^2,Y^2,YX)$:
$(16|3)_0 = k1 \oplus kXY$ *and* $(16|3)_1 = kX \oplus kY$.

(18;λ) $k\langle X,Y\rangle/(X^2,Y^2,YX-\lambda XY)$, *where* $\lambda \in k$ *with* $\lambda \neq 1,0,-1$:
$(18;\lambda|2)_0 = k1 \oplus kXY$ *and* $(18;\lambda|2)_1 = kX \oplus kY$.

(19) $k\langle X,Y\rangle/(Y^2,X^2+YX,YX+XY)$:
$(19|1)_0 = k1 \oplus kXY$ *and* $(19|1)_1 = kX \oplus kY$.

Moreover, $(18;\lambda|2) \cong (18;\lambda_1|2)$ *if and only if* $\lambda_1 = \lambda$ *or* $\lambda\lambda_1 = 1$.

Proof:
(a) is obvious since $k^*\backslash k^{*2} = \emptyset$ when k is algebraically closed.

For (b), we shall write the generators of $(28;\theta,\eta,\lambda,\kappa)$ as x, y and z which were given as X, Y and Z respectively in Theorem 2.4.1. This is to distinguish them from the generators X and Y of the superalgebras $(7), (8), (12), (16), (18;\lambda)$ and (19) given above. Then $(28;\theta,\eta,\lambda,\kappa|1)_0 =$

$k1 + kx$ and $(28; \theta, \eta, \lambda, \kappa|1)_1 = ky + kz$ with $x^2 = xy = xz = yx = zx = 0$, $y^2 = \theta x$, $yz = \eta x$, $zy = \lambda x$ and $z^2 = \kappa x$. Note that at least one of θ, η, λ and κ is not zero.

We first consider the case : $\theta = \kappa = 0$. In this case, $\eta \neq 0$ or $\lambda \neq 0$. We may assume that $\eta \neq 0$ (otherwise, we can switch y and z). By replacing x with ηx, one can assume $\eta = 1$ and hence $yz = x$. Note that $y^2 = z^2 = 0$. Either $\lambda = 0$, $\lambda = -1$, $\lambda = 1$ or $\lambda \neq -1, 0, 1$. If $\lambda = 0$ then $(28; \theta, \eta, \lambda, \kappa|1) = (28; 0, 1, 0, 0|1)$ is isomorphic to $(16|3)$ via $y \mapsto X$ and $z \mapsto Y$. Similarly, if $\lambda = -1$ then $(28; \theta, \eta, \lambda, \kappa|1) = (28; 0, 1, -1, 0|1)$ is isomorphic to $(12|2)$, if $\lambda = 1$ then $(28; \theta, \eta, \lambda, \kappa|1) = (28; 0, 1, 1, 0|1)$ is isomorphic to $(7|3)$, and if $\lambda \neq -1, 0, 1$ then $(28; \theta, \eta, \lambda, \kappa|1) = (28; 0, 1, \lambda, 0|1)$ is isomorphic to $(18; \lambda|2)$.

Now we consider the case: $\theta \neq 0$ or $\kappa \neq 0$. We may assume that $\theta \neq 0$ (otherwise, we can switch y and z). By replacing x with θx, we may assume that $\theta = 1$ and hence $y^2 = x$. Since k is algebraically closed, there is an $\alpha \in k$ such that $\alpha^2 + (\eta + \lambda)\alpha + \kappa = 0$. Now let $z_0 = \alpha y + z$. Then $(28; \theta, \eta, \lambda, \kappa|1)_1 = ky + kz_0$, $z_0^2 = (\alpha y + z)^2 = \alpha^2 y^2 + \alpha yz + \alpha zy + z^2 = (\alpha^2 + (\eta + \lambda)\alpha + \kappa)x = 0$, $xz_0 = z_0 x = 0$, $yz_0 = (\alpha + \eta)x$ and $z_0 y = (\alpha + \lambda)x$. Hence, by replacing z with z_0, we may assume $\kappa = 0$. Thus we have $x^2 = xy = xz = yx = zx = 0$, $y^2 = x$, $yz = \eta x$, $zy = \lambda x$ and $z^2 = 0$. Either $\eta = 0$ and $\lambda = 0$, $\eta = 0$ and $\lambda \neq 0$ or $\eta \neq 0$. If $\eta = 0$ and $\lambda = 0$, then $yz = zy = 0$. In this case, $(28; \theta, \eta, \lambda, \kappa|1) = (28; 1, 0, 0, 0|1)$ is isomorphic to $(8|3)$ via $y \mapsto X$ and $z \mapsto Y$. If $\eta = 0$ and $\lambda \neq 0$, then by replacing z with $\lambda^{-1}z$ we may assume $\lambda = 1$. Hence we have $yz = 0$ and $zy = x$. Let $y_1 = z$ and $z_1 = y - z$. Then $(28; \theta, \eta, \lambda, \kappa|1)_1 = ky_1 + kz_1$, $y_1^2 = 0$, $y_1 z_1 = x$, $z_1 y_1 = 0$ and $z_1^2 = (y-z)^2 = y^2 - yz - zy + z^2 = x - 0 - x + 0 = 0$. By replacing y and z with y_1 and z_1 respectively, one can see that $(28; \theta, \eta, \lambda, \kappa|1) = (28; 1, 0, 1, 0|1) \cong (28; 0, 1, 0, 0|1) \cong (16|3)$ by the first case. Finally, suppose $\eta \neq 0$. Then by replacing z with $\eta^{-1}z$ we may assume $\eta = 1$. Hence $yz = x$. Either $\lambda = 1$, $\lambda = -1$ or $\lambda \neq \pm 1$. If $\lambda = 1$ then $(28; \theta, \eta, \lambda, \kappa|1) = (28; 1, 1, 1, 0|1) \cong (7|3)$ via $x \mapsto \frac{1}{2}XY$, $y \mapsto \frac{1}{2}(X + Y)$ and $z \mapsto Y$. If $\lambda = -1$ then $y^2 = x$, $yz = x$, $zy = -x$ and $z^2 = 0$. In this case, $(28; \theta, \eta, \lambda, \kappa|1) = (28; 1, 1, -1, 0|1)$ is isomorphic to $(19|1)$ via $y \mapsto X$ and $z \mapsto Y$. If $\lambda \neq \pm 1$, let $y_2 = -(1+\lambda)y+z$ and $z_2 = -(1+\lambda)^{-1}z$. Then $\{y_2, z_2\}$ is a k-basis of $(28; \theta, \eta, \lambda, \kappa|1)_1$. Now we have $y_2^2 = (-(1+\lambda)y+z)^2 = (1+\lambda)^2y^2 - (1+\lambda)yz - (1+\lambda)zy + z^2 = ((1+\lambda)^2 - (1+\lambda) - (1+\lambda)\lambda)x = 0$, $y_2z_2 = (-(1+\lambda)y+z)(-(1+\lambda)^{-1}z) = yz - (1+\lambda)^{-1}z^2 = x$, $z_2y_2 = (-(1+\lambda)^{-1}z)(-(1+\lambda)y+z) = zy - (1+\lambda)^{-1}z^2 = \lambda x$ and $z_2^2 = (1+\lambda)^{-2}z^2 = 0$. By replacing y and z with y_2 and z_2, respectively, one can get that $(28; \theta, \eta, \lambda, \kappa|1) = (28; 1, 1, \lambda, 0|1) \cong (28; 0, 1, \lambda, 0|1) \cong (18; \lambda|2)$ from the first case.

It follows from Proposition 2.2.1 that the superalgebras listed in the corollary are non-isomorphic.

By switching X and Y, one can see $(18; \lambda|2) \cong (18; \lambda^{-1}|2)$. Thus if $\lambda_1 = \lambda$ or $\lambda\lambda_1 = 1$ then $(18; \lambda|2) \cong (18; \lambda_1|2)$. Conversely, assume $f : (18; \lambda|2) \to (18; \lambda_1|2)$ is a superalgebra isomorphism. Let X and Y be the generators of $(18; \lambda|2)$, and X_1 and Y_1 be the generators of $(18; \lambda_1|2)$. Then $f(X) = \alpha_{11}X_1 + \alpha_{12}Y_1$ and $f(Y) = \alpha_{21}X_1 + \alpha_{22}Y_1$ for some $\alpha_{ij} \in k$, $1 \leq i, j \leq 2$, with $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \neq 0$. Now we have $0 = f(X^2) = f(X)^2 = (\alpha_{11}X_1 + \alpha_{12}Y_1)^2 = \alpha_{11}\alpha_{12}(X_1Y_1 + Y_1X_1) = \alpha_{11}\alpha_{12}(1+\lambda_1)X_1Y_1$, and hence $\alpha_{11}\alpha_{12} = 0$. Similarly, from the equation $f(Y^2) = f(Y)^2$ one gets $\alpha_{21}\alpha_{22} = 0$. Either $\alpha_{11} \neq 0$ or $\alpha_{11} = 0$. If $\alpha_{11} \neq 0$, then $\alpha_{12} = 0$, $\alpha_{22} \neq 0$ and $\alpha_{21} = 0$. Now from the equation $f(Y)f(X) = f(YX) = f(\lambda XY) = \lambda f(Y)f(X)$, one obtains $\lambda = \lambda_1$. If $\alpha_{11} = 0$, then $\alpha_{12}\alpha_{21} \neq 0$ and $\alpha_{22} = 0$. Now from the equation $f(Y)f(X) = \lambda f(X)f(Y)$, one obtains $\lambda\lambda_1 = 1$. Thus we have proved that $(18; \lambda|2) \cong (18; \lambda_1|2)$ if and only if $\lambda_1 = \lambda$ or $\lambda\lambda_1 = 1$. $\square$

2.5 Case k is Algebraically Closed

In this section we collect results from the previous sections, which, when combined with the results from [12] gives us a complete classification of four dimensional superalgebras in the case that k is algebraically closed and has $\mathrm{ch}(k) \neq 2$. We note here that even if k is not algebraically closed then a four dimensional superalgebra must be either a four dimensional algebra endowed with the trivial $\mathbb{Z}_2$-grading or isomorphic to one of the superalgebras described in Proposition 2.2.12, Theorem 2.3.1 or Theorem 2.4.1. The superalgebra described in Proposition 2.2.12 is denoted by $(9|3)$ (see Example 2.2.11).

Although the results from the previous sections give a full classification of four dimensional superalgebras with non-trivial $\mathbb{Z}_2$-gradings, we would like a complete classification of superalgebras, even the trivially $\mathbb{Z}_2$-graded superalgebras. For example these results are needed before starting on the geometric classification (which is done in Chapter 3). Thus, for this section we make the additional assumption that k is algebraically closed. We use the results of [12] to classify the trivially $\mathbb{Z}_2$-graded superalgebras and specialise our results from the previous sections to the case where k is algebraically closed. This yields us the following theorem.

Theorem 2.5.1 *(Algebraic classification of 4-dimensional superalgebras)*
Assume that k is algebraically closed and that $\mathrm{ch}(k) \neq 2$. Let A be a 4-dimensional superalgebra. Then A is isomorphic to one of the following superalgebras. Moreover each pair of classes is non-isomorphic.

$(1):$ $(1|0)$, $(1|1)$, $(1|2)$,
$(2):$ $(2|0)$, $(2|1)$, $(2|2)$, $(2|3)$,
$(3):$ $(3|0)$, $(3|1)$, $(3|2)$, $(3|3)$,
$(4):$ $(4|0)$, $(4|1)$,
$(5):$ $(5|0)$, $(5|1)$,
$(6):$ $(6|0)$, $(6|1)$, $(6|2)$,
$(7):$ $(7|0)$, $(7|1)$, $(7|2)$, $(7|3)$,

$(8):$ $(8|0)$, $(8|1)$, $(8|2)$, $(8|3)$,
$(9):$ $(9|0)$, $(9|1)$, $(9|2)$, $(9|3)$,
$(10):$ $(10|0)$, $(10|1)$,
$(11):$ $(11|0)$, $(11|1)$, $(11|2)$, $(11|3)$,
$(12):$ $(12|0)$, $(12|1)$, $(12|2)$,
$(13):$ $(13|0)$, $(13|1)$,
$(14):$ $(14|0)$, $(14|1)$, $(14|2)$, $(14|3)$,
$(15):$ $(15|0)$, $(15|1)$, $(15|2)$, $(15|3)$,
$(16):$ $(16|0)$, $(16|1)$, $(16|2)$, $(16|3)$,
$(17):$ $(17|0)$, $(17|1)$, $(17|2)$,
$(18;\lambda):$ $(18;\lambda|0)$, $(18;\lambda|1)$, $(18;\lambda|2)$, *where* $\lambda \in k$ *with* $\lambda \neq 1, 0, -1$,
$(19):$ $(19|0)$, $(19|1)$.
Furthermore, $(18;\lambda|0) \cong (18;\lambda_1|0)$ *if and only if* $\lambda_1 = \lambda$ *or* $\lambda\lambda_1 = 1$, $(18;\lambda|1) \cong (18;\lambda_1|1)$ *if and only if* $\lambda = \lambda_1$, *and* $(18;\lambda|2) \cong (18;\lambda_1|2)$ *if and only if* $\lambda_1 = \lambda$ *or* $\lambda\lambda_1 = 1$.

Proof:
This follows from the results of [12], Proposition 2.2.12, Theorem 2.3.1, Theorem 2.4.1 and Corollary 2.4.7. □

Remark 2.5.2 *Compare these results with those obtained by Gabriel in [12]. It is interesting to note that each* 4*-dimensional algebra from his classification results admits at least one non-trivial* $\mathbb{Z}_2$*-grading. Some of the algebras admit only one such non-trivial* $\mathbb{Z}_2$*-grading. However some admit up to three non-isomorphic non-trivial* $\mathbb{Z}_2$*-gradings.*

Theorem 2.5.1 above lays the foundations for the geometric classification in the following chapter, since the isomorphism classes of 4-dimensional superalgebras are in one-to-one correspondence with G_4-orbits in Salg_4. Before we move onto the geometric classification problem, we list the superalgebra automorphism groups for the superalgebras listed in Theorem 2.5.1, since they will also be needed in the following chapter.

2.6 Automorphism groups

In this section we calculate the automorphism groups of the algebras described in Section 2.5, where we assumed that k is algebraically closed and $\mathrm{ch}(k) \neq 2$. We will use the results from this section to calculate the dimensions of orbits in the variety Salg_4. We shall describe the varieties Salg_n in the next section.

We shall choose a basis for each each superalgebra, $\{e_1 = 1, e_2, e_3, e_4\}$, and find the constants $a_{21}, \ldots, a_{44}$ for which ϕ is an automorphism of the given superalgebra, where ϕ is defined by $\phi(e_1) = e_1, \phi(e_2) = a_{21}e_1 + a_{22}e_2 + a_{23}e_3 + a_{24}e_4, \phi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3 + a_{34}e_4, \phi(e_4) = a_{41}e_1 + a_{42}e_2 + a_{43}e_3 + a_{44}e_4$. Since we shall choose homogeneous bases, then for superalgebras with $\dim A_0 = 3$ we must have $a_{24} = a_{34} = a_{41} = a_{42} = a_{43} = 0$; for the superalgebras with $\dim A_0 = 2$ we must have $a_{23} = a_{24} = a_{31} = a_{32} = a_{41} = a_{42} = 0$; and for those with $\dim A_0 = 1$ we must have $a_{21} = a_{31} = a_{41} = 0$, in order for the given map to be homogeneous. We shall not mention these constants in these cases. In the following, we give the values or relations amongst these constants for each to give a superalgebra automorphism.

$(1|0) : e_1 = (1, 1, 1, 1), e_2 = (1, 0, 0, 0), e_3 = (0, 1, 0, 0), e_4 = (0, 0, 1, 0)$

Then, in this case, one can show that there are only finitely many possibilities for the constants. (This suffices for our purposes — to list the possibilities takes a while).

$(1|1) : e_1 = (1, 1, 1, 1), e_2 = (1, 0, 0, 0), e_3 = (0, 1, 0, 0), e_4 = (0, 0, 1, -1)$

Then $a_{21} = a_{31} = 0, a_{44} = \pm 1$ and either

- $a_{22} = a_{33} = 1, a_{23} = a_{32} = 0$; or
- $a_{22} = a_{33} = 0, a_{23} = a_{32} = 1$

$(1|2) : e_1 = (1, 1, 1, 1), e_2 = (1, 1, 0, 0), e_3 = (1, -1, 0, 0), e_4 = (0, 0, 1, -1)$

Then either

- $a_{21} = a_{34} = a_{43} = 0, a_{22} = 1, a_{33} = \pm 1, a_{44} = \pm 1$; or

- $a_{21} = 1, a_{22} = -1, a_{33} = a_{44} = 0, a_{34} = \pm 1, a_{43} = \pm 1$

$(2|0) : e_1 = (1, 1, 1), e_2 = (1, 0, 0), e_3 = (0, 1, 0), e_4 = (0, 0, X)$
Then $a_{21} = a_{24} = a_{31} = a_{34} = a_{41} = a_{42} = a_{43} = 0, a_{44} \neq 0$ and either

- $a_{22} = a_{33} = 1, a_{23} = a_{32} = 0$; or
- $a_{22} = a_{33} = 0, a_{23} = a_{32} = 1$

$(2|1) : e_1 = (1, 1, 1), e_2 = (1, 0, 0), e_3 = (0, 1, 0), e_4 = (0, 0, X)$
Then $a_{21} = a_{31} = 0, a_{44} \neq 0$ and either

- $a_{22} = a_{33} = 1, a_{23} = a_{32} = 0$; or
- $a_{22} = a_{33} = 0, a_{23} = a_{32} = 1$

$(2|2) : e_1 = (1, 1, 1), e_2 = (1, 1, 0), e_3 = (0, 0, X), e_4 = (1, -1, 0)$
Then $a_{21} = a_{23} = a_{31} = a_{32} = 0$ and $a_{22} = 1, a_{33} \neq 0, a_{44} = \pm 1$
$(2|3) : e_1 = (1, 1, 1), e_2 = (1, 1, 0), e_3 = (1, -1, 0), e_4 = (0, 0, X)$
Then $a_{21} = a_{34} = a_{43} = 0$ and $a_{22} = 1, a_{33} = \pm 1, a_{44} \neq 0$
$(3|0) : e_1 = (1, 1), e_2 = (1, 0), e_3 = (X, 0), e_4 = (0, Y)$
Then $a_{23} = a_{24} = a_{31} = a_{32} = a_{41} = a_{42} = 0$ and either

- $a_{21} = a_{34} = a_{43} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0$; or
- $a_{33} = a_{44} = 0, a_{21} = 1, a_{22} = -1, a_{34} \neq 0, a_{43} \neq 0$

$(3|1) : e_1 = (1, 1), e_2 = (1, 0), e_3 = (X, 0), e_4 = (0, Y)$
Then $a_{21} = a_{23} = a_{31} = a_{32} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0$
$(3|2) : e_1 = (1, 1), e_2 = (1, 0), e_3 = (X, 0), e_4 = (0, Y)$
Then either

- $a_{21} = 0, a_{22} = 1, a_{34} = a_{43} = 0, a_{33} \neq 0, a_{44} \neq 0$; or
- $a_{21} = 1, a_{22} = -1, a_{33} = a_{44} = 0, a_{34} \neq 0, a_{43} \neq 0$

$(3|3) : e_1 = (1, 1), e_2 = (X, Y), e_3 = (1, -1), e_4 = (X, -Y)$
Then $a_{21} = a_{34} = a_{43} = 0$ and either

- $a_{33} = -1, a_{22} = -a_{44} \neq 0$; or
- $a_{33} = 1, a_{22} = a_{44} \neq 0$

$(4|0) : e_1 = (1,1), e_2 = (1,0), e_3 = (0,X), e_4 = (0,X^2)$

Then $a_{21} = a_{23} = a_{24} = a_{31} = a_{32} = a_{41} = a_{42} = a_{43} = 0, a_{22} = 1, a_{44} = a_{33}^2 \neq 0, a_{34}$ is unconstrained

$(4|1) : e_1 = (1,1), e_2 = (1,0), e_3 = (0,X^2), e_4 = (0,X)$

Then $a_{21} = a_{23} = a_{31} = a_{32} = 0, a_{22} = 1, a_{33} = a_{44}^2 \neq 0$

$(5|0) : e_1 = 1, e_2 = X, e_3 = X^2, e_4 = X^3$

Then $a_{21} = a_{31} = a_{32} = a_{41} = a_{42} = a_{43} = 0, a_{22} \neq 0, a_{33} = a_{22}^2, a_{34} = 2a_{22}a_{23}, a_{44} = a_{22}a_{33} = a_{22}^3, a_{23}$ and a_{24} are unconstrained

$(5|1) : e_1 = 1, e_2 = X^2, e_3 = X, e_4 = X^3$

Then $a_{21} = a_{43} = 0, a_{33} \neq 0, a_{22} = a_{33}^2, a_{44} = a_{22}a_{33} = a_{33}^3, a_{34}$ is unconstrained

$(6|0) : e_1 = (1,1), e_2 = (1,0), e_3 = (0,X), e_4 = (0,Y)$

Then $a_{21} = a_{23} = a_{24} = a_{31} = a_{32} = a_{41} = a_{42} = 0, a_{22} = 1$ and $a_{33}, a_{34}, a_{43}, a_{44}$ are unconstrained apart from $a_{33}a_{44} - a_{34}a_{43} \neq 0$

$(6|1) : e_1 = (1,1), e_2 = (1,0), e_3 = (0,X), e_4 = (0,Y)$

Then $a_{21} = a_{23} = a_{31} = a_{32} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0$

$(6|2) : e_1 = (1,1), e_2 = (1,0), e_3 = (0,X), e_4 = (0,Y)$

Then $a_{21} = 0, a_{22} = 1$ and $a_{33}, a_{34}, a_{43}, a_{44}$ are unconstrained apart from $a_{33}a_{44} - a_{34}a_{43} \neq 0$

$(7|0) : e_1 = 1, e_2 = X, e_3 = Y, e_4 = XY$

Then $a_{21} = a_{31} = a_{41} = a_{42} = a_{43} = 0, a_{24}, a_{34}$ are unconstrained and either

- $a_{22} \neq 0, a_{33} \neq 0, a_{23} = a_{32} = 0, a_{44} = a_{22}a_{33}$; or
- $a_{23} \neq 0, a_{32} \neq 0, a_{22} = a_{33} = 0, a_{44} = a_{23}a_{32}$

$(7|1) : e_1 = 1, e_2 = X + Y, e_3 = 2XY, e_4 = X - Y$

Then $a_{21} = a_{31} = a_{32} = 0, a_{33} = a_{22}^2 = a_{44}^2 \neq 0$ we have cases $a_{22} = a_{44}$ or $a_{22} = -a_{44}$ and a_{23} is unconstrained

$(7|2): e_1 = 1, e_2 = X, e_3 = Y, e_4 = XY$

Then $a_{21} = a_{43} = 0, a_{22} \neq 0, a_{33} \neq 0, a_{44} = a_{22}a_{33}, a_{34}$ is unconstrained

$(7|3): e_1 = 1, e_2 = XY, e_3 = X, e_4 = Y$

Then $a_{21} = 0$ and either

- $a_{34} = a_{43} = 0, a_{33} \neq 0, a_{44} \neq 0, a_{22} = a_{33}a_{44}$; or
- $a_{33} = a_{44} = 0, a_{34} \neq 0, a_{43} \neq 0, a_{22} = a_{34}a_{43}$

$(8|0): e_1 = 1, e_2 = X, e_3 = X^2, e_4 = Y$

Then $a_{21} = a_{31} = a_{32} = a_{34} = a_{41} = a_{42} = 0, a_{33} = a_{22}^2 \neq 0, a_{44} \neq 0, a_{23}, a_{24}, a_{43}$ are unconstrained

$(8|1): e_1 = 1, e_2 = X, e_3 = X^2, e_4 = Y$

Then $a_{21} = a_{31} = a_{32} = 0, a_{33} = a_{22}^2 \neq 0, a_{44} \neq 0, a_{23}$ is unconstrained

$(8|2): e_1 = 1, e_2 = X^2, e_3 = Y, e_4 = X$

Then $a_{21} = a_{23} = a_{31} = 0, a_{22} = a_{44}^2 \neq 0, a_{33} \neq 0, a_{32}$ is unconstrained

$(8|3): e_1 = 1, e_2 = X^2, e_3 = X, e_4 = Y$

Then $a_{21} = a_{43} = 0, a_{22} = a_{33}^2 \neq 0, a_{44} \neq 0, a_{34}$ is unconstrained

$(9|0): e_1 = 1, e_2 = X, e_3 = Y, e_4 = Z$

Then $a_{21} = a_{31} = a_{41} = 0, a_{22}, a_{23}, a_{24}, a_{32}, a_{33}, a_{34}, a_{42}, a_{43}, a_{44}$ are unconstrained apart from $\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \neq 0$

$(9|1): e_1 = 1, e_2 = X, e_3 = Y, e_4 = Z$

Then $a_{21} = a_{31} = 0, a_{44} \neq 0, a_{22}, a_{23}, a_{32}, a_{33}$ are unconstrained apart from $a_{22}a_{33} - a_{23}a_{32} \neq 0$

$(9|2): e_1 = 1, e_2 = X, e_3 = Y, e_4 = Z$

Then $a_{21} = 0, a_{22} \neq 0, a_{33}, a_{34}, a_{43}, a_{44}$ are unconstrained apart from $a_{33}a_{44} - a_{34}a_{43} \neq 0$

$(9|3): e_1 = 1, e_2 = X, e_3 = Y, e_4 = Z$

Then there are no constraints other than $\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \neq 0$

$(10|0):$

Now $\mathrm{M}_2(k)$ is a central simple algebra, so each automorphism must be inner by the Skolem-Noether theorem. For $A \in \mathrm{GL}_2(k)$, $\phi_A(X) = AXA^{-1}$,

$\phi_A = \mathrm{id} \Leftrightarrow A = \lambda I_2$ for some $\lambda \neq 0$. Thus $\mathrm{Aut}(\mathrm{M}_2(k)) = \mathrm{GL}_2(k)/k^* = \mathrm{PGL}_2(k)$

$(10|1) : e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Then either

- $a_{21} = 0, a_{22} = 1, a_{34} = a_{43} = 0, a_{33} \neq 0, a_{44} = a_{33}^{-1}$; or
- $a_{21} = 1, a_{22} = -1, a_{33} = a_{44} = 0, a_{34} \neq 0, a_{43} = a_{34}^{-1}$

$(11|0) : e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Then $a_{31} = a_{32} = a_{41} = a_{42} = 0, a_{23}, a_{24}$ are unconstrained and either

- $a_{21} = 0, a_{22} = 1, a_{34} = a_{43} = 0, a_{33} \neq 0, a_{44} \neq 0$; or
- $a_{21} = 1, a_{22} = -1, a_{33} = a_{44} = 0, a_{34} \neq 0, a_{43} \neq 0$

$(11|1) : e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Then $a_{21} = a_{31} = a_{32} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0, a_{23}$ is unconstrained

$(11|2) : e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Then either

- $a_{21} = 0, a_{22} = 1, a_{34} = a_{43} = 0, a_{33} \neq 0, a_{44} \neq 0$; or

- $a_{21} = 1, a_{22} = -1, a_{33} = a_{44} = 0, a_{34} \neq 0, a_{43} \neq 0$

$(11|3) : e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Then $a_{21} = a_{43} = 0, a_{34}$ is unconstrained, and either

- $a_{33} = 1, a_{22} \neq 0, a_{44} = a_{22}$; or

- $a_{33} = -1, a_{22} \neq 0, a_{44} = -a_{22}$

$(12|0) : e_1 = 1, e_2 = X, e_3 = Y, e_4 = XY$

Then $a_{21} = a_{31} = a_{41} = a_{42} = a_{43} = 0, a_{44} = a_{22}a_{33} - a_{23}a_{32} \neq 0, a_{24}, a_{34}$ are unconstrained

$(12|1) : e_1 = 1, e_2 = X, e_3 = Y, e_4 = XY$

Then $a_{21} = a_{43} = 0, a_{44} = a_{22}a_{33} \neq 0, a_{34}$ is unconstrained

$(12|2) : e_1 = 1, e_2 = XY, e_3 = X, e_4 = Y$

Then $a_{21} = 0, a_{22} = a_{33}a_{44} - a_{34}a_{43} \neq 0$

$(13|0) : e_1 = \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), e_2 = \left(0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), e_3 = \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right), e_4 = \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$

Then $a_{21} = a_{23} = a_{31} = a_{32} = a_{41} = a_{42} = a_{43} = 0, a_{22} = a_{33} = 1, a_{44} \neq 0, a_{24} = -a_{34}$

$(13|1) : e_1 = \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), e_2 = \left(0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), e_3 = \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right), e_4 = \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$

Then $a_{21} = a_{23} = a_{31} = a_{32} = 0, a_{22} = a_{33} = 1, a_{44} \neq 0$

$(14|0) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $a_{21} = a_{24} = a_{31} = a_{32} = a_{34} = a_{41} = a_{42} = a_{43} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0, a_{23}$ is unconstrained

$(14|1) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $a_{21} = a_{31} = a_{32} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0, a_{23}$ is unconstrained

$(14|2) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Then $a_{21} = a_{23} = a_{31} = a_{32} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0$

$(14|3) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $a_{21} = a_{34} = a_{43} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0$

$(15|0) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $a_{21} = a_{24} = a_{31} = a_{32} = a_{34} = a_{41} = a_{42} = a_{43} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0, a_{23}$ is unconstrained

$(15|1) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $a_{21} = a_{31} = a_{32} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0, a_{23}$ is unconstrained

$(15|2) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $a_{21} = a_{23} = a_{31} = a_{32} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0$

$(15|3) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $a_{21} = a_{34} = a_{43} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0$

$(16|0) : e_1 = 1, e_2 = X, e_3 = Y, e_4 = XY$

Then $a_{21} = a_{23} = a_{31} = a_{32} = a_{41} = a_{42} = a_{43} = 0, a_{22} \neq 0, a_{33} \neq 0, a_{44} = a_{22}a_{33}, a_{24}, a_{34}$ are unconstrained

$(16|1) : e_1 = 1, e_2 = X, e_3 = Y, e_4 = XY$

Then $a_{21} = a_{43} = 0, a_{22} \neq 0, a_{33} \neq 0, a_{44} = a_{22}a_{33}, a_{34}$ is unconstrained

$(16|2) : e_1 = 1, e_2 = Y, e_3 = X, e_4 = XY$

Then $a_{21} = a_{43} = 0, a_{22} \neq 0, a_{33} \neq 0, a_{44} = a_{22}a_{33}, a_{34}$ is unconstrained

$(16|3) : e_1 = 1, e_2 = XY, e_3 = X, e_4 = Y$

Then $a_{21} = a_{34} = a_{43} = 0, a_{33} \neq 0, a_{44} \neq 0, a_{22} = a_{33}a_{44}$

$(17|0) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Then $a_{21} = a_{31} = a_{32} = a_{41} = a_{42} = 0, a_{22} = 1, a_{23}, a_{24}, a_{33}, a_{34}, a_{43}, a_{44}$ are unconstrained, apart from $a_{33}a_{44} - a_{34}a_{43} \neq 0$

$(17|1) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Then $a_{21} = a_{31} = a_{32} = 0, a_{22} = 1, a_{33} \neq 0, a_{44} \neq 0, a_{23}$ is unconstrained

$(17|2) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Then $a_{21} = 0, a_{22} = 1, a_{33}, a_{34}, a_{43}, a_{44}$ are unconstrained, apart from $a_{33}a_{44} - a_{34}a_{43} \neq 0$

$(18; \lambda|0)$ for $\lambda \neq -1$: $e_1 = 1, e_2 = X, e_3 = Y, e_4 = XY$

If $\lambda \neq 1$: then $a_{21} = a_{23} = a_{31} = a_{32} = a_{41} = a_{42} = a_{43} = 0, a_{22} \neq 0, a_{33} \neq 0, a_{44} = a_{22}a_{33}, a_{24}, a_{34}$ are unconstrained

If $\lambda = 1$: then this is case $(7|0)$

$(18; \lambda|1)$ for $\lambda \neq -1$: $e_1 = 1, e_2 = X, e_3 = Y, e_4 = XY$

Then $a_{21} = a_{43} = 0, a_{22} \neq 0, a_{33} \neq 0, a_{44} = a_{22}a_{33}, a_{34}$ is unconstrained

$(18; \lambda|2)$ for $\lambda \neq -1$: $e_1 = 1, e_2 = XY, e_3 = X, e_4 = Y$

If $\lambda \neq 1$: then $a_{21} = a_{34} = a_{43} = 0, a_{33} \neq 0, a_{44} \neq 0, a_{22} = a_{33}a_{44}$

If $\lambda = 1$: then this is case $(7|3)$

$(19|0) : e_1 = 1, e_2 = XY, e_3 = X, e_4 = Y$

Then $a_{21} = a_{23} = a_{24} = a_{31} = a_{41} = a_{43} = 0, a_{33} \neq 0, a_{22} = a_{33}^2, a_{44} = a_{33}, a_{32}, a_{34}, a_{42}$ are unconstrained

$(19|1) : e_1 = 1, e_2 = XY, e_3 = X, e_4 = Y$

Then $a_{21} = a_{43} = 0, a_{33} \neq 0, a_{22} = a_{33}^2, a_{44} = a_{33}, a_{34}$ is unconstrained

Chapter 3

Geometric Classification

In this chapter we attempt the geometric classification problem for 4-dimensional superalgebras. While we make significant progress towards a geometric classification theorem of 4-dimensional superalgebras, the problem is not completely solved. We must assume that our ground field, k, is algebraically closed to apply the techniques of algebraic geometry. We additionally assume $\mathrm{ch}(k) \neq 2$, in which case the problem becomes determining which superalgebra structures listed in Theorem 2.5.1 are generic. To make sense of these ideas, we define a new variety whose points represent superalgebra structures on an n-dimensional vector space, we call this variety Salg_n. We then study the geometry of this variety, as this helps us attack the geometric classification problem.

