N-DETACHABLE PAIRS IN 3-CONNECTED MATROIDS I: UNVEILING X

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Abstract. Let M be a 3-connected matroid, and let N be a 3-connected minor of M. We say that a pair $\{x_1, x_2\} \subseteq E(M)$ is N-*detachable* if one of the matroids $M/x_1/x_2$ or $M\backslash x_1\backslash x_2$ is both 3-connected and has an N-minor. This is the first in a series of three papers where we describe the structures that arise when it is not possible to find an N-detachable pair in M. In this paper, we prove that if M has no N-detachable pairs, then either M has a 3-separating set, which we call X, with certain strong structural properties, or M has one of three particular 3-separators that can appear in a matroid with no N-detachable pairs.

1. Introduction

Let M be a 3-connected matroid, and let N be a 3-connected minor of M. We say that a pair $\{x_1, x_2\} \subseteq E(M)$ is N-*detachable* if either

(a) $M/x_1/x_2$ is 3-connected and has an N-minor, or
(b) $M\backslash x_1\backslash x_2$ is 3-connected and has an N-minor.

This is the first in a series of three papers where we describe the structures that arise when it is not possible to find an N-detachable pair. As a consequence, we show that if M has at least ten more elements than N, then either M has an N-detachable pair after possibly performing a single Δ-Y or Y-Δ exchange, or M is essentially N with a single spike attached. More precisely, we have the following theorem. Formal definitions of Δ-Y exchange and "spike-like 3-separator" are given in Section 2.

Theorem 1.1. *Let M be a 3-connected matroid, and let N be a 3-connected minor of M such that $|E(N)| \geq 4$ and $|E(M)| - |E(N)| \geq 10$. Then either*

(i) *M has an N-detachable pair,*
(ii) *there is a matroid M' obtained by performing a single Δ-Y or Y-Δ exchange on M such that M' has an N-detachable pair, or*
(iii) *there is a spike-like 3-separator P of M such that at most one element of $E(M) - E(N)$ is not in P.*

Date: April 18, 2018.

1991 *Mathematics Subject Classification.* 05B35.

Key words and phrases. matroid representation, excluded minor, 3-connected, Splitter Theorem.

The authors were supported by the New Zealand Marsden Fund.

In fact, we prove a stronger result that requires only that $|E(M)| - |E(N)| \geq 5$, but a handful of additional highly structured outcomes involving particular 3-separators of bounded size arise. Describing these requires some preparation and we defer the full statement of the stronger theorem until the third paper.

These papers had their genesis in the Ph.D. thesis of Alan Williams [21] where the problem of finding a detachable pair without worrying about keeping a minor was solved. In essence, the strategy here follows the strategy of [21], but with the additional responsibility of always taking care to keep the minor.

Background and motivation. The proof of Theorem 1.1 is long; much longer than we originally anticipated. Without a solid motivation, the case for going to the trouble of proving it is weak indeed. In fact, the motivation is clear. It comes from a desire to find exact excluded-minor characterisations of certain minor-closed classes of representable matroids. What follows is a discussion of that motivation.

To some extent, progress in matroid theory can be measured by success in finding excluded-minor characterisations of classes of matroids. Results to date include Tutte's excluded-minor characterisation of binary and regular matroids [17]; Bixby's and, independently, Seymour's excluded-minor characterisation of ternary matroids [1, 15]; Geelen, Gerards and Kapoor's excluded-minor characterisation of $GF(4)$-representable matroids [7]; and Hall, Mayhew and van Zwam's excluded-minor characterisation of the near-regular matroids, that is, the matroids representable over all fields with at least three elements [8]. Recently Geelen, Gerards and Whittle announced a proof of Rota's Conjecture [6]. However, their techniques are extremal and give no insight into how one might find the exact list of excluded minors for such classes. Extending the range of known exact excluded-minor theorems for basic classes of matroids remains a problem of genuine interest and, indeed, a significant challenge that tests the state of the art of techniques in matroid theory.

At this stage we need to note that regular matroids, and many other naturally arising classes of representable matroids such as near-regular, dyadic and $\sqrt[6]{1}$-matroids [19], can be described as classes of matroids representable over an algebraic structure called a *partial field*. Of course, a field is an example of a partial field, and classes of matroids representable over partial fields enjoy many of the properties that hold for matroids representable over fields.

The immediate problem that looms large is that of finding the excluded minors for the class of $GF(5)$-representable matroids. While this problem is beyond the range of current techniques, a road map for an attack is outlined in [14]. In essence, this road map reduces the problem to a finite sequence of problems of the following type. We have the class of $\mathbb{P}$-representable matroids for some fixed partial field $\mathbb{P}$. We have a 3-connected matroid N

with the property that every $\mathbb{P}$-representation of N extends *uniquely* to a $\mathbb{P}$-representation of any 3-connected $\mathbb{P}$-representable matroid having N as a minor. Such a matroid N is called a *strong stabilizer* for the class of $\mathbb{P}$-representable matroids. With these ingredients, the goal is to bound the size of an excluded minor for the class of $\mathbb{P}$-representable matroids having the strong stabilizer N as a minor. This situation is a more general version of the one that arises in the proof of Rota's Conjecture for $GF(4)$. There, the partial field is $GF(4)$ and the fixed minor N is $U_{2,4}$.

For all of the classes described above we may attempt to generalise the strategy developed by Geelen, Gerards and Kapoor. We have an excluded minor M, with strong stabilizer N. We wish to bound the size of M relative to N. Assume, for a contradiction, that M is large relative to N. It is proved in [7, 20] that in this case, up to duality, one can find a pair of elements $x, y \in E(M)$ such that $M\backslash x$, $M\backslash y$ and $M\backslash x\backslash y$ have N-minors and are 3-connected up to series pairs. Finding such a pair is the first step in the proofs given in [7, 8]. But there is the rub. The possible presence of series pairs leads to a major complication in the subsequent analysis. The current proofs for the excluded-minor characterisations of both $GF(4)$-representable and near-regular matroids could be significantly shortened if we could replace "3-connected up to series pairs" by "3-connected" in the initial step. That is precisely what Theorem 1.1 enables us to do.

If we are to succeed in finding the excluded minors for the classes of matroids that would lead to an exact solution to Rota's Conjecture for $GF(5)$, eliminating unnecessary technicalities in the analyses becomes more than just a convenience; it becomes absolutely essential. Eliminating unnecessary technicalities is what this paper achieves. It gives a feasible first step on the way to an explicit characterisation of the excluded minors for these classes.

Note that outcomes (ii) and (iii) of Theorem 1.1 do not limit its applicability for finding excluded-minor characterisations of matroids representable over partial fields. It is known that excluded minors for a partial field are closed under the Δ-Y exchange [12]. Moreover, it is not difficult to show that excluded minors have bounded-size spike-like 3-separators.

Theorem 1.1 has already been applied to make further progress on excluded-minor problems. For a fixed matroid N, a matroid M is *N-fragile* if, for all $e \in E(M)$, at most one of $M\backslash e$ and M/e has an N-minor. It is shown in [4] that if M is a sufficiently large excluded minor for a partial field $\mathbb{P}$ with a strong stabilizer N as a minor, then at most two elements can be removed from M to obtain an N-fragile matroid. The proof of this result makes essential use of Theorem 1.1. In essence, this reduces the problem of bounding the size of an excluded minor to understanding the class of $\mathbb{P}$-representable N-fragile matroids. In general this appears to be a difficult problem, but progress has been made for two genuinely interesting classes.

The Hydra-5 partial field captures the first layer of the hierarchy of $GF(5)$-representable partial fields mentioned above. The 2-regular partial field has the property 2-regular-representable matroids are representable over all fields of size at least four. It turns out that $U_{2,5}$ is a strong stabilizer for both these partial fields. Moreover, the $U_{2,5}$-fragile matroids that are either 2-regular or are Hydra-5 representable are known [5]. Using this, it is possible to obtain an explicit bound for the size of an excluded minor for either of these partial fields [4]. The current bound is too large to enable an exhaustive search for the excluded minors. It is hoped that, in the not too distant future, we can refine this bound and obtain an explicit list of the excluded minors. There would be some satisfaction in achieving this. Finding the excluded minors would be an important first step on the way to getting the excluded minors for $GF(5)$. Having said that, it seems likely that, in the end, combinatorial explosion will make the full solution impossible. Nonetheless it is interesting to know just where the boundary of infeasibility lies.

On the other hand, obtaining the excluded minors for the 2-regular matroids would be a significant step towards understanding the matroids representable over all fields of size at least 4. We know that this class contains the class of 2-regular matroids. The excluded minors for 2-regular that belong to the class would be interesting indeed and it is likely that they could be exploited to obtain an explicit description of the class of matroids representable over all fields of size at least four. Indeed it would also be a significant step towards understanding the classes that arise when one considers matroids representable over sets of fields that contain $GF(4)$. This would generalise analogous results for $GF(2)$ and $GF(3)$.

The structure of these papers. We now outline the approach taken to prove Theorem 1.1 in this series of papers. As is traditional, we begin by recalling Seymour's Splitter Theorem.

Theorem 1.2 (Seymour's Splitter Theorem [16])**.** *Let M be a 3-connected matroid that is not a wheel or a whirl, and let N be a 3-connected proper minor of M. Then there exists an element $e \in E(M)$ such that M/e or $M\backslash e$ is 3-connected and has an N-minor.*

By Seymour's Splitter Theorem, we may assume, up to duality, that there is an element $d \in E(M)$ such that $M\backslash d$ is 3-connected and has an N-minor. Let $d' \in E(M\backslash d)$ such that $M\backslash d\backslash d'$ has an N-minor. If $M\backslash d\backslash d'$ is 3-connected, then $\{d, d'\}$ is an N-detachable pair. On the other hand, if $M\backslash d\backslash d'$ is not 3-connected, then $M\backslash d\backslash d'$ has a 2-separation (Y, Z) where the N-minor is primarily contained in one side of the 2-separation, Z say.

The main result of this first paper of the series shows that if M were to have no N-detachable pairs, and $|Y| \geq 4$, then either Y contains a 3-separating set X with a number of strong structural properties, or $Y \cup d$ contains one of the handful of particular 3-separators that can appear in a matroid with no N-detachable pairs (we describe these particular 3-separators in Section 5). On the journey towards the proof of this result, we prove a number of lemmas about the existence of N-detachable pairs when M or M^* contains one of a few special structures: namely, triangles (Section 3), a $U_{3,5}$-restriction (Section 4), or a single-element extension of a flan (Section 6). The proof of the main result is in Section 7.

In the second paper, we further analyse this structured set X, and show that if we cannot find an N-detachable pair, then $X \cup d$ is contained in one of the handful of particular 3-separators that can appear in a matroid with no N-detachable pairs. In the third paper, the main hurdle that remains is handling the case where for any pair $\{d, d'\}$ such that $M \backslash d \backslash d'$ has an N-minor, the 2-separation (Y, Z) in $M \backslash d \backslash d'$ has $|Y| < 4$.

2. Preliminaries

The notation and terminology in the paper follow Oxley [11]. We write $x \in \mathrm{cl}^{(*)}(Y)$ to denote that either $x \in \mathrm{cl}(Y)$ or $x \in \mathrm{cl}^*(Y)$. The phrase "by orthogonality" refers to the fact that a circuit and a cocircuit cannot intersect in exactly one element. For a set X and element e, we write $X \cup e$ instead of $X \cup \{e\}$, and $X - e$ instead of $X - \{e\}$. We say that X meets Y if $X \cap Y \neq \emptyset$. We denote $\{1, 2, \dots, n\}$ by $[n]$.

Spike-like 3-separators. Let M be a matroid with ground set E. We say that a 4-element set $Q \subseteq E$ is a *quad* if it is both a circuit and a cocircuit of M.

Definition 2.1. Let $P \subseteq E$ be an exactly 3-separating set of M. If there exists a partition $\{L_1, \dots, L_t\}$ of P with $t \geq 3$ such that

(a) $|L_i| = 2$ for each $i \in \{1, \dots, t\}$, and
(b) $L_i \cup L_j$ is a quad for all distinct $i, j \in \{1, \dots, t\}$,

then P is a *spike-like* 3-*separator* of M.

To illustrate the necessity for outcome (iii) of Theorem 1.1 we describe the construction of a matroid that satisfies neither (i) nor (ii) of the theorem. Let F_7 be a copy of the Fano matroid with a triangle $\{x, y, z\}$. Let F_7' be the matroid obtained from F_7 by adding elements y' and z' in parallel with y and z respectively, and relabelling the element x as t. Now let S be a spike with tip t, where $r(S) \geq 4$, and let $T = \{t, y', z'\}$ be a leg of S. Let $M = P_T(F_7', S) \backslash T$, the generalised parallel connection of S and F_7' along T with the elements T removed. Then M has no F_7-detachable pairs. Alternatively, let F_7'' be the matroid obtained from F_7 by adding elements y' and z' in parallel with y and z respectively, and freely adding the element t on the line spanned by $\{x, y, z\}$. Then, similarly, $P_T(F_7'', S) \backslash T$ has no F_7-detachable pairs.

We will see three more particular 3-separators, and how they can give rise to matroids without any N-detachable pairs, in Section 5.

