2020 IEEE 7th International Conference on Industrial Engineering and Applications

A New Robust Optimization Approach to Deal with Dependent Uncertain Parameters

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***Abstract*—In the optimization problems with uncertain parameters, a solution is said to be robust if it is feasible with high probability regarding the realization of uncertain parameters. In this paper, a new robust approach is developed for the linear problems in which the model parameters are dependent on each other. The proposed approach converts the linear model to an equivalent integer linear programming one using the primal and dual theorem. The results of the paper indicate the ability of the new approach in fixing some inconsistency of the common robust optimization approach for the mentioned problem.**

Keywords-deterministic robust optimization; dependent uncertain parameters; consistence robust solution; primal and dual relations

I. Introduction

The robust optimization approaches encompass techniques and methodologies that support decision-making against parameter uncertainty to find a robust solution [1].In the concept of robustness, a range of solutions should be considered, which are obtained by taking different possible values of uncertain parameters. On one hand of this range, there is a thoroughly conservative solution that remains feasible for any realization of the uncertain parameters as a "pessimistic solution." At the opposite end, there is an "optimistic solution," i.e., the solution which could be obtained if the uncertain parameters are realized in the state wherein we have the best value of the objective function. In reality, the conservative approach is appropriate for engineering applications such as robust control theory; for example, a doomed satellite launch or a destroyed unmanned robot results in a significant adverse effect that cannot be connived. However, in business practices, the argumentative event, such as low demand, does not lead to high consequences as an engineering practice. Business managers usually seek a feasible solution with a high level of confidence and also high value for a considered objective function. Indeed, they are not interested in complete protection against uncertain events at the expense of rigorous deterioration in the objective function [1]. Therefore, the solutions which lie on a spectrum between these two extremes may be more desirable for decision-makers. Each of the solutions in this range can be regarded as a robust solution with a different degree of robustness.

In reality, the uncertainty in the model parameters comes from different resources such as the uncertainty of system outputs related to approximations in the modeling process [2, 3], production tolerances and actuator imprecision intentionally exposed for being the model simple and cost and time effective and changes in environmental and operating conditions of the model in the future that could not be predicted precisely before that [4, 5].

In mathematical optimization, the uncertainty in the objective function and the feasible region of solutions are sometimes distinguished. When the uncertainty only influences the feasibility of solutions, then robust optimization approaches aim to seek a feasible solution for any realization of unknown coefficient within a smaller realistic set, which is called "uncertainty set." This set sometimes is centered on the nominal values of uncertain parameters. The choice of uncertainty set plays an important role in determining the level of deterioration in the objective function and the level of protection against uncertainty and feasibility of the resulting solutions [1, 6, 7]. When the uncertainty exists in parameters of an objective function, the robust optimization approach pursues a solution for any realization of unknown parameters. In this case, a common approach is to optimize the worst-case of the objective function. Also, this can be done by incorporating the objective function as a new constraint into the model constraints. Moreover, the researchers examined both single objective function[8, 9] and multi-objective function[10, 11] for achieving robust solutions. In the proposed robust approach, a worst-case analysis of objective functions has been considered.

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II. Literature Review

The duality has played a key role in the development of robust models [12, 13]. Some researchers have investigated the relationship between primal and corresponding dual of robust optimization problems. For example, it has been shown that the worst of primal is equivalent to the dual best [14]. In the proposed robust approach, this relationship is utilized to convert a MinMax combinatorial problem to an equivalent MILP.

It is worth to note that the robust concepts and techniques have originally rooted in engineering practices, and as a preliminary effort in this field, the ‘‘Taguchi method’’ is a well-known practice that considers the robustness in the design of products[15]. In the operation research context, the research of [16] could be referred to as pioneering publications.

To deal with uncertainty in the robust optimization approaches, generally, three main categories exist in the literature: deterministic, probabilistic, and possibilistic. In the deterministic type, the domain of uncertain parameters is considered. This type of uncertainty arises when there is no more information about uncertain parameters except their domain of variation [12, 17].In the Probabilistic robust optimization, the likelihood of uncertain events is considered, and the probabilistic theory is extended to the robust optimization problem. In this field, the connections of the robust optimization with moment information [18, 19], two-stage optimization approach [20, 21], multi-stage optimization approach [22, 23], and risk theory [24-26] have been studied. Moreover, the degree of an uncertain event is measured using fuzzy measures in the possibilistic type [27]. It is worth mentioning that in this paper, the deterministic type of uncertainty is studied, and the dependency between uncertain parameters is taken into account for avoiding inconsistency in the model results.