3.1 Preliminaries

It is assumed that the reader need not have any familiarity with algebraic geometry, although this would be helpful. We do however assume a knowledge of basic topology. In this section we shall briefly review some of the ideas from algebraic geometry and fix the terminology that we shall use. Again, we work over a fixed ground field k with $\mathrm{ch}(k) \neq 2$, and we now must additionally assume that k is algebraically closed.

We point out that an we make a non-standard definition of varieties. Typically varieties are defined to be irreducible, however in keeping with the literature on the idea of geometric classification of algebraic structures, e.g. [12, 21] we do not impose this constraint. As remarked earlier, the main goal of the geometric classification problem is to determine the irreducible components.

Our review of algebraic geometry is largely a synthesis from [7, 15, 25, 30]. The material we present is mostly standard, (except we use a slightly different definition of variety from that in most modern treatments) so many other books would give the reader an adequate introduction to the subject. We shall omit proofs and references for the results which are standard. Both [16, 29] on linear algebraic groups, also provide a quick introduction into algebraic geometry, which would be appropriate for our purposes. The book [10] is quite different to most standard algebraic geometry texts, in that it approaches the subject from the functorial viewpoint. While it explains some of the approaches used in Gabriel's paper [12], it is not really appropriate for a light introduction to the subject.

Before we begin our review of algebraic geometry, we define some basic notions. After the review of algebraic geometry, our last topic in this section is a review of the work done on Alg_n — the variety of n-dimensional algebras.

Definition 3.1.1 *Given a group G (whose operation we denote by juxtaposition and whose identity we denote by e) and a set X, we say that G* **acts on** *X or X* **has a G-action** *when there is a map $\phi : G \times X \to X$ such that*

$$\phi(e, x) = x \text{ for all } x \in X \text{ ; and}$$

$$\phi(g, \phi(h, x)) = \phi(gh, x) \text{ for all } g, h \in G, x \in X$$

For brevity we shall write $\phi(g, x) = g \cdot x$, in which case these two conditions become:

$$e \cdot x = x \text{ for all } x \in X \text{ ; and}$$

$$g \cdot (h \cdot x) = (gh) \cdot x \textit{ for all } g, h \in G, x \in X$$

The **orbits in** X **under the action of** G *are the sets* $G \cdot x = \{g \cdot x : g \in G\}$.

When we have a map between two sets both with actions of some group G, we are interested in how it interacts with the action of G. In the case that the map "preserves" the G-action on both sets we make the following definition.

Definition 3.1.2 *Suppose we have a group G and two sets X and Y both equipped with an action of the group G, when we have a map* $f : X \to Y$ *such that* $f(g \cdot x) = g \cdot f(x)$ *we call the map,* f, G**-equivariant**.

Now we begin the material on introductory Algebraic Geometry.

Definition 3.1.3 *Affine* n*-space,* $\mathbb{A}^n$ *is the topological space, which is* k^n *as a set, and is endowed with the Zariski topology which we shall define below (see Definition 3.1.7). When thinking of* k^n *in this manner, we shall use the notation* $\mathbb{A}^n$ *to indicate this.*

We set $P = k[X_1, \ldots, X_n]$ and view P as a set of k-valued functions on $\mathbb{A}^n$, where $f \in P$ assigns to the point $(a_1, \ldots, a_n) \in \mathbb{A}^n$ the value $f(a_1, \ldots, a_n)$. We say that f **vanishes at** $(a_1, \ldots, a_n)$ or $(a_1, \ldots, a_n)$ **is a zero of** f if $f(a_1, \ldots, a_n) = 0$.

Definition 3.1.4 *Given* $f \in P$ *the* **vanishing set of** f *is defined to be*

$$V(f) = \{p \in \mathbb{A}^n : f(p) = 0\}$$

and more generally, if S *is a set of polynomials we define the* **vanishing set of** S *to be*

$$V(S) = \{p \in \mathbb{A}^n : f(p) = 0 \quad \forall f \in S\}$$

If I is the ideal generated by S then $V(I) = V(S)$, so we usually just consider vanishing sets of ideals (and lose no generality in doing so). Note that when R is a ring and $r \in R$ we denote by (r) the ideal in R generated by the element r. In this case, we have $V(f) = V((f))$.

Definition 3.1.5 *A subset of some affine space, $\mathbb{A}^n$, is called an* **algebraic set** *if it is equal to the vanishing set, $V(S)$, of some set of polynomials, S.*

So V is a function, mapping subsets of P to algebraic sets (which are subsets of $\mathbb{A}^n$).

We remark that using some of the ideas introduced later, it should not be too difficult to see that a set of finitely many polynomials will always suffice to "cut out" an algebraic set.

When indexed sets of functions are being used and the range of the indices is clear, we shall omit listing the range of indices in the vanishing set for notational convenience. For example, if f_i^j are functions with indices $i = 1, \ldots n, j = 1, \ldots m$, then we shall simply write $V(\{f_i^j\})$ instead of the more cumbersome $V\left(\{f_i^j\}_{\substack{i=1,\ldots,n\\ j=1,\ldots,m}}\right)$.

Lemma 3.1.6 *The function V has the following properties:*

(a) $V(0) = V((0)) = \mathbb{A}^n$, $V(1) = V(P) = \emptyset$

(b) $V(I \cap J) = V(I) \cup V(J)$

(c) $V(\bigcup_{\alpha \in A} I_\alpha) = \bigcap_{\alpha \in A} V(I_\alpha)$

(d) If $I \subseteq J$ then $V(I) \supseteq V(J)$

Properties (a) to (c) show that the collection of algebraic sets form the closed sets for some topology on $\mathbb{A}^n$ — this is the Zariski topology. More formally, we make the following definition.

Definition 3.1.7 *The* **Zariski topology** *on $\mathbb{A}^n$ is defined by taking the open sets to be complements of the algebraic sets in $\mathbb{A}^n$. Now, if $f \in P$ we have a special open subset of $\mathbb{A}^n$ defined by such a function, we set $D(f) = \{p \in \mathbb{A}^n : f(p) \neq 0\}$, we call such a subset a* **distinguished open subset** .

Note that $D(f)$ is the complement of the algebraic set $V(f)$. Also notice that the collection of distinguished open subsets forms a basis for the Zariski topology.

Examples 3.1.8 *(a) The Zariski topology on $\mathbb{A}^1$: The closed sets in $\mathbb{A}^1$ are simply the zeros of a polynomial in one variable, say x, $f(x) = 0$. But it is easy to check that there can be only finitely many such zeros for a polynomial in one variable. Thus the closed sets are $\emptyset, \mathbb{A}^1$ and any finite set of points. Thus, the Zariski topology on $\mathbb{A}^1$ is simply the cofinite topology on $\mathbb{A}^1$.*

(b) The Zariski topology on $\mathbb{A}^2$: A closed set in $\mathbb{A}^2$ is $\emptyset, \mathbb{A}^2$ or a finite union of points and "curves", where by curves we mean the set of zeros of some polynomial of two variables, say x and y, $f(x, y) = 0$. The open sets in $\mathbb{A}^2$ are then simply the complements of these sets.

Remark 3.1.9 *When $k = \mathbb{R}$ or $k = \mathbb{C}$, the Zariski topology on k^n is very different to the metric topology on k^n. The Zariski topology on k^n is T_1 since any point has an open set not containing that point, but it is not T_2 since any two open subsets must intersect. In fact, it is due to this lack of separation, that the notion of irreducibility is useful when using the Zariski topology, viz Remark 3.1.17. It is also quasi-compact, that is, every open cover of the space has a finite subcover (this is essentially the same idea as that of compactness, yet in the definition of compactness people sometimes require that the space be T_2, hence the distinction).*

Definition 3.1.10 *In a ring R, the* **radical of an ideal** *I is the set $\sqrt{I} = \{r \in R : r^n \in I$ for some $n \in \mathbb{N}\}$. An ideal I is called* **radical** *if it is equal to its own radical, that is $I = \sqrt{I}$. An ideal I is called* **prime** *if $ab \in I$ implies that either $a \in I$ or $b \in I$.*

Notice that all prime ideals are radical ideals.

Definition 3.1.11 *On a subset X of $\mathbb{A}^n$, one defines the* **ideal of functions vanishing on** *X as $I(X) = \{f \in P : f(p) = 0 \quad \forall p \in X\}$.*

One can check that this is indeed an ideal of P. Moreover one can check that each ideal $I(X)$ is a radical ideal in P.

So I is a function, mapping subsets of $\mathbb{A}^n$ to radical ideals of P.

Lemma 3.1.12 *The function I has the following properties:*

(a) $I(\emptyset) = P, I(\mathbb{A}^n) = (0)$

(b) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$

(c) If $X_1 \subseteq X_2$ *then* $I(X_1) \supseteq I(X_2)$

Since V maps subsets of P to subsets of $\mathbb{A}^n$ and I maps subsets of $\mathbb{A}^n$ to subsets of P, the composites $V \circ I$ and $I \circ V$ are defined. It is interesting to see what we can say about these composites. This is what we consider in the following lemma.

Lemma 3.1.13 *Now if J is an ideal of P and $X, Y \subseteq \mathbb{A}^n$ where Y is an algebraic set in $\mathbb{A}^n$, then we have the following statements:*

(a) $I(V(J)) = \sqrt{J}$

(b) $V(I(X)) = \overline{X}$ *(the closure being taken in the Zariski topology on $\mathbb{A}^n$)*

(c) $V(I(Y)) = Y$

We remark that for (a) it is relatively easy to show that $\sqrt{J} \subseteq I(V(J))$. For the other inclusion we use Hilbert's Nullstellensatz, which we give below. We also note that part (c) follows immediately from part (b), since algebraic sets are the closed sets in the Zariski topology.

Lemma 3.1.14 *(Hilbert's Nullstellensatz)*
Let k be an algebraically closed field and let J be an ideal of $P = k[X_1, \ldots, X_n]$ and let $f \in P$ be a polynomial which vanishes at all points of $V(J)$. Then $f^r \in J$ for some $r \in \mathbb{N}$.

Now, combining several results seen so far, Lemma 3.1.6, Lemma 3.1.12 and Lemma 3.1.13, we get the following result:

Proposition 3.1.15 *The assignment $I \mapsto V(I)$ sets up an inclusion reversing bijective correspondence between radical ideals of P and algebraic subsets of $\mathbb{A}^n$.*

This is an interesting result — it relates the topological structure of the space $\mathbb{A}^n$ with the algebraic structure of $P = k[X_1, \ldots, X_n]$.

Definition 3.1.16 *A non-empty subset X of a topological space Y is called* **irreducible** *if it cannot be written as the union of two proper closed subsets (where the topology on X is the subspace topology induced from the topology of Y). Equivalently, X is irreducible if $\emptyset \neq X = X_1 \cup X_2$ with X_1, X_2 closed, then $X_1 = X$ or $X_2 = X$.*

Remark 3.1.17 *We mention that this notion is not very useful in a T_2-space, as the only irreducible subsets as single points (see exercise 1.2.2 on p3 in [29])*

Lemma 3.1.18 *We have the following results about irreducibility:*

(a) A set X is irreducible if and only if any two open subsets of X intersect

(b) Any non-empty open subset of an irreducible set is irreducible and dense

(c) For a subset Y of X, Y is irreducible if and only if its closure in X $\overline{Y}$, is irreducible

(d) The image of an irreducible set under a continuous map is irreducible

Under the correspondence between radical ideals of P and algebraic subsets of $k[X_1, \ldots X_n]$ mentioned in Proposition 3.1.15, prime ideals correspond to irreducible algebraic subsets.

Definition 3.1.19 *For an algebraic set $X \subseteq \mathbb{A}^n$ we define the* **coordinate ring** *of X to be $A(X) = P/I(X)$.*

We interpret elements of the coordinate ring as functions on X. If we can only observe values that a function takes on X, then it is natural to consider two functions to be the same if their values agree on all of X. When this is the case, these two functions represent the same element of the coordinate ring, that is if $f(p) = g(p)$ for all $p \in X$ then $f + I(X) = g + I(X)$ in the coordinate ring $A(X)$.

Remark 3.1.20 *As seen from the above definition, from an algebraic set one can uniquely construct an object called its coordinate ring. However it is also possible to uniquely reconstruct an algebraic set from its coordinate ring — we omit the details. So, in some sense the coordinate ring encodes all the information about the algebraic set. We shall make this comment a lot more formal soon, but first we need the notion of morphisms of algebraic sets.*

Definition 3.1.21 *If* $X \subseteq \mathbb{A}^n$ *and* $Y \subseteq \mathbb{A}^m$ *are algebraic sets, then* $f : X \to Y$ *is a* **morphism of algebraic sets** *if*

$$f(X_1, \ldots, X_n) = (f_1(X_1, \ldots, X_n), \ldots, f_m(X_1, \ldots, X_n))$$

where each $f_i(X_1, \ldots, X_n)$ *is a polynomial in* $X_1, \ldots X_n$ *and each point of* X *is mapped to a point of* Y. *A morphism* $f : X \to Y$ *of algebraic sets is said to be an* **isomorphism** *if there exists another morphism* $g : Y \to X$ *such that* $g \circ f = \mathrm{id}_X$ *and* $f \circ g = \mathrm{id}_Y$. *In this case* X *and* Y *are said to be* **isomorphic**.

Remark 3.1.22 *In fact with this definition of morphisms of algebraic sets, one obtains the category of algebraic sets, and the definition of isomorphism given above is simply the general category theoretic definition of an isomorphim between two objects.*

We also remind the reader that in a general category, a bijective morphism need not be an isomorphism. The category of algebraic sets provides an example of where the two ideas need not coincide. Consider the following example given in [30]: let C *be the curve given by* $y^2 = x^3$ *in the* xy*-plane,* $\mathbb{A}^2$*, then the morphism* f *from* $\mathbb{A}^1$ *(with coordinate* t*) to the curve* C *given by* $t \mapsto (t^2, t^3)$ *is bijective, but is not an isomorphism.*

It is standard to show, although we omit the details, that from a morphism $f : X \to Y$ between algebraic sets, there is an induced k-algebra homomorphism of coordinate rings, $A(f) : A(Y) \to A(X)$, but in the opposite direction. I have suggestively used the notation $A(f)$ above, because the assignment $X \mapsto A(X)$ extends to a contravariant functor between the category of algebraic sets and morphisms between them as defined in Definition 3.1.21, and the category of finitely generated k-algebras with no nilpotent elements and k-algebra homomorphisms. In fact, we have the following result:

Proposition 3.1.23 *The functor A, described above, is an (anti-)equivalence of categories.*

Thus from a finitely generated k-algebra with no nilpotent elements B, one can construct an algebraic set which has B for its coordinate ring.

As a corollary of the above result we find that two algebraic sets are isomorphic if and only if their coordinate rings are isomorphic. So, in this sense, the coordinate ring contains all the information about an algebraic set.

Remark 3.1.24 *The category of irreducible algebraic sets and morphisms between them and the category of finitely generated k-algebras which are integral domains and k-algebra homomorphisms are also (anti-)equivalent.*

Definition 3.1.25 *In a topological space, a subset is* **locally closed** *if it is open in its closure or equivalently if it is the intersection of an open and a closed set.*

Definition 3.1.26 *If $X \subseteq \mathbb{A}^n$ is locally closed, a function $f : X \to k$ is* **regular at a point** *$p \in X$ if there is an open neighbourhood U with $p \in U \subseteq X$ and polynomials $g, h \in k[X_1, \ldots, X_n]$ such that h is nowhere zero on U, and $f = g/h$ on U. We say that f is* **regular on** *X if it is regular at every point of X. The* **set of regular maps** *X is denoted by $\mathcal{O}(X) = \{f : X \to k : f \text{ is regular on } X\}$.*

So regular functions are functions which are locally quotients of polynomials. This need not be true globally, however. Think about the following example given in [30]. Consider the algebraic set $X = V(wx-yz) \subseteq \mathbb{A}^4$, and the subset $U = D(y) \cup D(w) = \{(w, x, y, z) \in X : w \neq 0 \text{ or } y \neq 0\}$. The subset U is a locally closed subset of $\mathbb{A}^4$. Now the function h defined by taking $h = \frac{x}{y}$ on $D(y)$ and $h = \frac{z}{w}$ on $D(w)$ is a regular function on U (note that this function is well-defined). But it cannot be written as a quotient of polynomials globally.

Definition 3.1.27 *A* **variety** *is a locally closed subset X of $\mathbb{A}^n$ endowed with its topology and the collection of $\mathcal{O}(U)$ for all U open in X.*

Note that this definition is not the standard one. Nowadays most people include irreducibility in the definition of a variety, but this is not convenient for our purposes. Following the literature on the geometric classification of algebraic structures e.g. [7, 12] we choose not to require this.

Definition 3.1.28 *A* **morphism** *$\phi : X \to Y$ between varieties is a continuous map such that for all $U \subseteq Y$ and all regular maps $\theta : U \to k$ the composition*

$$\phi^{-1}(U) \xrightarrow{\phi} U \xrightarrow{\theta} k$$

is regular. This property is sometimes stated as "ϕ pulls back regular functions to regular functions".

It is useful to notice that a morphism must in particular be a continuous map.

Algebraic sets give us an examples of a special kind of variety, which we define now.

Definition 3.1.29 *An* **affine variety** *is one which is isomorphic to a closed subset of $\mathbb{A}^n$ for some n.*

We give a proof of the following result, since it is important in the later sections of the chapter. It does however follow from a more general result. We do not state or prove the more general version since we have no need for it — Lemma 3.1.30 will suffice for our purposes.

Lemma 3.1.30 *Suppose we are given varieties $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ and $f = (f_1, \ldots, f_m) : X \to Y$ a function between these varieties such that each component f_i of the funcion is a rational function with non-vanishing denominator, i.e. for all $i \in \{1, \ldots, m\}$, $f_i(x_1, \ldots, x_n) = \frac{p_i(x_1,\ldots,x_n)}{q_i(x_1,\ldots,x_n)}$ with $q_i(x_1, \ldots, x_n)$ non-vanishing. Then f is a morphism of varieties.*

Proof:
Suppose $h(y_1, \ldots, y_m) = 0$ where $h(y_1, \ldots, y_m)$ is a polynomial. Then the preimage $f^{-1}(V(h))$ is given by the points in X satisfying $(h \circ f)(x_1, \ldots, x_n) = 0$, i.e.

$$h\left(\frac{p_1(x_1, \ldots, x_n)}{q_1(x_1, \ldots, x_n)}, \ldots, \frac{p_m(x_1, \ldots, x_n)}{q_m(x_1, \ldots, x_n)}\right) = 0 \tag{†}$$

Since each $q_i(x_1, \ldots, x_n)$ is non-zero we may multiply through by sufficiently large powers of the q_i to clear the denominators in the above. It follows that we get some polynomial $\tilde{h}(x_1, \ldots, x_n)$ such that $\tilde{h}(x_1, \ldots, x_n) = 0 \Leftrightarrow (h \circ f)(x_1, \ldots, x_n) = 0$. Thus $f^{-1}(V(h)) = V(\tilde{h})$, hence f is continuous.

Suppose that $U \subseteq Y$ is open and $\theta : U \to k$ is regular. Then given $x \in f^{-1}(U)$ let $y = f(x)$. By hypothesis y has a neighbourhood V such that $\theta|_{U \cap V} = g/h$ with $g, h \in k[y_1, \ldots, y_m]$ with h non-vanishing on V. Then $f^{-1}(V)$ is a neighbourhood of x such that

$$\theta \circ f|_{f^{-1}(U) \cap f^{-1}(V)} = \frac{g\left(\frac{p_1(x_1,\ldots,x_n)}{q_1(x_1,\ldots,x_n)}, \ldots, \frac{p_m(x_1,\ldots,x_n)}{q_m(x_1,\ldots,x_n)}\right)}{h\left(\frac{p_1(x_1,\ldots,x_n)}{q_1(x_1,\ldots,x_n)}, \ldots, \frac{p_m(x_1,\ldots,x_n)}{q_m(x_1,\ldots,x_n)}\right)} \tag{‡}$$

we can multiply through the numerator and denominator by suitably large powers of q_i to obtain polynomials on the numerator and denominator. After doing this, say we obtain $\theta \circ f|_{f^{-1}(U) \cap f^{-1}(V)} = \frac{g'}{h'}$, where $g', h' \in k[x_1, \ldots, x_n]$.

Now, we know that h is non-vanishing on V, i.e. $V(h) \cap V = \emptyset$, so then

$$\begin{aligned} V(h) \cap V = \emptyset &\Rightarrow f^{-1}(V(h) \cap V) = f^{-1}(\emptyset) = \emptyset \\ &\Rightarrow f^{-1}(V(h)) \cap f^{-1}(V) = \emptyset \\ &\Rightarrow V(\tilde{h}) \cap f^{-1}(V) = \emptyset \end{aligned}$$

Notice that h' and $\tilde{h}$ differ by a factor of terms $q_i^{n_i}$, i.e. $h' = \prod_{i=1}^n q_i^{n_i}\tilde{h}$ (since, in general, we will need to multiply (‡) through by more factors of q_i than (†) to ensure that *both* numerator and denominator are polynomials). But since the q_i are non-vanishing $V(h') = V(\tilde{h})$, so $V(h') \cap f^{-1}(V) = \emptyset$ too, i.e. h' is non-vanishing on $f^{-1}(V)$.

Thus we have shown that the composite $\theta \circ f$ is regular, i.e. f pulls back regular functions to regular functions. Hence f is a morphism. $\square$

Actually, using the more general version of this lemma (where we only require that the component functions be regular functions, not necessarily rational functions with non-vanishing denominator) we can give an even better characterization of morphism.

Lemma 3.1.31 *$f = (f_1, \ldots, f_m) : X \to Y$ is a morphism of varieties if and only if the components f_i are regular functions on X.*

Proof:
The generalised version of Lemma 3.1.30 gives the sufficiency. To see the necessity, suppose that f is a morphism. Now with π_i the projection function defined as follows $\pi_i(x_1, \ldots, x_m) = x_i$, we see that π_i is a regular function and thus $f_i = \pi_i \circ f$ must be regular.

$\square$

Definition 3.1.32 *A topological space X is called* **Noetherian** *if it satisfies the descending chain condition for closed subsets: that is, for any sequence $Y_1 \supseteq Y_2 \supseteq \ldots$ of closed subsets, there is an integer $r \in \mathbb{N}$ such that $Y_r = Y_{r+1} = \ldots$.*

We shall soon show that affine space, $\mathbb{A}^n$ is a Noetherian topological space.

Remark 3.1.33 *The definition of a Noetherian ring is very similar to the definition of a Noetherian algebra, given in the previous chapter. A ring R is* **Noetherian** *if it satisfies the ascending chain condition for ideals: that is, for any sequence $I_1 \subseteq I_2 \subseteq \ldots$ of ideals there is an integer $r \in \mathbb{N}$ such that $I_r = I_{r+1} = \ldots$.*

Lemma 3.1.34 *(The Hilbert Basis Theorem)*
If a ring R is Noetherian, then $R[X]$ is also Noetherian.

By inductively applying this result, we get the following.

Corollary 3.1.35 *$k[X_1, \ldots, X_n]$ is Noetherian*

Examples 3.1.36 *We have the following examples of Noetherian spaces:*

(a) $\mathbb{A}^n$ is a Noetherian topological space. Suppose $Y_1 \supseteq Y_2 \supseteq \ldots$ is a descending chain of closed subsets, then $I(Y_1) \subseteq I(Y_2) \subseteq \ldots$ is an ascending chain of ideals in $P = k[X_1, \ldots, X_n]$. However, P is a Noetherian ring, so this chain must eventually stabilize, say $I(Y_r) = I(Y_{r+1}) = \ldots$. Then using $Y_i = V(I(Y_i))$ we see that $Y_r = Y_{r+1} = \ldots$. So any descending chain of closed subsets must stabilize, giving the desired result.

(b) If a topological space is Noetherian, then so is any closed subspace.

Lemma 3.1.37 *In a Noetherian topological space X, every non-empty closed subset Y can be expressed as a finite union $Y = Y_1 \cup \ldots \cup Y_r$ of irreducible closed subsets Y_i. If we require that $Y_i \nsubseteq Y_j$ for $i \neq j$ then the Y_i are uniquely determined. They are called the* **irreducible components** *of Y.*

Remark 3.1.38 *We have the following observations on the above result:*

(a) There are only **finitely many** *irreducible components,*

(b) This decomposition is unique up to reordering,

(c) The irreducible components are the maximal irreducible subsets of the space,

(d) The irreducible components are closed (since if X is irreducible then so is $\overline{X}$)

As a corollary of the above result we see that every algebraic set can be written as a finite union of irreducible algebraic sets.

Definition 3.1.39 *If X is a topological space, we define the* **dimension** *of X (denoted $\dim X$) to be the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of distinct irreducible closed subsets of X. We define the* **dimension** *of a variety to be its dimension as a topological space.*

Definition 3.1.40 *In a ring R, the* **height** *of a prime ideal, $\mathfrak{p}$, is the supremum of all integers n such that there exists a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_n = \mathfrak{p}$ of distinct prime ideals. We define the* **Krull dimension** *of R to be the supremum of the heights of all prime ideals in R.*

Lemma 3.1.41 *We have the following facts about dimension:*

(a) For an algebraic set X, the dimension of X is equal to the Krull dimension of its coordinate ring $A(X)$

(b) The dimension of $\mathbb{A}^n$ is n

(c) If $U \neq \emptyset$ is open in an irreducible variety X, then $\dim U = \dim X$

(d) If $X = \bigcup_{i=1}^{n} U_i$ with the U_i irreducible, then $\dim X = \max_{i \in \{1,\ldots,n\}}\{\dim U_i\}$

(e) If $X \subseteq Y$ then $\dim X \leq \dim Y$, moreover if X is closed and Y is irreducible, then $X \subset Y$ implies $\dim X < \dim Y$

The following result may be well known, however we include a short proof. It is a result that, if it were not true, then something would be wrong with our notion of variety. Essentially it says that it doesn't matter how we view an r-dimensional affine space — as an affine space in its own right or as a vector subspace of a larger affine space — they both share the same properties as varieties.

Lemma 3.1.42 *An r-dimensional vector subspace W of $\mathbb{A}^n$ with $n > r$ is isomorphic as a variety to $\mathbb{A}^r$. In particular this means that W is irreducible and as a variety has dimension r.*

Proof:
A vector subspace W of $\mathbb{A}^n$ is the solution space of a system of homogeneous linear equations in the n unknowns $x_1, \dots, x_n$. Since W is r-dimensional, exactly $n - r$ of these linear equations are linearly independent.

If the homogeneous linear system is $\mathbf{Ax} = \mathbf{0}$, then by reducing the coefficient matrix $\mathbf{A}$ to row-echelon form and permuting some of the X_i if necessary, we may assume that the linear equations are $f_i = x_i + \sum_{j=n-r+1}^{n} a_{ij} x_j$ for $1 \leq i \leq n - r$.

The map $\phi : k[x_1, \dots, x_n] \to k[x_1, \dots, x_n]$ defined by

$$\phi(x_i) = \begin{cases} f_i, & 1 \leq i \leq r, \\ x_i, & r < i \leq n \end{cases}$$

is an automorphism. One can check that this map sends the prime ideal $(x_1, \dots, x_{n-r})$ to the prime ideal $(f_1, \dots, f_{n-r})$. Hence ϕ induces an isomorphism $k[x_1, \dots, x_n]/(x_1, \dots x_{n-r}) \cong k[x_1, \dots, x_n]/(f_1, \dots, f_{n-r})$. We compose this with the following isomorphisms $k[x_1, \dots, x_r] \cong k[x_{n-r}, \dots, x_n] \cong k[x_1, \dots, x_n]/(x_1, \dots, x_{n-r})$ to obtain the isomorphism $k[x_1, \dots, x_r] \cong k[x_1, \dots, x_n]/(f_1, \dots f_{n-r})$.

Now notice that $k[x_1, \dots, x_n]/(f_1, \dots, f_{n-r})$ is the coordinate ring of W and $k[x_1, \dots, x_r]$ is the coordinate ring of $\mathbb{A}^r$. The isomorphism of coordinate rings induces an isomorphism of the varieties W and $\mathbb{A}^r$, as mentioned after Proposition 3.1.23, (which implies that W and $\mathbb{A}^r$ are homeomorphic as topological spaces with the Zariski topology).

Since $\mathbb{A}^r$ is irreducible, so must W. Also the isomorphism between W and $\mathbb{A}^r$ shows that chains of closed irreducible subsets in W, with length n, correspond to such chains in $\mathbb{A}^r$ also having length n, and vice versa. This implies that, as a variety, the dimension of W and $\mathbb{A}^r$ must be the same. But by Lemma 3.1.41 (b), $\dim \mathbb{A}^r = r$. $\square$

Definition 3.1.43 *We define the* **local dimension at** $x \in X$ *as* $\dim_x X = \min\{\dim U : U$ *is a neighbourhood in* X *of* $x\}$

Lemma 3.1.44 $\dim_x X = \max\{\dim Z : Z$ *is an irreducible component of* X *containing* $x\}$

Definition 3.1.45 *A morphism* $f : X \to Y$ *is* **dominating** *if its image is dense in* Y*, i.e.* $Y = \overline{f(X)}$*.*

We shall now list several results from Mumford's book [25], which we shall use later. Note that Mumford adopts the more common definition of variety and requires his varieties to be irreducible. This should be kept in mind while reading these next few results.