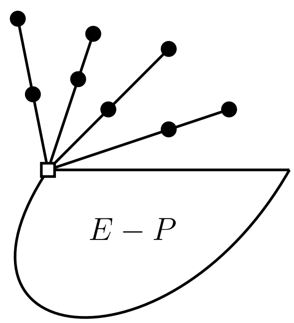


FIGURE 1. An example of a spike-like 3-separator in a matroid with rank $r(E-P)+3$

Connectivity. Let M be a matroid with ground set E. The *connectivity function* of M, denoted by λ_M, is defined as follows, for all subsets X of E:

$$\lambda_M(X) = r(X) + r(E-X) - r(M).$$

A subset X or a partition $(X, E-X)$ of E is *k-separating* if $\lambda_M(X) \leq k-1$. A k-separating partition $(X, E-X)$ is a *k-separation* if $|X| \geq k$ and $|E-X| \geq k$. A k-separating set X, a k-separating partition $(X, E-X)$ or a k-separation $(X, E-X)$ is *exact* if $\lambda_M(X) = k-1$. The matroid M is *n-connected* if, for all $k < n$, it has no k-separations. When a matroid is 2-connected, we simply say it is *connected*.

The connectivity functions of a matroid and its dual are equal; that is, $\lambda_M(X) = \lambda_{M^*}(X)$. In fact, it is easily shown that

$$\lambda_M(X) = r(X) + r^*(X) - |X|.$$

The next lemma is a consequence of the easily verified fact that the connectivity function is submodular. We write "by uncrossing" to refer to an application of this lemma.

Lemma 2.1. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.*

Lemma 2.2. *Let e be an element of a matroid M, and let X and Y be disjoint sets whose union is $E(M) - \{e\}$. Then $e \in \mathrm{cl}(X)$ if and only if $e \notin \mathrm{cl}^*(Y)$.*

Lemma 2.3. *Let X be an exactly 3-separating set in a 3-connected matroid M, and suppose that $e \in E(M) - X$. Then $X \cup e$ is 3-separating if and only if $e \in \mathrm{cl}^{(*)}(X)$.*

Lemma 2.4. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid M. Suppose $|X| \geq 3$ and $x \in X$. Then $x \in \mathrm{cl}^{(*)}(X - x)$.*

Lemma 2.5. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid M, with $|X| \geq 3$ and $x \in X$. Then $(X - x, Y \cup x)$ is exactly 3-separating if and only if x is in one of $\mathrm{cl}(X - x) \cap \mathrm{cl}(Y)$ and $\mathrm{cl}^*(X - x) \cap \mathrm{cl}^*(Y)$.*

If (X, Y) and $(X - x, x \cup Y)$ are exactly 3-separating partitions in a 3-connected matroid, then we say x is a *guts element* if $x \in \mathrm{cl}(X - x) \cap \mathrm{cl}(Y)$, and x is a *coguts element* if $x \in \mathrm{cl}^*(X - x) \cap \mathrm{cl}^*(Y)$. We also say x is *in the guts of* (X, Y) or x is *in the coguts of* (X, Y), respectively.

Lemma 2.6. *Let (X, Y) be a 3-separation in a 3-connected matroid. Then $\mathrm{cl}(X) \cap \mathrm{cl}^*(X) \cap Y = \emptyset$.*

A k-separation $(X, E - X)$ of a matroid M with ground set E is *vertical* if $r(X) \geq k$ and $r(E - X) \geq k$. We also say a partition $(X, \{z\}, Y)$ of E is a *vertical 3-separation* when $(X \cup \{z\}, Y)$ and $(X, Y \cup \{z\})$ are both vertical 3-separations and $z \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. Note that, given a vertical 3-separation (X, Y) and some $z \in Y$, if $z \in \mathrm{cl}(X)$, then $(X, \{z\}, Y)$ is a vertical 3-separation, by Lemmas 2.3 and 2.4.

A vertical k-separation in M^* is known as a cyclic k-separation in M. More specifically, a k-separation $(X, E - X)$ of M is *cyclic* if $r^*(X) \geq k$ and $r^*(E - X) \geq k$; or, equivalently, if X and $E - X$ contain circuits. We also say that a partition $(X, \{z\}, Y)$ of E is a *cyclic 3-separation* if $(X, \{z\}, Y)$ is a vertical 3-separation in M^*.

We say that a partition $(X_1, X_2, \ldots, X_m)$ of $E(M)$ is a *path of 3-separations* if $(X_1 \cup \cdots \cup X_i, X_{i+1} \cup \cdots \cup X_m)$ is a 3-separation for each $i \in [m - 1]$. Observe that a vertical, or cyclic, 3-separation $(X, \{z\}, Y)$ is an instance of a path of 3-separations.

A proof of the following is in [20]. We use this lemma, and its dual, freely without reference.

Lemma 2.7. *Let M be a 3-connected matroid and let $z \in E(M)$. The following are equivalent:*

(i) *M has a vertical 3-separation $(X, \{z\}, Y)$.*
(ii) *$\mathrm{si}(M/z)$ is not 3-connected.*

A *segment* in a matroid M is a subset S of $E(M)$ such that $M|S \cong U_{2,k}$ for some $k \geq 3$, while a *cosegment* of M is a segment of M^*.

Lemma 2.8. *Let M be a 3-connected matroid and let S be a segment with at least four elements. If $s \in S$, then $M \backslash s$ is 3-connected.*

The next two lemmas will be referred to by name.

Lemma 2.9 (Bixby's Lemma [2])**.** *Let e be an element of a 3-connected matroid M. Then either M/e is 3-connected up to parallel pairs, or $M \backslash e$ is 3-connected up to series pairs.*

Lemma 2.10 (Tutte's Triangle Lemma [18])**.** *Let $\{a, b, c\}$ be a triangle in a 3-connected matroid M. If neither $M \backslash a$ nor $M \backslash b$ is 3-connected, then M has a triad which contains a and exactly one element from $\{b, c\}$.*

A proof of the following is in [20].

Lemma 2.11. *Let C^* be a rank-3 cocircuit of a 3-connected matroid M. If $x \in C^*$ has the property that $\mathrm{cl}_M(C^*) - x$ contains a triangle of M/x, then $\mathrm{si}(M/x)$ is 3-connected.*

Proofs of the following two lemmas appear in [3].

Lemma 2.12. *Let M be a 3-connected matroid with $r(M) \geq 4$. Suppose that C^* is a rank-3 cocircuit of M. If there exists some $x \in C^*$ such that $x \in \mathrm{cl}(C^* - x)$, then $\mathrm{co}(M\backslash x)$ is 3-connected.*

Lemma 2.13. *Let (X, Y) be a 3-separation of a 3-connected matroid M. If $X \cap \mathrm{cl}(Y) \neq \emptyset$ and $X \cap \mathrm{cl}^*(Y) \neq \emptyset$, then $|X \cap \mathrm{cl}(Y)| = 1$ and $|X \cap \mathrm{cl}^*(Y)| = 1$.*

Suppose M is a 3-connected matroid, there is an element $d \in E(M)$ such that $M\backslash d$ is 3-connected, and $X \subseteq E(M\backslash d)$ is exactly 3-separating in $M\backslash d$. We say that d *blocks* X if X is not 3-separating in M, and d *fully blocks* X if neither X nor $X \cup d$ is 3-separating in M. If d blocks X, then $d \notin \mathrm{cl}(E(M\backslash d) - X)$, so $d \in \mathrm{cl}^*(X)$ by Lemma 2.2. It is easily shown that d fully blocks X if and only if $d \notin \mathrm{cl}(X) \cup \mathrm{cl}(E(M\backslash d) - X)$.

Full closure. A set X in a matroid M is *fully closed* if it is closed and coclosed; that is, $\mathrm{cl}(X) = X = \mathrm{cl}^*(X)$. The *full closure* of a set X, denoted $\mathrm{fcl}(X)$, is the intersection of all fully closed sets that contain X. It is easily seen that the full closure is a well-defined closure operator, and that one way of obtaining the full closure of a set X is to take the closure of X, then the coclosure of the result, and repeat until neither the closure nor coclosure introduces new elements. We frequently use the following straightforward lemma.

Lemma 2.14. *Let (X, Y) be a 2-separation in a connected matroid M where M contains no series or parallel pairs. Then $(\mathrm{fcl}(X), Y - \mathrm{fcl}(X))$ is also a 2-separation of M.*

Fans. Let M be a 3-connected matroid. A subset F of $E(M)$ having at least three elements is a *fan* if there is an ordering $(f_1, f_2, \ldots, f_k)$ of the elements of F such that

(a) $\{f_1, f_2, f_3\}$ is either a triangle or a triad, and
(b) for all $i \in [k-3]$, if $\{f_i, f_{i+1}, f_{i+2}\}$ is a triangle, then $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triad, while if $\{f_i, f_{i+1}, f_{i+2}\}$ is a triad, then $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triangle.

An ordering of F satisfying (a) and (b) is a *fan ordering* of F. If F has a fan ordering $(f_1, f_2, \ldots, f_k)$ where $k \geq 4$, then f_1 and f_k are the *ends* of F, and $f_2, f_3, \ldots, f_{k-1}$ are the *internal elements* of F. A fan ordering is unique, up to reversal, when $k \geq 5$.

Let F be a fan with ordering $(f_1, f_2, \ldots, f_k)$ where $k \geq 4$, and let $i \in [k]$ if $k \geq 5$, or $i \in \{1, 4\}$ if $k = 4$. An element f_i is a *spoke element* of F if

$\{f_1, f_2, f_3\}$ is a triangle and i is odd, or if $\{f_1, f_2, f_3\}$ is a triad and i is even; otherwise f_i is a *rim element*.

The next lemma follows easily from Bixby's Lemma.

Lemma 2.15. *Let M be a 3-connected matroid that is not a wheel or a whirl. Suppose M has a fan F of at least four elements, and let f be an end of F.*

(i) *If f is a spoke element, then $\mathrm{co}(M\backslash f)$ is 3-connected and $\mathrm{si}(M/f)$ is not 3-connected.*
(ii) *If f is a rim element, then $\mathrm{si}(M/f)$ is 3-connected and $\mathrm{co}(M\backslash f)$ is not 3-connected.*

A fan F is *maximal* if it is not properly contained in any other fan. Oxley and Wu [13, Lemma 1.5] proved the following result concerning the ends of a maximal fan.

Lemma 2.16. *Let M be a 3-connected matroid that is not a wheel or a whirl. Suppose M has a maximal fan F of at least four elements, and let f be an end of F.*

(i) *If f is a spoke element, then $M\backslash f$ is 3-connected.*
(ii) *If f is a rim element, then M/f is 3-connected.*

Retaining an N-minor. Let M and N be matroids. Throughout, when we say that M *has an N-minor*, we mean that M has an isomorphic copy of N as a minor. Let $X \subseteq E(M)$. To simplify exposition, we say *M has an N-minor with $|X \cap E(N)| \le 1$*, for example, to mean that M has an isomorphic copy N' of N as a minor such that $|X \cap E(N')| \le 1$.

For a matroid M with a minor N, we say an element $e \in E(M)$ is *N-contractible* if M/e has an N-minor, and e is *N-deletable* if $M\backslash e$ has an N-minor. We also say a set $X \subseteq E(M)$ is *N-contractible* if M/X has an N-minor, and X is *N-deletable* if $M\backslash X$ has an N-minor. An element $e \in E(M)$ is *doubly N-labelled* if both M/e and $M\backslash e$ have N-minors.

The next lemma has a straightforward proof.

Lemma 2.17. *Let (X, Y) be a 2-separation of a connected matroid M and let N be a 3-connected minor of M. Then $\{X, Y\}$ has a member U such that $|U \cap E(N)| \le 1$. Moreover, if $u \in U$, then*

(i) *M/u has an N-minor if M/u is connected, and*
(ii) *$M\backslash u$ has an N-minor if $M\backslash u$ is connected.*

The dual of the following is proved in [3, 4].

Lemma 2.18. *Let N be a 3-connected minor of a 3-connected matroid M. Let $(X, \{z\}, Y)$ be a cyclic 3-separation of M such that $M\backslash z$ has an N-minor with $|X \cap E(N)| \le 1$. Let $X' = X - \mathrm{cl}^*(Y)$ and $Y' = \mathrm{cl}^*(Y) - z$. Then*

(i) *each element of X' is N-deletable; and*
(ii) *at most one element of $\mathrm{cl}^*(X) - z$ is not N-contractible, and if such an element x exists, then $x \in X' \cap \mathrm{cl}(Y')$ and $z \in \mathrm{cl}^*(X' - x)$.*

Suppose C and D are disjoint subsets of $E(M)$ such that $M/C\backslash D \cong N$. We call the ordered pair (C, D) an *N-labelling of M*, and say that each $c \in C$ is *N-labelled for contraction*, and each $d \in D$ is *N-labelled for deletion*. We also say a set $C' \subseteq C$ is *N-labelled for contraction*, and $D' \subseteq D$ is *N-labelled for deletion*. An element $e \in C \cup D$ or a set $X \subseteq C \cup D$ is *N-labelled for removal*.

Let (C, D) be an N-labelling of M, and let $c \in C$, $d \in D$, and $e \in E(M) - (C \cup D)$. Then, we say that the ordered pair $((C - c) \cup d, (D - d) \cup c)$ is obtained from (C, D) by *switching the N-labels on c and d*. Similarly, $((C - c) \cup e, D)$ (or $(C, (D - d) \cup e)$, respectively) is obtained from (C, D) by *switching the N-labels on c* (respectively, d) *and e*.

The following straightforward lemma, which gives a sufficient condition for retaining a valid N-labelling after an N-label switch, will be used freely.