To measure the robustness of a solution, some robustness measures have been introduced in the literature. The *robust counterpart* is a robustness measure that explores a solution amongst feasible solutions with the best objective function for all values of uncertain parameters [28]. This measure is used for deterministic uncertainty type as a conservative approach. On the other hand, *expectancy measures* are another robustness measure that takes into account a probabilistic function of uncertain model and obtains a robust solution by optimizing this function. *Conditional expected value* [29] and *variance* [30] are examples of these measures. Another measure is the *probabilistic threshold of robustness,* which considers the probability distribution of an uncertain model for searching a robust solution within a predefined threshold [29]. *Statistical feasibility robustness* is another measure of robustness that explores a feasible solution with a predetermined probability. The *chance constraint programming method* is an example of this robustness measure [31]. Finally, the *possibilistic uncertainty measure* takes into account uncertainty that is identified subjectively. For this type of uncertainty, membership function is usually defined using a fuzzy set theory [27].

Generally, to implement the ideas of the robustness, two approaches exist. One approach is the *"simplification strategy"* introduced by [32], whereas another approach is *"simulation optimization technique"* nominated by [33]. The first approach is used for modeling problems mathematically, and it is applicable if the probabilistic distribution or plausible membership function of an uncertain model can be determined or estimated. On the other hand, techniques such as Monte-Carlo simulation are applied to simulate the real-world problem in the second approach. The latter is helpful for some problems that an explicit mathematical programming model cannot model either, or their mathematical models are complicated enough to be implemented efficiency by the available standard procedures. In this paper, a linear problem with shared uncertain parameters is addressed, and a *mixed-integer linear programming* (MILP) model is proposed for taking into account the concept of robustness. Thus, the proposed model in this study is categorized as a "*simplification strategy*."

III. Study Design

In this paper, it is assumed that uncertain events are common between the parameters of the model. Therefore, it is not reasonable to consider these parameters individually. Here, each event is defined as a random variable, whereas only its domain is known. The models introduced by [17],[34], and [12] are among the robust models which regard the domain of uncertain parameters. [17] developed a robust model under the uncertain condition with a fully conservative robust approach, wherein this solution remains feasible for all realization of uncertain parameters. The advantage of the Soyster approach is that a linear programming model remains linear after applying robustness modifications. However, that conservative and worst-case analysis approach does not yield good value for the objective function of the problem. [34] proposed a less conservative robust model that made a tradeoff between feasibility and optimality of the solutions. Indeed, that method converted a linear programming problem to a nonlinear one and made it more challenging to solve. One of the applicable robust optimization approaches is [12] that has been great fund interest between researchers and practitioners. Moreover, this approach has the ability to doing tradeoff efficiently between feasibility and optimality of solutions. However, that approach does not consider repetitive uncertain parameters. Thus, we utilized the idea of robustness in [12] and developed a new robust optimization approach for the problem with dependent uncertain parameters.

The rest of this paper is organized as follows. First, after a brief description of the [12] approach, the challenges of this approach are discussed. Then, the new robust approach, which overcomes the mentioned shortcomings of the [12] is developed. After that, numerical examples are conducted to show the validation of the proposed approach. A discussion section follows these lines. Finally, the results of the paper and future research opportunities are presented.

IV. Assumptions and Analysis

A. A Model with Shared Parameters

Before explaining the proposed robust optimization approach, we first consider a simple production planning model. Also, we show some inconsistency in the robust solution of the model based on [12] approach. This model is as follow:

Problem (1)

$$Max\ c_1d_1x_1+c_2d_2x_2+c_3d_3x_3+c_4d_4x_4 \quad (1)$$

$$s.t.$$

$$a_1d_1x_1+a_2d_2x_2+a_3d_3x_3+a_4d_4x_4 \le C \quad (2)$$

$$\forall i,\ 0 \le x_i \le 1 \quad (3)$$

The model aims to maximize the net profit of production in a firm with total production capacity equal to *C*. Here, d_i (i=1, 2, 3, 4) stands for the uncertain demand of product *i*. Moreover, x_i denotes the percentage of product *i* demand, which is prepared by the firm as the model variable. Also, c_i and a_i are the net profit per unit and the capacity usage per unit for the product *i*. In the following, after explaining [12] approach, we show how this approach results in an incompatible robust solution for the production planning problem. [12] approach considers a linear programming model as follow:

$$Max\ Cx \quad (4)$$

$$s.t.$$

$$Ax \le b \quad (5)$$

$$l \le x \le u \quad (6)$$

In this model, it is assumed that J_i is the set of coefficients in row *i* of matrix *A* that are subject to uncertainty. Each entry $a_{ij} \mid j \in J_i$ in row *i* is a symmetric and bounded random variable that takes values in an interval $[a_{ij}-\hat{a}_{ij}, a_{ij}+\hat{a}_{ij}]$. Associated with the uncertain variable a_{ij}, the random variable $\eta_{ij}=(a_{ij}-a_{ij})/a_{ij}$ is regarded, which is assumed to follow an unknown but symmetric distribution in the interval [-1, 1]. Following the above assumptions and denoting Γ_i as the desired number of uncertain parameters in row *i* which are regarded as critical parameters, [12] introduced the equivalent robust model of the above linear model as follow:

$$Max\ C'X \quad (7)$$

$$s.t.$$

$$\sum_j a_{ij}x_j +$$

$$\max_{S_i}\{\sum_{j\in s_i}\hat{a}_{ij}y_i+(\Gamma_i-\lfloor\Gamma_i\rfloor)\hat{a}_{it_i}y_t\} \le b_i \quad \forall i \quad (8)$$

$$S_i=\{s_i \cup \{t_i \mid s_i \subseteq J_i, |s_i|=\lfloor\Gamma_i\rfloor, t_i \in J_i \backslash s_i\}$$

$$-y_j \le x_j \le y_j \quad (9)$$

$$L \le X \le U \quad (10)$$

$$Y \ge 0 \quad (11)$$

Also, using the primal and dual relations, [12] showed that the above robust model is equivalent to the linear model as below:

$$Max\ C'X \quad (12)$$

$$s.t.$$

$$\sum_j a_{ij}x_j + Z_i\Gamma_i + \sum_{j\in J_i} P_{ij} \le b_i \quad \forall i \quad (13)$$

$$Z_i + P_{ij} \ge \hat{a}_{ij}y_j \quad \forall i, j \in J_i \quad (14)$$

$$-y_j \le x_j \le y_j \quad \forall j \quad (15)$$

$$L \le X \le U \quad (16)$$

$$P_{ij} \ge 0 \quad \forall i,j \quad (17)$$

$$Z_i \ge 0 \quad \forall i \quad (18)$$

$$y_j \ge 0 \quad \forall j \quad (19)$$

The idea behind this model is to choose Γ_i uncertain parameters in the constraint *i* and set their values equal to their upper or lower bounds such that the constraint imposes the most restriction on the feasible space of solutions. In this way, the feasibility of the obtained solution is ensured with more probability than a thoroughly conservative approach.

Notably, the variable P_{ij} in the constraints (13) and (14) denotes whether parameter *j* in constraint *i* take its upper bound or lower bound as a critical value or its mean value. In this regard, using the dual-primal relations for developing this model as introduced in [12], if P_{ij} is positive in the obtained robust solution, then this means that the parameter *j* of constraint *i* takes a critical value.

Considering the above discussion, we come back to *Problem (1)* and explain the arisen inconsistency of [12] approach for this problem. Let the value of parameters in *Problem (1)* is as Table I. Furthermore, we assume *capacity*=90. Moreover, to convert *Problem (1)* in the form of [12] model, we define the parameters $e_i=c_i.d_i$ and $f_i=a_i.d_i$ and substitute them in *Problem* (1). Doing so and using the interval multiplier operation, the values of e_i and f_i for different products will be as Table II.

TABLE I. Value of the Parameters in Problem (1)

i	**1**	**2**	**3**	**4**
a_i	4	3	2	3
c_i	8	4	3	4
d_i	8	7	8	9
$\hat{d}_i$	3	2	4	4

TABLE II. Value of the Modified Parameters in *Problem* (1)

i	**1**	**2**	**3**	**4**
e_i	40	28	24	36
$\hat{e}_i$	15	8	12	16
f_i	32	21	16	27
$\hat{f}_i$	12	6	8	12

Now, to model *Problem (1)* using [12] approach, two other modifications are done: the objective function is shifted to the problem constraints as $z \geq c_1 d_1 x_1 + c_2 d_2 x_2 + c_3 d_3 x_3 + c_4 d_4 x_4$; the objective function of the problem is considered $Max\ z$. Now, if *Problem (1)* is solved using [12] approach, the robust solution of the problem $\Gamma_1 = 2$ (the converted objective function constraint) and $\Gamma_2 = 2$ (the capacity constraint) is as Table III.

TABLE III. THE ROBUST SOLUTION OF *PROBLEM* (1) USING BERTSIMAS AND SIM (2004) APPROACH

i	1	2	3	4
x_i	0.532	1	0.665	1
y_i	0.532	1	0.665	1
p_{1i}	0	0.025	0	2.658
p_{2i}	0.38	0	0	1.975

Taking into account the value of $p_{1i} \mid \mathrm{i} = 1,2,3,4$ in Table (3), it is clear that the parameters e_2 and e_4 take their upper bound value in the optimal robust solution. This is corresponding to considering the lower bound values for d_2 and d_4. However, the parameters e_1 and e_3 take their mean value, and that is corresponding to setting d_1 and d_2 equal to their mean value. Similarly, the values of $p_{2i} \mid \mathrm{i} = 1,2,3,4$ Table (3) denote that the parameters f_1 and f_4 take their upper bound value, which is corresponding to d_1 and d_4 at their upper bounds. Subsequently, f_2 and f_3 appear with their mean value, which means d_2 and d_3 takes their mean values in the second constraint. These results show an inconsistency or a conflict in the robust solution because d_i 's appear with different values in the different constraints.