Lemma 3.1.46 *([25, Chapter* 1 *§8 Proposition* 1*])*

If $f : X \to Y$ *is any morphism, let* $Z = \overline{f(X)}$*. Then* Z *is irreducible and the restricted morphism* $f' : X \to Z$ *is dominating.*

Lemma 3.1.47 *([25, Chapter* 1 *§8 Theorem* 3*])*

Let $f : X \to Y$ *be a dominating morphism of varieties and let* $r = \dim X - \dim Y$*. Then there exists a non-empty open set* $U \subset Y$ *such that:*

(i) $U \subseteq f(X)$

(ii) for all irreducible subsets $W \subseteq Y$ *such that* $W \cap U \neq \emptyset$*, and for all components* Z *of* $f^{-1}(W)$ *such that* $Z \cap f^{-1}(U) \neq \emptyset$

$$\dim Z = \dim W + r$$

$$\textit{or}\ \operatorname{codim}(Z \textit{ in } X) = \operatorname{codim}(W \textit{ in } Y)$$

Definition 3.1.48 *A function* $f : X \to \mathbb{Z}$ *is said to be* **upper semicontinuous** *if the set* $\{x \in X : f(x) \geq n\}$ *is closed in* X *for all* $n \in \mathbb{Z}$*.*

Lemma 3.1.49 *(essentially [25, Chapter* 1 *§8 Corollary* 3*])*

If $f : X \to Y$ *is a morphism of varieties then the function* $x \mapsto \dim_x f^{-1}(f(x))$ *is upper semicontinuous.*

Definition 3.1.50 *If V is a vector space and W a subset of V, then W is called a* **cone in** *V if W contains the zero vector and is closed under scalar multiplication.*

In Section 3.4 we shall require use of a lemma given in [8] which is apparently well-known. We give this lemma now. In [7], a sketch of a proof is given, where it is derived as a special case of Lemma 3.1.49 above.

Lemma 3.1.51 *Suppose X is a variety, V a vector space and we are given subsets $V_x \subseteq V$ for all $x \in X$. Suppose that*

(a) each V_x is a cone in V

(b) $\{(x, v) : v \in V_x\}$ is closed in $X \times V$

Then the map $x \mapsto \dim V_x$ is upper semicontinuous.

We shall on occasion want to talk about products of varieties, Crawley-Boevey notes in [7] that given two varieties X, Y then $X \times Y$ has the structure of a variety. All we shall do here is indicate how one can naturally view the product of locally closed subsets $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ as a locally closed subset $X \times Y \subseteq \mathbb{A}^{n+m}$.

Suppose that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are varieties, then they are both the intersection of open and closed sets and since the distinguished open subsets $D(f)$ form a basis for the Zariski topology we must be able to write each as follows: $X = V(\{e_i\}) \cap \Big(\bigcup_\beta D(f_\beta)\Big)$ and $Y = V(\{g_j\}) \cap \Big(\bigcup_\gamma D(h_\gamma)\Big)$. (Notice that we know there must only be finitely many of the indices i and j). The product of these is then the variety in $\mathbb{A}^{n+m}$ defined by $X \times Y = V(\{e_i, g_j\}) \cap \Big(\bigcup_\beta D(f_\beta)\Big) \cap \Big(\bigcup_\gamma D(h_\gamma)\Big)$.

It is important to realize, however, that the topology on the product variety is not the product topology from the topologies on each variety. For example, $\mathbb{A}^1 \times \mathbb{A}^1$ with the product topology has only points, horizontal and vertical lines for its closed sets, whereas the topology on the product variety $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$, has these and many more closed sets, in addition. Its closed sets are the vanishing sets of polynomials of two variables. In this case, where the varieties are, in fact, algebraic sets, this can be further explained by the fact that the coordinate ring of the product of the algebraic sets is isomorphic to the tensor product of the coordinate rings.

Lemma 3.1.52 *The product of two irreducible varieties is irreducible.*

Finally we give a quick review of the work already done on Alg_n the variety of n-dimensional algebras. This variety was studied in detail by Gabriel in [12], however it was known and used in other papers (see, for example, [11]) before this time. Since then, there has been more work done on it. However its study does not seem to have the popularity that the study of module varieties does.

General properties of Alg_n are given in [7, 12, 17]. The paper [18] also contains a short introduction to the variety Alg_n. The geometric classification problem for algebras of dimension n is equivalent to finding the irreducible components of Alg_n. Classification of algebras of dimension ≤ 4 is given in [12]. The case of algebras of dimension 5 is given in [21] and [13] lists some special irreducible components ("rigid" components) for Alg_6 working over the field $\mathbb{C}$.

Our review of this material will be brief since we shall study these ideas, for the case of superalgebras, in more detail in the body of the chapter. We remind the reader that when we say "algebra" without qualification, it shall mean a unitary associative algebra. On several occasions we will mention non-unitary associative algebras, in which case we shall make this clear.

On an n-dimensional vector space V, a (unitary associative) algebra structure on V gives rise to the set of structure constants $(\alpha_{ij}^k) \in \mathbb{A}^{n^3}$. Choose a basis for V, say $\{e_1, \ldots, e_n\}$, the **structure constants** are then determined by the multiplication on V so that

$$e_i e_j = \sum_{k=1}^{n} \alpha_{ij}^k e_k$$

Conversely, the structure constants induces an algebra structure on V, where the multiplication of basis vectors is given by the previous formula. Multiplication is then extended to the whole of V by linearity. It is important to notice that isomorphic algebras can give rise to different structure constants — even a single algebra structure on V may give rise to different structure constants, when using different bases. Structure constants correspond to an n-dimensional algebra with some given basis. Although an n-dimensional algebra with a given basis gives rise to a unique set of structure constants in $\mathbb{A}^{n^3}$, this correspondence isn't one-to-one, since a different basis on the same algebra may give rise to the same structure constants.

The structure constants must obey certain equations to reflect the fact that they represent *associative, unitary* algebra structures. In terms of the basis elements, these equations can be written:

$$e_1 e_i = e_i$$

$$e_i e_1 = e_i$$

$$(e_i e_j) e_k = e_i (e_j e_k)$$

Which translate into the following relations amongst the structure constants:

$$\alpha_{1i}^{j} - \delta_i^j = 0 \tag{$\triangle$.1}$$

$$\alpha_{i1}^{j} - \delta_i^j = 0 \tag{$\triangle$.2}$$

$$\sum_{l=1}^{n} (\alpha_{ij}^{l} \alpha_{lk}^{m} - \alpha_{il}^{m} \alpha_{jk}^{l}) = 0 \tag{$\triangle$.3}$$

Definition 3.1.53 *We define:*

(a) Alg_n *— the variety of (unitary associative) algebras, with the identity fixed at the first element of the basis — to be the variety in* $\mathbb{A}^{n^3}$*, which is cut out by the above 3 sets of equations, (△.1)–(△.3).*

(b) $\mathcal{S}_n$ *— the variety of associative (and perhaps non-unitary) algebras — to be the variety in* $\mathbb{A}^{n^3}$ *cut out by the equations in (△.3) above.*

(c) Alg'_n *— the variety of (unitary associative) algebras, without requiring the identity to be the first member of (or even in) the basis — to be* $\mathrm{Alg}'_n = \{(\alpha_{ij}^k \in \mathcal{S}_n : (\alpha_{ij}^k)$ *defines a unitary algebra* $\}$

Notice that Alg'_n was the variety originally studied by Gabriel in [12] and others. However following [18] we are more interested in the related variety Alg_n.

Lemma 3.1.54 *We have the following results about the above varieties:*
(a) Alg_n *and* $\mathcal{S}_n$ *are algebraic sets and hence affine varieties.*
(b) Alg'_n *is an affine variety.*

Proof:
Part (a) is easy. The proof of part (b) can be found in [12] or in [7]. The proof given in [7] is essentially identical to our proof of Lemma 3.2.4. □

We can equivalently write the multiplication of the algebra as an element $\mu \in \mathrm{Hom}(V \otimes V, V)$. Thus the structure constants $(\alpha_{ij}^k) \in \mathcal{S}_n$ or Alg'_n give rise to such an element and it must obey the following equation:

$$\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$$

If the structure constants, (α_{ij}^k), actually belong to Alg_n then this element must obey the following two equations also:

$$\mu(e_1 \otimes e_i) = e_i$$

$$\mu(e_i \otimes e_1) = e_i$$

Conversely, an element $\mu \in \mathrm{Hom}(V \otimes V, V)$ obeying the first equation above, along with a choice of basis, gives rise to structure constants (α_{ij}^k) which is a point in $\mathcal{S}_n$ or perhaps Alg_n'. If this element μ obeys all three equations above, then the structure constants (α_{ij}^k) is a point in Alg_n. We shall, on occasion, use this alternate notation instead of the structure constants.

Since Alg_n was defined with the identity fixed as the first element of the basis, when we consider an action on Alg_n we must use maps which send the first element of the basis to itself. Thus a subgroup G_n of GL_n acts on Alg_n not the whole group as one might expect. We can describe G_n for $n \geq 2$ as follows:

$$G_n = \left\{ \begin{pmatrix} 1 & b^T \\ 0 & \Sigma \end{pmatrix} : \Sigma \in \mathrm{GL}_{n-1}, b \in k^{n-1} \right\}$$

Now, there is an action of G_n on Alg_n. If we let $\Lambda = (\lambda_i^j) \in G_n$ and $(\nu_i^j) = \Lambda^{-1}$ this can be described as follows:

$$\Lambda \cdot (\alpha_{ij}^k) = \left(\sum_{l,p,q=1}^{n} \nu_l^k \alpha_{pq}^l \lambda_i^p \lambda_j^q \right)$$

or in the alternate notation:

$$\Lambda \cdot \mu = \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1})$$

There are also actions of GL_n on $\mathcal{S}_n$ and Alg_n' which are given by the same formulae as above, except now allowing $\Lambda \in \mathrm{GL}_n$. Any of these actions may be referred to as the "**transport of structure actions**" .

Lemma 3.1.55 *The orbits under G_n in Alg_n and the orbits under GL_n in Alg_n' can be identified with the isomorphism classes of n-dimensional (unitary associative) algebras.*

Thus, for an n-dimensional algebra, A, we shall speak of its orbit in either Alg_n denoted by $G_n \cdot A$, or in Alg_n' denoted by $\mathrm{GL}_n \cdot A$, and mean the orbit in Alg_n or Alg'_n respectively, which is identified with the isomorphism class of A as mentioned in the lemma above.

We now introduce a very important idea in the study of these varieties — the idea of degeneration. It is very useful when trying to determine the irreducible components of these varieties.

Definition 3.1.56 *For n-dimensional algebras A and B, if there exists a point $(\alpha_{ij}^k) \in G_n \cdot B$ such that $(\alpha_{ij}^k) \in \overline{G_n \cdot A}$ then we say that A* **degenerates** *to B and denote this by $A \to B$.*

This notion of degeneration extends to a partial order ($\leq_{degr}$) on the isomorphism classes of n-dimensional algebras. We write $B \leq_{degr} A$ if and only if A degenerates to B.

Depending on which variety we use (Alg_n or Alg'_n) we could potentially end up with two different degeneration partial orders on the isomorphism classes of n-dimensional algebras. However, it would be nice to think that the idea of degeneration was intrinsic to the algebra, and these two partial orders would thus be the same. This is indeed the case, as remarked in [18]. So, in this way, these two varieties share a similar geometry.

Definition 3.1.57 *If A and B are n-dimensional algebras, a* **specialization of** *A* **to** *B is the following situation: one makes a change of basis in A to a "variable" basis, i.e. one involving some unknown t, such that the point of Alg_n (or Alg'_n) obtained by structural transport is given by polynomial functions in t and lies in the orbit of A for $t \neq 0$, yet at $t = 0$ lies in the orbit B.*

If there is a specialization from A to B, then there must also be a degeneration from A to B (we prove this in the case of superalgebras in Corollary 3.4.3. It should be clear how to alter that proof to apply to the algebra case).

Lemma 3.1.58 *([12, Proposition 2.2])*

In Alg'_n there is one closed orbit, which is the orbit which is identified with isomorphism class of $k[X_1, \ldots, X_{n-1}]/(X_1, \ldots, X_{n-1})^2$.

The next lemma follows from the previous lemma without too much work. (The ideas for this proof are contained in the proof of Proposition 3.4.5).

Lemma 3.1.59 Alg'_n *is connected for all* n

The above two results were proved for Alg'_n in [12], but the proofs carry over to the case of Alg_n too. The proof of the first result follows from these facts:

- each algebra structure in Alg_n degenerates to the algebra structure on $k[X_1, \ldots, X_{n-1}]/(X_1, \ldots, X_{n-1})^2$
- thus the orbit of $k[X_1, \ldots, X_{n-1}]/(X_1, \ldots, X_{n-1})^2$ has minimal dimension
- orbits of minimal dimension are closed

We would like to think that the following result also holds in Alg_n but we have not checked through the details.

Lemma 3.1.60 *([12, Corollary 2.5])*
The orbit of an n*-dimensional algebra* A *is open in* Alg'_n *if* $H^2(A, A) = 0$, *(where* $H^2(A, A)$ *is the Hochschild cohomology group of the algebra* A*).*

The converse of the above lemma need not hold in general however, e.g. in the case of 5-dimensional algebras Mazzola mentions that algebra (26) in his classification has an open orbit, yet $H^2((26), (26))$ is two-dimensional.

To show that A does not degenerate to B, it is enough to show a closed subset containing the orbit of A, disjoint from the orbit of B. So we are interested in closed subsets of Alg_n or Alg'_n which are stable under the actions of G_n or GL_n respectively. The following provides us with some such subsets. Since Alg_n is a closed subvariety of Alg'_n the analogous sets in Alg_n are also closed.

Lemma 3.1.61 *([12, Proposition 2.7])*

The following subsets of Alg'_n *are Zariski-closed (where* s *is any fixed value):*

(a) $\{A \in \mathrm{Alg}'_n : \dim J(A) \geq s\}$

(b) $\{A \in \mathrm{Alg}'_n : \dim Z(A) \geq s\}$

(c) $\{A \in \mathrm{Alg}'_n :$ *number of blocks* $\leq s\}$

(d) $\{A \in \mathrm{Alg}'_n : A$ *is basic, i.e.* $A/J(A) \cong k^t$ *for some* $t\}$

For the degeneration diagrams and a list of the irreducible components of Alg'_n for $n \leq 4$ see [12]. For a list of the irreducible components in the case $n = 5$ see [21].

This concludes our preliminaries section.

3.2 The variety Salg_n and its properties

In the preliminaries, we summarised the work done on Alg_n. From this, it should be clear that the idea of using structure constants in some variety to represent an algebraic object is very useful. The question is then, how do we modify the analysis used to study Alg_n so that we have some sort of structure constants to represent an n-dimensional superalgebra structure on V? To answer this, we remark that a superalgebra $A = A_0 \oplus A_1$ is the same as the pair (A, σ) where A is an algebra and σ is an algebra involution on A. Given a superalgebra $A = A_0 \oplus A_1$ the $\mathbb{Z}_2$-grading induces the **main involution**, given by $\sigma(a_0 + a_1) = a_0 - a_1$ where $a_i \in A_i$. Conversely any algebra involution σ induces a $\mathbb{Z}_2$-grading on A via $A_0 = \{a \in A : \sigma(a) = a\}, A_1 = \{a \in A : \sigma(a) = -a\}$ (and the main involution induced from this $\mathbb{Z}_2$-grading is σ).

The algebra involution σ on an algebra A (as a linear map from A to itself) may be described by the set of constants $(\gamma_i^j) \in \mathbb{A}^{n^2}$ satisfying $\sigma(e_i) = \sum_{j=1}^n \gamma_i^j e_j$. It follows then, that to each superalgebra, (A, σ), we can associate a set of augmented **structure constants** $(\alpha_{ij}^k, \gamma_i^j) \in \mathbb{A}^{n^3+n^2}$ where (α_{ij}^k) are the structure constants determined by the algebra structure of A and (γ_i^j) the constants determined by the $\mathbb{Z}_2$-grading in the above manner. For brevity we simply refer to the $(\alpha_{ij}^k, \gamma_i^j)$ as "structure constants" from here on. However it is not true that an arbitrary set of augmented structure constants can give rise to a superalgebra. The structure constants must obey certain relations to reflect how we have defined a superalgebra.

As a superalgebra (A, σ) must in particular be a unitary associative algebra, we have a multiplicative identity which we always take to be the first element of our basis, e_1. Then to be a unitary associative algebra we have the following conditions:

$$e_1 e_i = e_i$$

$$e_i e_1 = e_i$$

$$(e_i e_j)e_k = e_i(e_j e_k)$$

Which translate into the following relations amongst the structure constants:

$$\alpha_{1i}^{j} - \delta_i^j = 0 \tag{3.1}$$

$$\alpha_{i1}^{j} - \delta_i^j = 0 \tag{3.2}$$

$$\sum_{l=1}^{n}(\alpha_{ij}^{l}\alpha_{lk}^{m} - \alpha_{il}^{m}\alpha_{jk}^{l}) = 0 \tag{3.3}$$

For σ to be an algebra involution means that:

$$\sigma(e_1) = e_1$$

$$\sigma(e_i e_j) = \sigma(e_i)\sigma(e_j)$$

$$\sigma^2(e_i) = e_i$$

These become the following relations in terms of the structure constants:

$$\gamma_1^j - \delta_1^j = 0 \tag{3.4}$$

$$\sum_{k=1}^{n}\alpha_{ij}^{k}\gamma_k^m - \sum_{k,l=1}^{n}\gamma_i^k\gamma_j^l\alpha_{kl}^{m} = 0 \tag{3.5}$$

$$\sum_{j=1}^{n}\gamma_i^j\gamma_j^k - \delta_i^k = 0 \tag{3.6}$$

It is precisely those structure constants obeying the relations (3.1)–(3.6) given above which give rise to superalgebras.

Definition 3.2.1 *The equations (3.1)–(3.6) given above cut out a variety in* $\mathbb{A}^{n^3+n^2}$ *which we shall call* Salg_n *— the* **variety of n-dimensional superalgebras**.

It shall be our interest for the rest of the chapter to study the geometry of Salg_n. We will see that the geometry of Salg_n is influenced by that of Alg_n — but Salg_n also has a more rich geometrical structure.

Definition 3.2.2 *We define* SA_n *— the* **variety of n-dimensional superalgebras not requiring existence of a unit** *— to be the subvariety of* $\mathbb{A}^{n^3+n^2}$ *cut out by equations (3.3), (3.5) and (3.6).*

One checks that if A is a unitary algebra and $\sigma : A \to A$ satisfies $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma^2 = \mathrm{id}_A$ then $\sigma(1_A) = 1_A$ (This follows from the more general fact that any invertible homomorphism $\sigma : A \to B$ between rings with unit must map the identity to the identity, i.e. $\sigma(1_A) = 1_B$), which after a little thought shows that $\mathrm{Salg}_n = \mathrm{SA}_n \cap V(\{\alpha^j_{1i} - \delta^j_i, \alpha^j_{i1} - \delta^j_i\})$. So we obtain the following result:

Lemma 3.2.3 Salg_n *is a closed subvariety of* SA_n.

A point $(\alpha^k_{ij}, \gamma^j_i)$ in SA_n clearly gives rise to elements $\mu \in \mathrm{Hom}(V \otimes V, V)$ and $\sigma \in \mathrm{Hom}(V, V)$ from the multiplication structure constants (α^k_{ij}) and the $\mathbb{Z}_2$-grading structure constants (γ^j_i) respectively. These elements obey the following equations:

$$\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu) \tag{3.7}$$

$$\mu \circ (\sigma \otimes \sigma) = \sigma \circ \mu \tag{3.8}$$

$$\sigma \circ \sigma = \mathrm{id} \tag{3.9}$$

If the point $(\alpha_{ij}^k, \gamma_i^j)$ not only belongs to SA_n but in fact belongs to Salg_n then the elements μ, σ described earlier satisfy the following three equations in addition to (3.7)–(3.9) above:

$$\mu(e_1 \otimes e_i) = e_i \tag{3.10}$$

$$\mu(e_i \otimes e_1) = e_i \tag{3.11}$$

$$\sigma(e_1) = e_1 \tag{3.12}$$

Conversely, assuming that V has a chosen basis, then any pair elements $\mu \in \mathrm{Hom}(V \otimes V, V)$ and $\sigma \in \mathrm{Hom}(V, V)$ obeying equations (3.7)–(3.9) above give rise to a point $(\alpha_{ij}^k, \gamma_i^j)$ in SA_n. If these elements obey equations (3.10)–(3.12) as well, then the point $(\alpha_{ij}^k, \gamma_i^j)$ actually belongs to Salg_n. We shall, on occasion, use this alternate notation to represent points in Salg_n or SA_n — it often turns out to be more straightforward to do calculations with this. (The proof of Lemma 3.2.6 uses this notation. While the proof could be done using the structure constants notation, it is far more cumbersome).

It is important to notice the way that we have defined Salg_n — requiring the identity to be fixed — is analogous to the way Alg_n is defined in [18], but is not analogous to the way Alg_n was defined in [12] (the definition given in [12] corresponds to our variety Alg'_n). We define Salg'_n to be the subset of SA_n which consists of superalgebras with unit, but not necessarily requiring the unit to be the first element (or even in) the basis (this is to distinguish between the two possible definitions for Salg_n). It is Salg'_n whose definition is analogous to the case treated in [12].

Lemma 3.2.4 Salg'_n *is an open affine subvariety of* SA_n.

Proof:
Our proof of this lemma follows from making minor alterations to the proof given in [7] for the algebra case. One should note that the alterations

are trivial, since the question of existence of an identity depends only on the underlying algebra of the given superalgebra. Here we use the alternate notation and consider points in SA_n as a pair $(\mu, \sigma) \in \mathrm{Hom}(V \otimes V, V) \times \mathrm{Hom}(V, V)$, the first element giving the multiplication of the superalgebra and the second giving the $\mathbb{Z}_2$-grading.

Suppose that an algebra A has multiplication given by $\mu \in \mathrm{Hom}(V \otimes V, V)$. Denote by l_a^μ and r_a^μ respectively the maps defined by left and right multiplication by an element $a \in A$, that is, $l_a^\mu(x) = \mu(a \otimes x), r_a^\mu(x) = \mu(x \otimes a)$. As noted in [7], A has 1 if and only if l_a^μ and r_a^μ are invertible, in which case the unit is $[l_a^\mu]^{-1}(a)$.

The set $D_a = \{(\mu, \sigma) \in \mathrm{SA}_n : \det(l_a^\mu)\det(r_a^\mu) \neq 0\}$ is open in SA_n and $\mathrm{Salg}'_n = \bigcup_a D_a$, by the above. Thus Salg'_n is open in SA_n.

If we denote by 1^μ the unit for the multiplication given by μ, then the map $\mathrm{Salg}'_n \to \mathbb{A}^n$ given by $(\mu, \sigma) \mapsto 1^\mu$ is a regular map, since on D_a it is equal to $\mu \mapsto [l_a^\mu]^{-1}(a)$ which is a quotient of polynomials, with non-vanishing denominators.

So $\mathrm{Salg}_n \cong \{((\mu, \sigma), x) \in \mathrm{SA}_n \times \mathbb{A}^n : x \text{ is a unit for } \mu\}$ via the morphisms $(\mu, \sigma) \mapsto ((\mu, \sigma), 1^\mu)$ and $((\mu, \sigma), x) \mapsto (\mu, \sigma)$. The set on the right is closed since for $x \in \mathbb{A}^n$ one can find c_j such that $x = \sum_{i=1}^n c_j e_j$. The conditions $\mu(x \otimes e_i) = e_i$ and $\mu(e_i \otimes x) = e_i$ then translate into $\sum_{j=1}^n c_j \alpha_{ji}^k = \delta_i^k$ and $\sum_{j=1}^n c_j \alpha_{ij}^k = \delta_i^k$. Thus Salg'_n is isomorphic to a closed subset of an affine space, so is an affine variety. □

Similarly to the situation remarked in [18], since for our definition of Salg_n we require that the identity be the first element in the basis of any superalgebra, a subgroup G_n of GL_n acts on Salg_n (not the full group GL_n as one may expect). This action is induced by considering what happens to the structure constants when one makes a basis change. As the identity must be the first element in the basis, this means that the first column of the matrix describing the basis change must be $\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}^T$ (identifying the given basis $\{e_1 = 1, e_2, \dots, e_n\}$ with the standard basis vectors for k^n). Hence we can describe G_n for $n \geq 2$ as follows: $G_n =$

$$\left\{\begin{pmatrix}1 & b^T \\ 0 & \Sigma\end{pmatrix} : \Sigma \in \mathrm{GL}_{n-1}, b \in k^{n-1}\right\}.$$

Remark 3.2.5 *If one so desired, our methods could be modified to study* Salg'_n *with it's action of* GL_n*. However, one would hope that the geometry of both spaces are very similar — in particular we would like the degeneration partial orders induced in each space to coincide (the degeneration partial order will be introduced in the Section 3.3). We would hope that such properties are intrinsic to the superalgebras and thus not depend on the way in which they are represented by a particular variety. We have not investigated this thoroughly, although in [18], it is remarked that this is the case for the degeneration partial orders in* Alg_n *and* Alg'_n*.*

Let $\Lambda = (\lambda_i^j) \in G_n$ and $(\nu_i^j) = \Lambda^{-1}$. Then we can describe the action of G_n on Salg_n as follows:

$$\Lambda \cdot (\alpha_{ij}^k, \gamma_i^j) = (\sum_{l,p,q=1}^{n} \nu_l^k \alpha_{pq}^l \lambda_i^p \lambda_j^q, \sum_{k,l=1}^{n} \nu_k^j \gamma_l^k \lambda_i^l) = (\alpha_{ij}'^k, \gamma_i'^j)$$

Firstly, recall that the formula for the inverse of a matrix means that we can express the entries ν_i^j of the matrix Λ^{-1} as a polynomial in the entries λ_i^j of the matrix Λ and $1/\det(\Lambda)$. Then the above formula expresses the new structure constants $\alpha_{ij}'^k, \gamma_i'^j$ in Salg_n as a polynomial in the old structure constants $\alpha_{ij}^k, \gamma_i^j$, the entries of the matrix $\Lambda \in G_n$ and $1/\det(\Lambda)$ which has non-vanishing denominator. Hence we may apply Lemma 3.1.30 to see that the action gives us a morphism $G_n \times \mathrm{Salg}_n \to \mathrm{Salg}_n$. The same reasoning also shows that the transport of structure action on Alg_n gives a morphism $G_n \times \mathrm{Alg}_n \to \mathrm{Alg}_n$.

Notice that if one uses the alternate notation, writing the multiplication as an element μ of $\mathrm{Hom}_k(V \otimes V, V)$ and the $\mathbb{Z}_2$-grading σ as an element of $\mathrm{Hom}_k(V, V)$, then the action of $\Lambda \in G_n$ on Salg_n is given by:

$$\Lambda \cdot (\mu, \sigma) = (\Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}), \Lambda \circ \sigma \circ \Lambda^{-1})$$

Then one sees instantly that the action of G_n on Salg_n is simply the transport of structure action from Alg_n on the first component (which gives the algebra structure) and conjugation on the second component (which gives the $\mathbb{Z}_2$-grading).

We may refer to the above action of G_n on Salg_n as the **transport of structure action**. However as it is the only action of G_n on Salg_n considered here, we shall often simply refer to it as the action of G_n on Salg_n

Lemma 3.2.6 *The transport of structure action on* Salg_n *is well-defined.*

Proof:
First we check that this action is well-defined. Let $\mu' = \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1})$ and $\sigma' = \Lambda \circ \sigma \circ \Lambda^{-1}$. Then as $\Lambda \cdot (\mu, \sigma) = (\mu', \sigma')$, showing that the action is well-defined amounts to showing that when the pair (μ, σ) satisfies equations (3.7)–(3.12), so too must (μ', σ').

Now note that since $\Lambda \in G_n$ then $\Lambda e_1 = e_1$ and $\Lambda^{-1} e_1 = e_1$ too.

$$\begin{aligned}
\mu'(e_1 \otimes e_i) &= (\Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}))(e_1 \otimes e_i) \\
&= (\Lambda \circ \mu)(\Lambda^{-1} e_1 \otimes \Lambda^{-1} e_i) \\
&= \Lambda(\mu(e_1 \otimes \Lambda^{-1} e_i)) \\
&= \Lambda(\Lambda^{-1} e_i) \\
&= e_i
\end{aligned}$$

$$\begin{aligned}
\mu'(e_i \otimes e_1) &= (\Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}))(e_i \otimes e_1) \\
&= (\Lambda \circ \mu)(\Lambda^{-1} e_i \otimes \Lambda^{-1} e_1) \\
&= \Lambda(\mu(\Lambda^{-1} e_i \otimes e_1)) \\
&= \Lambda(\Lambda^{-1} e_i) \\
&= e_i
\end{aligned}$$

$$
\begin{aligned}
\mu' \circ (\mu' \otimes \mathrm{id}) &= \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \circ ((\Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1})) \otimes \mathrm{id}) \\
&= \Lambda \circ \mu \circ ((\Lambda^{-1} \circ \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1})) \otimes (\Lambda^{-1} \circ \mathrm{id})) \\
&= \Lambda \circ \mu \circ ((\mu \circ (\Lambda^{-1} \otimes \Lambda^{-1})) \otimes (\mathrm{id} \circ \Lambda^{-1})) \\
&= \Lambda \circ \mu \circ (\mu \otimes \mathrm{id}) \circ (\Lambda^{-1} \otimes \Lambda^{-1} \otimes \Lambda^{-1}) \\
&= \Lambda \circ \mu \circ (\mathrm{id} \otimes \mu) \circ (\Lambda^{-1} \otimes \Lambda^{-1} \otimes \Lambda^{-1}) \\
&= \Lambda \circ \mu \circ (\Lambda^{-1} \otimes (\mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}))) \\
&= \Lambda \circ \mu \circ (\Lambda^{-1} \otimes (\Lambda^{-1} \circ \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}))) \\
&= \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \circ (\mathrm{id} \otimes (\Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}))) \\
&= \mu' \circ (\mathrm{id} \otimes \mu')
\end{aligned}
$$

$$
\begin{aligned}
\sigma'(e_1) &= (\Lambda \circ \sigma \circ \Lambda^{-1})(e_1) \\
&= \Lambda(\sigma(\Lambda^{-1}(e_1))) \\
&= \Lambda(\sigma(e_1)) \\
&= \Lambda(e_1) \\
&= e_1
\end{aligned}
$$

$$
\begin{aligned}
\mu' \circ (\sigma' \otimes \sigma') &= \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \circ ((\Lambda \circ \sigma \circ \Lambda^{-1}) \otimes (\Lambda \circ \sigma \circ \Lambda^{-1})) \\
&= \Lambda \circ \mu \circ ((\Lambda^{-1} \circ \Lambda \circ \sigma \circ \Lambda^{-1}) \otimes (\Lambda^{-1} \circ \Lambda \circ \sigma \circ \Lambda^{-1})) \\
&= \Lambda \circ \mu \circ ((\sigma \circ \Lambda^{-1}) \otimes (\sigma \circ \Lambda^{-1})) \\
&= \Lambda \circ \mu \circ (\sigma \otimes \sigma) \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \\
&= \Lambda \circ \sigma \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \\
&= \Lambda \circ \sigma \circ \Lambda^{-1} \circ \Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}) \\
&= \sigma' \circ \mu'
\end{aligned}
$$

$$\begin{aligned}\sigma' \circ \sigma' &= \Lambda \circ \sigma \circ \Lambda^{-1} \circ \Lambda \circ \sigma \circ \Lambda^{-1} \\ &= \Lambda \circ \sigma \circ \sigma \circ \Lambda^{-1} \\ &= \Lambda \circ \mathrm{id} \circ \Lambda^{-1} \\ &= \Lambda \circ \Lambda^{-1} \\ &= \mathrm{id}\end{aligned}$$

This shows that the action is well-defined.Finally, we show that this is indeed an action of G_n on Salg_n. Note, I_n the $n \times n$ identity matrix, is the identity of the group G_n.

$$\begin{aligned}I_n \cdot (\mu, \sigma) &= (I_n \circ \mu \circ (I_n^{-1} \otimes I_n^{-1}), I_n \circ \sigma \circ I_n^{-1}) \\ &= (I_n \circ \mu \circ (I_n \otimes I_n), I_n \circ \sigma \circ I_n) \\ &= (\mu, \sigma)\end{aligned}$$

Let $\Gamma, \Delta \in G_n$

$$\begin{aligned}\Gamma \cdot (\Delta \cdot (\mu, \sigma)) &= \Gamma \cdot (\Delta \circ \mu \circ (\Delta^{-1} \otimes \Delta^{-1}), \Delta \circ \sigma \circ \Delta^{-1}) \\ &= (\Gamma \circ (\Delta \circ \mu \circ (\Delta^{-1} \otimes \Delta^{-1})) \circ (\Gamma^{-1} \otimes \Gamma^{-1}), \Gamma \circ (\Delta \circ \sigma \circ \Delta^{-1}) \circ \Gamma^{-1}) \\ &= (\Gamma \circ \Delta \circ \mu \circ (\Delta^{-1} \otimes \Delta^{-1}) \circ (\Gamma^{-1} \otimes \Gamma^{-1}), \Gamma \circ \Delta \circ \sigma \circ \Delta^{-1} \circ \Gamma^{-1}) \\ &= (\Gamma \circ \Delta \circ \mu \circ ((\Delta^{-1} \circ \Gamma^{-1}) \otimes (\Delta^{-1} \circ \Gamma^{-1})), \Gamma \circ \Delta \circ \sigma \circ \Delta^{-1} \circ \Gamma^{-1}) \\ &= ((\Gamma \circ \Delta) \circ \mu \circ ((\Gamma \circ \Delta)^{-1} \otimes (\Gamma \circ \Delta)^{-1}), (\Gamma \circ \Delta) \circ \sigma \circ (\Gamma \circ \Delta)^{-1}) \\ &= (\Gamma \circ \Delta) \cdot (\mu, \sigma)\end{aligned}$$

This completes the proof. □

Lemma 3.2.7 *The orbits of* Salg_n *under the action of* G_n *can be identified with the isomorphism classes of* n*-dimensional superalgebras.*

Proof:
Suppose A is a superalgebra with structure constants $(\alpha_{ij}^k, \gamma_i^j)$ and B is a superalgebra with structure constants $(\beta_{ij}^k, \epsilon_i^j)$. The result follows since $A \cong B$ if and only if $(\alpha_{ij}^k, \gamma_i^j)$ and $(\beta_{ij}^k, \epsilon_i^j)$ belong to the same orbit — this is what we shall show. If $A \cong B$, say the isomorphism (of superalgebras) is given by the linear map represented by the matrix Λ, then $(\beta_{ij}^k, \epsilon_i^j) = \Lambda \cdot (\alpha_{ij}^k, \gamma_i^j)$ so that the structure constants of A and B are in the same orbit. Conversely if the structure constants of A and B belong to the same orbit, then there exists $\Lambda \in G_n$ such that $(\beta_{ij}^k, \epsilon_i^j) = \Lambda \cdot (\alpha_{ij}^k, \gamma_i^j)$, then one can see that the linear map represented by the matrix Λ gives an isomorphism of superalgebras showing $A \cong B$. $\square$

For an n-dimensional superalgebra A, we will sometimes use $G_n \cdot A$ to represent the orbit in Salg_n which the isomorphism class of A can be identified with. If in some basis the superalgebra A has structure constants $(\alpha_{ij}^k, \gamma_i^j)$ then $G_n \cdot A = G_n \cdot (\alpha_{ij}^k, \gamma_i^j)$.