Lemma 2.19. *Let M be a 3-connected matroid, let N be a 3-connected minor of M with $|E(N)| \geq 4$, and let (C, D) be an N-labelling of M. Suppose $\{d, e\}$ is a parallel pair in M/c, for some $c \in C$. Let (C', D') be obtained from (C, D) by switching the N-labels on d and e; then (C', D') is an N-labelling.*

Delta-wye exchange. Let M be a matroid with a triangle $\Delta = \{a, b, c\}$. Consider a copy of $M(K_4)$ having Δ as a triangle with $\{a', b', c'\}$ as the complementary triad labelled such that $\{a, b', c'\}$, $\{a', b, c'\}$ and $\{a', b', c\}$ are triangles. Let $P_\Delta(M, M(K_4))$ denote the generalised parallel connection of M with this copy of $M(K_4)$ along the triangle Δ. Let M' be the matroid $P_\Delta(M, M(K_4))\backslash\Delta$ where the elements a', b' and c' are relabelled as a, b and c respectively. This matroid M' is said to be obtained from M by a *Δ-Y exchange* on the triangle Δ. Dually, a matroid M'' is obtained from M by a *Y-Δ exchange* on the triad $\{a, b, c\}$ if $(M'')^*$ is obtained from M^* by a Δ-Y exchange on $\{a, b, c\}$.

3. Triangles and triads

Let M be a 3-connected matroid and let N be a 3-connected minor of M. If, for all distinct $a, b \in T$, none of $M/a/b$, $M/a\backslash b$, $M\backslash a/b$, and $M\backslash a\backslash b$ have an N-minor, then T is an *N-grounded triangle*. Similarly, a triad T^* of M is an *N-grounded triad* if, for all distinct $a, b \in T^*$, none of $M/a/b$, $M/a\backslash b$, $M\backslash a/b$, and $M\backslash a\backslash b$ have an N-minor.

When $|E(N)| \geq 4$, no element of an N-grounded triangle is N-contractible. As we use this straightforward fact frequently, we state it as a lemma below.

Lemma 3.1. *Let M be a 3-connected matroid with a 3-connected minor N where $|E(N)| \geq 4$. If T is an N-grounded triangle of M with $x \in T$, then M/x does not have an N-minor.*

Proof. Let $T = \{x, y, z\}$. Since $\{y, z\}$ is a parallel pair in M/x, and $|E(N)| \geq 4$, if M/x has an N-minor, then $M/x\backslash y$ has an N-minor. Thus T is not N-grounded; a contradiction. □

If M has a triangle or triad that is not N-grounded, then the next theorem states that, up to a single Δ-Y or Y-Δ exchange, we can find an N-detachable pair. Thus, subject to this theorem, we can focus on the case where every triangle or triad of M is N-grounded.

Theorem 3.2. *Let M be a 3-connected matroid, and let N be a 3-connected minor of M with $|E(N)| \geq 4$, where $|E(M)| - |E(N)| \geq 5$. Then either*

(i) *M has an N-detachable pair, or*
(ii) *there is a matroid M' obtained by performing a single Δ-Y or Y-Δ exchange on M such that M' has an N-detachable pair, or*
(iii) *each triangle or triad of M is N-grounded.*

Proof. Suppose M has a triangle or triad T that is not N-grounded. First, suppose that M is a wheel or a whirl. By taking the dual, if necessary, we may assume that T is a triangle. Let $T = \{x, y, z\}$ where y is a rim element and x and z are spoke elements with respect to a fan ordering of $E(M)$. If M is a wheel (respectively, a whirl), then $M/y\backslash z$ is a wheel (respectively, a whirl) of rank $r(M)-1$. In particular, $M/y\backslash z$ is 3-connected since $|E(M)| > 6$. Let M' be the matroid obtained from M by performing a Δ-Y exchange on T. Then $M/y\backslash z \cong M'/y/x$. If one of M/y, $M\backslash x$ and $M\backslash z$ has an N-minor, then N is a minor of a wheel or a whirl of rank $r(M) - 1$, so $M'/y/x$ has an N-minor, and $\{y, z\}$ is an N-detachable pair of M', satisfying (ii). Since T is not N-grounded, we may now assume that M/x or M/z has an N-minor. Without loss of generality, say M/z has an N-minor. Since $\{x, y\}$ is a parallel pair in M/z and N is simple, it follows that $M\backslash x$ has an N-minor, so (ii) holds.

Now, suppose T is contained in a maximal fan F of size at least five. We start by proving the following claim:

3.2.1. *Suppose there are distinct elements $c \in E(M)$ and $d \in F$ such that $M/c\backslash d$ is 3-connected and has an N-minor. Then* (ii) *holds.*

Subproof. Since $M\backslash d$ is 3-connected, Lemma 2.15 implies that if d is an end of F, it is a spoke element. Now d is either an internal element or a spoke of F, so it is contained in a triangle T_1. Let M' be the matroid obtained from M by performing a Δ-Y exchange on T_1. Then $M\backslash d$ is isomorphic to M'/d. Hence $M'/d/c$ is 3-connected and has an N-minor, as required. ◁

By 3.2.1 and its dual, we can now look for a pair of elements, at least one of which is in F, whose removal in any way preserves 3-connectivity and an N-minor. Lemma 2.16 provides one candidate element for removal; to find the second, we require that the resulting matroid, after the element is removed, is not a wheel or a whirl.

3.2.2. *The triangle or triad T is contained in a maximal fan F' with ordering $(x_1, x_2, \dots, x_\ell)$, for $\ell \geq 5$, such that, up to duality, $\{x_1, x_2, x_3\}$ is a triangle, and $M\backslash x_1$ is 3-connected and not a wheel or a whirl.*

Subproof. We have that T is contained in a maximal fan F of size at least five. We may assume, by reversing the ordering if necessary, that $T \subseteq F - x_\ell$, and, by duality, that x_1 is a spoke element of F, so $\{x_1, x_2, x_3\}$ is a triangle. Then, by Lemma 2.16, $M\backslash x_1$ is 3-connected.

Towards a contradiction, suppose $M\backslash x_1$ is a wheel or a whirl. Then x_2 is in a triangle of $M\backslash x_1$ that meets x_3 or x_4, by orthogonality with the triad $\{x_2, x_3, x_4\}$ of $M\backslash x_1$. If $\{x_2, x_3\}$ is contained in a triangle of $M\backslash x_1$, then $\{x_1, x_2, x_3\}$ is contained in a 4-element segment of M that intersects the triad $\{x_2, x_3, x_4\}$ in two elements, which contradicts orthogonality. So $M\backslash x_1$ has a triangle $\{x_2, x_4, q\}$, say, where $q \in E(M\backslash x_1) - x_3$.

Suppose $|F| > 6$. Then $\{x_4, x_5, x_6\}$ is a triad, and, by [13, Lemma 3.4], the only triangle of M containing x_4 is $\{x_3, x_4, x_5\}$. Since $\{x_2, x_4, q\}$ is also a triangle of M, this is a contradiction. So $|F| = 5$.

Now (x_1, x_3, x_2, x_4, q) is a fan ordering of M, and this fan contains T. It follows from orthogonality that $\{x_4, q\}$ is not contained in a triad, so this fan ordering extends to a maximal fan F' where q is an end. As $M\backslash q$ is 3-connected by Lemma 2.16, if $M\backslash q$ is not a wheel or a whirl, then 3.2.2 holds for the fan F'.

So we may assume that $M\backslash q$ is a wheel or a wheel. Now $(x_1, x_2, x_3, x_4, x_5)$ is a fan ordering in $M\backslash q$ that extends to a fan ordering $(x_1, x_2, \dots, x_\ell)$ of $E(M\backslash q)$. Observe that $\ell \geq 8$ and ℓ is even. In $M\backslash x_1$, there is a fan with ordering (q, x_2, x_4, x_3, x_5) that extends to a fan ordering of $E(M\backslash x_1)$. So there is a triad containing $\{x_3, x_5\}$, and it meets $\{x_6, x_7\}$ by orthogonality, but if it contains x_6, then $\{x_3, x_4, x_5, x_6\}$ is a cosegment that intersects the triangle $\{x_5, x_6, x_7\}$ in two elements; a contradiction. So $\{x_3, x_5, x_7\}$ is a triad. If $\ell > 8$, then this triad intersects the triangle $\{x_7, x_8, x_9\}$ in a single element; a contradiction. So $|E(M)| = 9$, and hence $r(M) = 4$. It now follows that q is in a triangle $\{q, x_6, x_8\}$. By circuit elimination, $\{x_2, x_4, x_6, x_8\}$ contains a circuit. As this set does not contain a triangle, $\{x_2, x_4, x_6, x_8\}$ is a circuit, so $M\backslash q$ is a wheel. Since $\{x_2, x_4, q\}$ and $\{x_6, x_8, q\}$ are circuits of M, it follows that M is binary. So M has no $U_{2,4}$-minor, in which case $|E(N)| \geq 5$, and $|E(M)| \geq 10$; a contradiction. $\lhd$

Let F_1 be the fan F' of 3.2.2 with ordering $(x_1, \dots, x_\ell)$. Now $M\backslash x_1$ is 3-connected, and is neither a wheel nor a whirl.

3.2.3. *There is an N-labelling such that x_1 is N-labelled for deletion, and either x_2 or x_3 is N-labelled for contraction.*

Subproof. First, observe that if either x_2 or x_3 is N-labelled for contraction, then, since $\{x_1, x_2, x_3\}$ is a triangle and $|E(N)| \geq 4$, it follows that x_1 is N-labelled for deletion up to an N-label switch with x_3 or x_2 respectively, using Lemma 2.19. So it suffices to show that either x_2 or x_3 is N-labelled for contraction.

Since F contains a triangle or triad T that is not N-grounded, there is an internal element x_j of F that is N-labelled for removal. Suppose x_2 is N-labelled for deletion. Then $\{x_3, x_4\}$ is a series pair in $M\backslash x_2$. It follows that, after possibly performing an N-label switch on x_3 and x_4, the element x_3 is N-labelled for contraction.

Similarly, if x_j is N-labelled for deletion for some $j \geq 3$, then, as $\{x_{j-1}, x_j\}$ is contained in a triad, x_{j-1} is N-labelled for contraction, up to a possible N-label switch. Likewise, if x_j is N-labelled for contraction, for some $j > 3$, then, there is a triangle containing $\{x_{j-1}, x_j\}$; after a possible N-label switch, x_{j-1} is N-labelled for deletion. By repeating this process, we obtain an N-labelling where either x_2 or x_3 is N-labelled for contraction, as required. This proves the claim. $\triangleleft$

Consider the matroid $M\backslash x_1$. By 3.2.3, this matroid has an N-labelling where either x_2 or x_3 is N-labelled for contraction. The set $F_1 - x_1$ is a 4-element fan that is contained in a maximal fan F_2, with ordering $(y_1, y_2, \ldots, y_t)$, for some $t \geq 4$. If x_2 is N-labelled for contraction and x_2 is an end of F_2, then, as x_2 is a rim, the matroid $M\backslash x_1/x_2$ is 3-connected by Lemma 2.16, and (ii) holds by 3.2.1.

So we may assume that either x_3 is N-labelled for contraction, or x_2 is not an end of F_2. In either case, F_2 has an internal element that is N-labelled for contraction. By Lemma 2.16, either y_1 is a spoke and $M\backslash x_1\backslash y_1$ is 3-connected, or y_1 is a rim and $M\backslash x_1/y_1$ is 3-connected. Using a similar argument as in 3.2.3, we can iteratively switch N-labels so that y_1 is N-labelled for deletion if it is a spoke, or N-labelled for contraction if it is a rim. It follows that (ii) holds.

Now suppose T is contained in a maximal 4-element fan F. Let (f_1, f_2, f_3, f_4) be a fan ordering of F where $\{f_1, f_2, f_3\}$ is a triangle. Since F contains T, which is not N-grounded, at least one of f_2 and f_3, is N-labelled for removal. Up to duality and switching labels on f_2 and f_3, we may assume that f_2 is N-labelled for deletion. Since $\{f_3, f_4\}$ is a series pair in $M\backslash f_2$, we may also assume, up to an N-label switch, that f_4 is N-labelled for contraction. Now M/f_4 is 3-connected, by Lemma 2.16, and has an N-minor. Let M' be the matroid obtained by Y-Δ exchange on the triad $\{f_2, f_3, f_4\}$. Then M/f_4 is isomorphic to $M'\backslash f_4$.

Now $\{f_1, f_2, f_3\}$ is a triangle of M/f_4 that does not meet a triad, so M/f_4 is not a wheel or a whirl. Hence, by the Splitter Theorem, there is an element $e \in E(M/f_4)$ such that either $M/f_4/e$ or $M/f_4\backslash e$ is 3-connected and has an N-minor. In the latter case, $M/f_4\backslash e$ is isomorphic to $M'\backslash f_4\backslash e$, so (ii) holds.

Finally, we may assume that T is a triangle that is not contained in a 4-element fan. Let $T = \{a, b, c\}$. We claim that, up to relabelling, $M\backslash a$ and $M\backslash b$ have N-minors. Indeed, if c is N-labelled for contraction, then, since

$\{a, b\}$ is a parallel pair in M/c, both $M\backslash a$ and $M\backslash b$ have N-minors. On the other hand, if T has no elements that are N-labelled for contraction, then, as T is not N-grounded, it has at most one element that is not N-labelled for removal, and, by labelling this element c, we have that $M\backslash a$ and $M\backslash b$ have N-minors.

Since there is no triad meeting T, Tutte's Triangle Lemma implies that at least one of $M\backslash a$ and $M\backslash b$ is 3-connected. Without loss of generality, let $M\backslash a$ be 3-connected. Now $M\backslash a$ has a proper N-minor, so if $M\backslash a$ is not a wheel or a whirl, then, by the Splitter Theorem, there is some element $x \in E(M\backslash a)$ such that $M\backslash a\backslash x$ or $M\backslash a/x$ is 3-connected and has an N-minor. In the first case, M has an N-detachable pair as required, so assume the latter. Let M' be the matroid obtained by a Δ-Y exchange on T. Then $M\backslash a$ is isomorphic to M'/a. In particular, M'/a has an N-minor. Thus $\{a, x\}$ is an N-detachable pair in M', satisfying (ii).