In the next section, a new robust optimization approach is proposed, which fixed the mentioned inconsistency, although it inherits the concept of robustness addressed by [12].

B. *The New Robust Optimization Model*

In the previous section, the robust optimization approach of [12] was discussed, and it was shown that this approach is not satisfactory for some problems with shared parameters. The previous section aimed to argue that in problems with shared uncertain parameters, the approaches which consider these parameters individually and ignores the dependency between them may lead to conflicts in their results. To the best of our knowledge, this situation has not been addressed by the previous studies in the robust optimization problems particularly when interval numbers express uncertain events.

To explain the new approach, it should be noted that in the linear optimization problems, the critical value of an uncertain parameter which causes the objective function to be worst is its upper bound or its lower bound. Therefore, we introduce two binary variables γ_1 and γ_2 for each uncertain parameter denoting whether the uncertain parameter takes its upper bound or its lower bound, respectively. Moreover, to explain the new approach, a maximization problem as following is considered:

$$Max\ z = CX \tag{20}$$

$$s.t$$

$$AX \leq b \tag{21}$$

$$X \geq 0 \tag{22}$$

In this problem, X is a $n \times 1$ vector of positive real variables, C is $1 \times n$ a vector, A is a $m \times n$ matrix, and b is a $n \times 1$ vector of parameters. It is assumed that each entry a_{ij} of the matrix A , each entry c_j of vector C , and each entry b_i of vector b take values in the intervals $[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$, $[c_j - \hat{c}_j, c_j + \hat{c}_j]$ and $[b_i - \hat{b}_i, b_i + \hat{b}_i]$ respectively.

It also is assumed that there are *k* uncertain events which could be shared between uncertain parameters. For each uncertain event *k*, two binary variables γ_1^k and γ_2^k are considered indicating whether that uncertain event appears in the robust solution with its upper bound or its lower bound. Moreover, we call γ_1^k as the *first robust binary variable of the uncertain event k,* which will be equal to 1 if that event takes its upper bound in the robust solution or will be 0 otherwise. Similarly, γ_2^k is called the *second robust binary variable of the uncertain event k* and will be equal to 1 if that event appears with its lower bound in the robust solution or will be 0 otherwise. Each element of the model parameters is assumed to be related to only one uncertain event. In this regard, the notations e^{c_j} , $e^{a_{ij}}$ and e^{b_i} are defined for indicating the related uncertain event of the element c_j , a_{ij} and b_i respectively. To distinguish the uncertain parameters which take their critical value from the ones that appear with their mean value in the robust solution, we define the matrix *A*, the vector *C,* and the vector *b,* which contain the elements a_{ij} , c_j and b_j respectively. Also, the matrix $\hat{A}$, the vector $\hat{C}$, and the vector $\hat{b}$ are introduced, which contain the elements $\hat{a}_{ij}$, $\hat{c}_j$ and $\hat{b}_i$ respectively. Remembering these notations, the matrix $\hat{A}^{\gamma_1}$, the vector $\hat{C}^{\gamma_1}$, and the vector $\hat{b}^{\gamma_1}$ are defined whose elements are respectively the corresponding elements of the matrix $\hat{A}$, $\hat{C}$ and $\hat{b}$ which are multiplied by the first robust binary variable of those elements. For example, the vector $\hat{C}^{\gamma_1}$ shows the vector whose *j*'th element is $\hat{c}_j \cdot \gamma_1^{e^{c_j}}$.

Similarly, the matrix $\hat{A}^{\gamma_2}$, the vector $\hat{C}^{\gamma_2}$, and the vector $\hat{b}^{\gamma_2}$ are defined whose elements are respectively the corresponding elements of matrix $\hat{A}$, $\hat{C}$ and $\hat{b}$ which are multiplied by the second robust binary variable of those elements. Also, the parameter set Ω_l is defined as a set containing the uncertain parameters of type *l* (for example, the parameters which are related to demand of products can be classified as the parameter set of demand). Taking into account the above notations, the proposed robust optimization model is presented as the following:

Model (1):

$$\underset{\substack{\sum_{k\in\Omega_j}\gamma_1^k+\gamma_2^k\le\Gamma_l\ \forall l\\ \gamma_1^k+\gamma_2^k\le 1\quad \forall k\in K\\ \gamma_1^k,\gamma_2^k\in\{0,1\}\,\forall k\in K}}{Min}\quad Max\ z=(C-C_1^{\gamma_1}+C_1^{\gamma_2})X \tag{23}$$