There are two interesting morphisms between Salg_n and Alg_n (the reader can use Lemma 3.1.30 to see that they are indeed morphisms). They arise from the observations that: any n-dimensional superalgebra may be regarded as an n-dimensional algebra and any n-dimensional algebra can be endowed with the trivial $\mathbb{Z}_2$-grading making it into an n-dimensional superalgebra.

The first morphism: $U : \mathrm{Salg}_n \to \mathrm{Alg}_n$ is defined by $(\alpha_{ij}^k, \gamma_i^j) \mapsto (\alpha_{ij}^k)$. This can be viewed as the composition of the projection onto the subset of $\mathbb{A}^{n^3+n^2}$ defined by $\gamma_i^j = 0$ followed by the natural identification of $\mathbb{A}^{n^3} \times \{0\}$ with $\mathbb{A}^{n^3}$. This is a "forgetful" map — it forgets the superalgebra structure on V and only remembers the algebra structure on V.

The second morphism: $i : \mathrm{Alg}_n \to \mathrm{Salg}_n$ is defined by $(\alpha_{ij}^k) \mapsto (\alpha_{ij}^k, \delta_i^j)$ where δ_i^j is the Kronecker delta function defined earlier. This takes an algebra structure on V and endows it with the trivial $\mathbb{Z}_2$-grading making it a superalgebra on V.

Notice that the subset of Salg_n consisting of superalgebras with the trivial $\mathbb{Z}_2$-grading is a closed subset of Salg_n and is given by $V(\{\gamma_i^j - \delta_i^j\}) \cap \mathrm{Salg}_n$. The morphism i above identifies Alg_n with this subset. This result is part of the following proposition.

Proposition 3.2.8 *The morphisms U and i described above are continuous closed maps. Moreover i provides an isomorphism of Alg_n with the closed subset of Salg_n consisting of the superalgebras with the trivial $\mathbb{Z}_2$-grading.*

Proof:
As morphisms we know instantly that both maps are continuous.

A closed set in Alg_n is of the form $C = V(\{f_1, \ldots, f_m\}) \cap \mathrm{Alg}_n$ where f_i for $1 \leq i \leq m$ is a polynomial in $\alpha_{11}^1, \ldots, \alpha_{nn}^n$. $i(C) = V(\{f_1, \ldots, f_m, \gamma_i^j - \delta_i^j\}) \cap \mathrm{Salg}_n$. Let $D = V(\{f_1, \ldots, f_m\}) \cap \mathrm{Salg}_n$ then $U(D) = V(\{f_1', \ldots, f_{m'}'\}) \cap \mathrm{Alg}_n$ where we obtain the f_i' from the f_i as follows. If f_i is a polynomial containing only γ_i^j then we omit it in this new set of polynomials, otherwise we obtain f_i' from f_i by setting $\gamma_i^j = 0$ and we let f_i' be the polynomial so obtained. We also omit repeats of any of the polynomials f_i'. Thus we obtain the set $\{f_1', \ldots, f_{m'}'\}$ of polynomials. This verifies that U and i are closed maps.

Let $W = V(\{\gamma_i^j - \delta_i^j\}) \cap \mathrm{Salg}_n$. This is the subset of Salg_n consisting of the trivially $\mathbb{Z}_2$-graded superalgebras. One can check that $U|_W \circ i = \mathrm{id}_{\mathrm{Alg}_n}$ and $i \circ U|_W = \mathrm{id}_W$. Thus it follows that i is an isomorphism of Alg_n with the subset of Salg_n consisting of superalgebras with the trivial $\mathbb{Z}_2$-grading.

$\square$

Suppose that A is a superalgebra and B is an algebra. We may formalise the notion of underlying algebra, which we have already mentioned, and shall say that A is a superalgebra on B, or B is the **underlying algebra** of A in the case that $U(A) = B$. This simply means that by forgeting the $\mathbb{Z}_2$-grading on A we are left with the algebra B.

Notice that we have G_n-actions on both Salg_n and Alg_n. Now, one can quickly check (this is probably easier in the (μ, σ) notation) that for $\Lambda \in G_n$ $U(\Lambda \cdot (\alpha_{ij}^k, \gamma_i^j)) = \Lambda \cdot U((\alpha_{ij}^k, \gamma_i^j))$ and $i(\Lambda \cdot (a_{ij}^k)) = \Lambda \cdot i((\alpha_{ij}^k))$, which shows that the morphisms U and i are G_n-equivariant.

After noting this G_n-equivariance of U, we may obtain a corollary to Proposition 3.2.8 by applying the following standard fact from General Topology: If $f : X \to Y$ is a closed continuous map, then for any $Z \subseteq X$ $f\left(\overline{Z}\right) = \overline{f(Z)}$. We proceed as follows:

$$\begin{aligned} U\left(\overline{G_n \cdot (\alpha_{ij}^k, \gamma_i^j)}\right) &= \overline{U(G_n \cdot (\alpha_{ij}^k, \gamma_i^j))} \\ &= \overline{G_n \cdot U(\alpha_{ij}^k, \gamma_i^j))} \\ &= \overline{G_n \cdot (\alpha_{ij}^k)} \end{aligned}$$

This proves the following:

Corollary 3.2.9 $U\left(\overline{G_n \cdot (\alpha_{ij}^k, \gamma_i^j)}\right) = \overline{G_n \cdot (\alpha_{ij}^k)}$.

Suppose that one has a superalgebra A with $\dim A_0 = i$ and $\mathbb{Z}_2$-grading given by the algebra involution σ. Now change to a homogeneous basis (say by a linear map represented by the matrix Λ), which clearly has $\mathbb{Z}_2$-grading σ' given by the linear map represented by the diagonal matrix with 1 for the first i entries and -1 for the last $n-i$ entries. From the above we have $\sigma' = \Lambda\sigma\Lambda^{-1}$ (identifying the $\mathbb{Z}_2$-gradings with their matrix representatives), so $\sigma = \Lambda^{-1}\sigma'\Lambda$. Now recall that the trace and determinant tr and $\det$ are given by polynomials in the entries of a matrix, and also the standard facts $\mathrm{tr}(AB) = \mathrm{tr}(BA), \det(AB) = \det(A)\det(B)$. Thus $\mathrm{tr}(\sigma) = \mathrm{tr}(\Lambda^{-1}\sigma'\Lambda) = \mathrm{tr}(\sigma'\Lambda\Lambda^{-1}) = \mathrm{tr}(\sigma') = i - (n-i) = 2i - n$ and $\det(\sigma) = \det(\Lambda^{-1}\sigma'\Lambda) = \det(\Lambda^{-1})\det(\sigma')\det(\Lambda) = \det(\sigma') = (-1)^{n-i}$.

We now define Salg_n^i to be the subset of Salg_n consisting of the superalgebras A with $\dim A_0 = i$. Obviously we have $\mathrm{Salg}_n = \bigcup_{i=1}^n \mathrm{Salg}_n^i$. Hence, from above, the trace and determinant are constant on Salg_n^i. It is clear that these subsets must be disjoint. We are interested in when these subsets are also closed. The following lemma gives some sufficient conditions for this to be the case.

Before stating the next couple of results we mention how vital the assumption that $\mathrm{ch}(k) \neq 2$ is to Lemma 3.2.10 and Proposition 3.2.12. These are very basic results about the geometry of Salg_n — the study of Salg_n over an algebraically closed field k with $\mathrm{ch}(k) = 2$ would require new techniques as the proofs of these two results do not work in the case $\mathrm{ch}(k) = 2$.

Lemma 3.2.10 *The sets* Salg_n^i *are closed subsets of* Salg_n *in the following situations:*

(a) $\mathrm{ch}(k) = p$ *and* $n \leq 2p$

(b) $\mathrm{ch}(k) = 0$ *(with no restriction on* n *in this case)*

(c) $n \leq 6$ *(for any algebraically closed field* k *with* $\mathrm{ch}(k) \neq 2$*)*

Proof:
Define $\mathrm{S}_n^i = V(\{\sum_{j=1}^n \gamma_j^j - (2i - n), \sum_\pi \mathrm{sgn}(\pi)\gamma_1^{\pi(1)} \ldots \gamma_n^{\pi(n)} - (-1)^{n-i}\}) \cap \mathrm{Salg}_n$ for $i \in \{1, \ldots, n\}$, (where $\mathrm{sgn}(\pi)$ denotes the signature of the permutation π, and the sum is taken over all permutations of $\{1, \ldots, n\}$). Thus the S_n^i are closed subsets of Salg_n. From the statements above, it is clear that $\mathrm{Salg}_n^i \subseteq \mathrm{S}_n^i$. The first polynomial $\sum_{j=1}^n \gamma_j^j$ represents the trace of the $\mathbb{Z}_2$-grading and the second $\sum_\pi \mathrm{sgn}(\pi)\gamma_1^{\pi(1)} \ldots \gamma_n^{\pi(n)}$ represesnts its determinant.

For the proof of part (a), consider the following. Let $i, j \in \{1, \ldots, n\}, i \neq j$. If i and j differ by $2p$ then both the traces and the determinants for Salg_n^i and S_n^j will agree, so $\mathrm{Salg}_n^i \subseteq \mathrm{S}_n^j$. If i and j differ by less than $2p$, then the traces of Salg_n^i and S_n^j will differ unless i and j differ by p, in which case, since p is odd (remember we are excluding the case $\mathrm{ch}(k) = 2$ throughout this thesis) the determinants will differ. Thus Salg_n^i and S_n^j are disjoint. From these comments one can see that we have the equality $\mathrm{Salg}_n^i = \mathrm{S}_n^i$ for all $i \in \{1, \ldots, n\}$ if and only if there are no two distinct integers $i, j \in \{1, \ldots, n\}$ which differ by $2p$. One can always be sure that this is condition is met when $n \leq 2p$. This completes the proof of (a).

For part (b), we have $\mathrm{ch}(k) = 0$. Here one simply needs to consider the traces on Salg_n^i and S_n^j, which must differ unless $i = j$, showing that the subsets Salg_n^i and $\mathrm{S}_\mathrm{n}^\mathrm{j}$ are disjoint unless $i = j$, that is $\mathrm{Salg}_n^i = \mathrm{S}_n^i$.

Finally, for part (c) we combine the results of (a) and (b). In the case of positive characteristic p, then as $p \geq 3$, from part (a) we know that these subsets are disjoint and closed for $n \leq 6$, while in the case of zero characteristic from part (b) we know that these subsets are disjoint and closed for any n. Combine these statements to see that regardless of the characteristic of the field k, the subsets Salg_n^i are all closed subsets when $n \leq 6$. $\square$

Remark 3.2.11 *Lemma 3.2.10 is likely to be general enough for us to use in all cases where determining irreducible components of* Salg_n *is currently practical. The irreducible components of* Alg_n *have so far only been described for* $n \leq 5$ *(with some special — "rigid" — components described in the case* $n = 6$*), and finding these irreducible components is a more basic question than finding the irreducible components of* Salg_n*. However, it is of theoretical interest to determine whether the subsets* Salg_n^i *are in fact closed subsets of* Salg_n *for all* n *and any field* k *with* $\mathrm{ch}(k) \neq 2$*, or if there is some field* k *of prime characteristic,* p*, and some integer,* n*, such that the variety* Salg_n *over the field* k *has one of its subsets* Salg_n^i *which is not closed. As we shall see, when the* Salg_n^i *are closed they form the connected components of* Salg_n*. Thus it would be interesting to know if the geometry of* Salg_n *can change in this manner for some integer,* n*, and field,* k*, of prime characteristic,* p*.*

Using the notation from the proof of Lemma 3.2.10 we have the following situation for the variety Salg_7 over an algebraically closed field of characteristic 3. $\mathrm{S}_7^1 = \mathrm{S}_7^7 = V(\{\sum_{j=1}^n \gamma_j^j - 1, \sum_\pi \mathrm{sgn}(\pi)\gamma_1^{\pi(1)} \ldots \gamma_n^{\pi(n)} - 1\}) \cap \mathrm{Salg}_7$. This is the smallest example of where the above lemma may not be applied. While it is clear that Salg_7^1 and Salg_7^7 are disjoint, it may be possible that $\overline{\mathrm{Salg}_7^1}$ and Salg_7^7 have some point in common. (Recall that we remarked earlier that Salg_n^n is closed — so $\overline{\mathrm{Salg}_n^n} = \mathrm{Salg}_n^n$ and thus we do know that $\overline{\mathrm{Salg}_7^7} = \mathrm{Salg}_7^7$ and Salg_7^1 are disjoint).

Proposition 3.2.12 Salg_n *is disconnected for* $n \geq 2$*.*

Proof:
By the comments above Lemma 3.2.10, for each superalgebra, the determinant of the $\mathbb{Z}_2$-grading is either -1 or 1. Since $\mathrm{ch}(k) \neq 2$, -1 and 1 are distinct elements of k, hence $X_{-1} = V(\{\sum_\pi \mathrm{sgn}(\pi)\gamma_1^{\pi(1)} \dots \gamma_n^{\pi(n)} - (-1)\}) \cap \mathrm{Salg}_n$ and $X_1 = V(\{\sum_\pi \mathrm{sgn}(\pi)\gamma_1^{\pi(1)} \dots \gamma_n^{\pi(n)} - 1\}) \cap \mathrm{Salg}_n$ are disjoint closed subsets whose union is Salg_n. But $X_{-1} = \mathrm{Salg}_n \setminus X_1$ and $X_1 = \mathrm{Salg}_n \setminus X_{-1}$, hence both are open sets too. Thus Salg_n is a union of two disjoint open subsets. Both subsets are non-empty for $n \geq 2$. Thus for $n \geq 2$, Salg_n is disconnected. $\square$

Assumption 3.2.13 *From here onwards, we make the assumption that* Salg_n^i *are closed subsets of* Salg_n*.*

The main examples which we are interested in are Salg_n for $n = 2, 3, 4$, and in these cases this assumption is satisfied by Lemma 3.2.10. The places where this assumption is used it should be obvious from the proof. This assumption is, however, not needed for the material on algebraic groups and their actions in the next section.

As we have mentioned in Remark 3.2.11, when Assumption 3.2.13 holds, the Salg_n^i are the connected components of Salg_n. We must however, postpone the proof of this fact until Section 3.4 when we will have sufficient tools to prove it.

Since some algebras and superalgebras arise several times, we shall name them for convenience.

Definition 3.2.14 *Define C_n to be the algebra $k[X_1, \dots, X_{n-1}]/(X_1, \dots, X_{n-1})^2$ and for $i = 1, \dots, n$, let $C_n(i)$ be the superalgebra structure, which has C_n as its underlying algebra and is given the $\mathbb{Z}_2$-grading, $C_n(i)_0 = \mathrm{span}\{1, X_1, \dots, X_{i-1}\}$, $C_n(i)_1 = \mathrm{span}\{X_i, \dots, X_{n-1}\}$. The algebra C_n and the superalgebras $C_n(i)$ for $i = 1, \dots, n$ all have dimension n.*

The following lemma shows that each superalgebra structure on C_n is isomorphic to one of the $C_n(i)$.

Lemma 3.2.15 *Consider the algebra C_n. There are n distinct isomorphism classes of superalgebras on this algebra, which are $C_n(1), \ldots, C_n(n)$.*

Proof:
Let $B = B_0 \oplus B_1$ be a superalgebra structure on C_n where $\dim B_0 = i + 1$ with $0 \leq i \leq n-1$ (so $\dim B_1 = n - i - 1$). Suppose B_0 has basis $\{1, u_1, \ldots, u_i\}$ and B_1 has basis $\{u_{i+1}, \ldots, u_{n-1}\}$. There must be scalars such that for $1 \leq j \leq n-1$, $u_j = \alpha_{j1}1 + \alpha_{j2}X_1 + \ldots + \alpha_{jn}X_{n-1}$.

Now let $u'_j = u_j - \alpha_{j1}1 = \alpha_{j2}X_1 + \ldots + \alpha_{jn}X_{n-1}$. Then $\{1, u'_1, \ldots, u'_i\}$ is also a basis for B_0.

If $\alpha_{j1} \neq 0$ for any $i+1 \leq j \leq n-1$ then $u_j = \alpha_{j1}1 + \sum_{i=1}^{n-1} \alpha_{ji+1}X_i$, so $u_j^2 = \alpha_{j1}^2 1 + 2\sum_{i=1}^{n-1} \alpha_{ji+1}X_i$. Since $u_j^2 \in B_0$ we must have $\sum_{i=1}^{n-1} \alpha_{ji+1}X_i \in B_0$, say $\sum_{i=1}^{n-1} \alpha_{ji+1}X_i = \beta_1 1 + \sum_{k=1}^{i} \beta_{k+1}u_k$ then $(\beta_1 + \alpha_{j1})1 + \sum_{k=1}^{i} \beta_{k+1}u_k - u_j = 0$, which contradicts the linear independence of the basis. So $\alpha_{j1} = 0$ for all $i+1 \leq j \leq n-1$.

It is easy to check that any two of $u'_1, \ldots, u'_i, u_{i+1}, \ldots, u_{n-1}$ have product zero (including a product involving two of the same terms). So we can define a map $\phi : B \to C_n(i+1)$ by $1 \mapsto 1, u'_1 \mapsto X_1, \ldots, u'_i \mapsto X_i, u_{i+1} \mapsto X_{i+1}, \ldots, u_{n-1} \mapsto X_{n-1}$. It is easy to see that this is a bijection, which preserves the algebra structure and $\mathbb{Z}_2$-grading, hence is an isomorphism of superalgebras. Thus a superalgebra structure on C_n must be isomorphic to one of those described in the lemma.

To conclude the proof, we note that the n superalgebra structures given in the lemma are clearly mutually non-isomorphic. $\square$

So for each i there is a unique (up to isomorphism) superalgebra structure A on $k[X_1, \ldots, X_{n-1}]/(X_1, \ldots, X_{n-1})^2$ which has $\dim A_0 = i$.

In the case of n-dimensional algebras, Gabriel showed that the closed orbit consists of algebras isomorphic to C_n. The closed orbits in Salg_n consist of superalgebras isomorphic to one of the superalgebras $C_n(i)$, as the following Proposition shows.

Proposition 3.2.16 *There are n closed orbits in Salg_n. They are all disjoint, $C_n(i)$ being the closed orbit in Salg_n^i.*

Proof:
Suppose $G_n \cdot A$ is a closed orbit, i.e. $\overline{G_n \cdot A} = G_n \cdot A$. As $U(A)$ is an n-dimensional algebra, $G_n \cdot U(A)$ is an orbit in Alg_n. Now by Corollary 3.2.9 $\overline{G_n \cdot U(A)} = U(\overline{G_n \cdot A}) = U(G_n \cdot A) = G_n \cdot U(A)$. Thus the orbit $G_n \cdot U(A)$ is closed in Alg_n but then, by the results of [12], $U(A)$ must be isomorphic to C_n. That is, A must be isomorphic to a superalgebra structure on C_n.

It remains to show that the orbits, $G_n \cdot C_n(i)$, corresponding to the isomorphism classes of the superalgebras $C_n(i)$ are, in fact, closed. Notice that $C_n = U(C_n(i))$ is the algebra structure whose isomorphism class corresponds to the closed orbit in Alg_n. That is, the orbit $G_n \cdot C_n$ is closed in Alg_n and thus $U^{-1}(G_n \cdot C_n)$ is closed in Salg_n. Now, by Assumption 3.2.13, Salg_n^i are closed disjoint subsets, thus $U^{-1}(G_n \cdot C_n) \cap \mathrm{Salg}_n^i$ is closed. However this set is the orbit $G_n \cdot C_n(i)$ (since Lemma 3.2.15 above showed that all superalgebra structures on algebra C_n with the degree zero component having dimension i are all isomorphic). The result follows. $\square$

Remark 3.2.17 *Recall that Gabriel showed that the orbit, $\mathrm{GL}_n \cdot A$ of an n-dimensional algebra is open when $H^2(A, A) = 0$ (see Lemma 3.1.60), the obvious generalization of this statement to the case of superalgebras being that, for an n-dimensional superalgebra A, the orbit, $G_n \cdot A$ is open in Salg_n when $H^2(A, A) = 0$. $H^2(A, A)$ now being interpreted as the Hochschild cohomology group of the* **superalgebra** *A. We have not made any progress on proving or disproving this statement, although it would be interesting to know if it holds.*

Lemma 3.2.18 *Suppose that A is a superalgebra with $\dim A_0 = i$ and there is only one isomorphism class of superalgebras on $U(A)$ which has $\dim_0 = i$. If the orbit $G_n \cdot U(A)$ is open in Alg_n then the orbit $G_n \cdot A$ is open in Salg_n*

Proof:
Since Salg_n^i are all disjoint closed subsets (by Assumption 3.2.13), they are

also each open. Now $U^{-1}(G_n \cdot U(A)))$ is the collection of superalgebra structures on $U(A)$. Since $G_n \cdot U(A)$ is open, so too must be $U^{-1}(G_n \cdot U(A))$, by the continuity of U. Now by the assumptions made $G_n \cdot A = U^{-1}(G_n \cdot U(A)) \cap \mathrm{Salg}_n^i$. Thus $G_n \cdot A$ is the intersection of two open sets, so is open itself. $\square$

Example 3.2.19 *This is indeed the case for several orbits in* Salg_4*. Using this result and the fact that the orbits of* (1) *and* (10) *are open in* Alg_4 *we discover that the orbits* $(1|0), (1|1), (1|2), (10|0)$ *and* $(10|1)$ *are open in* Salg_4*.*

Finally, we shall introduce the notion of degeneration. This idea is very important for the remainder of the chapter.

Definition 3.2.20 *For* n*-dimensional superalgebras* A *and* B*, if* $(\alpha_{ij}^k, \gamma_i^j) \in G_n \cdot B$ *and* $(a_{ij}^k, \gamma_i^j) \in \overline{G_n \cdot A}$ *then we say that* A **degenerates to** B *and denote this by* $A \to B$*. In some places the terminology* A **dominates** B *is used instead of* A *degenerates to* B*. As we shall see in the next section, this extends to a well defined partial order on the isomorphism classes of* n*-dimensional superalgebras called the* **degeneration partial order***. We define* $B \leq_{degr} A$ *if and only if* A *degenerates to* B*. Clearly, whenever* $(\alpha_{ij}^k, \gamma_i^j) \in G_n \cdot A$*, then we also have* $(\alpha_{ij}^k, \gamma_i^j) \in \overline{G_n \cdot A}$ *since* $G_n \cdot A \subseteq \overline{G_n \cdot A}$*. A degeneration of this form is referred to as a* **trivial degeneration***, any degeneration not of this form is called a* **non-trivial degeneration***.*

Intuitively, if the superalgebra A degenerates to the superalgebra B (where $B \ncong A$ that is, this is a proper degeneration) then we think of the orbit $G_n \cdot B$ as consisting of some of those points outside the orbit $G_n \cdot A$, but which are "close to" some of the points in the orbit $G_n \cdot A$. This is supported by observing that the orbit $G_n \cdot B$ belongs to the boundary of $G_n \cdot A$ (i.e. the set $\overline{G_n \cdot A} \backslash G_n \cdot A$) as we shall see in the next section. Another observation supporting this intuition is that some degenerations may be obtained by taking a sequence of points in the orbit $G_n \cdot A$ whose "limit" lies in the orbit $G_n \cdot B$ (see Corollary 3.4.3).

Now we shall present a section on group actions, before returning to our primary interest of studying the degenerations in Salg_n.

3.3 Algebraic groups and their actions

Definition 3.3.1 *Let G be a variety, which additionally has the structure of a group. If the maps for multiplication $\mu : G \times G \to G$ given by $\mu(x, y) = xy$ and inversion $\iota : G \to G$ given by $\iota(x) = x^{-1}$ are morphisms of varieties, then we call G an* **algebraic group**. *The algebraic groups are group objects in the category of varieties.*

Remark 3.3.2 *The reader who has met topological or lie groups before should see the analogy. In these cases G is required to be a topological space, respectively a differentiable manifold, and the multiplication and inversion maps are required to be continuous, respectively differentiable. These are the group objects in the categories of topological spaces and differentiable manifolds respectively.*

Examples 3.3.3 *We have already mentioned two algebraic groups: G_n and GL_n. They are examples of algebraic groups, due to the formulae for matrix multiplication and inversion. There are many more examples that one can easily construct, using the fact thay any subgroup of GL_n which is closed in the Zariski Topology is an algebraic group.*

Recall from Section 3.1 the definition of a group acting on a set. This notion transfers straight across to the situation where G is an algebraic group and X is a variety. In this case the notion is the most interesting when the map giving the action of G on X, $\phi : G \times X \to X$ is a morphism of varieties, because in this case we can relate the structures of G and X as varieties. In this case, we say that the action is **algebraic**. For example if G is irreducible and it acts algebraically on a variety X, then we know that the G-orbits in X are irreducible also.

We have already seen examples of this kind of action, the structure transport action of G_n on Salg_n (and Alg_n) by the remarks in Section 3.3. Since these are the actions which primarily interest us, we will assume that all actions are algebraic. Given an algebraic group, G, and a variety,

X, we shall simply say that G acts on X when we really mean that G acts *algebraically* on X.

The constant maps $h_x : G \to X$, $h_g : X \to G$ defined by $g \mapsto x$ for fixed $x \in X$ and $x \mapsto g$ for fixed $g \in G$ respectively, are morphisms of varieties. Now with the identity maps $\mathrm{id}_G : G \to G$, $\mathrm{id}_X : X \to X$, one can construct the product maps of id_G with h_x and h_g with id_X to define morphisms $i_x : G \to G \times X$ and $i_g : X \to G \times X$, which can be described by $g \mapsto (g, x)$ and $x \mapsto (g, x)$ respectively. Then by composition with the action map $\phi : G \times X \to X$ we get morphisms $\phi_x : G \to X$ (the **orbit map**) described by $g \mapsto g \cdot x$ and $\phi_g : X \to X$ (the **translation map**) described by $x \mapsto g \cdot x$.

As an easy application of the maps constructed in the last paragraph, we have the following small result, which we shall use later in the section.

Lemma 3.3.4 *If an algebraic group G acts algebraically on a variety X and $U \subseteq X$ is open, then $g \cdot U$ is open for any $g \in G$.*

Proof:
Use the translation maps $\phi_g, \phi_{g^{-1}}$, and the fact that morphisms are continuous.

$\square$

In the theory of algebraic groups, the convention is to refer to an algebraic group, which is irreducible as a variety, as a **connected algebraic group**. This is because irreducibility has a different meaning in the context of group representations. This makes sense as, for an algebraic group, its irreducible components coincide with its connected components.

Lemma 3.3.5 *G_n and GL_n are connected algebraic groups with dimensions $n^2 - n$ and n^2 respectively.*

Proof:
We give the proof for G_n since these facts are better known for GL_n. In

any case, one can easily modify the arguments given here to prove the statements for GL_n.

Let $M'_n = \left\{ \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} : a_{ij} \in k \right\}$. Note that by cofactor expansion, the determinant of a matrix in M'_n is equal to the determinant of the lower right $(n-1) \times (n-1)$ submatrix, i.e. $\begin{vmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}$. It is clear that G_n is a subset of M'_n. In fact, G_n is the distinguished open subset $D(\det) = \{X \in M'_n : \det(X) \neq 0\}$ of M'_n. Thus G_n is a variety.

We notice that the formualae for matrix multiplication and inversion of a matrix are morphisms of varieties, by Lemma 3.1.30. Thus G_n is an algebraic group. Another way to see this result is to use the fact mentioned above in Example 3.3.3 — that any subgroup of GL_n which is closed in the Zariski topology is an algebraic group. Now G_n is a subgroup of GL_n and it is closed in the Zariski topology, being the set $V(\{a_{i1} - \delta_1^i\}) \cap \mathrm{GL}_n$. So we reach the same conclusion.

