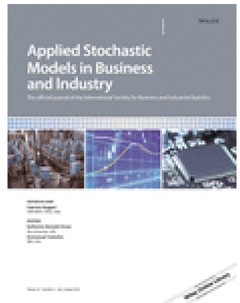

Discussion of Virtual age, is it real?

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Discussion of Virtual age, is it real?

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Abstract. The paper by Finkelstein and Cha centres around the attempt to model a system subject to failures after it has experienced a damage and/or failure episode. The paper discusses the validity of the assumptions of virtual age models, and provides helpful examples which elucidate the uses and shortcomings of models using a time transformation to mimic the effect of damage and ageing.

We first recast their examples in a uniform notation. We make comments on the nature of empirical vs. physical models, and the advantages and disadvantages of these. We close with a brief note about our own shock/repair model and its relationship to the time transformed failure models discussed by Finkelstein and Cha.

This interesting paper by Finkelstein and Cha centres around the attempt to model a system subject to failures after it has experienced a damage and/or failure episode. The paper discusses the validity of the assumptions of virtual age models, and provides helpful examples which elucidate the uses and shortcomings of models using a time transformation to mimic the effect of damage and ageing.

We have only a few points to make, and begin by briefly restating the problem and the logic used by these authors.

1 The modelling setting

Prior to the failure the ('baseline') survival function may be modelled by some parametric family $\bar{F}_b(t|\theta_b)$. Let us assume that such a suitable $\bar{F}_b(t|\theta_b)$ has been identified, and that its properties empirically match the observed distribution of first failures of the system.

The question then arises of how to model the survival function $\bar{F}_s(\cdot)$ after the episode at t^*. One possibility is to construct a model $\bar{F}_s(t|t > t^*;\theta_s)$ from a completely different parametric family from $\bar{F}_b(t|\theta_b)$: the system has been damaged and may have been repaired, and consequently it can be expected to behave differently. The identification of the appropriate $\bar{F}_s$ requires a search in the class of all possible survival functions, and the estimation of its parameters.

However there may be strong physical reasons to suspect that $\bar{F}_b$ and $\bar{F}_s$ might come from the same parametric family – conveniently removing the need to search model space for a suitable function $\bar{F}_s$. We might find, empirically, that the distribution of subsequent failures is well represented by $\bar{F}_b(t|\theta_s)$, but with changed values of the parameters θ_s compared to the baseline values θ_b.

A more general and attractive modelling approach proposed by the authors notes that if both $\bar{F}_b$ and $\bar{F}_s$ are strictly decreasing functions, then it is always possible to write

$$\bar{F}_s(t) = \bar{F}_b(W(t)) \tag{1}$$

for some monotonically increasing function $W(t)$. The modelling focus then shifts from identifying a suitable $\bar{F}_s$ to instead identifying the age transformation function $W(t)$. This fully general construction, which the authors call the 'age correspondence' formulation, will be practical and an advantageous restatement of the original problem only if a suitably simple function $W(t)$ can be found that matches the data.

Accelerated age models, where $W(t) \geq t$, include the transformation of the exponential failure model into the Weibull model by setting $W(t) = at^b$ (for $a \geq 1$, $b \geq 1$).

Virtual age models in contrast simply add an offset $\tau(t^*) \geq -t^*$ to the age of the system: $W(t) = \tau(t^*) + t$ at times $t \geq t^*$. Thus the item behaves at time $t \geq t^*$ as if it had been operating without failure or damage, but was actually of age $\tau(t^*) + t$. It is the motivation and applicability of these virtual age models that concern the authors most in their paper.