$s.t.$

$$(A-A_1^{\gamma_1}+A_1^{\gamma_2}).X\le(b-b_1^{\gamma_1}+b_1^{\gamma_1}) \tag{24}$$

$$X\ge 0 \tag{25}$$

In *model* (1) for given values of Γ_j s, the aim is to find a solution by selecting up to Γ_j uncertain parameters in the set Ω_j and set the value of them equal to their upper bound or lower bound such that it leads to the worst optimal value of the objective function. This idea will cover the concepts of robustness in the proposed approach. Before explaining the solution approach proposed for converting this problem to an equivalent mixed-integer linear programming one, we claim such an approach is similar to [12] approach when there are no shared uncertain events in the problem. Actually, in [12] approach, the aim is to choose Γ uncertain parameters in their critical value such that the solution space becomes the most restricted. In this way, the feasibility of the robust solution is ensured with high probability regardless of the realization of the uncertain events. Also, it is known from the operational research concepts that if the feasible space of the solutions becomes more restricted in an optimization problem, the value of the objective function will not be better, and it is possible to become worse. So the new approach results in a worst-case analysis regarding the sets Ω_j 's as the "*uncertainty set*". Hence, ***Lemma* (1)** is stated as below:

***Lemma* (1)**: the results of the new approach are the same as [12] approach when there are no shared parameters in the problem.

Now, the procedure used for solving *Model* (1) is described. *Model* (1) is a two-level combinatorial optimization model. In the first level, all combinations of *first* and *second robust binary variables* for all uncertain events are examined such that for each combination, the relations $\sum_{k\in\Omega_l}\gamma_1^k+\gamma_2^k\le\Gamma_l\ \ \forall l$, $\forall k,\gamma_1^k+\gamma_2^k\le 1$ and $\gamma_1^k,\gamma_2^k\in\{0,1\}\forall k\in K$ are satisfied. In these relations, K is the set of all uncertain events. Moreover, it should be pointed out that uncertain events in a solution could be considered in just one form of its upper bound, its lower bound, or its mean value. Therefore, the constraint $\gamma_1^k+\gamma_1^k\le 1,\forall k$ should be held in the robust model.

Model (2):

$$Max\ z=(C-C_1^{\gamma_1}+C_1^{\gamma_2})X \tag{26}$$

$s.t.$

$$(A-A_1^{\gamma_1}+A_1^{\gamma_2}).X\le b-b_1^{\gamma_1}+b_1^{\gamma_2} \tag{27}$$

$$X\ge 0 \tag{28}$$

Finally, each of the combinations that have the minimum value of the objective function z is the robust solution of the proposed approach. Hence, for solving *Model* (2), the vector X acts as the variable of *Model* (2) and the values of γ_1^k and γ_2^k are regarded as known parameters. Now, to convert such nonlinear and combinatorial problem to a linear one, first, the Dual of *Model* (2) is considered as *Model* (3):

Model (3)

$$Min V=w.(b-b_1^{\gamma_1}+b_1^{\gamma_2}) \tag{29}$$

$s.t.$

$$w.(A-A_1^{\gamma_1}+A_1^{\gamma_2})\ge(C-C_1^{\gamma_1}+C_1^{\gamma_2}) \tag{30}$$

$$w\ge 0 \tag{31}$$

It is known from the duality theorem that if a primal problem is bounded and feasible (as we suppose for our problem), then the dual problem is also bounded and has a finite solution, and these two solutions are equal. So we can substitute *Model* (2) with *Model* (3) into *Model* (1). This yields to *Model* (4) as follow:

Model (4)

$$\underset{\substack{\sum_{k\in\Omega_l}\gamma_1^k+\gamma_2^k\le\Gamma_l\ \forall l\\ \gamma_1^k+\gamma_2^k\le 1\ \forall k\in K\\ \gamma_1^k,\gamma_2^k\in\{0,1\}\,\forall k\in K}}{Min}\quad Min V=w.(b-b_1^{\gamma_1}+b_1^{\gamma_2}) \tag{32}$$

$s.t.$

$$w.(A-A_1^{\gamma_1}+A_1^{\gamma_2})\ge(C-C_1^{\gamma_1}+C_1^{\gamma_2}) \tag{33}$$

$$w\ge 0 \tag{34}$$

The Min-Min model of *Model* (4) is equivalent to *Model* (5), which are shown as the following. This model is a mixed-integer non-linear model.