As a vector space M'_n is isomorphic to $\mathbb{A}^{n^2-n}$, so they are isomorphic as varieties (by Lemma 3.1.42) and hence have the same dimensions, and M'_n is irreducible. So M'_n has dimension $n^2 - n$. The set G_n has dimension $n^2 - n$ also, as it is an open subset of M'_n. It follows G_n is also irreducible, since any open subspace of an irreducible space is irreducible. □

By Lemma 3.3.5 above, we may apply a lemma from [7] on group actions, to the structure transport action of G_n on Salg_n.

Lemma 3.3.6 *Let G be a connected algebraic group acting on a variety X, then:*

(a) *Each orbit $G \cdot x$ is locally closed and irreducible*

(b) $\dim G \cdot x = \dim G - \dim \mathrm{Stab}_G(x)$

(c) *$\overline{G \cdot x} \backslash G \cdot x$ is a union of orbits of dimension $< \dim G \cdot x$*

Proof:
We use the proof from [7], giving extra details.

For part (a): $G \cdot x$ is the image of G under the orbit map $\phi_x : G \to X$ given by $g \mapsto g \cdot x$. Since this map is a morphism and hence continuous, it follows that $G \cdot x$ is irreducible. It then follows that $\overline{G \cdot x}$ is irreducible. By Lemma 3.1.46 the restricted morphism $\phi'_x : G \to \overline{G \cdot x}$ is dominating and by applying part (i) of Lemma 3.1.47 there exists a non-empty open set $\emptyset \neq U \subseteq X$ such that $U \subseteq G \cdot x$. Thus there is $\emptyset \neq U \subseteq G \cdot x$ with U open in $\overline{G \cdot x}$. Now $G \cdot U = \bigcup_{g \in G} g \cdot U \subseteq G \cdot x$ which is G-stable and hence equals $G \cdot x$. Each $g \cdot U$ is open in $\overline{G \cdot x}$ since U is open (using Lemma 3.3.4). Thus $G \cdot U$ is open in $\overline{G \cdot x}$, i.e. $G \cdot x$ is open in $\overline{G \cdot x}$, hence is locally closed.

For part (b), we continue on using Lemma 3.1.47, however it is part (ii) which assists us here. Now since $\emptyset \neq U \subseteq G \cdot x$, U must contain an element of the form $h \cdot x$ where $h \in G$, set $W = \{h \cdot x\}$ then for each component Z of $\phi^{-1}(\{h \cdot x\}) = h \cdot \mathrm{Stab}_G(x)$, $\dim Z = \dim\{h \cdot x\} + \dim G - \dim \overline{G \cdot x}$. We now note that as $G \cdot x$ is open in $\overline{G \cdot x}$ which is irreducible, $\dim G \cdot x = \dim \overline{G \cdot x}$, $\dim\{h \cdot x\} = 0$ (being a single point) and as $\dim h \cdot \mathrm{Stab}_G(x)$ is the maximum of the dimensions of its components, we discover $\dim h \cdot \mathrm{Stab}_G(x) = \dim G - \dim G \cdot x$. Finally, by considering G to be acting on itself by left multiplication, we may use the translation maps ϕ_h and $\phi_{h^{-1}}$ (which are morphisms, so in particular are continuous) to see that $\mathrm{Stab}_G(x)$ and $h \cdot \mathrm{Stab}_G(x)$ have the same dimensions. Hence $\dim \mathrm{Stab}_G(x) = \dim G - \dim G \cdot x$, getting the required statement upon rearrangement.

This leaves us with part (c). Firstly, fix an element $g \in G$, then note: 1. if U is an open neighbourhood of y then $g \cdot U$ is an open neighbourhood of $g \cdot y$; 2. if V is an open neighbourhood of $g \cdot y$ then $g^{-1} \cdot V$ is an open neighbourhood of y; and 3. $y \notin G \cdot x \Leftrightarrow g \cdot y \notin G \cdot x$. So we have:

$$\begin{aligned}
& y \in \overline{G \cdot x} \backslash G \cdot x \\
\Leftrightarrow\ & y \in \overline{G \cdot x}, y \notin G \cdot x \\
\Leftrightarrow\ & \text{Every open neighbourhood of } y \text{ intersects } G \cdot x \text{ and } y \notin G \cdot x \\
\Leftrightarrow\ & \text{Every open neighbourhood of } g \cdot y \text{ intersects } G \cdot x \text{ and } g \cdot y \notin G \cdot x \\
\Leftrightarrow\ & g \cdot y \in \overline{G \cdot x} \backslash G \cdot x
\end{aligned}$$

In particular, we see that $y \in \overline{G \cdot x} \backslash G \cdot x \Rightarrow g \cdot y \in \overline{G \cdot x} \backslash G \cdot x$ for any $g \in G$, and thus $G \cdot y = \bigcup_{g \in G} g \cdot y \subseteq \overline{G \cdot x} \backslash G \cdot x$. Secondly, since $G \cdot z$ is open in $\overline{G \cdot z}$ which is irreducible, we find $\dim G \cdot z = \dim \overline{G \cdot z}$. If $G \cdot y \subset \overline{G \cdot x}$ and $y \notin G \cdot x$, then $\overline{G \cdot y} \subset \overline{G \cdot x}$ and since $\overline{G \cdot y}$ is closed and $\overline{G \cdot x}$ is irreducible, we have $\dim \overline{G \cdot y} < \dim \overline{G \cdot x}$, hence $\dim G \cdot y < \dim G \cdot x$. Combine these two arguments to get the required statement. □

Remark 3.3.7 *We make the following remark of how to interpret the stabiliser subgroup in the case of the G_n-action on Salg_n. Suppose one has a point $(\alpha_{ij}^k, \gamma_i^j)$ of Salg_n which is in the orbit $G_n \cdot A$ for some superalgebra A. Recall that when one represents points in Salg_n using the alternate notation $(\mu, \sigma) \in \mathrm{Hom}(V \otimes V, V) \times \mathrm{Hom}(V, V)$, the transport of structure action can be described as follows: for $\Lambda \in G_n$ $\Lambda \cdot (\mu, \sigma) = (\Lambda \circ \mu \circ (\Lambda^{-1} \otimes \Lambda^{-1}), \Lambda \circ \sigma \circ \Lambda^{-1})$. Then the matrix Λ, viewed as a linear map from V to V, is an automorphism of A (as a superalgebra) if and only if it satisfies $\Lambda \circ \mu = \mu \circ (\Lambda \otimes \Lambda)$ and $\Lambda \circ \sigma = \sigma \circ \Lambda$. Then an automorphism Λ of A is in the stabiliser $\mathrm{Stab}_{G_n}((\alpha_{ij}^k, \gamma_i^j))$, and conversely, an element of this stabiliser gives an automorphim of the superalgebra A. In fact, because of this correspondence the stabiliser of a point $(\alpha_{ij}^k, \gamma_i^j)$ in Salg_n is isomorphic to the automorphism group of the superalgebra whose isomorphism class is identified with the orbit $G_n \cdot (\alpha_{ij}^k, \gamma_i^j)$.*

Whenever we have a connected algebraic group G acting on a variety X, we have the idea of degeneration. The action of G on X partitions the variety into equivalence classes under the equivalence relation

$x \equiv y \Leftrightarrow \exists\, g \in G$ such that $y = g \cdot x$. The equivalence classes are the G-orbits. Because of this, we shall use the notation $[x] = G \cdot x$ for brevity, while stating and proving results about this more general notion of degeneration.

Definition 3.3.8 *We say that $[x]$* **degenerates** *to $[y]$ if $y \in \overline{G \cdot x}$ and will write $[x] \to [y]$. By appealing to Lemma 3.3.6 we can show that this idea of degeneration is not only well-defined on the G-orbits of X, but it also gives rise to a partial order on the G-orbits in X. This is the content of the following corollary. We define $[y] \leq_{degr} [x]$ if and only if $[x]$ degenerates to $[y]$. (Note that in some places the degeneration partial order is defined to be the opposite to this. This happens for example in [33]).*

Some people define the idea of degeneration as: $[x]$ degenerates to $[y]$ if $G \cdot y \subseteq \overline{G \cdot x}$. Using part (c) of Lemma 3.3.6, one can see that this is an equivalent definition. It does provide a useful way to visualize the notion of degeneration — that an orbit is contained in the closure of some other orbit.

Corollary 3.3.9 *When G is a connected algebraic group acting on a variety X, $\leq_{degr}$ is a partial order on the G-orbits of X.*

Proof:
By part (c) of Lemma 3.3.6, $\overline{G \cdot x}$ is a union of orbits. So if $y \in \overline{G \cdot x}$, then $G \cdot y \subseteq \overline{G \cdot x}$. From this statement, we deduce that $[y] \leq_{degr} [x]$ if and only if $[y'] \leq_{degr} [x']$ for any $y' \in G \cdot y, x' \in G \cdot x$, which shows that $\leq_{degr}$ is a well-defined relation on the G-orbits.

Clearly $x \in \overline{G \cdot x}$, thus $[x] \leq_{degr} [x]$.

Suppose $[x] \leq_{degr} [y]$ and $[y] \leq_{degr} [z]$, so from above we have $G \cdot x \subseteq \overline{G \cdot y}$ and $G \cdot y \subseteq \overline{G \cdot z}$. Since we clearly have $x \in G \cdot x$, combine this with the previous statements to obtain $x \in G \cdot x \subseteq \overline{G \cdot y} \subseteq \overline{G \cdot z}$, i.e. $[x] \leq_{degr} [z]$.

Suppose $[x] \leq_{degr} [y]$ and $[y] \leq_{degr} [x]$, so $G \cdot x \subseteq \overline{G \cdot y}$ and $G \cdot y \subseteq \overline{G \cdot x}$. Now assume that $G \cdot x \neq G \cdot y$. By part (c) of Lemma 3.3.6 $G \cdot x \subseteq \overline{G \cdot y}$

and $G \cdot x \neq G \cdot y$ implies $\dim G \cdot x < \dim G \cdot y$. Similarly $G \cdot y \subseteq \overline{G \cdot x}$ and $G \cdot x \neq G \cdot y$ implies $\dim G \cdot y < \dim G \cdot x$. Combining these, we get $\dim G \cdot x < \dim G \cdot y < \dim G \cdot x$, which is clearly absurd, hence $G \cdot x = G \cdot y$. □

The above result can be used to show that the idea of degeneration in Salg_n which was introduced at the end of the previous section, extends to a partial order on the isomorphism classes of n-dimensional superalgebras.

Lemma 3.3.10 *When G is a connected algebraic group acting on a variety X, the irreducible components of X are stable under the action of G.*

Proof:
We remind the reader that saying G is a connected algebraic group means precisely that G is irreducible as a variety. Now suppose Y is an irreducible component of X, then $G \times Y$ is irreducible. Now letting $\phi : G \times X \to X$ be the morphism giving the action of G on X, then ϕ is a continuous map, so $G \cdot Y = \phi(G \times Y)$ is also irreducible. Clearly, $Y \subseteq G \cdot Y$. However, by maximality of Y we must have $Y = G \cdot Y$, i.e. Y is stable under the action of G. □

Corollary 3.3.11 *When G is a connected algebraic group acting on a variety, the irreducible components are closures of a single orbit or closures of an infinite family of orbits.*

Proof:
From Lemma 3.3.10, irreducible components are G-stable. We also know that components are closed, hence each component can be taken to be the closure of a union of orbits. If there are only finitely many orbits in the union, then by using $\overline{A \cup B} = \overline{A} \cup \overline{B}$ we see that the component is not irreducible unless it is the closure of a single orbit. This gives the required statement. □

In the case of the G_n transport of structure action on Alg_n Flanigan goes further, and in [11] proves a result describing algebraic properties of the algebras belonging to some infinite family, whose orbits give rise to an irreducible component as described above.

In the following we shall abuse the terminology, and refer to the situation when some structure is contained in the closure of the union of the orbits of an infinite family of orbits, as a degeneration. We see an example of this in Alg_4 in the results of Gabriel, where the structure (19) is contained in the closure of the union of orbits of the family of structures $(18; \lambda)$. It is important to notice, however, that this is not a degeneration as defined earlier. Similarly, when an infinite family of orbits is contained in another infinite family of structures, we may also wish to refer to this as a degeneration too. We have an example of this given by Mazzola's work on Alg_5 in **[21]**, where the orbits of the infinite family of structures $(35; \lambda)$ is contained in the closure of the union of the orbits in the infinite family of structures $(13; \lambda)$. Finally, one may wish to refer to the case where an infinite family of structures is contained in the closure of a single orbit as a degeneration. This idea is less of an abuse of terminolgy than the others mentioned above, however, since we could consider it to be an infinite family of degenerations (in the original sense), one to each of the orbits in the infinite family. Although an abuse of terminology, it is useful to extend the notion of degeneration in this way, as it helps with determining the irreducible components.

Corollary 3.3.12 *When G is a connected algebraic group acting on a variety X, we have the following statements regarding the notions of degeneration and irreducible components:*

(a) If $[x] \to [y]$ then $[y]$ belongs to all the irreducible components to which $[x]$ belongs (and possibly more too)

(b) If there is no degeneration to $[x]$, then its closure is an irreducible component

(c) If $\cup_\lambda[x(\lambda)]$ is irreducible and there is no degeneration to $\cup_\lambda[x(\lambda)]$ then its closure is an irreducible component

Proof:
For part (a) $G_n \cdot y \subseteq \overline{G_n \cdot x}$, so that any irreducible component containing $G_n \cdot x$ must also contain $G_n \cdot y$.

For parts (b) and (c), consider what happens if $\overline{G_n \cdot x}$ (respectively $\overline{\cup_\lambda G_n \cdot x(\lambda)}$) is not an irreducible component. Then, as an irreducible set, it must be contained in some irreducible component implying that $[x]$ (respectively $\cup_\lambda[x(\lambda)]$) is contained in the closure of an orbit, or in the closure of the union of an infinite family of orbits. This means that there is a degeneration to $[x]$ (respectively $\cup_\lambda[x(\lambda)]$), contrary to our assumption.

$\square$

Remark 3.3.13 *This leads one to wonder when a union of a family of orbits is irreducible, so that we may apply part (c) of the above. This might not be true for arbitrary actions of algebraic groups on a variety. However the infinite families which arise in* Alg_4 *and* Alg_5 *can be shown to be irreducible. We illustrate this idea using the superalgebras* $(18; \lambda|i)$*. Firstly fix* i *as either* $0, 1$ *or* 2*. Use the basis* $e_1 = 1, e_2 = X, e_3 = Y, e_4 = XY$ *of* $(18; \lambda|i)$ *then for the member of the family with parameter value* $\lambda \neq -1$ *we have that the structure constant,* $\alpha_{23}^4 = \lambda$*. Hence, using this basis, we obtain a set of points in* Salg_4*. Call this set* S *— one point from each orbit corresponding to a member of the family* $(18; \lambda|i)$*. This set of points can be identified with* $k\backslash\{-1\}$ *which is irreducible in* $\mathbb{A}^1$ *(being the distinguished open* $D(x+1)$ *of* $\mathbb{A}^1$*), thus the set of points,* S*, is also irreducible. Now denote by* $\phi : G_n \times \mathrm{Salg}_n \to \mathrm{Salg}_n$ *the morphism arising from the transport of structure action of* G_n *on* Salg_n*. The union of the orbits of* $(18; \lambda|i)$ *is given by* $\phi(G_n \times S)$*, which, exactly as in Lemma 3.3.10, is seen to be irreducible. So we have shown that the union of orbits of superalgebras* $(18; \lambda|i)$ *for* $i = 0, 1, 2$ *are irreducible. The infinite families in* Alg_5 *can be shown to be irreducible in a similar manner.*

The above corollary tells us that the irreducible components are the orbits or infinite families of orbits, which no other orbit or infinite family of orbits degenerates to. So if one knows all degenerations between orbits and infinite families of orbits, then it is a trivial matter to determine the irreducible components. Unfortunately, the problem of determining all these degenerations is usually difficult. The problem of determining the irreducible components is somewhat easier, but can still be difficult too.

Definition 3.3.14 *An n-dimensional superalgebra A (respectively, a family of superalgebras $A(\lambda)$) is called* **generic**, *if the closure of its orbit in* Salg_n — $\overline{G_n \cdot A}$ *(respectively, the closure of the union of the family of orbits —* $\overline{\bigcup_\lambda G_n \cdot A(\lambda)}$*), is an irreducible component of* Salg_n*.*

Remark 3.3.15 *A superalgebra, A, whose orbit is open is always generic. Since it must lie in some irreducible component (being an irreducible set by part (a) of Lemma 3.3.6) and, as an open subset of any irreducible set is dense, we must have that $\overline{G_n \cdot A}$ is the entire component.*

However the observations in Corollary 3.3.12 applies more generally and can also aid us in finding the irreducible components. For example, after finding that no algebras degenerate to (17) *in* Alg_4*, by applying the closed continuous map U, we discover that no superalgebras can degenerate to any of* $(17|i)$ *for* $i = 0, 1, 2$ *in* Salg_4*. Then, by using the observations given in Corollary 3.3.12, we see that* $(17|i)$ *for* $i = 0, 1, 2$ *give rise to irreducible components of* Salg_4*, hence these algebras are also generic.*

The last two lemmas of this section are concerned with calculating the dimensions of the orbits in Salg_n. We explain how to read these tables now. Each row corresponds to a different algebra structure and the columns of the table are for different $\mathbb{Z}_2$-gradings on that given underlying algebra structure. Thus the underlying algebra structure of the superalgebra determines which row you look in, and which particular $\mathbb{Z}_2$-grading is used to obtain the given superalgebra structure determines which column you look under. We illustrate this by using an example. To find the dimension of the stabilizer of a point in the orbit of $(3|2)$ we look in the row labelled $(3|\cdot)$ and then look under the column labelled 2 to see that the dimension of the required stabilizer is 2.

Lemma 3.3.16 *The following gives the dimensions of the stabilizers of points in the orbits in* Salg_4*:*

Stabilizer dimensions				
$\cdot$	*0*	*1*	*2*	*3*
$(1\|\cdot)$	*0*	*0*	*0*	
$(2\|\cdot)$	*1*	*1*	*1*	*1*
$(3\|\cdot)$	*2*	*2*	*2*	*1*
$(4\|\cdot)$	*2*	*1*		
$(5\|\cdot)$	*3*	*2*		
$(6\|\cdot)$	*4*	*2*	*4*	
$(7\|\cdot)$	*4*	*2*	*3*	*2*
$(8\|\cdot)$	*5*	*3*	*3*	*3*
$(9\|\cdot)$	*9*	*5*	*5*	*9*
$(10\|\cdot)$	*3*	*1*		
$(11\|\cdot)$	*4*	*3*	*2*	*2*
$(12\|\cdot)$	*6*	*3*	*4*	
$(13\|\cdot)$	*2*	*1*		
$(14\|\cdot)$	*3*	*3*	*2*	*2*
$(15\|\cdot)$	*3*	*3*	*2*	*2*
$(16\|\cdot)$	*4*	*3*	*3*	*2*
$(17\|\cdot)$	*6*	*3*	*4*	
$(18;\lambda\|\cdot)$	*4*	*3*	*2*	
$(19\|\cdot)$	*4*	*2*		

Proof:
If the point $(\alpha_{ij}^k, \gamma_i^j)$ is in the orbit, $G_4 \cdot A$, which is identified with the isomorphism class of superalgebra A, then $\mathrm{Stab}_{G_4}((\alpha_{ij}^k, \gamma_i^j)) \cong \mathrm{Aut}(A)$ where the automorphism group is the group of automorphisms of the superalgebra A as mentioned in Remark 3.3.7. See Section 2.6 for a description of these automorphism groups.

The statements given in Lemma 3.1.41 are also useful when computing the dimension of the automorphism groups. We also remark that $\dim \mathrm{PGL}_n(k) = n^2 - 1$, so that $\dim \mathrm{PGL}_2(k) = 2^2 - 1 = 3$ (see for example [14]) $\square$

Proposition 3.3.17 *The following gives the dimensions of the orbits in* Salg_4*:*

Orbit dimensions				
$\cdot$	*0*	*1*	*2*	*3*
$(1\|\cdot)$	*12*	*12*	*12*	
$(2\|\cdot)$	*11*	*11*	*11*	*11*
$(3\|\cdot)$	*10*	*10*	*10*	*11*
$(4\|\cdot)$	*10*	*11*		
$(5\|\cdot)$	*9*	*10*		
$(6\|\cdot)$	*8*	*10*	*8*	
$(7\|\cdot)$	*8*	*10*	*9*	*10*
$(8\|\cdot)$	*7*	*9*	*9*	*9*
$(9\|\cdot)$	*3*	*7*	*7*	*3*
$(10\|\cdot)$	*9*	*11*		
$(11\|\cdot)$	*8*	*9*	*10*	*10*
$(12\|\cdot)$	*6*	*9*	*8*	
$(13\|\cdot)$	*10*	*11*		
$(14\|\cdot)$	*9*	*9*	*10*	*10*
$(15\|\cdot)$	*9*	*9*	*10*	*10*
$(16\|\cdot)$	*8*	*9*	*9*	*10*
$(17\|\cdot)$	*6*	*9*	*8*	
$(18;\lambda\|\cdot)$	*8*	*9*	*10*	
$(19\|\cdot)$	*8*	*10*		

Proof:
We have calculated the dimensions of the automorphism groups, or equivalently, the dimensions of stabilizers of any point in each orbit in

Lemma 3.3.16 above. We know that the dimension of G_4 is 12 from Lemma 3.3.5. By using part (b) of Lemma 3.3.6, we can calculate the dimension of the orbit $G_4 \cdot (\alpha_{ij}^k, \gamma_i^j)$ by subtracting the dimension of the stabilizer, $\mathrm{Stab}_{G_4}((\alpha_{ij}^k, \gamma_i^j))$, from the dimension of G_4 which is 12.

□

Remark 3.3.18 *We remark that to calculate the dimensions of the orbits in the case where we don't require the identity to be fixed (i.e. the orbits in* Salg_4' *and in which case* GL_4 *acts on this variety) we can subtract the dimensions of the stabilizers found in Lemma 3.3.16 from* 16 *(*16 *being the dimension of* GL_4 *by Lemma 3.3.5). If we then compare the dimensions of the orbits of the trivially* $\mathbb{Z}_2$*-graded superalgebras* $(i|0)$ *for* $i = 1, \ldots, 18; \lambda, 19$*, thus calculated, with those given by Gabriel in [12], we find that the two sets of numbers do not agree. In fact the orbit dimensions that Gabriel gives are exactly one less than the orbit dimensions we calculate in each case. This is strange. Since Gabriel did not give the proof of these facts in [12] it is difficult to find an explanation for this difference. However in Mazzola's paper [21] on classifying algebras of dimension five, the orbit dimensions are calculated by subtracting the dimension of the automorphism groups from 25 (25 being the dimension of* GL_5*) — this would tend to suggest that our methodology for calculating orbit dimensions is correct.*

3.4 Degenerations in Salg_n

Recall the notion of degeneration between two superalgebra, which was introduced at the end of Section 3.2.

In this section we concern ourselves with conditions determining when a degeneration of superalgebras in Salg_n can or cannot exist. When looking for conditions for the non-existence of degenerations between a given pair of superalgebras, it would be helpful to have some invariants of the superalgebra which are "rigid" in the sense that if there is a degeneration of superalgebras $A \to B$, then the superalgebras A and B must have the same value for the invariant. Unfortunately, the only such invariant that we know of is $\dim_0$ (using Assumption 3.2.13, which states that the sets Salg_n^i are closed subsets of Salg_n and the fact that these subsets are disjoint). The next best thing is a property of a superalgebra which any degeneration of this superalgebra must inherit, or some property which cannot increase or decrease upon degeneration. Such properties are analogous to those described in [12, Proposition 2.7] (given here as Lemma 3.1.61), which states, for example, the fact that the dimension of the radical cannot decrease upon degeneration. Later in the section we determine several properties which any degeneration of a given superalgebra must share.

Lemma 3.4.1 *Let $\Omega : k \to \mathrm{Salg}_n$ be a polynomial function and $U \subseteq \mathrm{Salg}_n$. If there are infinitely many points of $\Omega(k)$ in U then $\Omega(k) \subseteq \overline{U}$.*

Proof:
First, note that we think of Ω as describing a curve in Salg_n. $\overline{U}$ is defined to be the intersection of all closed sets containing U. A closed set is the vanishing set of polynomials (intersected with Salg_n), so it is enough to show that any polynomial vanishing on U must also vanish on all of $\Omega(k)$.

By applying the appropriate projections to Ω, we may write $\alpha_{ij}^k = a_{ij}^k(t)$ and $\gamma_i^j = g_i^j(t)$ (letting the indeterminate be t), to describe the coordinates of this curve.

It is standard that $\Omega^{-1}(U) = \{t \in k : \Omega(t) \in U\}$, but notice that this set gives the t values such that the curve Ω lies inside the set U. We consider a polynomial function in $(\alpha_{ij}^k, \gamma_i^j)$, which vanishes on U, $f(\alpha_{ij}^k, \gamma_i^j) = 0$. Since f vanishes on U it must vanish at the points of $\Omega(k)$ lying inside U. So we have $t \in \Omega^{-1}(U) \Rightarrow f(a_{ij}^k(t), g_i^j(t)) = 0$. Note that $f(a_{ij}^k(t), g_i^j(t))$ is a polynomial in t, suppose the degree $\deg(f(a_{ij}^k(t), g_i^j(t))) = d$ (which must be finite).

It is impossible to have $d \geq 1$, since if $d \geq 1$ then $f(a_{ij}^k(t), g_i^j(t)) = 0$ has at most d zeros, which contradicts the fact we assumed it to vanish on all of $\Omega(k) \cap U$, which has infinitely many points.

Thus $d = 0$, hence $f(a_{ij}^k(t), g_i^j(t))$ must be a constant. The only way that $f(a_{ij}^k(t), g_i^j(t)) = 0$ is satisfied for points in $\Omega^{-1}(U)$ is if $f(a_{ij}^k(t), g_i^j(t))$ is the zero polynomial, in which case $f(a_{ij}^k(t), g_i^j(t)) = 0$ is satisfied for all $t \in k$. This completes the proof. $\square$

Definition 3.4.2 *If A and B are n-dimensional superalgebras, a* **specialization** *of A to B is the following situation: one makes a change of basis in A to a "variable" basis, i.e. one involving some unknown t, such that the point of Salg_n obtained by structural transport is given by some polynomial functions in t and lies in the orbit of A for $t \neq 0$, yet at $t = 0$ lies in the orbit B. We think of B as being obtained by a formal limit of the basis change in A.*

A specialization of superalgebras A to B is a more restrictive notion than a specialization of algebras, since not only must there be a specialization of the underlying algebras, this must occur in such a way that under the specialization, the $\mathbb{Z}_2$-grading on A also tends to the $\mathbb{Z}_2$-grading on B. This is usually a non-trivial constraint, so some specializations between algebras may not give rise to specializations of superalgebras on these algebras. Or perhaps one must use different specializations for different superalgebra structures on the same underlying algebra.

With this idea of specialization we obtain a useful corollary of the above lemma.

Corollary 3.4.3 *A specialization of A to B implies that A degenerates to B.*

Proof:
Clearly the specialization gives us a curve $\Omega : k \to \mathrm{Salg}_n$. We let the set U in Lemma 3.4.1 be the orbit $G_n \cdot A$. Now, as k is algebraically closed, it has infinitely many elements. Thus so does k^*. Then $\Omega(k^*) \subseteq G_n \cdot A$ so $G_n \cdot A$ contains infinitely many elements of $\Omega(k)$. So we may apply Lemma 3.4.1. Now note that $\Omega(0)$ gives structure constants for a point in the orbit $G_n \cdot B$. Hence, by Lemma 3.4.1 the point in the orbit $G_n \cdot B$ given by $\Omega(0)$ lies in the closure of the orbit of A — this means that A degenerates to B. $\square$

Remark 3.4.4 *Let A be a superalgebra with $\dim A_0 = i$, in other words $A \in \mathrm{Salg}_n^i$. Suppose the bases of A_0 and A_1 are given by $\{1, e_2, \ldots, e_i\}$ and $\{e_{i+1}, \ldots, e_n\}$ respectively. The specialization described by Gabriel in [12] given by $1 \mapsto 1, e_2 \mapsto te_2, \ldots, e_n \mapsto te_n$ and letting $t \to 0$ implies that any algebra degenerates to the algebra C_n. This specialization does not alter the $\mathbb{Z}_2$-grading, which implies (by Corollary 3.4.3) any superalgebra in Salg_n^i degenerates to the superalgebra $C_n(i)$ in Salg_n^i. Stated another way, the closure of any orbit in Salg_n^i contains the orbit of the superalgebra $C_n(i)$ in Salg_n^i (which is the closed orbit in Salg_n^i).*

Earlier in Section 3.2 we mentioned that Salg_n^i are the connected components of Salg_n. Using Corollary 3.4.3 above, we can now prove this to be the case.

Proposition 3.4.5 *The set $\{\mathrm{Salg}_n^i\}_{i=1}^n$ are the connected components of Salg_n.*

Proof:
Any irreducible component is connected, because a disconnected space is reducible.

$\mathbb{A}^m$ is a Noetherian space, and by Assumption 3.2.13 Salg_n^i is a closed subset of $\mathbb{A}^m$ (for $m = n^3 + n^2$), so by Remark 3.1.38 (a) Salg_n^i has a finite number of irreducible components. However, irreducible components are closed and they must all contain the orbit of the superalgebra $C_n(i)$ by the statements before this proposition. Hence the irreducible components have a non-empty intersection.

Thus Salg_n^i is a finite union of its irreducible components, these are connected and have non-empty intersection. Now apply the following result from General Topology to deduce that Salg_n^i is connected: if the family $\{X_i : i \in I\}$ of connected subsets of a topological space has non-empty intersection, then its union, $\bigcup_{i\in I} X_i$ is connected. $\square$

Remark 3.4.6 *To prove Proposition 3.4.5 above, we needed to assume that* $\{\mathrm{Salg}_n^i\}_{i=1}^n$ *are closed subsets of* Salg_n*. One can actually see that, in fact,* $\{\mathrm{Salg}_n^i\}_{i=1}^n$ *are the connected components of* Salg_n *if and only if* $\{\mathrm{Salg}_n^i\}_{i=1}^n$ *are closed subsets. Proposition 3.4.5 shows one of the directions, and for the converse we note that connected components are closed (another fact from General Topology).*

Given n-dimensional superalgebras A and B, to show that A cannot degenerate to B, it is sufficient to exhibit a closed set in Salg_n containing the orbit $G_n \cdot A$ which is disjoint from $G_n \cdot B$. Note that if there are two disjoint closed sets in Salg_n one containing the orbit $G_n \cdot A$ and the other containing the orbit of $G_n \cdot B$, then there cannot be any degenerations between A and B. We now look for some necessary conditions for a degeneration of superalgebras to exist.

Remark 3.4.7 *We have seen some conditions necessary (but not sufficient) for the existence of a degeneration earlier in the chapter, perhaps given in a different context. These are useful to show when there is no degeneration between two superalgebras. We point these out now. In the following, suppose that* A *and* B *are* n*-dimensional superalgebras.*

(a) If $U(A)$ *doesn't degenerate to* $U(B)$ *as algebras, then* A *cannot degenerate to* B *as superalgebras. This follows, as a degeneration of* A *to* B *as superalgebras implies a degeneration of* $U(A)$ *to* $U(B)$ *as algebras, by using the* G_n*-equivariant map* U*.*

(b) However, in the case of trivially $\mathbb{Z}_2$-graded superalgebras this condition is clearly also sufficient. Since Salg_n^n is isomorphic to Alg_n (see Proposition 3.2.8), and the isomorphism is G_n-equivariant, it follows that A degenerates to B in Salg_n^n if and only if $U(A)$ degenerates to $U(B)$ in Alg_n.