It remains to consider the case where $M\backslash a$ is a wheel or a whirl. Since M has no 4-element fans, for every triad T^* of $M\backslash a$, we have that $T^* \cup a$ is a cocircuit of M. By orthogonality, $T - a$ has non-empty intersection with each such T^*. If a wheel or whirl has rank more than four, then no two elements meet every triad. So $r(M\backslash a) \leq 4$, and thus $|E(M\backslash a)| \leq 8$. Thus, in the only remaining case $|E(M)| = 9$ and $|E(N)| = 4$, so $N \cong U_{2,4}$. Since $M\backslash a$ has an N-minor, $M\backslash a$ is the rank-4 whirl.

Let d be a spoke of $M\backslash a$. Then it is easily verified that $M\backslash d$ is 3-connected and has an N-minor. Moreover, $M\backslash d$ is not a wheel or a whirl, and d is in distinct triangles T_1 and T_2 of M. By the Splitter Theorem, there is some element $x \in E(M\backslash d)$ such that $M\backslash d\backslash x$ or $M\backslash d/x$ is 3-connected and has an N-minor. In the first case, M has an N-detachable pair as required. In the latter case, observe that x is not contained in either T_1 or T_2. Say $x \notin T_1$. Letting M' be the matroid obtained by a Δ-Y exchange on T_1, we observe that $M\backslash d/x \cong M'/d/x$ is 3-connected and has an N-minor, so (ii) holds. $\square$

4. 5-element planes

In this section, we show that when M has a $U_{3,5}$ restriction, and there are certain elements whose removal preserves an N-minor, then M has an N-detachable pair. For $P \subseteq E(M)$, we say that P is a 5-*element plane* if $M|P \cong U_{3,5}$. We also say P is a 5-*element coplane* if $M^*|P \cong U_{3,5}$. The proofs of the first two lemmas are routine.

Lemma 4.1. *Let M be a 3-connected matroid with $P \subseteq E(M)$ such that $M|P \cong U_{3,6}$. Then $M\backslash p$ is 3-connected for each $p \in P$.*

Lemma 4.2. *Let M be a 3-connected matroid with a set P such that $M|P \cong U_{3,5}$, and $|E(M)| \geq 6$. If P contains a triad T^*, then $M\backslash p$ is 3-connected for each $p \in P - T^*$.*

Lemma 4.3. *Let M be a 3-connected matroid with $P \subseteq E(M)$ such that $M|P \cong U_{3,5}$. Suppose that $\mathrm{cl}(P)$ contains no triangles and P contains no triads. If $M\backslash p$ is not 3-connected for some $p \in P$, then there is a labelling $\{p_1, p_2, p_3, p_4\}$ of $P - p$ such that $M\backslash p_i\backslash p_j$ is 3-connected for each $i \in \{1, 2\}$ and $j \in \{3, 4\}$.*

Proof. Let $P = \{p, p_1, p_2, p_3, p_4\}$, and suppose $M\backslash p$ is not 3-connected. If p is in a triad, then this triad is contained in P, by orthogonality; a contradiction. So M has a cyclic 3-separation $(A, \{p\}, B)$, where (A, B) is a 2-separation of $M\backslash p$. Without loss of generality, we may assume that $\{p_1, p_2\} \subseteq A$. If $p_3 \in A$ or $p_4 \in A$, then $p \in \mathrm{cl}(A)$, so $(A \cup p, B)$ is 2-separating in M; a contradiction. So $\{p_3, p_4\} \subseteq B$. Let $A' = A - \{p_1, p_2\}$ and $B' = B - \{p_3, p_4\}$. Since A and B contain circuits and $\mathrm{cl}(P)$ contains no triangles, $|A'|, |B'| \geq 2$. Now, $(A', \{p_1, p_2\}, \{p\}, \{p_3, p_4\}, B')$ is a path of 3-separations of M where p_1 and p_2, and p_3 and p_4, are guts elements. Again using that $\mathrm{cl}(P)$ contains no triangles, it follows that $r(A'), r(B') \geq 3$. Furthermore, each p_i is not in a triad, by orthogonality. Thus, by Bixby's Lemma, $M\backslash p_i$ is 3-connected for $i \in \{1, 2, 3, 4\}$; and, moreover, $M\backslash p_i\backslash p_j$ is 3-connected up to series pairs for $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Suppose that $\{p_i, p_j\}$ is in a 4-element cocircuit C^* of M. Then $E(M) - C^*$ is closed, so C^* meets A' and B', and contains an element of $P - \{p_i, p_j\}$. But this implies $|C^*| \geq 5$; a contradiction. This proves Lemma 4.3. □

Lemma 4.4. *Let M be a 3-connected matroid with $P \subseteq E(M)$ such that $M|P \cong U_{3,6}$, and $X \subseteq P$ such that $|X| = 4$. Suppose that $\mathrm{cl}(P)$ contains no triangles. Then there are distinct elements $x_1, x_2 \in X$ such that $M\backslash x_1\backslash x_2$ is 3-connected.*

Proof. Pick any distinct $x_1, x_2 \in X$. By Lemma 4.1, $M\backslash x_1$ is 3-connected, and $M|(P - x_1) \cong U_{3,5}$. If $M\backslash x_1\backslash x_2$ is 3-connected, then the lemma holds, so we may assume otherwise. Observe that P contains no triads, by orthogonality. Now, by Lemma 4.3, $M\backslash x_1\backslash p\backslash p'$ is 3-connected for $p, p' \in P - \{x_1, x_2\}$, where we can choose p and p' such that $p \in X$. In particular, $M\backslash x_1\backslash p$ is 3-connected for $\{x_1, p\} \subseteq X$, as required. □

The following lemma is useful for finding candidates for contraction in a 4-element cocircuit, particularly in the case where the cocircuit is independent.

Lemma 4.5. *Let M be a 3-connected matroid and let C^* be a 4-element cocircuit of M. If there are distinct elements $c', c'' \in C^*$ such that neither c' nor c'' is in a triangle, then M/c is 3-connected for some $c \in C^*$.*

Proof. Let $C^* = \{c_1, c_2, c_3, c_4\}$ and suppose that c_1 is one of two elements that is not contained in a triangle. If M/c_1 is not 3-connected, then M has a vertical 3-separation $(X, \{c_1\}, Y)$. We may assume that $c_2 \in X$ and $c_3, c_4 \in Y$. Suppose that c_2 is not in a triangle. If X is a triad, then by the dual of Lemma 2.12, M/c_2 is 3-connected as required. If X is not a triad, then either X is a cosegment with at least four elements, or X contains a circuit. In the first case, M/c_2 is 3-connected by the dual of Lemma 2.8. In the second case, as $c_2 \in \mathrm{cl}^*(Y \cup c_1)$, the circuit contained in X does not contain c_2. Now $(X - c_2, \{c_2\}, Y \cup c_1)$ is a cyclic 3-separation of M, so M/c_2 is once again 3-connected, by Bixby's Lemma. So we may assume that c_2 belongs to some triangle T.

As C^* is a cocircuit, $T \cap (C^* - c_2) \neq \emptyset$ by orthogonality, so we may assume that $c_3 \in T$ and c_4 is the other element of C^* that is not contained in a triangle. As $c_2 \notin \mathrm{cl}(Y)$, we have $|T \cap X| = 2$, so $(Y - c_3, \{c_1\}, X \cup c_3)$ is a vertical 3-separation of M. Note that $(Y - c_3) \cap C^* = \{c_4\}$. Again, if $Y - c_3$ is not a triad or a cosegment, then $(Y - \{c_3, c_4\}, \{c_4\}, X \cup \{c_1, c_3\})$ is a cyclic 3-separation of M, and M/c_4 is 3-connected by Bixby's Lemma. On the other hand, if $Y - c_3$ is a triad, then M/c_4 is 3-connected by the dual of Lemma 2.12; while if $Y - c_3$ is a cosegment with at least four elements, then M/c_4 is 3-connected by the dual of Lemma 2.8. □

The next two results show the existence of N-detachable pairs when M has a subset P such that $M|P \cong U_{3,5}$. The first handles the case where P is 3-separating, whereas the second handles the case where P is not 3-separating.

Proposition 4.6. *Let M be a 3-connected matroid with $|E(M)| \geq 9$ and $r(M) \geq 5$, and let N be a 3-connected minor of M where $|E(N)| \geq 4$ and every triangle or triad of M is N-grounded. Suppose there exists some exactly 3-separating set $P \subseteq E(M)$ such that $M|P \cong U_{3,5}$, and there are distinct elements $d^*, p \in P$ such that*

(a) *either P or $P - p$ is a cocircuit, and*
(b) *$M/d^*/p'$ has an N-minor for all $p' \in P - \{d^*, p\}$.

Then M has an N-detachable pair.

Proof. First, observe that for any $p' \in P - \{d^*, p\}$, the set $P - \{d^*, p'\}$ is contained in a parallel class in $M/d^*/p'$. Since $|E(N)| \geq 4$, the matroid $M\backslash q_1\backslash q_2$ has an N-minor for any distinct $q_1, q_2 \in P - \{d^*, p'\}$. By an appropriate choice of p', it follows that $M\backslash q_1\backslash q_2$ has an N-minor for all distinct $q_1, q_2 \in P - d^*$.

Let $P = \{p_1, p_2, p_3, p_4, p_5\}$, where $p_4 = d^*$ and $p_5 = p$. For each $i \in [4]$, M/p_i has an N-minor, so P does not contain an N-grounded triangle. Similarly, $M\backslash p_i$ has an N-minor for each $i \in \{1, 2, 3, 5\}$, so P does not contain an N-grounded triad. Suppose $\mathrm{cl}(P)$ contains an N-grounded triangle T. Then $T \subseteq \mathrm{cl}(P) - \{p_1, p_2, p_3, p_4\}$. Since $M/p_1/p_4$ has an N-minor and $\mathrm{cl}(P) - \{p_1, p_4\}$ is contained in a parallel class in $M/p_1/p_4$, there is an N-labelling (C, D) such that $T \subseteq C$; a contradiction. So $\mathrm{cl}(P)$ does not contain any triangles.

By Lemma 4.3, we may assume that $M\backslash p_i$ is 3-connected for each $i \in [5]$. Since P does not contain any triads, either P is a cocircuit, or P contains a 4-element cocircuit. Towards a contradiction, we now assume that M does not have an N-detachable pair.

4.6.1. *For each $i \in [3]$, there exists a cocircuit $\{p_i, p_i', p_5, z_i\}$ of M, where $p_i' \in P - \{p_i, p_5\}$ and $z_i \in E(M) - P$, and M/z_i is 3-connected.*

Subproof. We claim that $\mathrm{co}(M \backslash p_5 \backslash p_i)$ is 3-connected for each $i \in [4]$. First, suppose that $P - p_5$ is a cocircuit. Then, for $i \in [4]$, $(P - \{p_i, p_5\}, \{p_5\}, E(M) - P)$ is a vertical 3-separation of $M \backslash p_i$. Thus, by Bixby's Lemma, $\mathrm{co}(M \backslash p_5 \backslash p_i)$ is 3-connected. Now suppose that P is a cocircuit. We will show that $\mathrm{co}(M \backslash p_5 \backslash p_i)$ is 3-connected for $i = 4$, but the argument is the same for $i \in [3]$. Let (X, Y) be a 2-separation of $M \backslash p_5 \backslash p_4$. We may assume that $\{p_1, p_2\} \subseteq X$. Now $\{p_1, p_2, p_3\}$ is a triad of $M \backslash p_5 \backslash p_4$, so either $(X \cup p_3, Y - p_3)$ is a 2-separation, or Y is a series pair. But $p_5 \in \mathrm{cl}(X \cup p_3)$, so, in the former case, $(X \cup \{p_3, p_5\}, Y - p_3)$ is a 2-separation of $M \backslash p_4$; a contradiction. Thus Y is a series pair, and it follows that $\mathrm{co}(M \backslash p_5 \backslash p_4)$ is 3-connected.

Let $i \in [3]$, and recall that $M \backslash p_i \backslash p_5$ has an N-minor. Since $\{p_i, p_5\}$ is not an N-detachable pair, it follows that p_5 is in a triad T^* of $M \backslash p_i$. By orthogonality, T^* contains an element $p_i' \in P - \{p_i, p_5\}$, so let $T^* = \{p_5, p_i', z_i\}$. If P is a cocircuit, then $T^* \nsubseteq P$, so $z_i \in E(M) - P$, whereas if $P - p_5$ is a cocircuit, then $p_5 \in \mathrm{cl}(E(M) - P)$, so, by orthogonality, $z_i \in E(M) - P$. Since T^* is not a triad of M, $\{p_i, p_i', p_5, z_i\}$ is a cocircuit.

Suppose M/z_i is not 3-connected. If z_i is in a triangle, then, by orthogonality with the cocircuit $\{p_i, p_i', p_5, z_i\}$, this triangle meets P; a contradiction. So $\mathrm{si}(M/z_i)$ is also not 3-connected. Let $(A, \{z_i\}, B)$ be a vertical 3-separation of M. Without loss of generality, $|A \cap P| \geq 3$, so $(A \cup P, \{z_i\}, B - P)$ is also a vertical 3-separation, by uncrossing. But then $z_i \in \mathrm{cl}^*(A \cup P) \cap \mathrm{cl}(B - P)$; a contradiction. So M/z_i is 3-connected. $\lhd$

4.6.2. *Suppose, up to relabelling $\{p_1, p_2, p_3\}$, that M has a cocircuit $\{p_1, p_2, p_5, z\}$, for some $z \in E(M) - P$. Then M has an N-detachable pair.*

Subproof. Let (C, D) be an N-labelling such that $\{p_3, p_4\} \subseteq C$; such an N-labelling exists since $M/p_3/p_4$ has an N-minor.