In the general formulation of (1), $W(t)$ is a non-negative function which is strictly increasing in the intervals $[0, t^*)$ and $[t^*, \infty)$. Thus if $w(t) \geq 1$ is a function on the interval $[0, \infty)$ and $\tau(t^*) \geq -t^*$ then

$$W(t) = \tau(t^*) I(t \geq t^*) + \int_0^t w(u)\,\mathrm{d}u\,. \tag{2}$$

If the cumulative hazard rate function corresponding to $\bar{F}(t)$ is $\Lambda(t)$ then (1) and (2) imply

$$\begin{aligned}
\bar{F}(t) &= e^{-(\Lambda(t)-\Lambda(0))} \quad t \geq 0 && (3)\\
\Lambda(t) - \Lambda(0) &= \int_0^t \lambda(u)\,\mathrm{d}u \\
&\stackrel{\text{def}}{=} \Lambda_b(W(t)) - \Lambda_b(W(0)) && (4)\\
\Lambda_b(t) &= \int_0^t \lambda_b(u)\,\mathrm{d}u && (5)\\
\lambda(t) &= \lambda_b(W(t))W'(t) \\
&= \lambda_b(W(t))\left(\tau(t^*)\delta(t-t^*) + w(t)\right) && (6)
\end{aligned}$$

where $\delta(\cdot)$ is the Dirac delta function, and $\lambda(\cdot)$ and $\lambda_b(\cdot)$ are the hazard rate functions corresponding to $\bar{F}(t)$ and $\bar{F}_b(t)$ respectively. $\Lambda(\cdot)$ and $\Lambda_b(\cdot)$ are the corresponding cumulative hazard rate functions.

There are then some simple cases

1. Simple ageing: $w(t) = 1$, $\tau(t^*) = 0$ and $W(t) = t$;
2. Accelerated ageing: $w(t) \geq 1$, $\tau(t^*) = 0$, $W(t) \geq t$;
3. Baseline $\longrightarrow$ Severe:

$$\begin{aligned}
w(t) &= 1 + I(t \geq t^*)\mu(t) \\
W(t) &= t + I(t \geq t^*)\left(\tau(t^*) + \int_{t^*}^t \mu(u)\,\mathrm{d}u\right)
\end{aligned}$$

for some function $\mu(t) \geq 0$;

4. The Virtual Age model: repair following a shock (and possible failure) at t^*

$$\begin{aligned} w(t) &= 1 \\ W(t) &= t + \tau(t^*)I(t \geq t^*) \end{aligned}$$

with

(a) Good as New repair: $\tau(t^*) = -t^*$

(b) Imperfect repair: $-t^* < \tau(t^*) < 0$

(c) Minimal repair: $\tau(t^*) = 0$

(d) Repair but with lasting shock damage: $\tau(t^*) > 0$

These models can be compared with the specification of various possible depths of repair in §4 of Chukova et al. (2004).

2 Physical and empirical models

The objection that the authors make to the virtual age model as a universally applicable approach is well made, and we are in full agreement. In particular the authors provide (following their Remark 2) a persuasive counter-example for a cold standby system made of n iid components with exponentially distributed lifetimes. The virtual age model is logically incompatible with this setting. Thus we agree with their statement in §2 that *'In order for the virtual age concept that arises in the problem of imperfect repair/maintenance to be sound and practically justified, there should be a clear description of the corresponding repair operations that result in (and conform with) relationship (1). In our opinion, this is not the case for most repairable systems except for a specific case of minimal repair.'*

Where a practical understanding of a physical system is available and is accurate, then a model built on that understanding may lead to the most successful description of the failures of the system: with or without repair. Such models may involve the incorporation of appropriate covariates and measures of (correlates of) wear or degradation. If the model fits the data well, without being overparameterised (a separate question), then we will be satisfied with our model and its performance.

However if a distribution with no particular physical motivation does a better job of fitting the empirical data than our best physically motivated model, we must admit to having an incomplete understanding of, or at least an incomplete model for, the system under consideration. In this situation one may use this discrepancy as a reason to revisit our assumptions about the system. If we are not in a position to be able to revisit our assumptions (maintaining tractability) we will be better off with the empirically motivated model.