(c) Since we are making Assumption 3.2.13 that the disjoint sets Salg_n^i are also closed subsets of Salg_n it follows that there cannot be a degeneration from A to B unless $\dim A_0 = \dim B_0$.

(d) We also remark that for $n \geq 3$, Salg_n^1 consists only of the closed orbit of the superalgebra $C_n(1)$. (See Proposition 2.2.12). So when $n \geq 3$, in the $dim_0 = 1$ case, we do not need to worry about degenerations in Salg_n^1. Since, in this case, there is only one orbit.

The above facts follow from considering either the algebra structure or the $\mathbb{Z}_2$-grading in isolation. For some more necessary conditions for the existence of a degeneration we must exploit both the algebra structure and the $\mathbb{Z}_2$-grading simultaneously.

We look for closed G_n-stable subsets defined by some superalgebraic properties. Finding such subsets is made difficult and proving such a subset is closed is awkward since a point in Salg_n has structure constants representing, in general, a superalgebra with a non-homogeneous basis, yet superalgebraic properties are usually given in terms of homogeneous elements.

The results which follow all require use of Lemma 3.1.51 given in the preliminaries section.

Lemma 3.4.8 *The following sets are closed in Salg_n:*

(a) $\{A \in \mathrm{Salg}_n : A_1^2 = \{0\}\}$

(b) $\{A \in \mathrm{Salg}_n : A_0$ *is commutative* $\}$

Proof:
Recall that at the beginning of this chapter we defined superalgebra structures on an n-dimensional vector space V, which we gave a basis $\{e_1, \ldots, e_n\}$.

For the set in part (a) we assign to a superalgebra A the following subset $W_A = \{v \otimes w : v, w \in A_1, vw = 0\}$ of $V \otimes V$. For the set in part (b) we assign to a superalgebra A the following subset $W'_A = \{v \otimes w : v, w \in A_0, vw = wv\}$ of $V \otimes V$. It is straightforward to check that these are both cones in $V \otimes V$.

Then we may write $v = \sum_{i=1}^n c_i e_i$ and $w = \sum_{i=1}^n d_i e_i$. Now from $v \otimes w \neq 0$ it is possible to recover v and w up to scalar multiple. This fact shall cause us no problems, however, since W_A and W'_A are cones in $V \otimes V$.

We show now that $\{(A, v \otimes w) : v, w \in A_1, vw = 0\}$ is closed in $\mathrm{Salg}_n \times (V \otimes V)$. If $v \otimes w = 0$ then either $v = 0$ or $w = 0$, in which case $c_i = 0$ for $i = 1, \ldots, n$ or $d_i = 0$ for $i = 1, \ldots, n$. So for $v \otimes w \neq 0$, $v \in A_1 \Leftrightarrow \sum_{i=1}^n c_i \gamma_i^j + c_j = 0$ for $j = 1, \ldots, n$; $w \in A_1 \Leftrightarrow \sum_{i=1}^n d_i \gamma_i^j + d_j = 0$ for $j = 1, \ldots, n$; and $vw = 0 \Leftrightarrow \sum_{i,j=1}^n c_i d_j \alpha_{ij}^k = 0$ for $1 \leq i, j \leq n$. We remark that if coordinates of v and w with respect to the given basis, i.e. $(c_i), (d_i)$, satisfy these equations, then so too must $(\lambda c_i), (\mu d_i)$ for any $\lambda, \mu \in k$. Thus it does not matter that we can only obtain v and w up to scalar multiple. Thus $\{(A, v \otimes w) : v, w \in A_1, vw = 0\} = V(\{c_i\}) \cup V(\{d_i\}) \cup V(\{\sum_{i=1}^n c_i \gamma_i^j + c_j, \sum_{i=1}^n d_i \gamma_i^j + d_j, \sum_{i,j=1}^n c_i d_j \alpha_{ij}^k\})$, which is closed in $\mathrm{Salg}_n \times (V \otimes V)$.

We show now that $\{(A, v \otimes w) : v, w \in A_0, vw = wv\}$ is closed in $\mathrm{Salg}_n \times (V \otimes V)$. If $v \otimes w = 0$ then either $v = 0$ or $w = 0$, in which case $c_i = 0$ for $i = 1, \ldots, n$ or $d_i = 0$ for $i = 1, \ldots, n$. So for $v \otimes w \neq 0$, $v \in A_0 \Leftrightarrow \sum_{i=1}^n c_i \gamma_i^j - c_j = 0$ for $j = 1, \ldots, n$; $w \in A_0 \Leftrightarrow \sum_{i=1}^n d_i \gamma_i^j - d_j = 0$ for $j = 1, \ldots, n$; and $vw = wv \Leftrightarrow \sum_{i,j=1}^n c_i d_j (\alpha_{ij}^k - \alpha_{ji}^k) = 0$ for $1 \leq i, j \leq n$. Thus $\{(A, v \otimes w) : v, w \in A_0, vw = wv\} = V(\{c_i\}) \cup V(\{d_i\}) \cup V(\{\sum_{i=1}^n c_i \gamma_i^j - c_j, \sum_{i=1}^n d_i \gamma_i^j - d_j, \sum_{i,j=1}^n c_i d_j (\alpha_{ij}^k - \alpha_{ji}^k)\})$, which is closed in $\mathrm{Salg}_n \times (V \otimes V)$.

So by Lemma 3.1.51 the maps $A \mapsto \dim W_A$ and $A \mapsto \dim W'_A$ are upper semicontinuous.

Now since Salg_n^i are closed subsets of Salg_n it suffices to show that the sets mentioned in the lemma intersected with Salg_n^i are closed in Salg_n^i for each $i = 1, \ldots, n$. That is, we may assume $\dim A_0 = i$. We note that $W_A \subseteq A_1 \otimes A_1$. Now if $A_1^2 = 0$, then $W_A = A_1 \otimes A_1$ which has dimension $(n-i)^2$. If $A_1^2 \neq \{0\}$, then $W_A \subset A_1 \otimes A_1$. We can see from the above, that for a given superalgebra A, W_A is closed in $V \otimes V$, and we note that $A_1 \otimes A_1$ is irreducible and has dimension $(n-i)^2$ (as a variety, see Lemma 3.1.42) as it is isomorphic to the $(n-i)^2$-dimensional affine space $\mathbb{A}^{(n-i)^2}$, thus $\dim W_A < (n-i)^2$ by Lemma 3.1.41. Thus the set $\{A \in \mathrm{Salg}_n^i : A_1^2 = \{0\}\} = \{A \in \mathrm{Salg}_n^i : \dim W_A \geq (n-i)^2\}$ which is a closed set by the upper semicontinuity. This proves part (a).

Similarly $W'_A \subseteq A_0 \otimes A_0$, and if A_0 is commutative then $W'_A = A_0 \otimes A_0$ which has dimension i^2. If A_0 is not commutative then $W'_A \subset A_0 \otimes A_0$ and so similarly as above $\dim W'_A < i^2$ (we just need to note that W'_A is closed and $A_0 \otimes A_0$ is irreducible). Thus the set $\{A \in \mathrm{Salg}_n^i : A_0$ is commutative $\} = \{A \in \mathrm{Salg}_n^i : \dim W'_A \geq i^2\}$ which is a closed set by the upper semicontinuity. This proves part (b). $\square$

Definition 3.4.9 *On a superalgebra, one can define a new multiplication by* $a \bullet b = a_0b_0 + a_1b_0 + a_0b_1 - a_1b_1$. *The* **graded center** *of a superalgebra is then defined to be* $Z_g(A) = \{a \in A : ab = b \bullet a \quad \forall b \in A\} = \{a \in A : ab = b_0a_0 + b_1a_0 + b_0a_1 - b_1a_1 \quad \forall b \in A\}$. *In the case that the graded centre coincides with the entire superalgebra i.e.* $Z_g(A) = A$ *we say that the superalgebra is* **supercommutative** *or* **graded commutative**.

The set of supercommutative superalgebras form a closed subset as the next lemma shows. Unfortunately, this result doesn't help us with showing the non-existence of any degenerations in dimension 4 or less.

Lemma 3.4.10 $\{A \in \mathrm{Salg}_n : A$ *is supercommutative* $\}$ *is a closed subset of* Salg_n.

Proof:
The proof of this fact is similar to the proof of the above lemma, so we only present a sketch.

We assign to a superalgebra A the following subset $W_A = \{v \otimes w : vw = w \bullet v\}$ of $V \otimes V$. Then with $v = \sum_{i=1}^n c_i e_i$ and $w = \sum_{i=1}^n d_i e_i$ the conditions for these two elements to supercommute becomes:

$\sum_{i,j,k,l=1}^n [(c_j + c_i\gamma_i^j)(d_l + d_k\gamma_k^l) + (c_j + c_i\gamma_i^j)(d_l - d_k\gamma_k^l) + (c_j - c_i\gamma_i^j)(d_l + d_k\gamma_k^l) + (c_j - c_i\gamma_i^j)(d_l - d_k\gamma_k^l)]\alpha_{jl}^m = \sum_{i,j,k,l=1}^n [(c_j + c_i\gamma_i^j)(d_l + d_k\gamma_k^l) + (c_j + c_i\gamma_i^j)(d_l - d_k\gamma_k^l) + (c_j - c_i\gamma_i^j)(d_l + d_k\gamma_k^l) - (c_j - c_i\gamma_i^j)(d_l - d_k\gamma_k^l)]\alpha_{lj}^m$ for all $m \in \{1, \ldots, n\}$

Thus the subset $\{(A, v \otimes w) : v \otimes w \in W_A\}$ is closed in $\mathrm{Salg}_n \times V \otimes V$. So by Lemma 3.1.51 the map $A \mapsto \dim W_A$ is an upper semicontinuous map. The superalgebra A is supercommutative if and only if W_A has dimension n^2 (which is as large as the dimension can possibly be). So the set of supercommutative superalgebras is equal to $\{A \in \mathrm{Salg}_n : \dim W_A \geq n^2\}$ which is closed. $\square$

For Salg_n^2 we have other closed subsets. Since $\dim A_0 = 2$, $J(A_0) = \{x \in A_0 : x^2 = 0\}$, notice that this is a vector subspace of A_0.

Lemma 3.4.11 *The following are closed sets in* Salg_n^2*:*

- $\{A \in \mathrm{Salg}_n^2 : \dim J(A_0) = 1\}$
- $\{A \in \mathrm{Salg}_n^2 : \dim J(A_0) = 1, J(A_0)A_1 = \{0\}\}$
- $\{A \in \mathrm{Salg}_n^2 : \dim J(A_0) = 1, A_1 J(A_0) = \{0\}\}$

Proof:
We give the proof for the second subset, since the proof for the third is very similar and the proof for the first subset follows by simplifying this proof.

For the second subset we assign to a superalgebra A the subset $W_A = \{v \otimes w : v \in A_0, w \in A_1, v^2 = 0, vw = 0\}$ of $V \otimes V$. This is clearly a cone. We also note $W_A \subseteq J(A_0) \otimes A_1$.

Suppose $v = \sum_{i=1}^n c_i e_i$, $w = \sum_{i=1}^n d_i e_i$. We discover $\{(A, v \otimes w) : v \otimes w \in W_A\} = V(\{c_i\}) \cup V(\{d_i\}) \cup V(\{\sum_{i=1}^n c_i \gamma_i^j - c_j, \sum_{i=1}^n c_i c_j \alpha_{ij}^k, \sum_{i=1}^n d_i \gamma_i^j + d_j, \sum_{i,j=1}^n c_i d_j \alpha_{ij}^k\})$. Which is closed in $\mathrm{Salg}_n \times (V \otimes V)$.

So by Lemma 3.1.51 $A \mapsto \dim W_A$ is an upper semi-continuous map.

Now, if $A \in \{A \in \mathrm{Salg}_n^2 : \dim J(A_0) = 1, J(A_0)A_1 = \{0\}\}$ then $\dim W_A = n - 2$.

If $A \notin \{A \in \mathrm{Salg}_n^2 : \dim J(A_0) = 1, J(A_0)A_1 = \{0\}\}$ then either $\dim J(A_0) = 0$ in which case $W_A = \{0\}$ and $\dim W_A = 0$ or $\dim J(A_0) = 1$ and $J(A_0)A_1 \neq \{0\}$ in which case $W_A \subset J(A_0) \otimes A_1$. In this case $\dim W_A < n - 2$ since W_A is closed, and $J(A_0) \otimes A_1 \cong A_1 \cong \mathbb{A}^{n-2}$ as vector spaces, so $J(A_0) \otimes A_1$ is an irreducible subset of dimension $n - 2$ by Lemma 3.1.42.

Hence $\{A \in \mathrm{Salg}_n^2 : \dim J(A_0) = 1, J(A_0)A_1 = \{0\}\} = \{A \in \mathrm{Salg}_n^2 : \dim W_A \geq n - 2\}$ which is closed by the upper semi-continuity. □

One can quickly check that if a superalgebra belongs to one of the closed sets described in Lemma 3.4.8, Lemma 3.4.10 or Lemma 3.4.11, then any isomorphic superalgebra must also belong to the same set. Thus these closed sets are stable under the action of G_n.

3.5 Degenerations in Salg_4

In this section we are interested in determining when 4-dimensional superalgebra structures do or do not degenerate to one another. Here we use the results derived in the previous section to help us.

The results of this section give us most of the degenerations in Salg_4. Before giving the degeneration diagrams we shall first explain how to interpret them. We follow this by giving a partial classification theorem for Salg_4 — we determine twenty irreducible components. There are, however, two other structures which may or may not give rise to irreducible components, and finally we give the details of the degenerations or the non-existence of degenerations, which were shown in the degeneration diagram.

As we shall soon see, there can be no degenerations amongst 4-dimensional superalgebras A and B with $\dim A_0 \neq \dim B_0$. Thus we can give the degeneration diagram for Salg_4 by giving the degeneration diagrams for each of the connected components Salg_4^i for $i = 1, 2, 3, 4$ separately. However we shall omit the diagram for Salg_4^1 since this consists of the solitary orbit of $(9|3)$.

Before giving these diagrams we shall explain the notations that we use in these diagrams.

We represent the orbits of isomorphism classes of superalgebras, by using the $(i|j)$ notation from Chapter 2; $(i|j)$ shall be used to denote the orbit $G_4 \cdot (i|j)$ in Salg_4.

The families of superalgebras $(18;\lambda|i)$, $i = 0, 1, 2$ consist of those superalgebras for all values of λ except -1, which in particular includes the values $\lambda = 0$ and $\lambda = 1$. In these cases these orbits coincide with some of the other orbits. This is because, as superalgebras, we have the following equalities or isomorphisms: $(18;0|0) = (16|0), (18;0|1) = (16|1), (18;0|2) = (16|3), (18;1|0) \cong (7|0), (18;1|1) \cong (7|2), (18;1|2) \cong (7|3)$.

In the degeneration diagram we use a dashed line to indicate a “degeneration" by a family of superalgebra structures; that is, when an orbit lies in the closure of the union of a family of orbits. This explains the use of the dashed lines through the families $(18;\lambda|i)$, $i = 0, 1, 2$. The fact that we use an arrow from $(18;\lambda|0)$ to $(8|0)$ and from $(18;\lambda|2)$ to $(8|3)$ is because there is a genuine degeneration, in the sense of Definition 3.2.20, from each of the orbits in these families to the orbits $(8|0)$ or $(8|3)$.

The dotted arrows (or dotted lines in the case of degenerations by a family of structures), are used to indicate those degenerations which we are unsure of — there may or may not be a degeneration between the indicated superalgebras.

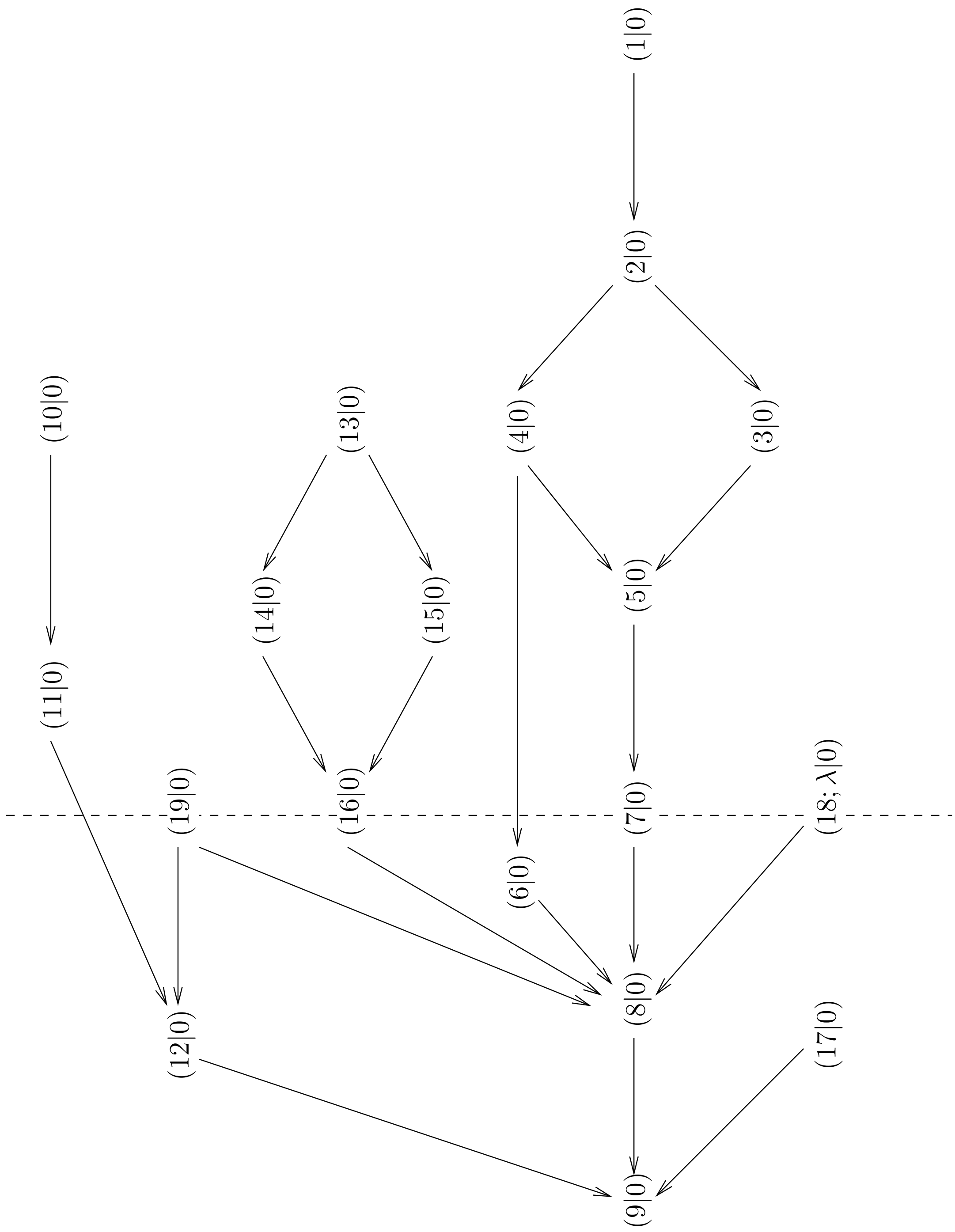

Figure 3.1: Degenerations in the component Salg_4^4

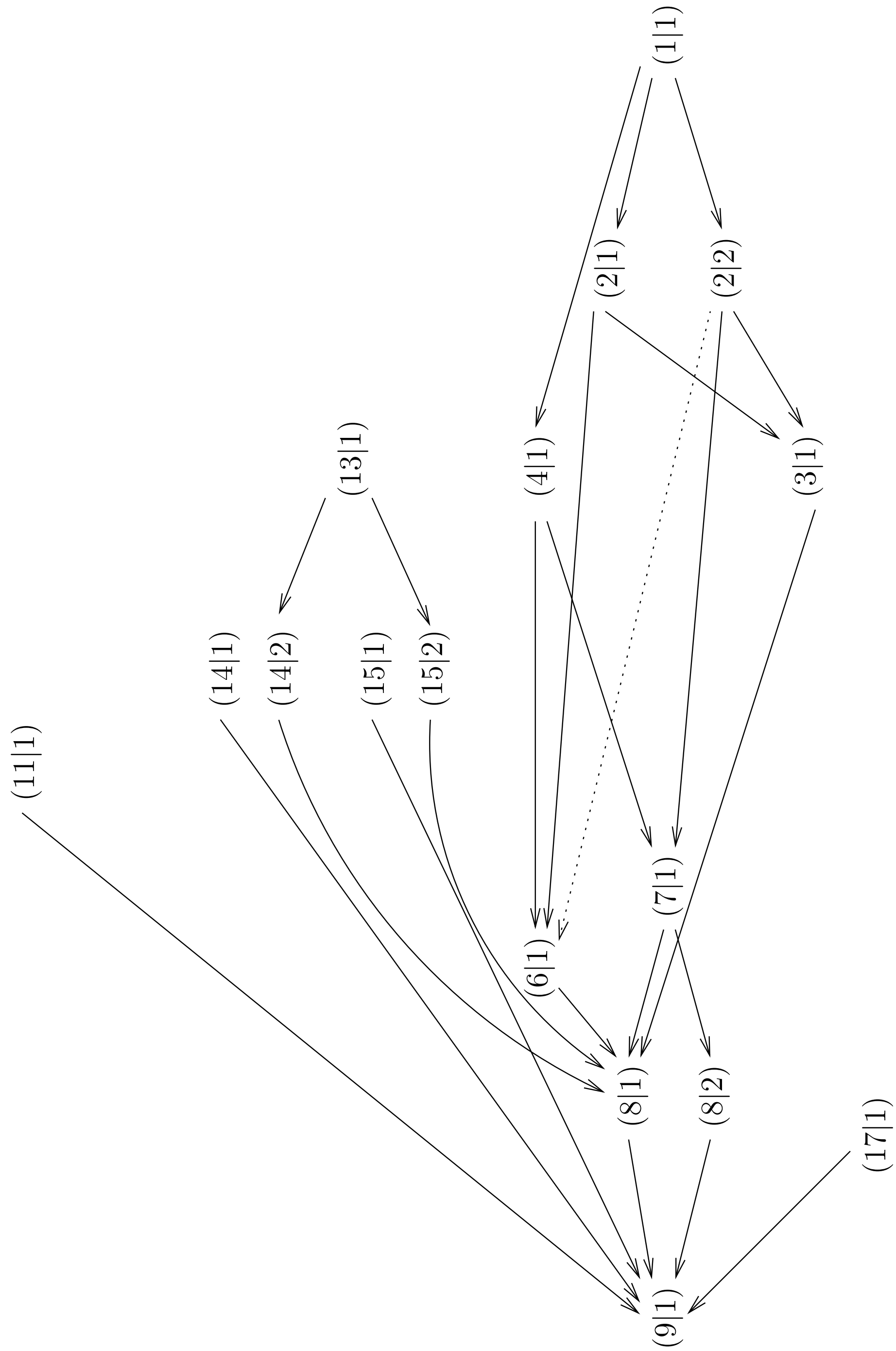

Figure 3.2: Degenerations in the component Salg_4^3

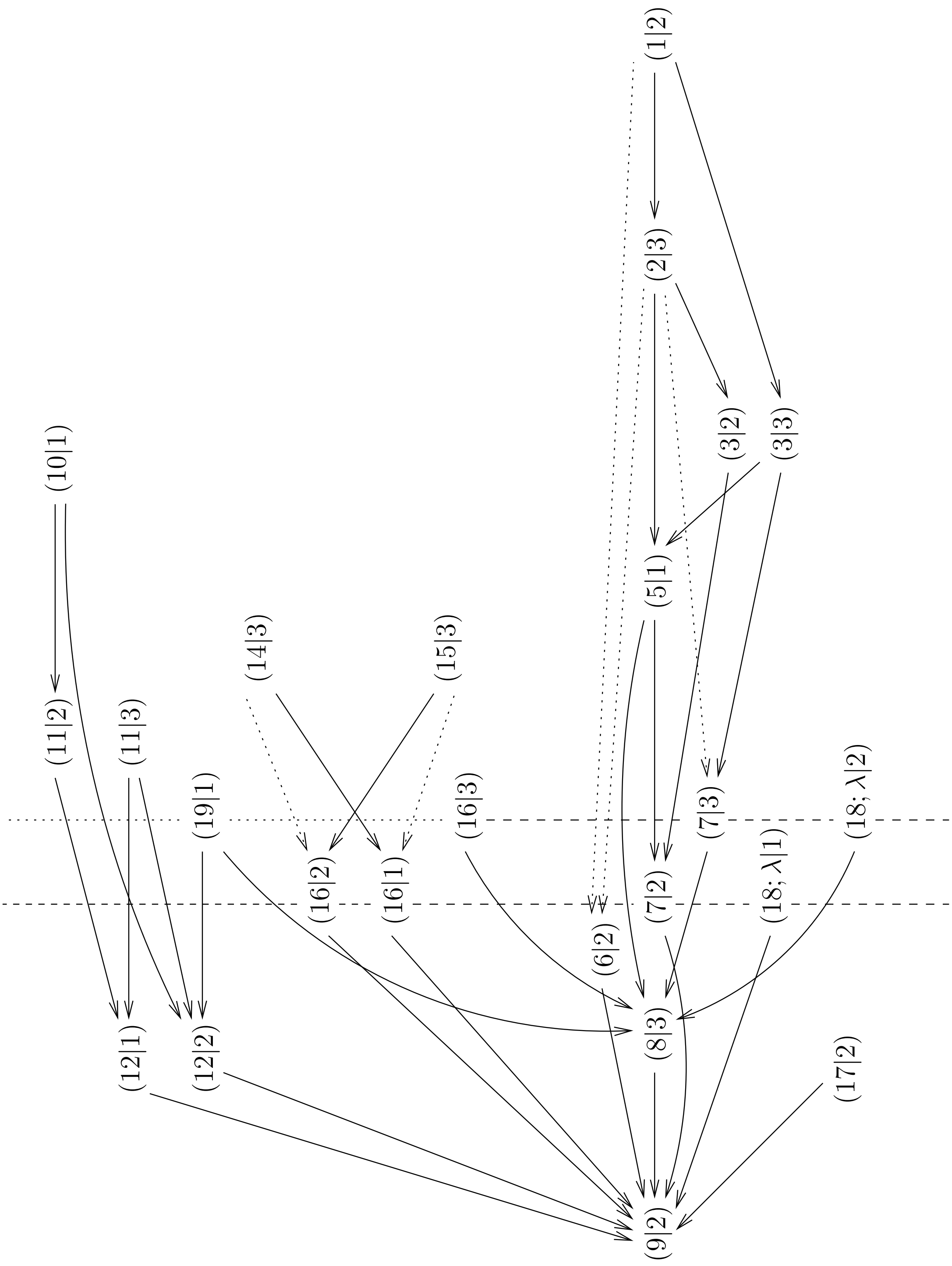

Figure 3.3: Degenerations in the component Salg_4^2

From this we get the following partial result classifying 4-dimensional superalgebras:

Theorem 3.5.1 *(Partial Geometric Classification of 4-dimensional Superalgebras) In Salg_4 there are at least twenty irreducible components. The following structures (or families of structures) are known to be generic:*

In Salg_4^4: $(1|0), (10|0), (13|0), (17|0), (18;\lambda|0)$

In Salg_4^3: $(1|1), (11|1), (13|1), (14|1), (15|1), (17|1)$

In Salg_4^2: $(1|2), (10|1), (11|3), (14|3), (15|3), (17|2), (18;\lambda|1), (18;\lambda|2)$

In Salg_4^1: $(9|3)$

Proof:
This follows from the degeneration diagrams and Corollary 3.3.12 which gives the relationship between the degeneration partial order and the irreducible components.

□

Remark 3.5.2 *The result above guarantees the existence of twenty irreducible components, however there could be up to two more irreducible components as well. It is the connected component Salg_4^2 in which we are unsure if we have found all of the irreducible components. It is not known whether the following two structures in Salg_4^2 are generic or not: $(6|2), (19|1)$ — so Salg_4^2 could have as few as eight irreducible components or as many as ten.*

We are unsure if $(18;\lambda|2)$ degenerates to $(19|1)$ or not. This is why the dashed line through $(18;\lambda|2)$ changes to a dotted line after passing through $(16|3)$. We point this out to the reader to ensure this important detail is not missed.

Remark 3.5.3 *Proposition 3.3.17 gives the dimensions of these orbits, which for the generic structures gives the dimensions of the components too. However, for the generic families $(18;\lambda|i)$ for $i = 0, 1, 2$, the dimension of the component must be at least one larger than the dimension of any single orbit in this family. Since* *the family depends on one parameter λ, we would suspect that the dimensions of these components of the generic families are exactly one larger than the dimension of any single orbit in this family. However, we have not proved this. To prove that this is indeed the case, it would suffice to show that there can be no closed irreducible set Y lying properly between $\overline{G_n \cdot (18; \lambda|i)}$ and $\overline{\bigcup_\lambda G_n \cdot (18; \lambda|i)}$, i.e. that it is impossible to have $\overline{G_n \cdot (18; \lambda|i)} \subset Y \subset \overline{\bigcup_\lambda G_n \cdot (18; \lambda|i)}$ when Y is closed and irreducible.*

We now provide the details which were used to obtain the degeneration diagrams just given:

We apply the following useful facts mentioned in Remark 3.4.7 in the previous section which shall help us here. Since $n = 4$ we may appeal Lemma 3.2.10 to see that Salg_4^i for $i = 1, 2, 3, 4$ are all closed disjoint subsets (and in fact by Proposition 3.4.5 are the connected components of Salg_4). Thus by part (c) of Remark 3.4.7 there cannot be a degeneration from A to B unless $\dim_0 A = \dim_0 B$. Thus we need only look at the degenerations amongst superalgebras belonging to the same subset Salg_4^i.

Another remark made in part (a) of Remark 3.4.7 is the following: If $U(A)$ doesn't degenerate to $U(B)$ as algebras, then A cannot degenerate to B as superalgebras. So we simply focus on degenerations from A to B, when there is a degeneration from $U(A)$ to $U(B)$ of underlying algebras. These two remarks represent large simplifications for us, as they greatly reduce the number of degenerations we must consider. Since two different superalgebras on the same underlying algebra have a trivial degeneration of the underlying algebra, we must however check to see if there are degenerations between different superalgebras on the same underlying algebra.

We also recall, any superalgebra in Salg_4^i degenerates to the superalgebra structure on $k[X, Y, Z]/(X, Y, Z)^2$ in Salg_4^i for $i = 1, 2, 3, 4$. The orbit of this superalgebra is the closed orbit in Salg_4^i. We will not mention this degeneration further since it always exists. We gave the specialization giving rise to this degeneration in Remark 3.4.4.

By Corollary 3.4.3, to show the existence of a degeneration, it suffices to exhibit a specialization. In this section to show the existence of degenerations we shall do this, except in one instance where we shall appeal to Lemma 3.4.1 directly.

We mention that all the specializations given in this section are "homogeneous", that is, the basis changes replace degree zero terms by degree zero terms, and similarly replace degree one terms by degree one terms. Corollary 3.4.3 applies equally well to non-homogeneous specializations, however, such specializations are more difficult to determine. In fact, there are some superalgebras which we haven't determined whether there is or is not a degeneration between (e.g. does $(1|2)$ degenerate to $(6|2)$?), but if the degeneration was to be obtained by a specialization it would necessarily have to be non-homogeneous. For an example of a degeneration obtained by a non-homogeneous specialization we have the following in the dimension 2 case, where each superalgebra is given the non-trivial $\mathbb{Z}_2$-grading:

$k \times k \to k[X]/(X^2)$ by $e_1 = (1,1), e_2 = (1,-1), e_1' = e_1, e_2' = te_1 + te_2$ let $t \to 0$

To show the non-existence of a degeneration we list the method which we use. There are several different methods. We give the name and a brief explanation for each below.