Model (5)

$$Min V=w.(b-b_1^{\gamma_1}+b_1^{\gamma_2}) \tag{35}$$

$s.t.$

$$w.(A-A_1^{\gamma_1}+A_1^{\gamma_2})\ge(C-C_1^{\gamma_1}+C_1^{\gamma_2}) \tag{36}$$

$$\sum_{k\in\Omega_l}\gamma_1^k+\gamma_2^k\le\Gamma_l\ \ \forall l \tag{37}$$

$$\gamma_1^k+\gamma_2^k\le 1\forall k\in K \tag{38}$$

$$w\ge 0 \tag{39}$$

Until now, a min-max constraint two-level model has been converted to an equivalent one-level model. However, this new model is still nonlinear because in the following terms, the nonlinear terms exist:

$$w.(A-A_1^{\gamma_1}+A_1^{\gamma_2})\ge(C-C_1^{\gamma_1}+C_1^{\gamma_2});\ w.(b-b_1^{\gamma_1}+b_1^{\gamma_2})\ .$$

Also, the nonlinear terms

$$w.(A-A_1^{\gamma_1}+A_1^{\gamma_2})\ge(C-C_1^{\gamma_1}+C_1^{\gamma_2})$$

are the consequence of multiplying the vector w by matrix $A_1^{\gamma_1}$ or $A_2^{\gamma_2}$. To get rid of such a nonlinearity term, a simple conversation could be used. Indeed, the nonlinear terms appeared in the forms of: $w_1.a_{1j}.\gamma_1^{a_{1j}},...,w_n.a_{nj}.\gamma_1^{a_{nj}},w_1.a_{1j}.\gamma_2^{a_{1j}},...,w_n.a_{nj}.\gamma_2^{a_{2j}}$, and $w_1.b_1.\gamma_1^{b_1},...,w_n.b_n.\gamma_1^{b_n},w_1.b_1.\gamma_2^{b_1},,...,w_n.b_n.\gamma_1^{b_n}$. Therefore, each nonlinear term composed of a positive variable multiplied by a binary variable. To linearize such non-linear terms, if we consider a term $y.z$ in which y is a positive variable and z is a binary variable, then the term $y.z$ could be replaced by the new variable u for which the following constraint is exact:

$$u = y\,z \tag{40}$$

$$y - M(1-z) \le u \le y + M(1-z) \tag{41}$$

$$u \le M\,z \tag{42}$$

Therefore, all nonlinear terms in *Model* (5) are converted to linear ones using the constraint (40), (41) and (42) and the resulting model is a mixed-integer linear programming model that covers the idea of robustness in the proposed approach.

C. Model Validation

As we claimed before, the proposed approach results in robust solutions similar to Bertsimas and Sim's approach particularly when there are no repeated or shared events in the model constraints. This could be examined by considering the following problem:

Problem (2):

$$max\ \ Z = 2x_1 + 3x_2 - 2x_3 + x_4 \tag{43}$$

$$s.t.$$

$$\tilde{a}_1.x_1 + \tilde{a}_2.x_2 + \tilde{a}_3.x_3 + \tilde{a}_4.x_4 \le 50 \tag{44}$$

$$b_1.x_1 + b_2.x_2 + b_3.x_3 + b_4.x_4 \le 60 \tag{45}$$

$$0 \le x_1,\ x_2,\ x_3, x_4 \le 2 \tag{46}$$

The value of the uncertain parameters in *Problem* (2) is considered as Table IV:

TABLE IV. THE VALUE OF THE PARAMETERS IN *PROBLEM* (2)

i	**1**	**2**	**3**	**4**
a_i	8	5	6	7
$\hat{a}_i$	2	4	3	5
b_i	6	4	8	7
$\hat{b}_i$	3	2	4	6

Considering P_{1i} and P_{2i} as the related variables of the uncertain parameters in the constraints (44) and (45) and solving this model using [12] approach, the following results as TableV will be obtained:

TABLE V. RESULTS OF THE *PROBLEM* (2) USING BERTSIMAS AND SIM (2004) APPROACH

Γ_1, Γ_2	1	2	3	4
Z	12	11.33	11	11
$P_{1i}>0$ *and* $P_{2i}>0$	P_{14}, P_{24}	$P_{12}, P_{14}, P_{21}, P_{22}$	$P_{11}, P_{12}, P_{14}, P_{21}, P_{22}, P_{23}$	$P_{11}, P_{12}, P_{13}, P_{14}\ P_{21}, P_{22}, P_{23}, P_{24}$

The results of Table V indicate that in the optimal robust solutions of *Problem* (2), when $\Gamma_1 = \Gamma_2 = 1$, the parameters a_4 and b_4 are in their critical values, when $\Gamma_1 = \Gamma_2 = 2$, a_2 , a_4 , b_1 and b_2 are in their critical values and so on.