An empirical model may lack such an understanding of a physically motivated model, but may nevertheless be highly successful in modelling failure time distributions. Meeker and Escobar put this very well:

> *'When there is little understanding of the chemical or physical processes leading to failure, it may be impossible to develop a model based on physical/chemical theory. An empirical model may be the only alternative. An empirical model may provide an excellent fit to the available data but may provide nonsense extrapolations (e.g., the quadratic models used in Section 17.5). In some situations there may be extensive empirical experience with particular combinations of variables and failure mechanisms and this experience may provide the needed justification for extrapolation to use conditions.'* (Chapter 18 Accelerated Test Models Meeker & Escobar 1998, p.469 Empirical Acceleration Models)

Finkelstein and Cha usefully identify the equivalent concept of the 'black-box' model for failure distributions, in which empirical observations of the behaviour of a system are used to identify a suitable family of modelling functions. In our discussion above the baseline survival function $\bar{F}_b$ is in this category. And as noted by the authors at the end of §2, such black-box distributions are of course the natural approach to component-wise failure distributions in a setting where a physical model of the cooperation of those components leads to a physically motivated failure model at system level.

Ultimately the test of any model is its utility – in particular that is regarded as valuable by practitioners in the field. Our view is that if the Virtual Age model is shown to be effective, the question of whether or not it is physically motivated is secondary from the practioner's point of view. If there is suitable physical information or insight available, that information may in many cases lead to a model superior to an empirical virtual age model, but that is not guaranteed to be the case. Moreover the criticism that the Virtual Age model encodes assumptions that cannot (always) be justified is a criticism from which all models suffer, and certainly applies to the selection of any particular parametric form for $W(t)$.

The authors spend some time (§4) discussing a simple shock model where the virtual age model is interpreted at the instant of failure as a scaling of the survival function, (10) and (11) being equivalent to

$$\bar{F}_s(t) = \begin{cases} \bar{F}_b(t) & 0 \le t < t^* \\ (1-p(t^*))\bar{F}_b(t^*) & t = t^* \\ \bar{F}_b(\tau(t^*)+t) & t > t^* \end{cases} \qquad \text{with} \qquad p(t^*) = 1 - \frac{\bar{F}_b(\tau(t^*)+t^*)}{\bar{F}_b(t^*)} .$$

The latter expression establishes the (bijective) relationship between $\tau(t^*)$ and $p(t^*)$. Given this relationship it follows that $\tau(t^*)$ is just as estimable as $p(t^*)$, and so we are not so convinced that this rescaling leads to any particular insight, especially because $\bar{F}_s(t)$ immediately switches to the time translation of $\bar{F}_b$ at $t > t^*$.

We welcome the author's use of the age correspondence construction $W(t)$ as a helpful reformulation of the general accelerated life model. We expect that this framework will assist in the construction of future models, not least in the explanation of the logic which accompanies these constructions.

An interesting question in this setting is the connection between accelerated life models as a representation of damage, as opposed to accelerated use in testing settings. For example where an item is used at a rate $r > 1$ times its conventional expected useage rate, and where we might assume $W(t) = rt$.

3 Response to the question about our own paper

In §4 the authors raise the question of whether our shock/repair model Arnold et al. (2016) can be cast in the same formulation as a virtual age translation.

If we consider our equation (1) in the case of a single shock at time $t = t^*$, then the survival function, conditional on survival beyond time t^*, can be written

$$\begin{aligned} \bar{F}(t|t > t^*) &= \frac{\bar{F}(t)}{\bar{F}(t^*)} \\ &= \frac{\bar{F}_b(t-\eta t^*)}{\bar{F}_b((1-\eta)t^*)} e^{-\phi(t-t^*)} \end{aligned}$$

where $0 \le \eta \le 1$ is a parameter controlling the depth of repair ($\eta = 0$ is minimal, $\eta = 1$ is good as new), and ϕ quantifies the permanent damage inflicted by each shock.

In the formulation of (1) and (2) above, our model is equivalent to the age correspondence model only in the particular case of a constant baseline hazard rate $\lambda_b(t) = \lambda_b$ and an age correspondence function

$$W(t) = (1-\eta)t^* + (1+\phi/\lambda_b)(t-t^*) \ ,$$

which is equivalent to imperfect repair $\tau(t^*) = -\eta t^*$ and

$$w(t) = \begin{cases} 1 & 0 \le t < t^* \ , \\ 1+\phi/\lambda_b & t \ge t^* \ . \end{cases}$$

The shock at $t = t^*$ thus increases the hazard rate by the multiplier $(1+\phi/\lambda_b)$ and is effectively an accelerated life model, rather than an instance of the virtual age model described in §4.

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