- By Lemma 3.3.6 part (c) the orbit dimension must strictly decrease upon proper degeneration. So a superalgebra cannot degenerate to another superalgebra of the same or greater dimension. We abbreviate this method by (OD). Note however that it is possible for a family of structures of a given dimension to "degenerate" to a structure of the same dimension. As an example of this, each orbit in $(18;\lambda|0)$ has dimension 8 as does the orbit $(19|0)$, yet the family $(18;\lambda|0)$ "degenerates" to $(19|0)$.

- For the other methods we use the closed G_n-stable subsets found in the previous section. If A belongs to one of these subsets, and B does not, then A cannot degenerate to B. We shall refer to this set of methods by which of the closed G_n-stable subsets we apply. The abbreviation we give to the method by applying one of the closed sets is listed below.

- (A) $\{A \in \mathrm{Salg}_n : A_1^2 = \{0\}\}$
- (B) $\{A \in \mathrm{Salg}_n : A_0$ is commutative $\}$
- (C) $\{A \in \mathrm{Salg}_4^2 : \dim J(A_0) = 1\}$
- (D) $\{A \in \mathrm{Salg}_4^2 : \dim J(A_0) = 1, J(A_0)A_1 = \{0\}\}$
- (E) $\{A \in \mathrm{Salg}_4^2 : \dim J(A_0) = 1, A_1 J(A_0) = \{0\}\}$

In the following, when $\alpha \neq 0$, we will use the shorthand, $\sqrt{\alpha}$ to denote *some element*, x, of k^*, such that $x^2 = \alpha$. (Such an element x always exists as k is algebraically closed. Moreover, if x is such an element, then so too is $-x$).

Case $\dim_0 = 4$:

Applying part (b) in Remark 3.4.7 from the previous section, we notice that the degeneration diagram of Salg_4^4 corresponds exactly to the degeneration diagram of Alg_4. These degenerations have been completely described by Gabriel in [12], where he gives the degeneration diagram.

Case $\dim_0 = 3$:

Existence of Degenerations:

$(1|1) \to (2|1) : e_1 = (1,1,1,1), e_2 = (0,1,0,0), e_3 = (0,0,1,1), e_4 = (0,0,1,-1), e_1' = e_1, e_2' = e_2, e_3' = e_3, e_4' = te_4$ let $t \to 0$

$(1|1) \to (2|2) : e_1 = (1,1,1,1), e_2 = (0,0,1,1), e_3 = (1,-1,0,0), e_4 = (0,0,1,-1), e_1' = e_1, e_2' = e_2, e_3' = te_3, e_4' = e_4$ let $t \to 0$

$(1|1) \to (4|1) : e_1 = (1,1,1,1), e_2 = (1,0,0,0), e_3 = (0,0,1,1), e_4 = (0,0,1,-1), e_1' = e_1, e_2' = e_2, e_3' = t^2 e_3, e_4' = te_4$ let $t \to 0$

$(2|1) \to (3|1) : e_1 = (1,1,1), e_2 = (1,1,0), e_3 = (1,-1,0), e_4 = (0,0,X), e'_1 = e_1, e'_2 = e_2, e'_3 = te_3, e'_4 = e_4$ let $t \to 0$

$(2|1) \to (6|1) : e_1 = (1,1,1), e_2 = (1,0,0), e_3 = (0,-1,1), e_4 = (0,0,X), e'_1 = e_1, e'_2 = e_2, e'_3 = te_3, e'_4 = e_4$ let $t \to 0$

$(2|2) \to (3|1) : e_1 = (1,1,1), e_2 = (1,1,0), e_3 = (0,0,X), e_4 = (1,-1,0), e'_1 = e_1, e'_2 = e_2, e'_3 = e_3, e'_4 = te_4$ let $t \to 0$

$(2|2) \to (7|1) : e_1 = (1,1,1), e_2 = (1,1,0), e_3 = (0,0,X), e_4 = (1,-1,0), e'_1 = e_1, e'_2 = \sqrt{2}te_2 + e_3, e'_3 = t^2e_2, e'_4 = \sqrt{-2}te_4$ let $t \to 0$

$(3|1) \to (8|1) : e_1 = (1,1), e_2 = (1,0), e_3 = (X,0), e_4 = (0,Y), e'_1 = e_1, e'_2 = te_2 + e_3, e'_3 = te_3, e'_4 = e_4$ let $t \to 0$

$(4|1) \to (6|1) : e_1 = (1,1), e_2 = (1,0), e_3 = (0,X^2), e_4 = (0,X), e'_1 = e_1, e'_2 = e_2, e'_3 = e_3, e'_4 = te_4$ let $t \to 0$

$(4|1) \to (7|1) : e_1 = (1,1), e_2 = (-1,1), e_3 = (0,X^2), e_4 = (0,X), e'_1 = e_1, e'_2 = t^2e_2 + e_3, e'_3 = t^2e_3, e'_4 = \sqrt{-2}te_4$ let $t \to 0$

$(6|1) \to (8|1) : e_1 = (1,1), e_2 = (-1,1), e_3 = (0,X), e_4 = (0,Y), e'_1 = e_1, e'_2 = te_2 + e_3, e'_3 = 2te_3, e'_4 = e_4$ let $t \to 0$

$(7|1) \to (8|1) : e_1 = 1, e_2 = X+Y, e_3 = XY, e_4 = X-Y, e'_1 = e_1, e'_2 = e_2, e'_3 = 2e_3, e'_4 = te_4$ let $t \to 0$

$(7|1) \to (8|2) : e_1 = 1, e_2 = X+Y, e_3 = XY, e_4 = X-Y, e'_1 = e_1, e'_2 = -2e_3, e'_3 = te_2, e'_4 = e_4$ let $t \to 0$

$(13|1) \to (14|2) : e_1 = \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), e_2 = \left(1, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right), e_3 = \left(1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right), e_4 = \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), e'_1 = e_1, e'_2 = e_2, e'_3 = te_3, e'_4 = e_4$ let $t \to 0$

$(13|1) \to (15|2) : e_1 = \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), e_2 = \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), e_3 = \left(1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right), e_4 = \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), e'_1 = e_1, e'_2 = e_2, e'_3 = te_3, e'_4 = e_4$ let $t \to 0$

$(14|2) \to (8|1) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e'_1 = e_1, e'_2 = te_2 + e_3, e'_3 = 2te_3, e'_4 = e_4$ let $t \to 0$

$(15|2) \to (8|1) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_4 =$

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_1' = e_1, e_2' = te_2 + e_3, e_3' = 2te_3, e_4' = e_4$ let $t \to 0$

Non-existence of Degenerations:

$(2|1) \not\to (2|2)$ (OD)

$(2|1) \not\to (4|1)$ (OD)

$(2|1) \not\to (7|1)$ (A)

$(2|1) \not\to (8|2)$ (A)

$(2|2) \not\to (2|1)$ (OD)

$(2|2) \not\to (4|1)$ (OD)

$(3|1) \not\to (7|1)$ (OD)

$(3|1) \not\to (8|2)$ (A)

$(6|1) \not\to (8|2)$ (A)

$(8|1) \not\to (8|2)$ (OD)

$(8|2) \not\to (8|1)$ (OD)

$(13|1) \not\to (8|2)$ (A)

$(13|1) \not\to (14|1)$ (B)

$(13|1) \not\to (15|1)$ (B)

$(14|1) \not\to (8|1)$ (OD)

$(14|1) \not\to (8|2)$ (OD)

$(14|1) \not\to (14|2)$ (OD)

$(14|2) \not\to (8|2)$ (A)

$(14|2) \not\to (14|1)$ (B)

$(15|1) \not\to (8|1)$ (OD)

$(15|1) \not\to (8|2)$ (OD)

$(15|1) \not\to (15|2)$ (OD)

$(15|2) \not\to (8|2)$ (A)

$(15|2) \not\to (15|1)$ (B)

Undetermined Degeneration:

$(2|2) \xrightarrow{?} (6|1)$

Case $\dim_0 = 2$:

Existence of Degenerations:

$(1|2) \to (2|3) : e_1 = (1,1,1,1), e_2 = (0,0,1,1), e_3 = (1,-1,0,0), e_4 = (0,0,1,-1), e_1' = e_1, e_2' = e_2, e_3' = e_3, e_4' = te_4$ let $t \to 0$

$(1|2) \to (3|3) : e_1 = (1,1,1,1), e_2 = (1,1,0,0), e_3 = (1,-1,1,-1), e_4 = (1,-1,0,0), e_1' = e_1, e_2' = te_2, e_3' = e_3, e_4' = te_4$ let $t \to 0$

$(2|3) \to (3|2) : e_1 = (1,1,1), e_2 = (1,1,0), e_3 = (1,-1,0), e_4 = (0,0,X), e_1' = e_1, e_2' = e_2, e_3' = te_3, e_4' = e_4$ let $t \to 0$

$(2|3) \to (5|1) : e_1 = (1,1,1), e_2 = (1,1,0), e_3 = (1,-1,0), e_4 = (0,0,X), e_1' = e_1, e_2' = t^2e_2, e_3' = te_3 + e_4, e_4' = t^3e_3$ let $t \to 0$

$(3|2) \to (7|2) : e_1 = (1,1), e_2 = (1,-1), e_3 = (X,Y), e_4 = (X,-Y), e_1' = e_1, e_2' = te_2, e_3' = e_3, e_4' = te_4$ let $t \to 0$

$(3|3) \to (5|1) : e_1 = (1,1), e_2 = (X,Y), e_3 = (1,-1), e_4 = (X,-Y), e_1' = e_1, e_2' = 2te_2, e_3' = te_3 + e_4, e_4' = 2t^2e_4$ let $t \to 0$

$(3|3) \to (7|3) : e_1 = (1,1), e_2 = (X,Y), e_3 = (1,-1), e_4 = (X,-Y), e_1' = e_1, e_2' = te_2, e_3' = te_3, e_4' = e_4$ let $t \to 0$

$(5|1) \to (7|2) : e_1 = 1, e_2 = X^2, e_3 = X, e_4 = X^3, e_1' = e_1, e_2' = e_2, e_3' = te_3, e_4' = te_4$ let $t \to 0$

$(5|1) \to (8|3) : e_1 = 1, e_2 = X^2, e_3 = X, e_4 = X^3, e_1' = e_1, e_2' = t^2e_2, e_3' = te_3, e_4' = e_4$ let $t \to 0$

$(7|3) \to (8|3) : e_1 = 1, e_2 = XY, e_3 = X+Y, e_4 = X-Y, e_1' = e_1, e_2' = 2e_2, e_3' = e_3, e_4' = te_4$ let $t \to 0$

$(10|1) \to (11|2) : e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_1' = e_1, e_2' = e_2, e_3' = te_3, e_4' = te_4$ let $t \to 0$

$(10|1) \to (12|2) : e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_1' = e_1, e_2' = t^2e_2, e_3' = te_3, e_4' = te_4$ let $t \to 0$

$(11|2) \to (12|1) : e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$
$e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_1' = e_1, e_2' = te_2, e_3' = e_3, e_4' = te_4$ let $t \to 0$

$(11|3) \to (12|1) : e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$
$e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_1' = e_1, e_2' = e_2, e_3' = te_3, e_4' = te_4$ let $t \to 0$

$(11|3) \to (12|2) : e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$
$e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_1' = e_1, e_2' = te_2, e_3' = te_3, e_4' = e_4$ let $t \to 0$

$(14|3) \to (16|1) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e_4 =$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, e_1' = e_1, e_2' = te_2, e_3' = e_3, e_4' = te_4$ let $t \to 0$

$(15|3) \to (16|2) : e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_4 =$
$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_1' = e_1, e_2' = te_2, e_3' = e_3, e_4' = te_4$ let $t \to 0$

$(16|3) \to (8|3) : e_1 = 1, e_2 = XY, e_3 = X + Y, e_4 = X - Y, e_1' = e_1, e_2' = e_2, e_3' = e_3, e_4' = te_4$ let $t \to 0$

$(18;\lambda|1) \to (7|2), (16|1)$: Since the orbits of $(7|2)$ and $(16|1)$ coincide with the orbits of $(18;1|1)$ and $(18;0|1)$ respectively, $(7|2)$ and $(16|1)$ are included in the closure of the union of the family of orbits $(18;\lambda|1)$.

$(18;\lambda|1) \to (16|2)$: Also $(16|2)$ is included in the closure of the union of the family of orbits $(18;\lambda|1)$. To see this, we look at the structure constants of $(18;t^{-1}|1)$ in the basis $e_1 = 1, e_2 = X, e_3 = Y, e_4 = YX$. This gives us a curve in Salg_4 which lies in the family of orbits of $(18;\lambda|1)$ for $t \neq 0$, yet lies in the orbit of $(16|2)$ when $t = 0$. By appealing to Lemma 3.4.1 directly the result follows.

$(18;\lambda|2) \to (7|3), (16|3)$: Similarly the orbits of $(7|3)$ and $(16|3)$ are included in the closure of the union of the family of orbits $(18;\lambda|2)$.

$(18;\lambda|2) \to (8|3) : e_1 = 1, e_2 = XY, e_3 = X+Y, e_4 = X-Y, e'_1 = e_1, e'_2 = (1+\lambda)e_2, e'_3 = e_3, e'_4 = te_4$ let $t \to 0$

$(19|1) \to (8|3) : e_1 = 1, e_2 = XY, e_3 = X+Y, e_4 = X-Y, e'_1 = e_1, e'_2 = e_2, e'_3 = e_3, e'_4 = te_4$ let $t \to 0$

$(19|1) \to (12|2) : e_1 = 1, e_2 = XY, e_3 = X, e_4 = Y, e'_1 = e_1, e'_2 = te_2, e'_3 = te_3, e'_4 = e_4$ let $t \to 0$

Non-existence of Degenerations:

$(2|3) \not\to (3|3)$ (OD)

$(3|2) \not\to (3|3)$ (OD)

$(3|2) \not\to (5|1)$ (OD)

$(3|2) \not\to (7|3)$ (OD)

$(3|2) \not\to (8|3)$ (A)

$(3|3) \not\to (3|2)$ (C)

$(5|1) \not\to (7|3)$ (OD)

$(6|2) \not\to (8|3)$ (OD)

$(7|2) \not\to (7|3)$ (OD)

$(7|2) \not\to (8|3)$ (OD)

$(7|3) \not\to (7|2)$ (D)

$(10|1) \not\to (11|3)$ (OD)

$(11|2) \not\to (11|3)$ (OD)

$(11|2) \not\to (12|2)$ (A)

$(11|3) \not\to (11|2)$ (C)

$(12|1) \not\to (12|2)$ (A)

$(12|2) \not\to (12|1)$ (OD)

$(14|3) \not\to (16|3)$ (OD)

$(14|3) \not\to (8|3)$ (A)

$(15|3) \not\to (16|3)$ (OD)

$(15|3) \not\to (8|3)$ (A)

$(16|1) \not\to (16|2)$ (OD)

$(16|1) \not\to (16|3)$ (OD)

$(16|1) \nrightarrow (8|3)$ (OD)
$(16|2) \nrightarrow (16|1)$ (OD)
$(16|2) \nrightarrow (16|3)$ (OD)
$(16|2) \nrightarrow (8|3)$ (OD)
$(16|3) \nrightarrow (16|1)$ (D)
$(16|3) \nrightarrow (16|2)$ (E)
$(18;\lambda|1) \nrightarrow (7|3), (16|3), (18;\lambda|2), (19|1)$ (A)
$(18;\lambda|1) \nrightarrow (8|3)$ (A)
$(18;\lambda|2) \nrightarrow (7|2), (16|1), (16|2), (18;\lambda|1)$ (D), (E)
$(19|1) \nrightarrow (12|1)$ (D)

Undetermined Degenerations:
$(1|2) \xrightarrow{?} (6|2)$
$(2|3) \xrightarrow{?} (6|2)$
$(18;\lambda|2) \xrightarrow{?} (19|1)$
$(2|3) \xrightarrow{?} (7|3)$
$(14|3) \xrightarrow{?} (16|2)$
$(15|3) \xrightarrow{?} (16|1)$

The first three of these undetermined degenerations are related to discovering whether $(6|2)$ or $(19|1)$ give rise to irreducible components in Salg_4^2.

Remark 3.5.4 *We close with the remark that in* Salg_4 *no two superalgebra structures* A *and* B *on the same underlying algebra can degenerate to each other, even if* $\dim_0 A = \dim_0 B$*. We have seen this from brute force checking of each case. Is it a general result that there can be no degeneration from a superalgebra to any other superalgebra having the same underlying algebra?*

Chapter 4

2-d and 3-d Superalgebras

In this chapter we repeat the analysis of the previous two chapters for superalgebras of dimensions 2 and 3. We still assume that k is a field with $\mathrm{ch}(k) \neq 2$. The first two sections are concerned with the algebraic classification of superalgebras of dimensions 2 and 3. We prove algebraic classification theorems for both of these cases, additionally assuming that k is algebraically closed for the case of superalgebras of dimension 3. In the final two sections we must additionally assume that k is algebraically closed and give the geometric classification theorems for these cases.

We again make use of the work of Gabriel in [12] on the varieties Alg_2 and Alg_3. It is interesting to compare the classification results that we derive with the classical ones just mentioned.

4.1 Dimension 2 case

We have two cases to consider: $\dim_0 = 2$ or $\dim_0 = 1$. As before, $(i|0)$ shall stand for the trivially $\mathbb{Z}_2$-graded superalgebra on algebra (i). This always has the form $(i|0)_0 = (i), (i|0)_1 = \{0\}$, so we do not describe the $\mathbb{Z}_2$-grading for these superalgebras in the following.

Theorem 4.1.1 *(Algebraic Classification of 2-dimensional superalgebras)*
Let k be a field with $\mathrm{ch}(k) \neq 2$.
(a) *Suppose A is a superalgebra with dimension 2. Then A is isomorphic to one of the following pairwise non-isomorphic families of superalgebras:*

$$\begin{array}{ll}
(1) & k \times k \\
 & (1|0) \\
 & (1|1)_0 = k(1,1), (1|1)_1 = k(1,-1) \\
(2) & k[X]/(X^2) \\
 & (2|0) \\
 & (2|1)_0 = k1, (2|1)_1 = kX \\
(3;\mu) & k(\sqrt{\mu}) \\
 & (3;\mu|0) \\
 & (3;\mu|1)_0 = k1, (3;\mu|1)_1 = X
\end{array}$$

(b)
(b.1) $(3;\mu|0) \cong (3;\mu_1|0)$ *if and only if* $\mu\mu_1^{-1} \in k^{*2}$
(b.2) $(3;\mu|1) \cong (3;\mu_1|1)$ *if and only if* $\mu\mu_1^{-1} \in k^{*2}$

Proof:
Firstly, we prove part (a). Suppose that A is a 2-dimensional superalgebra. In the case that $\dim A_0 = 2$ we have bases for A_0 and A_1 as $\{1, x\}$ and $\{\}$ respectively, whereas in the case that $\dim A_0 = 1$, we have bases for A_0 and A_1 as $\{1\}$ and $\{x\}$ respectively. In each case we may assume $x^2 = \alpha \in k$, which follows at once in the second case. In the first case, if $x^2 = \alpha + \beta x$, notice that $(x - \frac{\beta}{2})^2 = \alpha + (\frac{\beta}{2})^2$ and so we may replace x by $x - \frac{\beta}{2}$.

We get the following cases depending on whether α is an element of $\{0\}$, k^{*2} or $k^* \backslash k^{*2}$. If $\alpha = 0$ then A is isomorphic to a superalgebra on (2), via $1 \mapsto 1, x \mapsto X$. If $\alpha = \gamma^2$ for some $\gamma \in k^*$, then A is isomorphic to a superalgebra on (1), via $1 \mapsto (1,1), x \mapsto \gamma(1,-1)$. Finally, if $\alpha \in k^* \backslash k^{*2}$ then A is isomorphic to a superalgebra on $(3;\alpha)$ via $1 \mapsto 1, x \mapsto X$.

It is straightforward to show that these families are non-isomorphic and to prove the assertions given in part (b). □

Corollary 4.1.2 *In the case that k is algebraically closed, superalgebras on algebra $(3;\mu)$ can never arise.*

Proof:
In this case $k^{*2} = k^*$, so that $k^* \backslash k^{*2} = \emptyset$. □

Automorphism groups

We shall now also calculate the automorphism groups of those superalgbras described in Corollary 4.1.2, which shall be used later to calculate the dimensions of the corresponding orbits in Salg_2.

We choose a basis for each superalgebra $\{e_1 = 1, e_2\}$ and determine the constants a_{21}, a_{22} for which ϕ gives an automorphism of the given superalgebra, where ϕ is defined by $\phi(e_1) = e_1, \phi(e_2) = a_{21}e_1 + a_{22}e_2$. As this map must be homogeneous, and since we will choose homogeneous bases, we must have $a_{21} = 0$ for $\dim A_0 = 1$. We omit mention of a_{21} in this case.

$(1|0)$: $e_1 = (1,1), e_2 = (1,0)$
Then either $a_{21} = 0, a_{22} = 1$; or $a_{21} = 1, a_{22} = -1$
$(1|1)$: $e_1 = (1,1), e_2 = (1,-1)$
Then $a_{22} = \pm 1$
$(2|0)$: $e_1 = 1, e_2 = X$
Then $a_{21} = 0, a_{22} \neq 0$
$(2|1)$: $e_1 = 1, e_2 = X$
Then $a_{22} \neq 0$

4.2 Dimension 3 case

We have three cases to consider: either $\dim_0 = 3$, $\dim_0 = 2$ or $\dim_0 = 1$.

The case $\dim_0 = 3$ has been dealt with in Gabriels paper when k is algebraically closed. The case $\dim_0 = 1$ is dealt with by using Proposition 2.2.12. We see that the only superalgebra with $\mathbb{Z}_2$-grading of this form is: $(4|2) = k[X,Y]/(X,Y)^2$ having the $\mathbb{Z}_2$-grading $(4|2)_0 = k1, (4|2)_1 = kX \oplus kY$. Thus the one remaining case we must deal with is $\dim_0 = 2$, which we do in the following proposition.

Proposition 4.2.1 *Let k be a field with $\mathrm{ch}(k) \neq 2$.*
(a) *Suppose A is a superalgebra with $\dim A_0 = 2$ and $\dim A_1 = 1$. Then A is isomorphic to one of the following pairwise non-isomorphic families of superalgebras:*

(1) $k \times k \times k$
$(1|1)_0 = k(1,1,1) \oplus k(1,1,0), (1|1)_1 = k(1,-1,0)$

(2) $k \times k[X]/(X^2)$
$(2|1)_0 = k(1,1) \oplus k(1,0), (2|1)_1 = k(0,X)$

(3) $k[X]/(X^3)$
$(3|1)_0 = k1 \oplus kX^2, (3|1)_1 = kX$

(4) $k[X,Y]/(X,Y)^2$
$(4|1)_0 = k1 \oplus kX, (4|1)_1 = kY$

(5) $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$
$(5|1)_0 = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus k\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, (5|1)_1 = k\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$(6;\mu)$ $k \times k(\sqrt{\mu})$
$(6;\mu|1)_0 = k(1,1) \oplus k(1,0), (6;\mu|1)_1 = k(0,X)$

(b) $(6;\mu|1) \cong (6;\mu_1|1)$ *if and only if* $\mu\mu_1^{-1} \in k^{*2}$

Proof:
First we prove part (a). Suppose that A is a 3-dimensional superalgebra with $\dim A_0 = 2$, then we have bases $\{1, x\}$ and $\{y\}$ for A_0 and A_1 respectively. As we have seen before, we may assume $x^2 \in k$, say $x^2 = \alpha$. We get the following three cases depending on whether α is an element of $\{0\}$, k^{*2} or $k^*\backslash k^{*2}$.

I. If $\alpha = 0$ then $x^2 = 0$, so $J(A_0) = kx$. By Nakayama's lemma $J(A_0)A_1 \subset A_1 \Rightarrow \dim J(A_0)A_1 = 0$, so $J(A_0)A_1 = \{0\}$. i.e. $xy = 0$.

II. If $\alpha = \beta^2$ where $\beta \in k^*$, then $(\beta^{-1}x)^2 = 1$. Replacing x with $\beta^{-1}x$, we may assume $\alpha = 1$, and hence A_0 has a basis $\{1, x\}$ with $x^2 = 1$. Let $e_1 = \frac{1}{2}(1+x)$ and $e_2 = \frac{1}{2}(1-x)$. Then $e_1^2 = e_1$ $e_2^2 = e_2$ and $e_1e_2 = e_2e_1 = 0$. Since A_0 is a commutative algebra, the opposite algebra $A_0^{op} = A_0$. Hence $A_0 \otimes A_0^{op} = A_0 \otimes A_0 = \mathrm{span}\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\} \cong k \times k \times k \times k$, and $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ is a set of orthogonal idempotents with the sum being equal to 1. Thus $A_0 \otimes A_0$ is semisimple. In this case, any $A_0 \otimes A_0$-module is semisimple and any simple $A_0 \otimes A_0$-module is of dimension 1. Since A_1 is an A_0-bimodule, A_1 is a left $A_0 \otimes A_0$-module with the action given by $(a \otimes b)x = axb$, $a,\ b \in A_0$, $x \in A_1$. Notice that A_1 is a simple $A_0 \otimes A_0$-module. Thus we may choose a k-basis $\{y\}$ for A_1. Now by the Wedderburn-Artin Theorem, one gets the following four cases for which one of the four idempotents does not annihilate y:

(a) $(e_1 \otimes e_1)y = y$.
(b) $(e_1 \otimes e_2)y = y$.
(c) $(e_2 \otimes e_1)y = y$.
(d) $(e_2 \otimes e_2)y = y$.

Cases (c) and (d) can be reduced to (a) or (b) by relabelling e_1 and e_2.

III. If $\alpha \in k^*\backslash k^{*2}$ then $A_0 \cong k[X]/(X^2 - \alpha)$ hence A_0 is an extension field of k. Any module over a field is free. Thus A_1 is a free module, suppose it has rank n. Now since $n \geq 1$ then $\dim A_1 = n \dim A_0 = 2n \geq 2$, which is impossible because $\dim A_1 = 1$. So case III does not arise.

We deal with these cases now:

I: We have $x^2 = 0, xy = 0, yx = \beta y, y^2 = \gamma + \delta x$. The equation $(yx)x = yx^2$ gives that $\beta^2 = 0$ and thus $\beta = 0$. The equation $xy^2 = (xy)y$ gives $\gamma = 0$. Thus $y^2 = \delta x$. Either $\delta = 0$ or $\delta \neq 0$. If $\delta = 0$ then $A \cong (4|1)$, via $1 \mapsto 1, x \mapsto X, y \mapsto Y$. If $\delta \neq 0$ then $A \cong (3|1)$, via $1 \mapsto 1, x \mapsto \delta^{-1}X^2, y \mapsto X$.

II (a): We have $e_ie_j = \delta_i^j e_i$ for $1 \leq i, j \leq 2$ and $e_1ye_1 = y$, from which we deduce $e_1y = ye_1 = y, e_2y = ye_2 = 0$, so $y^2 = \gamma e_1$. Either $\gamma = 0$, $\gamma \in k^{*2}$ or $\gamma \in k^*\backslash k^{*2}$. If $\gamma = 0$ then $A \cong (2|1)$, via $e_1 \mapsto (0,1), e_2 \mapsto (1,0), y \mapsto (0,X)$. If $\gamma = \delta^2$ for some $\delta \in k^*$ then $A \cong (1|1)$, via $e_1 \mapsto (1,1,0), e_2 \mapsto (0,0,1), y \mapsto \delta(1,-1,0)$. If $\gamma \in k^*\backslash k^{*2}$ then $A \cong (6;\gamma|1)$, via $e_1 \mapsto (0,1), e_2 \mapsto (1,0), y \mapsto (0,X)$.

II (b): We have $e_ie_j = \delta_i^j e_i$ for $1 \leq i, j \leq 2$ and $e_1ye_2 = y$, from which we deduce $e_1y = y, e_2y = 0, ye_1 = 0, ye_2 = y$ and $y^2 = (e_1ye_2)(e_1ye_2) = 0$. In this case $A \cong (5|1)$, via $e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, y \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

It follows from Gabriel's results that (1)–(5) are non-isomorphic. Using ideas similar to those in Chapter 2 one can show that $(6;\mu)$ is non-isomorphic with (1)–(5), as algebras and hence also as superalgebras.

Finally, we deal with part (b). Suppose $\mu = \delta^2\mu_1$ with $\delta \in k^*$. Then $f(1,1) = (1,1), f(1,0) = (1,0), f(0,X) = \delta(0,X_1)$ gives a superalgebra isomorphism $(6;\mu|1) \cong (6;\mu_1|1)$. Conversely suppose $(6;\mu|1) \cong (6;\mu_1|1)$, then we must have $(6;\mu|1)_0 \cong (6;\mu_1|1)_0$ as algebras, that is $k(\sqrt{\mu}) \cong k(\sqrt{\mu_1})$. Thus by Lemma 2.2.5 it follows that $\mu\mu_1^{-1} \in k^{*2}$. $\square$

Corollary 4.2.2 *In the case that k is algbebraically closed, superalgebras on algebra $(6;\mu)$ do not occur.*

Proof:
In this case $k^{*2} = k^*$, so that $k^*\backslash k^{*2} = \emptyset$. $\square$

As before $(i|0)$ shall stand for the trivially $\mathbb{Z}_2$-graded superalgebra on algebra (i). This always has the form $(i|0)_0 = (i), (i|0)_1 = \{0\}$, so we do not describe the $\mathbb{Z}_2$-grading for these superalgebras in the following.

Theorem 4.2.3 *(Algebraic Classification of* 3*-dimensional superalgebras)*
Let k be an algebraically closed field with $\mathrm{ch}(k) \neq 2$. Suppose A is a superalgebra with dimension 3. Then A is isomorphic to one of the following pairwise non-isomorphic families of superalgebras:

(1) $k \times k \times k$

$(1|0)$

$(1|1)_0 = k(1,1,1) \oplus k(1,1,0), (1|1)_1 = k(1,-1,0)$

(2) $k \times k[X]/(X^2)$

$(2|0)$

$(2|1)_0 = k(1,1) \oplus k(1,0), (2|1)_1 = k(0,X)$

(3) $k[x]/(X^3)$

$(3|0)$

$(3|1)_0 = k1 \oplus kX^2, (3|1)_1 = kX$

(4) $k[X,Y]/(X,Y)^2$

$(4|0)$

$(4|1)_0 = k1 \oplus kX, (4|1)_1 = kY$

$(4|2)_0 = k1, (4|2)_1 = kX \oplus kY$

(5) $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$

$(5|0)$

$(5|1)_0 = k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus k \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, (5|1)_1 = k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Proof:
This follows from combining the results of Gabriel in [12], the comments at the beginning of this section, Proposition 4.2.1 and Corollary 4.2.2. □

Automorphism groups

We shall now also calculate the automorphism groups of those superalgbras described in Theorem 4.2.3, which shall be used later to calculate the dimensions of the corresponding orbits in Salg_3.

We choose a basis for each superalgebra $\{e_1 = 1, e_2, e_3\}$ and determine the constants $a_{21}, \ldots, a_{33}$ for which ϕ gives an automorphism of the given superalgebra, where ϕ is defined by $\phi(e_1) = e_1, \phi(e_2) = a_{21}e_1 + a_{22}e_2 + a_{23}e_3, \phi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3$. As this map must be homogeneous, and since we will choose homogeneous bases, we must have $a_{23} = a_{31} = a_{32} = 0$ for $\dim A_0 = 2$; and $a_{21} = a_{31} = 0$ for $\dim A_0 = 1$. We shall not mention these constants in these cases.