Since $\{p_1, p_2, p_5\}$ is contained in a parallel class in $M/p_3/p_4$, we may assume, up to switching the N-labels on p_5 and p_1 or p_2, that p_1 and p_2 are N-labelled for deletion. Moreover, as $\{z, p_5\}$ is a series pair in $M \backslash p_1 \backslash p_2$, we may also assume, by a possible N-label switch on p_5 and z, that z is N-labelled for contraction. In particular, $\{z, p_3\}$ is an N-contractible pair.

Since P is exactly 3-separating and $z \in \mathrm{cl}^*(P)$, Lemma 2.6 implies that $z \notin \mathrm{cl}(P)$. So P or $P - p_5$ is a rank-3 cocircuit in M/z. By Lemma 2.11, $\mathrm{si}(M/z/p_3)$ is 3-connected. Now either $M/z/p_3$ is 3-connected, or $\{z, p_3\}$ is contained in a 4-element circuit. In the former case, M has an N-detachable pair. So we may assume that $\{z, p_3\}$ is contained in a 4-element circuit C_z. By orthogonality, C_z meets $\{p_1, p_2, p_5\}$; moreover, since $z \notin \mathrm{cl}(P)$, we have $|C_z \cap P| = 2$. So $C_z = \{z, p_3, p'', f\}$ where $p'' \in \{p_1, p_2, p_5\}$ and $f \in E(M) - (P \cup z)$.

Note that p_3 and z are N-labelled for contraction. Thus, after possibly switching the N-labels on p'' and f, the element f is N-labelled for deletion. Let $p''' \in \{p_1, p_2\} - p''$, and note that p''' is also N-labelled for deletion. As $(P \cup z, \{f\}, E(M) - (P \cup \{z, f\}))$ is a vertical 3-separation and f is not in an N-grounded triad, the matroid $M\backslash f$ is 3-connected and has an N-minor. Note that $f \notin \mathrm{cl}^*(P)$, so P does not contain any triads in $M\backslash f$. Thus, by Lemma 4.3, $M\backslash f\backslash p'''$ is 3-connected, so $\{f, p'''\}$ is an N-detachable pair. $\lhd$

Now, by 4.6.1 and 4.6.2, we may assume that $\{p_1, p_4, p_5, z_1\}$, $\{p_2, p_4, p_5, z_2\}$, and $\{p_3, p_4, p_5, z_3\}$ are cocircuits of M.

Suppose $z_i = z_j$ for some distinct $i, j \in [3]$. Then, by the cocircuit elimination axiom, $\{p_i, p_j, p_4, p_5\}$ contains a cocircuit; in fact, since P does not contain any N-grounded triads, this set is a 4-element cocircuit. Since P is not a cocircuit, $P - p_5$ is also a cocircuit, by hypothesis. But now $P - p_5$ is 3-separating and $p_5 \in \mathrm{cl}(\{p_1, p_2, p_3, p_4\}) \cap \mathrm{cl}^*(\{p_1, p_2, p_3, p_4\})$; a contradiction. So $z_i \neq z_j$ for all distinct $i, j \in [3]$.

For $j \in \{2, 3\}$, the partition $(P, \{z_1\}, \{z_j\}, E(M) - (Z \cup \{z_1, z_j\}))$ is a path of 3-separations where z_1 and z_j are coguts elements. In particular, $z_j \in \mathrm{cl}^*(E(M) - (Z \cup \{z_1, z_j\}))$, so $z_j \notin \mathrm{cl}(P \cup z_1)$. We now fix an N-labelling such that p_1 and p_5 are N-labelled for deletion and p_2 is N-labelled for contraction (such an N-labelling exists since $M/p_2/p_4$ has an N-minor and $\{p_1, p_3, p_5\}$ is contained in a parallel class in this matroid). We may also assume that z_1 is N-labelled for contraction, since $\{z_1, p_4\}$ is a series pair in $M\backslash p_2\backslash p_4$. Recall that M/z_1 is 3-connected. By Lemma 2.11, $\mathrm{si}(M/z_1/p_2)$ is 3-connected. Thus, either $\{z_1, p_2\}$ is an N-detachable pair, or $\{z_1, p_2\}$ is contained in a 4-element circuit C_1. By orthogonality, C_1 meets $\{p_1, p_4, p_5\}$ and $\{p_4, p_5, z_2\}$. Since $z_1 \notin \mathrm{cl}(P)$, we have $|C_1 \cap P| = 2$. If $p_4 \in C_1$ or $p_5 \in C_1$, then $C_1 = \{z_1, p_2, p_\ell, z_3\}$ for $\ell \in \{4, 5\}$, so $z_3 \in \mathrm{cl}(P \cup z_1)$; a contradiction. On the other hand, if $\{p_4, p_5\} \cap C_1 = \emptyset$, then $\{p_1, z_2\} \subseteq C_1$, so $C_1 = \{p_1, p_2, z_1, z_2\}$ and $z_2 \in \mathrm{cl}(P \cup z_1)$; a contradiction. This completes the proof. $\square$

Proposition 4.7. *Let M be a 3-connected matroid with a 3-connected matroid N as a minor, where $|E(N)| \geq 4$ and every triangle or triad of M is N-grounded. Suppose there exists $P \subseteq E(M)$ such that $M|P \cong U_{3,5}$ and P is not 3-separating, and there are elements $d^*, p \in P$ such that*

(a) *M/d^* is 3-connected,*
(b) *$M/d^*/p'$ has an N-minor for all $p' \in P - \{d^*, p\}$, and*
(c) *for any $p' \in P - d^*$ and distinct elements $u, v \in \mathrm{cl}^*(P - d^*) - P$, either $M\backslash p'\backslash u$ or $M\backslash p'\backslash v$ has an N-minor.*

Then M contains an N-detachable pair.

Proof. Pick $p \in P$ such that $M/d^*/p'$ has an N-minor for each $p' \in P - \{d^*, p\}$. Since $P - \{d^*, p'\}$ is contained in a parallel class in $M/d^*/p'$ and $|E(N)| \geq 4$, the matroid $M\backslash q_1\backslash q_2$ has an N-minor for any distinct $q_1, q_2 \in$

$P - \{d^*, p'\}$. As p' is chosen arbitrarily among $P - \{d^*, p\}$, it follows that $M\backslash q_1\backslash q_2$ has an N-minor for all distinct $q_1, q_2 \in P - d^*$.

As $M\backslash p'$ has an N-minor for each $p' \in P - d^*$, P does not contain an N-grounded triad. Suppose $\mathrm{cl}(P)$ contains an N-grounded triangle T. Then T does not meet $P - \{d^*, p\}$, since M/p' has an N-minor for each $p' \in P - \{d^*, p\}$. There exist distinct $p', p'' \in P - \{d^*, p\}$ such that $M/p'/p''$ has an N-minor, and T is contained in a parallel class in this matroid. But this contradicts the fact that T is N-grounded, so $\mathrm{cl}(P)$ does not contain any triangles.

4.7.1. *If there are distinct elements $q, q', q'' \in P - d^*$ such that neither $\{q, q'\}$ nor $\{q, q''\}$ is contained in a 4-element cocircuit of M, then M has an N-detachable pair.*

Subproof. Recall that $\mathrm{cl}(P)$ does not contain an N-grounded triangle and P does not contain an N-grounded triad. Thus, by Lemma 4.3, either M has an N-detachable pair, or $M\backslash q$ is 3-connected. By the dual of Lemma 4.5, if there are distinct elements q' and q'' in $P - \{d^*, q\}$ that are not contained in a triad of $M\backslash q$, then either $\{q, q'\}$ or $\{q, q''\}$ is an N-detachable pair. $\triangleleft$

4.7.2. *There is a labelling $\{p_1, p_2, p_3, p_4\}$ of $P - d^*$ such that one of the following holds:*

(i) *$\{p_1, p_2, p_3, u\}$ and $\{p_2, p_3, p_4, v\}$ are cocircuits of M, with $u, v \in E(M) - P$, or*
(ii) *$\{p_1, p_2, p_3, u\}$, $\{d^*, p_2, p_4, u_2\}$ and $\{d^*, p_3, p_4, u_3\}$ are cocircuits of M, with $u, u_2, u_3 \in E(M) - P$, or*
(iii) *each of $\{d^*, p_1, p_3\}$, $\{d^*, p_1, p_4\}$, $\{d^*, p_2, p_3\}$, $\{d^*, p_2, p_4\}$, and $\{d^*, p_3, p_4\}$ is contained in a 4-element cocircuit of M.*

Subproof. By orthogonality, a 4-element cocircuit that intersects P must contain at least three elements of P; in fact, since P is not 3-separating, such a cocircuit contains exactly three elements of P.

If there are no cocircuits containing a 3-element subset of $\{p_1, p_2, p_3, p_4\}$, then by repeated applications of 4.7.1, it follows that (iii) holds. On the other hand, if there are two cocircuits of M containing distinct 3-element subsets of $\{p_1, p_2, p_3, p_4\}$, then (i) holds. So assume that $\{p_1, p_2, p_3, u\}$ is a cocircuit of M for $u \in E(M) - P$, and every other 4-element cocircuit meeting P contains d^*. If neither $\{p_2, p_4\}$ nor $\{p_3, p_4\}$ is contained in a 4-element cocircuit, then M has an N-detachable pair by 4.7.1; so we may assume that $\{p_3, p_4, v\}$ is a 4-element cocircuit for some $v \in E(M) - P$. But by repeating this argument with $\{p_1, p_4\}$ and $\{p_2, p_4\}$, we deduce that $\{p_2, p_4, v'\}$ is a cocircuit for some $v' \in E(M) - P$. Since (ii) holds in this case, this completes the proof. $\triangleleft$

Let u and v be elements in $E(M) - P$ contained in distinct 4-element cocircuits that intersect P in three elements. If $u = v$, then P contains a cocircuit by the cocircuit elimination axiom, contradicting the fact that P is not 3-separating. So $u \neq v$.

4.7.3. *Let $u, v \in E(M) - P$ be distinct elements in 4-element cocircuits C^*_u and C^*_v, respectively, where $C^*_u \subseteq P \cup u$ and $C^*_v \subseteq P \cup v$. Then $\mathrm{si}(M/u/v)$ is 3-connected.*

Subproof. Suppose (X, Y) is a 2-separation of $M/u/v$ where neither X nor Y is contained in a parallel class. We may assume that $|X \cap P| \geq 3$ and that X is closed. Thus $P \subseteq X$. But $\{u, v\} \subseteq \mathrm{cl}^*(P) \subseteq \mathrm{cl}^*(X)$, so $(X \cup \{u, v\}, Y)$ is a 2-separation of M; a contradiction. $\lhd$

4.7.4. *Let C^*_u be a 4-element cocircuit with $u \in C^*_u \subseteq P \cup u$, for $u \in E(M) - P$, and let $p' \in P - C^*_u$. Then $\mathrm{si}(M/p'/u)$ is 3-connected.*

Subproof. Suppose $\mathrm{si}(M/p'/u)$ is not 3-connected, and let (X, Y) be a 2-separation in $M/p'/u$ where neither X nor Y is a parallel pair. We may assume that $|X \cap (P - p')| \geq 2$ and that X is closed. Since $r_{M/p'}(P - p') = 2$, we have $P - p' \subseteq X$. Since $p' \notin C^*_u$, we have $u \in \mathrm{cl}^*_{M/p'}(P - p')$, and $(X \cup u, Y)$ is a 2-separation of M/p'; a contradiction. $\lhd$

4.7.5. *If* 4.7.2(i) *holds, then M has an N-detachable pair.*

Subproof. Let u and v be elements in $E(M) - P$ such that $\{p_1, p_2, p_3, u\}$ and $\{p_2, p_3, p_4, v\}$ are cocircuits of M. Recall that $M \backslash p_2 \backslash p_3$ has an N-minor. Let (C, D) be an N-labelling such that $\{p_2, p_3\} \subseteq D$. Since $\{p_1, u\}$ and $\{p_4, v\}$ are series pairs in $M \backslash p_2 \backslash p_3$, we may assume that $\{u, v\} \subseteq C$.

If $M/u/v$ is 3-connected, then $\{u, v\}$ is an N-detachable pair. By 4.7.3, $\mathrm{si}(M/u/v)$ is 3-connected. Since u and v are N-labelled for contraction, each is not in an N-grounded triangle. So we may assume there is a 4-element circuit C_{uv} of M containing $\{u, v\}$. By orthogonality, C_{uv} contains at least one element in P. Let $C_{uv} = \{u, v, p', z\}$ for some $p' \in P$ and $z \in E(M) - \{u, v, p'\}$.

We claim that $z \notin P$. Let $Z = E(M) - (P \cup \{u, v\})$. Since $\lambda(P) = 3$ and $u, v \in \mathrm{cl}^*(P)$, we have $r(Z) = r(M) - 2$. Suppose $z \in P$. Then $r(P \cup \{u, v\}) \leq 4$, so $(Z, P \cup \{u, v\})$ is a 3-separation. Next we show that $(Z, \{d^*\}, (P - d^*) \cup \{u, v\})$ is a vertical 3-separation. Clearly $d^* \in \mathrm{cl}(P - d^*)$. If d^* is in a cocircuit containing v and elements of $P - d^*$, then cocircuit elimination with $\{v, p_2, p_3, p_4\}$ implies that P contains a cocircuit; a contradiction. So $d^* \notin \mathrm{cl}^*((P - d^*) \cup v)$; thus $d^* \in \mathrm{cl}(Z \cup u)$. But if $d^* \notin \mathrm{cl}(Z)$, then $u \in \mathrm{cl}(Z \cup d^*)$ by the Mac Lane-Steinitz exchange condition, contradicting that $u \in \mathrm{cl}^*(\{p_1, p_2, p_3\})$. So $d^* \in \mathrm{cl}(Z)$, and $(Z, \{d^*\}, (P - d^*) \cup \{u, v\})$ is a vertical 3-separation implying that M/d^* is not 3-connected, contradicting (a).