This problem could also be solved by the proposed approach considering the following model:

$$Min \quad V = 50w_1 + 60w_2 + 2w_3 + 2w_4 + 2w_5 + 2w_6$$

$$s.t.$$

$$(w_1, w_2, w_3, w_4, w_5, w_6)\left(A - B + C\right) \ge (2 \quad 3 \quad -2 \quad 1)$$

$$A = \begin{pmatrix} 8 & 5 & 6 & 7 \\ 6 & 4 & 8 & 7 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2\gamma_1^1 & 4\gamma_1^2 & 3\gamma_1^3 & 5\gamma_1^4 \\ 3\gamma_1^5 & 2\gamma_1^6 & 4\gamma_1^7 & 6\gamma_1^8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 2\gamma_2^1 & 4\gamma_2^2 & 3\gamma_2^3 & 5\gamma_2^4 \\ 3\gamma_2^5 & 2\gamma_2^6 & 4\gamma_2^7 & 4\gamma_2^8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_1^k + \gamma_1^k \le 1 \qquad k \in \{1,2,3,4,5,6,7,8\}$$

$$\gamma_2^k + \gamma_2^k \le 1 \qquad \forall k \in \{1,2,3,4,5,6,7,8\}$$

$$\sum_{k \in \{1,2,3,4\}} \gamma_1^k + \gamma_2^k = \Gamma_1$$

$$\sum_{k \in \{5,6,7,8\}} \gamma_1^k + \gamma_2^k = \Gamma_2$$

$$\gamma_1^k, \gamma_2^k = \{0,1\} \ \ \forall k \in \{1,2,3,4,5,6,7,8\}$$

$$(w_1, w_2, w_3, w_4, w_5, w_6) \ge 0$$

In this model, we have nonlinear terms $w_k.\gamma_1^k, w_k.\gamma_2^k \quad \forall k = \{1,2,3,4,5,6,7,8\}$. For the above terms, the variables $z_1^k = w_k.\gamma_1^k, z_2^k = w_k.\gamma_2^k$ are defined, and the constraints (55) - (58) are added to the above model for the sake of linearization:

$$w_k - M(1-\gamma_1^k) \le z_1^k \le w_k + M(1-\gamma_1^k) \qquad \forall k \in \{1,2,3,4,5,6,7,8\}$$
$$w_k - M(1-\gamma_2^k) \le z_2^k \le w_k + M(1-\gamma_2^k) \qquad \forall k \in \{1,2,3,4,5,6,7,8\}$$
$$z_1^k \le M.\gamma_1^k \qquad \forall k = \{1,2,3,4,5,6,7,8\}$$
$$z_2^k \le M.\gamma_2^k \qquad \forall k = \{1,2,3,4,5,6,7,8\}$$

Solving *Problem* (2) by the proposed approach yields the results as Table VI.

TABLE VI. RESULTS OF *PROBLEM* (2) USING THE PROPOSED APPROACH

Γ_1, Γ_2	1	2	3	4
V	12	11.33	11	11
$\gamma_u^k \mid \gamma_u^k = 1$ $u = 1,2$ $k = 1, 2,3,4$	γ_2^4, γ_1^8	γ_2^2, γ_1^4 γ_1^5, γ_1^6	$\gamma_2^1, \gamma_2^2, \gamma_2^4$ $\gamma_1^5, \gamma_2^6, \gamma_2^7$	$\gamma_1^2, \gamma_2^2, \gamma_1^3, \gamma_2^4$ $\gamma_1^5, \gamma_2^6, \gamma_1^7, \gamma_1^8$

It is seen in Table VI that the results of the proposed approach are the same as [12] approach. For example, in both approaches, when $\Gamma_1 = \Gamma_2 = 2$ the uncertain parameters a_2, a_4, b_1 and b_2 are appeared with their critical value in the optimal robust solution. Moreover, the objective function value of both approaches is the same for different values of Γ_1 and Γ_2. However, this problem has not any shared events. To show the validity of the model for the problem with the shared uncertain events, we come back to *Problem (1)* and show how the proposed approach could solve the challenges of inconsistency addressed in that problem.

In *Problem* (1), the parameters d_i's were uncertain. For each d_i, two binary variables γ_1^i and γ_2^i could be defined that has a description as before. Using *Model* (5), the robust model of *Problem* (1) will be as follow:

$$Min \qquad V = C w_1 + w_2 + w_3 + w_4 + w_5 \tag{9}$$
$$s.t.$$
$$(w_1, w_2, w_3, w_4, w_5).(A - B + C) \ge (c_1.d_1 - c_1.\hat{d}_1.\gamma_1^1 + c_1.\hat{d}_1.\gamma_2^1,$$
$$c_2.d_2 - c_2.\hat{d}_2.\gamma_1^2 + c_2.\hat{d}_2.\gamma_2^2, c_3.d_3 - c_3.\hat{d}_3.\gamma_1^3 + c_3.\hat{d}_3.\gamma_2^3,$$
$$c_4.d_4 - c_4.\hat{d}_4.\gamma_1^4 + c_4.\hat{d}_4.\gamma_2^4)$$
$$where:$$
$$A = \begin{pmatrix} a_1.d_1 & a_2.d_2 & a_3.d_3 & a_4.d_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$B = \begin{pmatrix} a_1.\hat{d}_1.\gamma_1^1 & a_2.\hat{d}_2.\gamma_1^2 & a_3.\hat{d}_3.\gamma_1^3 & a_4.\hat{d}_4.\gamma_1^4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{60}$$
$$C = \begin{pmatrix} a_1.\hat{d}_1.\gamma_2^1 & a_2.\hat{d}_2.\gamma_2^2 & a_3.\hat{d}_3.\gamma_2^3 & a_4.\hat{d}_4.\gamma_2^4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sum_{j \in \{1,2,3,4\}} \gamma_1^k + \gamma_2^k \le \Gamma \tag{61}$$
$$w_j \ge 0 \quad \forall j = \{1,2,3,4,5\}, \quad \gamma_1^k, \gamma_1^k = \{0,1\} \quad \forall k = \{1,2,3,4\} \tag{62}$$