$(1|0)$: $e_1 = (1, 1, 1), e_2 = (1, 0, 0), e_3 = (0, 1, 0)$
Then either

- $a_{21} = a_{31} = 0, a_{22} = a_{33} = 1, a_{23} = a_{32} = 0$; or
- $a_{21} = a_{31} = 0, a_{22} = a_{33} = 0, a_{23} = a_{32} = 1$; or
- $a_{21} = 0, a_{31} = 1, a_{22} = 0, a_{23} = 1, a_{32} = a_{33} = -1$; or
- $a_{21} = 0, a_{31} = 1, a_{22} = 1, a_{23} = 0, a_{32} = a_{33} = -1$; or
- $a_{21} = 1, a_{31} = 0, a_{32} = 0, a_{22} = -1, a_{33} = 1, a_{23} = -1$; or
- $a_{21} = 1, a_{31} = 0, a_{32} = 1, a_{22} = -1, a_{33} = 0, a_{23} = -1$

$(1|1)$: $e_1 = (1, 1, 1), e_2 = (1, 1, 0), e_3 = (1, -1, 0)$
Then $a_{21} = 0, a_{22} = 1, a_{33} = \pm 1$
$(2|0)$: $e_1 = (1, 1), e_2 = (1, 0), e_3 = (0, X)$
Then $a_{21} = a_{23} = a_{31} = a_{32} = 0, a_{22} = 1, a_{33} \neq 0$
$(2|1)$: $e_1 = (1, 1), e_2 = (1, 0), e_3 = (0, X)$
Then $a_{21} = 0, a_{22} = 1, a_{33} \neq 0$
$(3|0)$: $e_1 = 1, e_2 = X^2, e_3 = X$
Then $a_{21} = a_{23} = a_{31} = 0, a_{33} \neq 0, a_{22} = a_{33}^2, a_{32}$ is unconstrained
$(3|1)$: $e_1 = 1, e_2 = X^2, e_3 = X$
Then $a_{21} = 0, a_{33} \neq 0, a_{22} = a_{33}^2$
$(4|0)$: $e_1 = 1, e_2 = X, e_3 = Y$

Then $a_{21} = a_{31} = 0, a_{22}, a_{23}, a_{32}, a_{33}$ are unconstrained apart from $a_{22}a_{33} - a_{23}a_{32} \neq 0$

$(4|1)$: $e_1 = 1, e_2 = X, e_3 = Y$

Then $a_{21} = 0, a_{22} \neq 0, a_{33} \neq 0$

$(4|2)$: $e_1 = 1, e_2 = X, e_3 = Y$

Then $a_{22}, a_{23}, a_{32}, a_{33}$ are unconstrained apart from $a_{22}a_{33} - a_{23}a_{32} \neq 0$

$(5|0)$: $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Then $a_{21} = a_{31} = a_{32} = 0, a_{22} = 1, a_{33} \neq 0, a_{23}$ is unconstrained

$(5|1)$: $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Then $a_{21} = 0, a_{22} = 1, a_{33} \neq 0$

4.3 Geometric Classification in Salg_2

In this section we discuss the geometry of Salg_2 and give the degeneration diagram for the corresponding orbits in this variety.

By Proposition 3.2.12 we know that this variety is disconnected, and by Proposition 3.4.5 we know that the connected components are Salg_2^i for $i = 1, 2$.

Lemma 4.3.1 *The following gives the dimensions of the stabilizers of points in the orbits in* Salg_2*:*

Stabilizer dimensions		
$\cdot$	*0*	*1*
$(1\vert\cdot)$	*0*	*0*
$(2\vert\cdot)$	*1*	*1*

Proof:
If the point $(\alpha_{ij}^k, \gamma_i^j)$ is in the orbit, $G_2 \cdot A$, which is identified with the isomorphism class of superalgebra A, then $\mathrm{Stab}_{G_2}((\alpha_{ij}^k, \gamma_i^j)) \cong \mathrm{Aut}(A)$ where the automorphism group is the group of automorphisms of the superalgebra A as mentioned in Remark 3.3.7. See Section 4.1 for a description of these automorphism groups.

The statements given in Lemma 3.1.41 are also useful when computing the dimension of the automorphism groups. □

Proposition 4.3.2 *The following gives the dimensions of the orbits in* Salg_2*:*

Orbit dimensions		
$\cdot$	*0*	*1*
$(1\vert\cdot)$	2	2
$(2\vert\cdot)$	*1*	*1*

Proof:
We have calculated the dimensions of the automorphism groups, or equivalently, the dimensions of stabilizers of any point in each orbit in Lemma 4.3.1 above. We know that the dimension of G_2 is 2 from Lemma 3.3.5. By using part (b) of Lemma 3.3.6, we can calculate the dimension of the orbit $G_2 \cdot (\alpha_{ij}^k, \gamma_i^j)$ by subtracting the dimension of the stabilizer, $\mathrm{Stab}_{G_2}((\alpha_{ij}^k, \gamma_i^j))$, from the dimension of G_2 which is 2.

□

We give the degeneration diagram of Salg_2 here. The brief explanations are given at the end of the section.

Salg_2^2 component

$$\begin{tikzcd} (2|0) & (1|0) \arrow[l] \end{tikzcd}$$

Salg_2^1 component

$$\begin{tikzcd} (2|1) & (1|1) \arrow[l] \end{tikzcd}$$

Figure 4.1: Degenerations in the variety Salg_2

Theorem 4.3.3 *(Geometric Classification of 2-dimensional Superalgebras)*
In Salg_2 there are two irreducible components. The following structures are generic:
In Salg_2^2*:* $(1|0)$
In Salg_2^1*:* $(1|1)$

Proof:
This follows from the degeneration diagram and Corollary 3.3.12 which

gives the relationship between the degeneration partial order and the irreducible components.

□

Notice that the irreducible components and the connected components coincide in this case.

Since the orbit of (1) is open in Alg_2, the orbit $(1|0)$ is open in Salg_2^2 and we can use Lemma 3.2.18 to see that the orbit of $(1|1)$ is also open.

Proposition 4.3.2 gives the dimensions of the orbits and hence the dimensions of the irreducible components also.

We give the details for the degeneration diagram now:

The degenerations in Salg_2^2 are as given in Gabriels paper. In fact, both degenerations $(1|0) \to (2|0)$ and $(1|1) \to (2|1)$ are obtained as the degenerations to the closed orbits $(2|0)$ and $(2|1)$. These can both be obtained using the following specialization: $e_1 = (1,1), e_2 = (1,-1), e_1' = e_1, e_2' = te_1$ let $t \to 0$.

4.4 Geometric Classification in Salg_3

In this section we discuss the geometry of Salg_3 and give the degeneration diagram for the corresponding orbits in this variety.

By Proposition 3.2.12 we know that this variety is disconnected, and by Proposition 3.4.5 we know that the connected components are Salg_3^i for $i = 1, 2, 3$.

Lemma 4.4.1 *The following gives the dimensions of the stabilizers of points in the orbits in* Salg_3*:*

Stabilizer dimensions			
$\cdot$	*0*	*1*	2
$(1\|\cdot)$	*0*	*0*	
$(2\|\cdot)$	*1*	*1*	
$(3\|\cdot)$	2	*1*	
$(4\|\cdot)$	*4*	2	*4*
$(5\|\cdot)$	2	*1*	

Proof:
If the point $(\alpha_{ij}^k, \gamma_i^j)$ is in the orbit, $G_3 \cdot A$, which is identified with the isomorphism class of superalgebra A, then $\mathrm{Stab}_{G_3}((\alpha_{ij}^k, \gamma_i^j)) \cong \mathrm{Aut}(A)$ where the automorphism group is the group of automorphisms of the superalgebra A as mentioned in Remark 3.3.7. See Section 4.2 for a description of these automorphism groups.

The statements given in Lemma 3.1.41 are also useful when computing the dimensions of the automorphism groups. $\square$

Proposition 4.4.2 *The following gives the dimensions of the orbits in* Salg_3*:*

Orbit dimensions			
$\cdot$	*0*	*1*	2
$(1\|\cdot)$	*6*	*6*	
$(2\|\cdot)$	*5*	*5*	
$(3\|\cdot)$	*4*	*5*	
$(4\|\cdot)$	2	*4*	2
$(5\|\cdot)$	*4*	*5*	

Proof:
We have calculated the dimensions of the automorphism groups, or equivalently, the dimensions of stabilizers of any point in each orbit in Lemma 4.4.1 above. We know that the dimension of G_3 is 6 from Lemma 3.3.5. By using part (b) of Lemma 3.3.6, we can calculate the dimension of the orbit $G_3 \cdot (\alpha_{ij}^k, \gamma_i^j)$ by subtracting the dimension of the stabilizer, $\mathrm{Stab}_{G_3}((\alpha_{ij}^k, \gamma_i^j))$, from the dimension of G_3 which is 6.

$\square$

We give the degeneration diagram of Salg_3 in Figure 4.2. The explanations are given at the end of the section.

Salg_3^3 component

$$
\begin{tikzcd}
 & & & (5|0) \arrow[llld] \\
(4|0) & & & \\
 & (3|0) \arrow[lu] & (2|0) \arrow[l] & (1|0) \arrow[l]
\end{tikzcd}
$$

Salg_3^2 component

$$
\begin{tikzcd}
 & & & (5|1) \arrow[llld] \\
(4|1) & & & \\
 & (3|1) \arrow[lu] & (2|1) \arrow[llu] & (1|1) \arrow[l] \arrow[ll, bend left]
\end{tikzcd}
$$

Salg_3^1 component

$$(4|2)$$

Figure 4.2: Degenerations in the variety Salg_3

Theorem 4.4.3 *(Geometric Classification of* 3*-dimensional Superalgebras)*
In Salg_3 *there are five irreducible components. The following structures are generic:*
In Salg_3^3*:* $(1|0), (5|0)$
In Salg_3^2*:* $(1|1), (5|1)$
In Salg_3^1*:* $(4|2)$

Proof:
This follows from the degeneration diagram and Corollary 3.3.12 which gives the relationship between the degeneration partial order and the irreducible components.

□

Proposition 4.4.2 gives the dimensions of the orbits and hence the dimensions of the irreducible components as well.

Since the orbits of (1) and (5) are open in Alg_3, the orbits $(1|0)$ and $(5|0)$ are open in Salg_3^3, and we can use Lemma 3.2.18 to see that the orbits $(1|1)$ and $(5|1)$ are also open. It may also pay to note that while the orbit $(4|2)$ is closed, it is also open as well.

We give the details for the degeneration diagram now:

The degenerations in Salg_3^3 are as given in Gabriels paper.

We remark that these degenerations can be obtained using those given below for orbits in Salg_3^2 and the remaining degeneration is as follows:

$(2|0) \to (3|0) : e_1 = (1,1), e_2 = (0,1), e_3 = (0,X), e_1' = e_1, e_2' = te_2 + e_3, e_3' = te_3$ let $t \to 0$

We remind the reader that we do not bother to mention the degeneration to the closed orbit since they always exist, the closed orbits in this case being $(4|0), (4|1), (4|2)$.

The degenerations in Salg_3^2 are as follows:

Existence of degenerations:

$(1|1) \to (2|1) : e_1 = (1,1,1), e_2 = (1,1,0), e_3 = (1,-1,0), e_1' = e_1, e_2' = e_2, e_3' = te_3$ let $t \to 0$

$(1|1) \to (3|1) : e_1 = (1,1,1), e_2 = (1,1,0), e_3 = (1,-1,0), e_1' = e_1, e_2' = t^2e_2, e_3' = te_3$ let $t \to 0$

Non-existence of degenerations:

$(2|1) \nrightarrow (3|1)$ (A)

(Where (A) is defined as in Section 3.5)

Since Salg_3^1 consists only of the closed orbit, there is no interesting behaviour to analyse there.

Chapter 5

Supermodules

In this chapter we define the varieties of supermodules over a superalgebra. This is the natural extension of the ordinary module varieties, studied in [7, 12, 17] amongst others, to the setting of superspaces.

This chapter is not intended to be rigorous, but merely to introduce this idea to the reader and discuss similarities with (i) the classical case of modules over an algebra and (ii) with the analysis developed to deal with superalgebras in Chapter 3. We mention the properties one is interested in studying and suggest several useful ideas in this regard. To conclude this chapter we give examples of 3-dimensional supermodules over the superalgebra $k[X]/(X^3)$ firstly when given the trivial $\mathbb{Z}_2$-grading and secondly when given the non-trivial $\mathbb{Z}_2$-grading (given in the previous chapter).

It is hoped that this discussion will stimulate interest in these varieties so that they will be studied in more detail in the future.

5.1 Supermodule varieties

First of all, we define the notion of a supermodule over a superalgebra.

Definition 5.1.1 *If $A = A_0 \oplus A_1$ is a superalgebra and M is an A-module, then M is an A-**supermodule** if there are subspaces M_0 and M_1 such that $M =$*

$M_0 \oplus M_1$ and $A_i \cdot M_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}_2$ or in full $A_0 \cdot M_0 \subseteq M_0$, $A_0 \cdot M_1 \subseteq M_1$, $A_1 \cdot M_0 \subseteq M_1$ and $A_1 \cdot M_1 \subseteq M_0$. The **dimension** *of M shall mean its dimension as a vector space over k.*

Notice that this means that both M_0 and M_1 are A_0-modules.

Now, we would like to know what kind of maps we should take between supermodules (over a superalgebra A) to obtain the category of supermodules (over a superalgebra A).

Definition 5.1.2 *Suppose that M and N are A-supermodules. Then an A-module map $f : M \to N$ is an A-***supermodule map** *if and only if $f(M_i) \subseteq N_i$ for $i = 0, 1$.*

Recall that a superalgebra can be described by giving an algebra and an algebra involution. We have a similar result for supermodules. If $A = A_0 \oplus A_1$ is a superalgebra with main involution σ, then every A-supermodule M gives rise to a linear map $\tau : M \to M$ defined by $\tau(m_0 + m_1) = m_0 - m_1$ which is an involution (that is, $\tau^2 = \tau \circ \tau = \mathrm{id}_M$) and satisfies $\tau(a \cdot m) = \sigma(a) \cdot \tau(m)$ for all $a \in A, m \in M$. Notice that the A-supermodule M must also be an A-module. Conversely, an A-module M and a linear map $\tau : M \to M$ with the above properties can make the A-module M into an A-supermodule.

For the definition of the supermodule varieties of a fixed k-dimension, over a given superalgebra, we must fix the basis of the superalgebra. (However a basis change of the superalgebra yields an isomorphic variety. Moreover this isomorpism is GL_m-equivariant with respect to the GL_m-action which we shall describe below). Suppose the basis of the superalgebra is $\{e_1 = 1, e_2, \ldots, e_n\}$ and it has structure constants $(\alpha_{ij}^k, \gamma_i^j)$.

Suppose M has dimension m as a vector space over k, and let the basis for M be $\{f_1, \ldots, f_m\}$, and suppose that the action of A on M is described as

$$e_i \cdot f_j = \sum_{k=1}^{n} \beta_{ij}^k f_k$$

and the involution τ is described as

$$\tau(f_i) = \sum_{j=1}^{n} \zeta_i^j f_j$$

Then $(\beta_{ij}^k, \zeta_i^j)$ gives us a point in $k^{nm^2+m^2} = k^{(n+1)m^2}$. For M to be an A-supermodule we require the following to be satisfied for all $a, b \in A, m \in M$:

$$(ab) \cdot m = a \cdot (b \cdot m)$$

$$1_A \cdot m = m$$

$$\tau(a \cdot m) = \sigma(a) \cdot \tau(m)$$

$$\tau^2 = id_M$$

These translate into the following conditions:

$$\sum_{l=1}^{n} \alpha_{ij}^l \beta_{lk}^m - \sum_{l=1}^{n} \beta_{jk}^l \beta_{il}^m = 0 \tag{5.1}$$

$$\beta_{1j}^k - \delta_j^k = 0 \tag{5.2}$$

$$\sum_{k=1}^{n} \beta_{ij}^k \zeta_k^m - \sum_{l,k=1}^{n} \gamma_i^l \zeta_j^k \beta_{lk}^m = 0 \tag{5.3}$$

$$\sum_{j=1}^{n} \zeta_i^j \zeta_j^k - \delta_i^k = 0 \tag{5.4}$$

Definition 5.1.3 *The equations (5.1)–(5.4) above, cut out a variety in $k^{(n+1)m^2}$ which we denote by Smod_m^A — the* **variety of A-supermodules of dimension** *m.*

There is a well-defined action of GL_m on Smod_m^A. Let $\Lambda = (\lambda_i^j) \in \mathrm{GL}_m$ and $(\nu_i^j) = \Lambda^{-1}$. Then this **transport of structure action** may be described as follows:

$$\Lambda \cdot (\beta_{ij}^k, \zeta_i^j) = (\sum_{l,m=1}^{n} \lambda_j^l \beta_{il}^m \nu_m^k, \sum_{k,l=1}^{n} \lambda_i^k \zeta_k^l \nu_l^j)$$

As we have seen in the case of superalgebras, this action gives rise to a morphism $\mathrm{GL}_m \times \mathrm{Smod}_m^A \to \mathrm{Smod}_m^A$ which means that the action is algebraic.

Suppose that A is an n-dimensional superalgebra and that M is an m-dimensional A-supermodule, then let V be an n-dimensional vector space and W an m-dimensional vector space. If we write the action map as an element ρ of $\mathrm{Hom}(V \otimes W, W)$ and the $\mathbb{Z}_2$-grading τ as an element of $\mathrm{Hom}(W, W)$, then the action of $\Lambda \in \mathrm{GL}_m$ on Smod_m^A is given by:

$$\Lambda \cdot (\rho, \tau) = (\Lambda \circ \rho \circ (\mathrm{id}_A \otimes \Lambda^{-1}), \Lambda \circ \tau \circ \Lambda^{-1})$$

which is simply the usual transport of structure action for modules on the first component and conjugation by G on the second component.

Now, the GL_m-action on Smod_m^A gives rise to the notion of orbits in this variety and the orbits under this action correspond to isomorphism classes of A-supermodules. If M is an A-supermodule, we write $\mathrm{GL}_m \cdot M$ to denote the orbit which is identified with the isomorphism class of M. Also the stabilizer of a point can be identified with the automorphism group of the supermodule, whose orbit the point belongs to.

As before, one is particularly interested in knowing which orbits are open and which are closed.

Since the action is algebraic, all the results from Section 3.3 on the actions of algebraic groups, immediately apply here too.

Again there is a notion of degeneration of supermodules, which can be defined in this setting as M **degenerates to** N, denoted by $M \to N$ if and only if there is a point in the orbit of N, $\mathrm{GL}_m \cdot N$, which belongs to the closure of the orbit of M, $\mathrm{GL}_m \cdot M$. This is seen to be equivalent to $\mathrm{GL}_m \cdot N \subseteq \overline{\mathrm{GL}_m \cdot M}$. As shown in Section 3.3, degeneration is very useful to determine the geometry of these varieties also.

Analogously to the classical cases of modules over an algebra and the case of superalgebras treated earlier, the main problem of the geometric classification of such varieties is to determine the “generic structures” or equivalently, those module structures whose orbits give rise to the irreducible components. Yet one should always first consider the more basic question of determining the connected components.

Now, with the main notions and problems for the geometric classification of supermodules over a given superalgebra, we finally suggest how to modify a few methods from Chapter 3, on the geometric classification of superalgebras so that they may apply to the situation here.

It should be fairly clear how to modify the proof of Lemma 3.2.10 to show that the sets of supermodules with $\dim M_0 = i, \dim M_1 = j$ with $i, j \geq 0, i + j = m$ are closed subsets (they are clearly disjoint). However, this only applies for $m \leq 2p - 1$ when $\mathrm{ch}(k) = p$, and thus only applies for $m \leq 5$ in general.

As before, one can define the idea of **specialization** of supermodules, and this idea is useful because it shows the existence of a degeneration between supermodules. More formally, if M and N are A-supermodules, then if there is a specialization from M to N, then M degenerates to N.

Next we define some useful maps. It is easy to check that they are all in fact GL_m-equivariant morphisms.

Noticing that any supermodule over a superalgebra A can be regarded as a module over the underlying algebra $U(A)$, simply by forgetting the $\mathbb{Z}_2$-grading of the supermodule — one finds another forgetful map. We also denote this by U. We use this perhaps slightly confusing notation to highlight the analogy with the superalgebra case (hopefully the confusion caused will be minimal). More formally we have $U : \mathrm{Smod}_m^A \to \mathrm{Mod}_m^{U(A)}$ defined by $(\beta_{ij}^k, \zeta_i^j) \mapsto (\beta_{ij}^k)$.

The fact that any module M over an algebra A can be regarded as a trivially $\mathbb{Z}_2$-graded supermodule over the trivially $\mathbb{Z}_2$-graded superalgebra $i(A)$, by endowing it with the following $\mathbb{Z}_2$-grading $M_0 = M, M_1 = \{0\}$ gives rise to another useful morphism. One can equivalently view this as endowing the module with the involution id_M to make it a supermodule over $i(A)$. We define $i : \mathrm{Mod}_m^A \to \mathrm{Smod}_m^{i(A)}$ by $(\beta_{ij}^k) \mapsto (\beta_{ij}^k, \delta_i^j)$.

However, there is no special reason why one should choose to place M in the degree zero component of the supermodule in the above paragraph. One could equally well make M into an $i(A)$-supermodule via the following $\mathbb{Z}_2$-grading $M_0 = \{0\}, M_1 = M$. (This is equivalent to using the involution $-\mathrm{id}_M$ to give M its $i(A)$-supermodule structure). For this reason, we refer to the $\mathbb{Z}_2$-grading described in the above paragraph as the trivial $\mathbb{Z}_2$-grading in degree zero, and the $\mathbb{Z}_2$-grading described in this paragraph as the trivial $\mathbb{Z}_2$-grading in degree one. Again one can define a morphism from this, $i' : \mathrm{Mod}_m^A \to \mathrm{Smod}_m^{i(A)}$ by $(\beta_{ij}^k) \mapsto (\beta_{ij}^k, -\delta_i^j)$.

The above morphisms show that the variety Mod_m^A can be identified with two closed subsets of $\mathrm{Smod}_m^{i(A)}$ where the A-modules are identified with $i(A)$-supermodules with one of the two trivial $\mathbb{Z}_2$-gradings. The fact that there are two closed subset of $\mathrm{Smod}_m^{i(A)}$ which are isomorphic to the variety Mod_m^A is really a consequence of a more general symmetry property of supermodules. If $M = M_0 \oplus M_1$ is an A-supermodule having $\dim M_0 = i, \dim M_1 = j$ (where $i, j \geq 0, i + j = m$) then one can define a new A-supermodule $M' = M'_0 \oplus M'_1$ by $M'_0 = M_1, M'_1 = M_0$. This A-supermodule M' has $\dim M'_0 = j, \dim M'_1 = i$. A little thought shows that all that one is doing is changing the sign on the involution. This remark suggests defining the following morphism, $s : \mathrm{Smod}_m^A \to \mathrm{Smod}_m^A$ by $(\beta_{ij}^k, \zeta_i^j) \mapsto (\beta_{ij}^k, -\zeta_i^j)$. One should quickly notice that s is an involution, that is, $s \circ s = \mathrm{id}_{\mathrm{Smod}_m^A}$ hence s is invertible. The morphism i' defined above is then simply $s \circ i$.

When the sets of supermodules in Smod_m^A with $\dim M_0 = i, \dim M_1 = j$ with $i, j \geq 0, i+j = m$ are closed subsets, the morphism s gives an isomorphism between the subset of supermodules with $\dim M_0 = i, \dim M_1 = j$ and the subset of supermodules with $\dim M_0 = j, \dim M_1 = i$, which shows that the geometry of these two subsets must coincide. Thus, in this case, one only needs to study $\lceil \frac{m+1}{2} \rceil$ of these subsets.

5.2 Examples of supermodule varieties

In this section consider the supermodules over superalgebras on $k[X]/(X^3)$. The trivially $\mathbb{Z}_2$-graded superalgebra is denoted by A where $A_0 = A, A_1 = \{0\}$ and the non-trivially $\mathbb{Z}_2$-graded superalgebra is denoted by B where $B_0 = k1 \oplus kX^2, B_1 = kX$. We give the isomorphism classes of 3-dimensional supermodules over both of these superalgebras. These give the orbits in the supermodule varieties and we then study the geometry of these varieties.

The following two propositions are stated without proof. The ideas used in earlier sections can be adapted to give proof of these results.

Proposition 5.2.1 *The 3-dimensional A-supermodules are isomorphic to one of the following, which are pairwise non-isomorphic:*

(1) $k[X]/(X^3)$:
$(1|0)_0 = k[X]/(X^3), (1|0)_1 = \{0\}$
$(1|1)_0 = \{0\}, (1|1)_1 = k[X]/(X^3)$

(2) $k[X]/(X^2) \oplus k$:
$(2|0)_0 = k[X]/(X^2) \oplus k, (2|0)_1 = \{0\}$
$(2|1)_0 = k(1,0) \oplus k(X,0), (2|1)_1 = k(0,1)$
$(2|2)_0 = k(0,1), (2|2)_1 = k(1,0) \oplus k(X,0)$
$(2|3)_0 = \{0\}, (2|3)_1 = k[X]/(X^2) \oplus k$

(3) $k \oplus k \oplus k$:
$(3|0)_0 = k \oplus k \oplus k, (3|0)_1 = \{0\}$
$(3|1)_0 = k(1,0,0) \oplus k(0,1,0), (3|1)_1 = k(0,0,1)$
$(3|2)_0 = k(0,0,1), (3|2)_1 = k(1,0,0) \oplus k(0,1,0)$
$(3|3)_0 = \{0\}, (3|3)_1 = k \oplus k \oplus k$

Proposition 5.2.2 *The 3-dimensional B-supermodules are isomorphic to one of the following, which are pairwise non-isomorphic:*

(1) $k[X]/(X^3)$
$(1|0)_0 = k1 \oplus kX^2, (1|0)_1 = kX$
$(1|1)_0 = kX, (1|1)_1 = k1 \oplus kX^2$

(2) $k[X]/(X^2) \oplus k$*:*
$(2|0)_0 = k(1,0) \oplus k(0,1), (2|0)_1 = k(X,0)$
$(2|1)_0 = k(X,0) \oplus k(0,1), (2|1)_1 = k(1,0)$
$(2|2)_0 = k(X,0), (2|0)_1 = k(1,0) \oplus k(0,1)$
$(2|3)_0 = k(1,0), (2|3)_1 = k(X,0) \oplus k(0,1)$

(3) $k \oplus k \oplus k$*:*
$(3|0)_0 = k \oplus k \oplus k, (3|0)_1 = \{0\}$
$(3|1)_0 = k(1,0,0) \oplus k(0,1,0), (3|1)_1 = k(0,0,1)$
$(3|2)_0 = k(0,0,1), (3|2)_1 = k(1,0,0) \oplus k(0,1,0)$
$(3|3)_0 = \{0\}, (3|3)_1 = k \oplus k \oplus k$

We now give the degeneration diagrams for these varieties. Figure 5.1 treats the module variety Smod_3^A, and Figure 5.2 treats the module variety Smod_3^B.

$\dim_0 = 3$ component

$$\begin{tikzcd} (3|0) & (2|0) \arrow[l] & (1|0) \arrow[l] \end{tikzcd}$$

$\dim_0 = 2$ component

$$\begin{tikzcd} (3|1) & (2|1) \arrow[l] \end{tikzcd}$$

$\dim_0 = 1$ component

$$\begin{tikzcd} (3|2) & (2|2) \arrow[l] \end{tikzcd}$$

$\dim_0 = 0$ component

$$\begin{tikzcd} (3|3) & (2|3) \arrow[l] & (1|1) \arrow[l] \end{tikzcd}$$

Figure 5.1: Degenerations in Smod_3^A

$\dim_0 = 3$ component

$(3|0)$

$\dim_0 = 2$ component

$$
\begin{tikzcd}
 & (2|0) \arrow[dl] & \\
(3|1) & & (1|0) \arrow[ul] \arrow[dl] \\
 & (2|1) \arrow[ul] &
\end{tikzcd}
$$

$\dim_0 = 1$ component

$$
\begin{tikzcd}
 & (2|2) \arrow[dl] & \\
(3|2) & & (1|1) \arrow[ul] \arrow[dl] \\
 & (2|3) \arrow[ul] &
\end{tikzcd}
$$

$\dim_0 = 0$ component

$(3|3)$

Figure 5.2: Degenerations in Smod_3^B

We briefly explain the methods used to obtain these two degeneration diagrams now:

Since the dimension of the modules is 3, we know from earlier statements that the sets of supermodules with $\dim M_0 = i, \dim M_1 = j$ with $i, j \geq 0, i + j = 3$ are closed subsets, which are clearly disjoint. Thus we only need to search for degenerations between modules having the same dimension for their homogeneous components.

For the variety Smod_3^A, the degenerations are obtained by the following specializations:

The specialization for the supermodule structures on (1) to the supermodule structures on (2): $f_1 = 1, f_2 = X, f_3 = X^2$, $f_1' = tf_1, f_2' = tf_2, f_3' = f_3$ let $t \to 0$

The specialization for the supermodule structures on (2) to the supermodule structures on (3): $f_1 = (1,0), f_2 = (X,0), f_3 = (0,1)$, $f_1' = tf_1, f_2' = f_2, f_3' = f_3$ let $t \to 0$

For the Smod_3^B, the degenerations are obtained by the following specializations:

The specialization for $(1|0) \to (2|0)$ and $(1|1) \to (2|2)$: $f_1 = 1, f_2 = X, f_3 = X^2$, $f_1' = tf_1, f_2' = tf_2, f_3' = f_3$ let $t \to 0$

The specialization for $(1|0) \to (2|1)$ and $(1|1) \to (2|3)$: $f_1 = 1, f_2 = X, f_3 = X^2$, $f_1' = tf_1, f_2' = f_2, f_3' = f_3$ let $t \to 0$

The specialization for the supermodule structures on (2) to the supermodule structures on (3): $f_1 = (1,0), f_2 = (X,0), f_3 = (0,1)$, $f_1' = tf_1, f_2' = f_2, f_3' = f_3$ let $t \to 0$

Finally consider associating to each supermodule M the following cones: $R_M = \{x \in M : x \in M_0, X \cdot x = 0\}$ and $S_M = \{x \in M : x \in M_1, X \cdot x = 0\}$ (where X is from $k[X]/(X^3)$). We use upper semicontinuity arguments as seen in Section 3.4. We then discover $\{M \in \mathrm{Smod}_3^B : \dim R_M \geq 2\}$ is a closed set containing the orbit $(2|1)$ and is disjoint from the orbit of $(2|0)$, and $\{M \in \mathrm{Smod}_3^B : \dim S_M \geq 1\}$ is a closed set containing the orbit of $(2|0)$ and is disjoint from the orbit of $(2|1)$. Thus there can be no degenerations between $(2|0)$ and $(2|1)$. A similar argument can be used to show that there are no degenerations between $(2|2)$ and $(2|3)$. Another way to show this fact is simply to notice that from comments in the previous section, the geometry of the subset with $\dim M_0 = 2$ is the same as the geometry of the subset with $\dim M_0 = 1$.

The final comments we make on the material in this section is that not only is it interesting to compare the varieties $\mathrm{Mod}_m^{U(A)}$ and Smod_m^A and their geometry, but it is also very interesting to compare the varieties Smod_m^A and Smod_m^B where A and B are non-isomorphic superalgebra structures on the same underlying algebra. In this way we see how the supermodules change when we endow a given algebra with different superalgebra structures. From our example above, we see that there is a dramatic change in the geometry of the two varieties of supermodules over the superalgebra $k[X]/(X^3)$ when we change the $\mathbb{Z}_2$-grading on $k[X]/(X^3)$.

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