Now $C_{uv} \cap P = \{p'\}$, so $p' \in \{p_2, p_3\}$, by orthogonality. Since $\{p', z\}$ is a parallel pair in $M/u/v$, by switching the N-labels on p' and z, we have that z is N-labelled for deletion.

In M/u, $\{v, p', z\}$ is a triangle, and, since $M \backslash z$ has an N-minor, z is not in an N-grounded triad. Thus Tutte's Triangle Lemma implies that $M/u \backslash z$ or $M/u \backslash v$ is 3-connected. Since $\{u, z\}$ and $\{u, v\}$ are not contained in triads, either $M\backslash z$ or $M\backslash v$ is 3-connected. Moreover, the same argument applies with the roles of u and v swapped, implying that either $M\backslash z$ or $M\backslash v$ is 3-connected.

Thus, if $M\backslash z$ is not 3-connected, then both $M\backslash u$ and $M\backslash v$ are 3-connected. Then, since $(M\backslash u)|P \cong (M\backslash v)|P \cong U_{3,5}$, it follows from Lemma 4.2 that $M\backslash u\backslash p_1$ and $M\backslash v\backslash p_1$ are 3-connected. Thus either $\{u, p_1\}$ or $\{v, p_1\}$ is an N-detachable pair, by (c).

Now we may assume that $M\backslash z$ is 3-connected. As $(M\backslash z)|P \cong U_{3,5}$, if P does not contain a triad of $M\backslash z$, then, by Lemma 4.3, M has an N-detachable pair. So suppose that z is in a 4-element cocircuit C^*_z with elements in P. Let $Q = P \cup \{u, v, z\}$. Observe that Q is 3-separating, as $r(Q) \leq 5$ due to the circuit $\{u, v, p', z\}$, and $r(E(M) - Q) = r(M) - 3$, as $r(E(M) - P) = r(M)$ and $\{u, v, z\} \subseteq \mathrm{cl}^*(P)$. If $d^* \notin C^*_z$, then

$$\lambda(Q - d^*) = r(Q - d^*) + r^*(Q - d^*) - |Q - d^*| \leq 5 + 4 - 7 = 2.$$

It follows, by Lemma 2.5, that d^* is a guts element in the path of 3-separations $(Q - d^*, \{d^*\}, E(M) - Q)$. But then M/d^* is not 3-separating; a contradiction. So $d^* \in C^*_z$. Now $T^* = C^*_z - z$ is a triad in $M\backslash z$ with $d^* \in T^*$. Let $p'' \in P - (T^* \cup p')$. By Lemma 4.2, $M\backslash z\backslash p''$ is 3-connected. Since $p'' \in P - \{d^*, p'\}$, $M\backslash z\backslash p''$ has an N-minor, so $\{z, p''\}$ is an N-detachable pair. $\triangleleft$

4.7.6. *If* 4.7.2(ii) *holds, then M has an N-detachable pair.*

Subproof. If $M\backslash p_1\backslash p_4$ is 3-connected, then M has an N-detachable pair, so assume otherwise. Suppose $\{p_1, p_4\}$ is not contained in a 4-element cocircuit. Then $\mathrm{co}(M\backslash p_1\backslash p_4)$ is also not 3-connected. Now $M\backslash p_1\backslash p_4$ has a 2-separation (X, Y) where $|X \cap \{p_2, p_3, d^*\}| \geq 2$ and X is fully closed. But it follows that $\{p_2, p_3, d^*\} \subseteq X$, and hence $(X \cup \{p_1, p_4\}, Y)$ is a 2-separation of M; a contradiction. So $\{p_1, p_4\}$ is contained in a 4-element cocircuit. If this cocircuit also contains either p_2 or p_3, then, up to relabelling $\{p_1, p_2, p_3, p_4\}$, we are in case 4.7.2(i). By 4.7.5, we may assume otherwise. So $\{d^*, p_i, p_4, u_i\}$ is a cocircuit for all $i \in [3]$.

Let $i \in [3]$ and $j \in [3] - i$ such that $p_j \neq p$. Then $M/d^*/p_j$ has an N-minor, and, as $P - \{d^*, p_j\}$ is contained in a parallel class in this matroid, $M/p_j\backslash p_i\backslash p_4$ also has an N-minor. Since $\{d^*, u_i\}$ is a series pair in $M/p_j\backslash p_i\backslash p_4$, it follows that $\{p_j, u_i\}$ is N-contractible in M. Now, by 4.7.4, either $\{p_j, u_i\}$ is an N-detachable pair, or $\{p_j, u_i\}$ is contained in a 4-element circuit $C_{i,j}$.

By orthogonality, $C_{i,j}$ meets $\{d^*, p_i, p_4\}$. If $C_{i,j} \subseteq P \cup u_i$, then $u_i \in \mathrm{cl}(P)$. Then $M|(P \cup u_i) \cong U_{3,6}$, and M has an N-detachable pair by Lemmas 4.1 and 4.2. So let $C_{i,j} = \{p_j, u_i, q_{i,j}, v_{i,j}\}$ where $q_{i,j} \in \{d^*, p_i, p_4\}$ and $v_{i,j} \in E(M) - (P \cup u_i)$ (for ease of notation, $q_{i,j} = q_{j,i}$ and $v_{i,j} = v_{j,i}$). If $v_{i,j} = u$, then by letting $j' \in [3] - \{i, j\}$, orthogonality between $C_{i,j}$ and the cocircuit $\{d^*, p_{j'}, p_4, u_{j'}\}$ implies that $q_{i,j} = p_i$, whereas orthogonality between $C_{i,j}$ and the cocircuit $\{d^*, p_j, p_4, u_j\}$ implies that $q_{i,j} \neq p_i$. So $v_{i,j} \neq u$. Observe that p_j is a member of the cocircuit $\{p_1, p_2, p_3, u\}$, and recall that $u \neq u_i$. Then, by orthogonality, $q_{i,j} = p_i$, so $\{p_i, p_j, u_i, v_{i,j}\}$ is a circuit. If $v_{i,j} \neq u_j$, then $\{p_j, p_4, d^*, u_j\}$ is a cocircuit that intersects this circuit in one element; a contradiction.

Without loss of generality we may now assume that $\{p_1, p_2, u_1, u_2\}$ is a circuit. It follows that $(P \cup \{u_1, u_2\}, E(M) - (P \cup \{u_1, u_2\}))$ is 3-separating in M. Since $M/d^*\backslash p_1\backslash p_2$ has an N-minor, and $\{p_3, u\}$ is a series pair in this matroid, $M/d^*/u$ has an N-minor up to an N-label switch. Suppose $M/d^*/u$ is not 3-connected. If $\{d^*, u\}$ is contained in a 4-element cocircuit, then, by orthogonality, this cocircuit is contained in $P \cup u$. It follows, by cocircuit elimination with $\{p_1, p_2, p_3, u\}$, that P contains a cocircuit; a contradiction. So $M/d^*/u$ has a 2-separation (U, V) for which we may assume $|P \cap U| \geq 2$ and U is fully closed. It follows that $P - d^* \subseteq U$, and hence $(U \cup u, V)$ is a 2-separation of M/d^*. But M/d^* is 3-connected, so this is contradictory. Hence $\{d^*, u\}$ is an N-detachable pair. $\lhd$

4.7.7. *If* 4.7.2(iii) *holds, then M has an N-detachable pair.*

Subproof. Consider $M\backslash p_1\backslash p_2$. If this matroid is 3-connected, then M has an N-detachable pair, so assume otherwise. If $\mathrm{co}(M\backslash p_1\backslash p_2)$ is not 3-connected, then there is a 2-separation (X, Y) of $M\backslash p_1\backslash p_2$ for which $|X \cap \{p_3, p_4, d^*\}| \geq 2$ and X is fully closed. It follows that $(X \cup \{p_1, p_2\}, Y)$ is a 2-separation of M; a contradiction. So $\{p_1, p_2\}$ is contained in a 4-element cocircuit of M. If this cocircuit also contains p_3 or p_4, then, up to relabelling, we are in case 4.7.2(ii). Hence, by 4.7.6, we may assume $\{d^*, p_1, p_2\}$ is contained in a 4-element cocircuit of M.

Now, for all distinct $i, j \in [4]$, we have that $\{d^*, p_i, p_j, u_{i,j}\}$ is a cocircuit for some $u_{i,j} \in E(M) - P$, where $u_{i,j} \neq u_{i',j'}$ if $i \neq i'$ or $j \neq j'$. (For ease of notation, we let $u_{i,j} = u_{j,i}$.) We may assume that $p = p_4$. Then, for all $i \in [3]$ we have that $M/d^*/p_i$ has an N-minor. For all distinct $j, j' \in [4] - i$, since $P - \{d^*, p_i\}$ is contained in a parallel class in $M/d^*/p_i$, the matroid $M/p_i\backslash p_j\backslash p_{j'}$ has an N-minor, and it follows that $M/p_i/u_{j,j'}$ has an N-minor. By 4.7.4, if $\{p_i, u_{j,j'}\}$ is not contained in a 4-element circuit, then M has an N-detachable pair. So we may assume that $\{p_i, u_{j,j'}\}$ is contained in a 4-element circuit, for all $i \in [3]$ and distinct $j, j' \in [4] - i$.

Let $\{i, j, j', \ell\} = [4]$ with $i \in [3]$. Consider the 4-element circuit containing $\{p_i, u_{j,j'}\}$. By orthogonality, this circuit meets $\{p_s, u_{i,s}, d^*\}$ for each $s \in [4] - i$. Hence, as the p_s's and $u_{i,s}$'s are distinct for $s \in [4] - i$, the circuit contains d^*. That is, $\{d^*, p_i, u_{j,j'}, v_{i,\ell}\}$ is a circuit for some $v_{i,\ell}$. Since $M/d^*/p_i$ has an N-minor, it follows that $\{p_t, u_{j,j'}\}$ is N-deletable for $t \in [4] - i$. If $v_{i,\ell} \in P$, then $u_{j,j'} \in \mathrm{cl}(P)$, so $M|(P \cup u_{j,j'}) \cong U_{3,6}$, and it follows from Lemma 4.1 that M has an N-detachable pair. So we may assume that $v_{i,\ell} \in E(M) - P$. Now, if $M\backslash u_{j,j'}$ is 3-connected, then, as $(M\backslash u_{j,j'})|P \cong U_{3,5}$, and $\{d^*, p_j, p_{j'}\}$ is a triad in $M\backslash u_{j,j'}$, it follows, by Lemma 4.2, that $\{p_\ell, u_{j,j'}\}$ is an N-detachable pair. So we may assume that $M\backslash u_{j,j'}$ is not 3-connected.

Again, we let $\{i, j, j', \ell\} = [4]$ with $i \in [3]$. Consider M/d^*. Recall that this matroid is 3-connected, and observe that $\{p_i, u_{j,j'}, v_{i,\ell}\}$ is a triangle, where $v_{i,\ell} \in E(M) - P$. Suppose that $\{p_i, u_{j,j'}, v_{i,\ell}\}$ is part of a 4-element fan. Then there is a triad of M that contains two elements of $\{p_i, u_{j,j'}, v_{i,\ell}\}$. But as p_i and $u_{j,j'}$ are N-deletable, neither is contained in an N-grounded triad, so this is contradictory.

Now, by Tutte's Triangle Lemma, either $M/d^*\backslash u_{j,j'}$ or $M/d^*\backslash v_{i,\ell}$ is 3-connected. If $M/d^*\backslash u_{j,j'}$ is 3-connected, then, as d^* is not in an triangle since it is N-contractible, $M\backslash u_{j,j'}$ is 3-connected; a contradiction. So $M/d^*\backslash v_{i,\ell}$ is 3-connected, and hence $M\backslash v_{i,\ell}$ is 3-connected.

Observe that, for $i \in [3]$ and $s \in [4] - i$, the matroid $M\backslash p_s\backslash v_{i,\ell}$ has an N-minor, since $M/d^*/p_i\backslash p_s$ has an N-minor and $\{u_{j,j'}, v_{i,\ell}\}$ is a parallel pair in this matroid. As $(M\backslash v_{i,\ell})|P \cong U_{3,5}$, if P does not contain a triad of $M\backslash v_{i,\ell}$, then, by Lemma 4.3, M has an N-detachable pair. So suppose $v_{i,\ell}$ is in a 4-element cocircuit C^* of M, where $C^* \subseteq P \cup v_{i,\ell}$. Then, by orthogonality, C^* contains d^* or p_i. Thus, there exists some $s \in [4] - i$ such that $p_s \notin C^*$. By Lemma 4.2, $M\backslash v_{i,\ell}\backslash p_s$ is 3-connected, so $\{v_{i,\ell}, p_s\}$ is an N-detachable pair. This completes the proof of 4.7.7. $\lhd$

The proposition now follows from 4.7.2 and 4.7.5–4.7.7. $\square$

5. Particular 3-separators

Throughout this series of papers, we will build up a collection of particular 3-separators. Any such particular 3-separator P will have the property that it can appear in a 3-connected matroid M, with a 3-connected minor N, and with no N-detachable pairs, where $E(M) - E(N) \subseteq P$. Recall that the first example we have seen is a spike-like 3-separator (see Definition 2.1). In this section, we define three more particular 3-separators, and, for each, describe the construction of a matroid containing the particular 3-separator, and with no N-detachable pairs. These particular 3-separators are illustrated in Figures 2 and 3.