Using the data in Table (1) and solving the shown model in (59) – (62), the results for different value of Γ will be as Table VII:

TABLE VII. RESULTS OF PROBLEM (1) USING THE PROPOSED APPROACH

Γ	1	2	3	4
V	112	97	85	77
γ_i^1 **=1**	γ_4^1	γ_1^1, γ_4^1	$\gamma_1^1, \gamma_3^1, \gamma_4^1$	$\gamma_1^1, \gamma_2^1, \gamma_3^1, \gamma_4^1$

As the results in Table VI indicate when $\Gamma = 2$ the objective function will be equal to 104 and the variable γ_1^1 and γ_1^4 will take value equal to one in the optimal solution which indicates the parameters d_1 and d_4 in the robust solution will take their lower bound values. It is clear that the value of the objective function has been improved using the new method. This can be explained by the fact that in [12] approach, Γ uncertain parameters considered by their critical values in both the objective function and the constraint. However, because the objective function and the constraint in *Problem (1)* are dependent, some parameters value that makes the objective function the worst may lead to a better feasible space for the variables and, in turn, make eligible points feasible.

V. DISCUSSION

In this section, some applications of the proposed approach in the real world problems are discussed. The first regarded example is the study of [35]. They considered a two objective model for minimizing the cost and risk of transportation in a network of hazard material supply chain. In that article, the risk was measured as the sum of risks in all paths of the network. Also, a threshold for acceptable risk of each path was considered. In that study, if the risk per shipment is considered as an uncertain parameter, it is clear that this parameter exists in both objective function and the constraints related to the maximum allowed risk of each path. The proposed robust approach suggested in this paper apply to this problem. So, the first application of the proposed approach is for the problems in which some limitations exist on the objective functions as the model constraints. Minimizing the cost of investment by imposing a limitation on the budgets of different investment areas is another example of this type [36].

The demand for products and services plays a vital role in business models. This parameter takes various forms in mathematical models of problems. In some problems, it affects variables such as the number of vehicles, the responded demands of customers, the reached level of quality or reliability, etc. Also, it may appear as a right-hand side parameter in the model constraints [37]. When the latter is the case, it should be noted that there is only one such uncertain parameter in a constraint — so defining the level of protection against uncertainty as Γ (the parameters related to the confidence level in [12] in a constraint with just one uncertain parameter is meaningless. Actually, in this case, this parameter can be regarded as a critical parameter or not. As we mentioned in section (3), the concept of "*uncertainty set"* can be more interpretable in this case. For example, the "*demand set*" could be regarded as a set that includes all customers' demand for products. Hence, by labeling these parameters as binary variables and considering the protection level Γ for the parameters in the *demand set*, the concept of robustness and protection level against uncertainty is seemed to be more desirable and applicable. Another example of the "*uncertainty set*" is the processing time of works on machines in the job shop and flow shop scheduling problems. This parameter might be uncertain and shared between the model constraints [38]. Again, by defining "*processing time set,*" the concept of robustness can be extended to these problems. The project scheduling problem (PSP) could be considered [39] as another type of problem with common uncertain parameters. In this problem, the duration of activities in a project network can be regarded as a shared uncertain parameter. In the precedence constraints of this problem, only one duration time exists. Also, the duration time of an activity is shared between the precedence constraints of all its processors. Therefore, in this problem, "*activity duration set*" could be defined and the protection against uncertainty could be considered for this set of parameters.

VI. Conclusion

In this research, a new robust optimization approach has been introduced for the problems with uncertain parameters in which these parameters are shared between the model constraints and objective function. The situations in which this model is useful were discussed, and the model validity was shown using numerical examples. It was shown that the suggested Min-Max strategy could prepare less conservative solutions than a fully conservative approach. It was also shown that this procedure yields the results the same as [12] approach when there is no shared parameter in a problem. However, in the problems with shared uncertain parameters, [12] had some inconsistencies, which can be fixed by the proposed approach. Also, we utilized the concept of "*uncertainty set*" that extends the concept of robustness to the sets with the same type of parameters. For future research, the implementation of the proposed approach for real-world problems is suggested.

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