Note that this is not a complete list of all such 3-separators that can give rise to a matroid without an N-detachable pair. Here, we first just consider those particular 3-separators that are either a single-element extension, or the dual of a single-element coextension, of a structure known as a flan, which we consider in the next section.

Let M be a 3-connected matroid with ground set E.

Definition 5.1. Let $P \subseteq E$ be a 6-element exactly 3-separating set such that $P = Q \cup \{p_1, p_2\}$ and Q is a quad. If there exists a labelling $\{q_1, q_2, q_3, q_4\}$ of Q such that

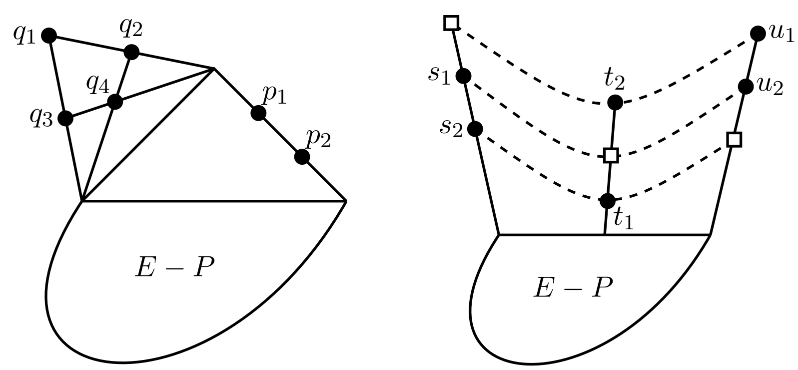
(a) $\{p_1, p_2, q_1, q_2\}$ and $\{p_1, p_2, q_3, q_4\}$ are the circuits of M contained in P, and
(b) $\{p_1, p_2, q_1, q_3\}$ and $\{p_1, p_2, q_2, q_4\}$ are the cocircuits of M contained in P,

then P is an *elongated-quad* 3*-separator* of M with *associated partition* $(Q, \{p_1, p_2\})$.

Definition 5.2. Let $P \subseteq E$ be a 6-element exactly 3-separating set of M. If there exists a labelling $\{s_1, s_2, t_1, t_2, u_1, u_2\}$ of P such that

(a) $\{s_1, s_2, t_2, u_1\}$, $\{s_1, t_1, t_2, u_2\}$, and $\{s_2, t_1, u_1, u_2\}$ are the circuits of M contained in P; and
(b) $\{s_1, s_2, t_1, t_2\}$, $\{s_1, s_2, u_1, u_2\}$, and $\{t_1, t_2, u_1, u_2\}$ are the cocircuits of M contained in P;

then P is a *skew-whiff* 3*-separator* of M.



(a) An elongated-quad 3-separator. (b) A skew-whiff 3-separator.

FIGURE 2. Two particular 3-separators. Each is a single-element extension of a 5-element flan, and is in a matroid with rank $r(E - P) + 2$.

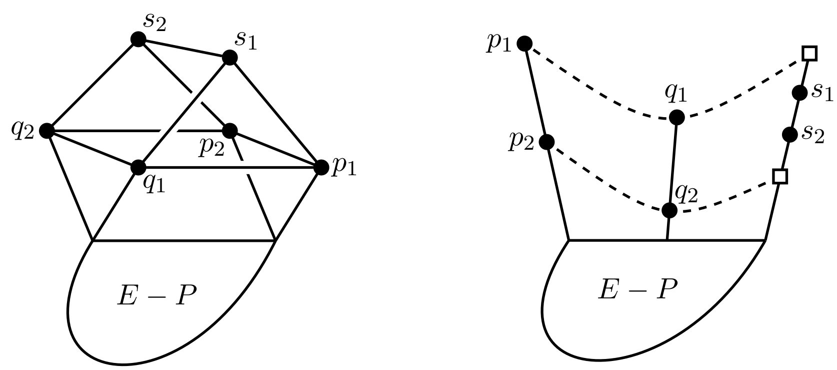
Definition 5.3. Let $P \subseteq E$ be an exactly 3-separating set with $P = \{p_1, p_2, q_1, q_2, s_1, s_2\}$. Suppose that

(a) $\{p_1, p_2, s_1, s_2\}$, $\{q_1, q_2, s_1, s_2\}$, and $\{p_1, p_2, q_1, q_2\}$ are the circuits of M contained in P; and
(b) $\{p_1, q_1, s_1, s_2\}$, $\{p_2, q_2, s_1, s_2\}$, $\{p_1, p_2, q_1, q_2, s_1\}$ and $\{p_1, p_2, q_1, q_2, s_2\}$ are the cocircuits of M contained in P.

Then P is a *twisted cube-like* 3*-separator* of M.

For a 3-connected matroid M, we say that a pair $\{x_1, x_2\} \subseteq E(M)$ is *detachable* if either $M/x_1/x_2$ or $M\backslash x_1\backslash x_2$ is 3-connected. In the former case, we will say that $\{x_1, x_2\}$ is a *contraction pair*; in the latter case, $\{x_1, x_2\}$ is a *deletion pair*.

Each particular 3-separator P that we have seen in this section can be used to construct a 3-connected matroid M with a 3-connected matroid N as a minor, such that M has no N-detachable pairs, and $E(M) - E(N) \subseteq P$. For the elongated-quad 3-separator and skew-whiff 3-separator, this follows from the fact that for such a 3-separator P, there is no detachable pair



(a) A twisted cube-like 3-separator of M. (b) A twisted cube-like 3-separator of M^*.

FIGURE 3. Geometric representations of a twisted cube-like 3-separator of M and M^*.

contained in P. On the other hand, the twisted cube-like 3-separator can contain a detachable pair, but appear in a matroid with no N-detachable pairs.

We first consider the elongated-quad 3-separator. Let M be a 3-connected matroid with a 3-separation (P,S) such that P is an elongated-quad 3-separator, the matroid $N = M\backslash P$ is 3-connected, $\mathrm{cl}(P)$ does not contain any triangles, and $\mathrm{cl}^*(P)$ does not contain any triads. Provided N is sufficiently structured to ensure that M/s and $M\backslash s$ have no N-minor for any $s \in S$, the matroid M has no N-detachable pairs, even after first performing a Δ-Y or Y-Δ exchange. Note that in this example $|E(M)| - |E(N)| = 6$.

It is also possible that $|E(M)| - |E(N)| = 5$. In this case, up to duality, the 3-connected N-minor can be obtained by extending M by an element e in the guts of (P,S), then restricting to $S \cup e$. Different cases arise depending on where in the guts needs to be "filled in" in order to obtain N. If P is labelled as in Figure 2(a), then e is in either

- $\mathrm{cl}(\{q_1,q_3\}) \cap \mathrm{cl}(\{q_2,q_4\}) \cap \mathrm{cl}(S)$,
- $\mathrm{cl}(\{p_1,p_2\}) \cap \mathrm{cl}(S)$, or
- $\mathrm{cl}(S) - (\mathrm{cl}(\{q_1,q_3\}) \cup \mathrm{cl}(\{q_2,q_4\}) \cup \mathrm{cl}(\{p_1,p_2\})$.

Here we have just focussed on cases where $|E(M)| - |E(N)| \geq 5$, though it is also possible that $|E(M)| - |E(N)| \in \{3,4\}$.

Similarly, a skew-whiff 3-separator can appear in a 3-connected matroid M with a 3-connected minor N such that $E(M) - E(N) \subseteq P$ and M has no N-detachable pairs, where $|E(M)| - |E(N)| \in \{3,4,5,6\}$.

Finally, we consider the twisted cube-like 3-separator. Let U_8 be the paving matroid on ground set $\{p_1,p_2,q_1,q_2,s_1,s_2,t_1,t_2\}$ whose non-spanning circuits are $\{t_1,t_2,p_1,q_1\}$, $\{t_1,t_2,p_2,q_2\}$, $\{p_1,p_2,q_1,q_2\}$, $\{p_1,p_2,s_1,s_2\}$, and $\{q_1,q_2,s_1,s_2\}$. Let U_8^+ be the single-element extension of U_8 by the element z such that z is in the span of the lines $\{t_1,t_2\}$, $\{q_1,p_1\}$, and $\{q_2,p_2\}$ and z

is not a loop. Label the triangle $T = \{t_1, t_2, z\}$. Let F_7^- be a copy of the non-Fano matroid with $E(F_7^-) \cap E(U_8^+) = T$ such that T is a triangle of F_7^-. Now let $M = P_T(U_8^+, F_7^-)\backslash z$, and observe that M is 3-connected and has an F_7^--minor (see Figure 4). In particular, $F_7^- \cong M/p_1\backslash\{s_1, s_2, p_2, q_2\}$, for example, so $|E(M)| - |E(F_7^-)| = 5$. Let $X = \{p_1, p_2, q_1, q_2, s_1, s_2\}$; the set X is a twisted cube-like 3-separator of M.

The matroid M has no F_7^--detachable pairs. To see this, first observe that neither $M\backslash y$ nor M/y has an F_7^--minor for any $y \in E(M) - X$. Due to the 4-element circuits and cocircuits contained in X, the only detachable pairs of M contained in X are the deletion pairs $\{p, q\} \in \{\{p_1, q_2\}, \{p_2, q_1\}\}$. But for any such $\{p, q\}$, the matroid $M\backslash p\backslash q$ has no F_7^--minor.

Note that although here we have used $N = F_7^-$ as the minor, other choices of N would work provided N is sufficiently structured and has a triangle $T = \{t_1, t_2, z\}$.

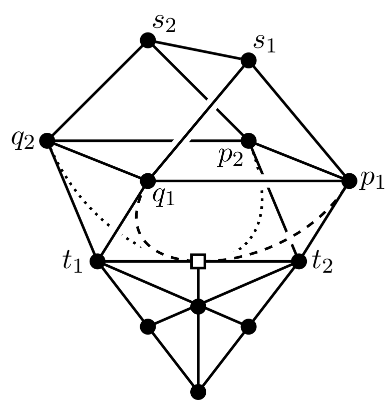


FIGURE 4. A matroid M with a twisted cube-like 3-separator and no F_7^--detachable pairs.

6. Flans

Let F be a set of elements in a 3-connected matroid M, with $t = |F| \geq 4$. Suppose F has an ordering $(f_1, f_2, \ldots, f_t)$ such that

(a) if $i \in [t-2]$ is odd, then $\{f_i, f_{i+1}, f_{i+2}\}$ is a triad; and
(b) if $i \in \{4, 5, \ldots, t\}$ is even, then $f_i \in \mathrm{cl}(\{f_1, f_2, \ldots, f_{i-1}\})$.

Then F is a *flan* of M, and $(f_1, f_2, \ldots, f_t)$ is a *flan ordering* (or just an *ordering*) of F. A flan F is *maximal* if it is not properly contained in another flan. Note that $\{f_1, f_2, \ldots, f_i\}$ is 3-separating for any $i \in [t]$.

A flan generalises the notion of a fan. We note that the definition of a flan used here is more restrictive than that often used in the literature (see [9, 10], for example).

In this section we consider the case where there is an N-deletable element $d \in E(M)$ such that $M\backslash d$ is 3-connected, but $M\backslash d$ has a flan F with at least five elements. In this case, we show that either M has an N-detachable pair, or $F \cup d$ is one of the 3-separators defined in Section 5. We start with a lemma that demonstrates that certain elements in a flan are candidates for contraction.

Lemma 6.1. *Let F be a maximal flan in a 3-connected matroid M, with $|F| \geq 5$ and $F \neq E(M)$. Let $i \in [3]$, let j be an odd integer such that $5 \leq j \leq |F|$, and suppose F has an ordering $(f_1, f_2, f_3, \dots, f_{|F|})$ such that f_i and f_j are not contained in triangles. Then*

(i) *M/f_i, M/f_j and $\mathrm{si}(M/f_i/f_j)$ are 3-connected;*
(ii) *if $j \geq 7$, then $M/f_i/f_j$ is 3-connected; and*
(iii) *if $|F| = 5$, then $M/f_i/f_j$ is 3-connected.*

Proof. Let $F' = \{f_1, f_2, \dots, f_j\}$. Suppose that $(F' - f_j, \{f_j\}, E(M) - F')$ is a cyclic 3-separation of M. Then $\mathrm{si}(M/f_j)$ is 3-connected by Bixby's Lemma. Since f_j is not contained in an triangle, M/f_j is 3-connected. On the other hand, if $(F' - f_j, \{f_j\}, E(M) - F')$ is not a cyclic 3-separation of M, then $(E(M) - F') \cup f_j$ is independent. Then $j = |F|$ and $F' = F$, otherwise f_{j+1} is in a circuit contained in $E(M) - F'$. If $|E(M) - F| < 3$, then, as M is 3-connected, F spans M, contradicting the maximality of F. It follows that $(E(M) - F) \cup f_j$ is a cosegment consisting of at least four elements, so M/f_j is 3-connected by the dual of Lemma 2.8.

Now, by the dual of Lemma 2.12, the matroids $\mathrm{si}(M/f_i)$ and $\mathrm{si}(M/f_i/f_j)$ are 3-connected. But f_i is not in a triangle, so M/f_i is 3-connected. This proves (i).

Now suppose $\{f_i, f_j\}$ is contained in a 4-element circuit C. By orthogonality, C must contain an element $f_i' \in \{f_1, f_2, f_3\} - f_i$ and an element $f_j' \in \{f_{j-2}, f_{j-1}\}$. If the elements $\{f_i, f_i', f_j, f_j'\}$ are distinct, then $f_j \in \mathrm{cl}(F' - f_j)$; a contradiction. It follows that if $j > 5$, then $\{f_i, f_j\}$ is not contained in a 4-element circuit, and thus $M/f_i/f_j$ is 3-connected. This proves (ii).

Now we may assume that $j = 5$, and $C \cap \{f_1, f_2, f_3, f_4, f_5\} = \{f_\ell, f_3, f_5\}$ for some $\ell \in \{1, 2\}$. Let $C - \{f_\ell, f_3, f_5\} = \{x\}$. Then $\{f_1, f_2, f_3, f_4, f_5, x\}$ is a flan. Thus, if $|F| = 5$, then $\{f_i, f_j\}$ is not contained in a 4-element circuit, and thus $M/f_i/f_j$ is 3-connected. This proves (iii). $\square$

The next proposition deals with the case where the flan has at least six elements.

Proposition 6.2. *Let M be a 3-connected matroid, and let N be a 3-connected minor of M, where $|E(N)| \geq 4$ and every triangle or triad of M is N-grounded. Suppose that $M\backslash d$ is 3-connected and has a maximal flan F with $|F| \geq 6$ and ordering $(f_1, f_2, f_3, \dots, f_{|F|})$, where $M\backslash d\backslash f_5$ has an N-minor with $|\{f_1, \dots, f_4\} \cap E(N)| \leq 1$. Then M has an N-detachable pair.*

Proof. Let $t = |F|$. Observe that $(\{f_1, f_2, f_3, f_4\}, E(M\backslash d) - \{f_1, \dots, f_5\})$ is a 2-separation of $M\backslash d\backslash f_5$.

6.2.1. *$M\backslash d/f_i/f_j$ has an N-minor for $i \in \{1, 2\}$ and $j \in \{5, 7\} \cap [t]$.*

Subproof. First consider $j = 5$. Since $\{f_3, f_4\}$ is a series pair in $M\backslash d\backslash f_5$, we have that $M\backslash d\backslash f_5/f_4$ is connected. Thus Lemma 2.17 implies that $M\backslash d\backslash f_5/f_4$, and hence $M\backslash d/f_4$ has an N-minor. By further applications of Lemma 2.17, we obtain that $M\backslash d/f_4\backslash f_3$ has an N-minor, and that $M\backslash d/f_4\backslash f_3/f_i$ has an N-minor for $i \in \{1, 2\}$. Since $M\backslash d\backslash f_3/f_i$ has an N-minor and $\{f_4, f_5\}$ is a series pair in this matroid, $M\backslash d\backslash f_3/f_i/f_5$, and in particular $M\backslash d/f_i/f_5$, has an N-minor, as required.

Now suppose $t \geq 7$, and consider $j = 7$. As $M\backslash d\backslash f_5$ has an N-minor and $M\backslash d\backslash f_5/f_i$ is connected, $M\backslash d\backslash f_5/f_i$ has an N-minor by Lemma 2.17. But $\{f_6, f_7\}$ is a series pair in this matroid, so we deduce that $M\backslash d/f_i/f_7$ has an N-minor. $\triangleleft$

6.2.2. *If $t \geq 7$, then M has an N-detachable pair.*

Subproof. Let $i \in \{1, 2\}$. By Lemma 6.1(ii), $M\backslash d/f_i/f_7$ is 3-connected, and, by 6.2.1, $M\backslash d/f_i/f_7$ has an N-minor. So either $\{f_i, f_7\}$ is an N-detachable pair, or d is in a parallel pair in $M/f_i/f_7$. Since neither f_i nor f_7 is in an N-grounded triangle, M has a 4-element circuit $\{d, f_i, f_7, t_i\}$ where, by orthogonality, $t_i \in \{f_3, f_4, f_5\}$. By circuit elimination, $\{f_1, f_2, t_1, t_2, f_7\}$ contains a circuit. But $f_7 \notin \mathrm{cl}(\{f_1, f_2, f_3, f_4, f_5\})$, so this circuit is $\{f_1, f_2, t_1, t_2\}$, and it follows that $\{t_1, t_2\} = \{f_3, f_4\}$.

We now work towards showing that either $\{f_5, f_7\}$ is an N-detachable pair, or $\{f_5, f_7\}$ is contained in a 4-element circuit of $M\backslash d$. We have that $\{d, f_4\} \subseteq \mathrm{cl}_{M/f_7}(\{f_1, f_2, f_3\})$, but $f_5 \notin \mathrm{cl}_{M/f_7}(\{f_1, f_2, f_3\}) = \mathrm{cl}_{M/f_7}(\{f_1, f_2, f_3, f_4, d\})$. Consider a triangle containing f_5 in M/f_7. It can contain at most one element in $\{f_1, f_2, f_3, f_4, d\}$. Thus, it cannot contain d, as $d \notin \mathrm{cl}_{M/f_7}(E(M/f_7) - \{f_1, f_2, f_3\})$ since d blocks the triad $\{f_1, f_2, f_3\}$.

We claim that $M\backslash d/f_5/f_7$ has an N-minor. Since $M\backslash d\backslash f_5$ has an N-minor and $M\backslash d\backslash f_5/f_2/f_3$ is connected, Lemma 2.17 implies that $M\backslash d\backslash f_5/f_2/f_3$ has an N-minor. Fix an N-labelling (C, D) with $\{f_2, f_3\} \subseteq C$ and $\{d, f_5\} \subseteq D$. Since $\{f_6, f_7\}$ is a series pair in $M\backslash d\backslash f_5$, we may assume that f_7 is N-labelled for contraction, up to an N-label switch on f_6 and f_7. Since $\{f_1, f_4\}$ is a parallel pair in $M/f_2/f_3$, we may also assume, up to an N-label switch on f_1 and f_4, that f_4 is N-labelled for deletion. In particular, observe that $M\backslash d\backslash f_4/f_7$ has an N-minor. Finally, $\{f_3, f_5\}$ is a series pair in $M\backslash d\backslash f_4$, so, after switching the N-labels on f_3 and f_5, the element f_5 is N-labelled for contraction, and f_3 is N-labelled for deletion. To summarise, $\{f_2, f_5, f_7\} \subseteq C$ and $\{d, f_3, f_4\} \subseteq D$. So $M\backslash d/f_5/f_7$ has an N-minor, as claimed.

Since f_7 is N-contractible, f_7 is not in a triangle of $M\backslash d$. Thus, by Lemma 6.1, $M\backslash d/f_7$ is 3-connected, and $(f_1, f_2, \dots, f_6)$ is a flan ordering in this matroid. Hence, either f_5 is in a triangle of $M\backslash d/f_7$, or, by another application of Lemma 6.1, $M\backslash d/f_5/f_7$ is 3-connected. In the latter case, as d is not in a triangle with f_5 in M/f_7, the matroid $M/f_5/f_7$ is 3-connected, so $\{f_5, f_7\}$ is an N-detachable pair. So we may assume that $\{f_5, f_7\}$ is contained in a 4-element circuit of $M\backslash d$.

By orthogonality, this circuit meets $\{f_3, f_4\}$. But if this circuit is contained in $\{f_1, f_2, \dots, f_7\}$, then $f_7 \in \mathrm{cl}(\{f_1, f_2, \dots, f_6\})$; a contradiction. It follows, by orthogonality, that $\{f_4, f_5, f_7, f_8\}$ is a circuit, where $f_8 \in E(M\backslash d) - \{f_1, f_2, \dots, f_7\}$. Note that $\{f_1, f_2, \dots, f_8\}$ is a flan. Recall the N-labelling (C, D) of M with $d \in D$ and $\{f_5, f_7\} \subseteq C$. Since $\{f_4, f_8\}$ is a parallel pair in $M/f_5/f_7$, the element f_8 is N-labelled for deletion after switching the N-labels on f_4 and f_8. In particular, $M\backslash d\backslash f_8$ has an N-minor.

Let $Z = E(M\backslash d) - \{f_1, \dots, f_8\}$, and observe that $(\{f_1, f_2, \dots, f_7\}, \{f_8\}, Z)$ is a path of 3-separations. Suppose $|Z| = 1$. Then $E(M\backslash d)$ is a 9-element flan, f_2 and f_7 are N-labelled for contraction with respect to the N-labelling (C, D), and it is easily verified that $M/f_2/f_7$ is 3-connected. So $\{f_2, f_7\}$ is an N-detachable pair. We now may assume that $|Z| \geq 2$.

We claim that $\mathrm{co}(M\backslash d\backslash f_8)$ is 3-connected. If $r(Z) \geq 3$, then $(\{f_1, f_2, \dots, f_7\}, \{f_8\}, Z)$ is a vertical 3-separation, and the claim follows from Bixby's Lemma. On the other hand, if $r(Z) \leq 2$ and $|Z| \geq 3$, then $(M\backslash d)|(Z \cup f_8) \cong U_{2,4}$, and $M\backslash d\backslash f_8$ is 3-connected by Lemma 2.8. Finally, if $|Z| = 2$, then $Z \cup \{f_7, f_8\}$ is a rank-3 cocircuit, and $\mathrm{co}(M\backslash d\backslash f_8)$ is 3-connected by Lemma 2.12, thus proving the claim. So either $\{d, f_8\}$ is an N-detachable pair, or f_8 is in a triad of $M\backslash d$.

We may now assume that f_8 is in a triad T^* of $M\backslash d$. Since $f_8 \notin \mathrm{cl}^*(\{f_1, f_2, \dots, f_7\})$, the triad T^* contains an element $q \in E(M) - \{f_1, f_2, \dots, f_8\}$. By orthogonality, T^* meets $\{f_4, f_5, f_7\}$. But if $T^* = \{f_4, f_8, q\}$, then T^* intersects the circuit $\{f_1, f_2, f_3, f_4\}$ in one element; a contradiction. Similarly, if $T^* = \{f_5, f_8, q\}$, then $\{d, f_5, f_8, q\}$ is a cocircuit of M that intersects the circuit $\{d, f_1, f_7, t_1\}$ (where $t_1 \in \{f_3, f_4\}$) in one element; a contradiction. We deduce that $T^* = \{f_7, f_8, q\}$. After relabelling q as f_9, we observe that $(f_1, f_2, \dots, f_9)$ is a flan ordering.

Next we claim that $M/f_i/f_9$ has an N-minor for $i \in \{1, 2\}$. Again we recall the N-labelling (C, D) from earlier, which has $\{f_2, f_7\} \subseteq C$ and $\{d, f_8\} \subseteq D$. Since $\{f_7, f_9\}$ is a series pair in $M\backslash d\backslash f_8$, the element f_9 is N-labelled for contraction after switching the N-labels on f_7 and f_9. So $M/f_2/f_9$ has an N-minor. Using a similar argument, but starting with an N-labelling (C', D') that has $\{f_1, f_3\} \subseteq C'$ and $\{d, f_5\} \subseteq D'$, one can show that $M/f_1/f_9$ has an N-minor.

Let $i \in \{1, 2\}$. Since each of f_i and f_9 is not contained in an N-grounded triangle, $M\backslash d/f_i/f_9$ is 3-connected, by Lemma 6.1(ii). Now either $\{f_i, f_9\}$ is an N-detachable pair, or d is in a parallel pair in $M/f_i/f_9$. Hence, we may assume that M has a 4-element circuit containing $\{d, f_i, f_9\}$. By orthogonality, this circuit meets $\{f_3, f_4, f_5\}$ and $\{f_5, f_6, f_7\}$, so $\{d, f_i, f_5, f_9\}$ is a circuit. As $i \in \{1, 2\}$ was chosen arbitrarily, we may now assume that $\{d, f_1, f_5, f_9\}$

and $\{d, f_2, f_5, f_9\}$ are circuits. By circuit elimination, $\{f_1, f_2, f_5, f_9\}$ contains a circuit. But this set does not contain a triangle, and $f_9 \notin \mathrm{cl}(\{f_1, f_2, f_5\})$, so this is contradictory. $\triangleleft$

6.2.3. *If $t = 6$, then M has an N-detachable pair.*

Subproof. For each $i \in \{1, 2\}$, the matroid $\mathrm{si}(M\backslash d/f_i/f_5)$ is 3-connected, by Lemma 6.1(i), and $M\backslash d/f_i/f_5$ has an N-minor, by 6.2.1. First, suppose that $\{d, f_i, f_5\}$ is not contained in a 4-element circuit, for some $i \in \{1, 2\}$. It follows that if $M\backslash d/f_i/f_5$ is 3-connected, then $M/f_i/f_5$ is 3-connected, and $\{f_i, f_5\}$ is an N-detachable pair. So assume that $\{f_i, f_5\}$ is contained in a 4-element circuit in $M\backslash d$. By orthogonality, this circuit must also contain an element of $\{f_1, f_2, f_3\} - f_i$ and an element of $\{f_3, f_4\}$. Thus, if it does not contain f_3, then $f_5 \in \mathrm{cl}(\{f_1, f_2, f_3, f_4\})$; a contradiction. It follows that this circuit is $\{f_i, f_3, f_5, q\}$ for some $q \in E(M\backslash d) - \{f_1, f_2, f_3, f_4, f_5\}$.

Since $M\backslash d/f_i/f_5$ has an N-minor, and $\{f_3, q\}$ is a parallel pair in this matroid, $\{d, q\}$ is N-deletable. Let $F' = \{f_1, f_2, f_3, f_4, f_5\}$. Now F' and $F' \cup q$ are 3-separating in $M\backslash d$, by Lemma 2.3. As $|F' \cap E(N)| \leq 1$ and $|E(N)| \geq 4$, we have $|E(M\backslash d) - F'| \geq 3$. We claim that $\mathrm{co}(M\backslash d\backslash q)$ is 3-connected. If $r(E(M\backslash d) - F') \leq 2$, then $(E(M\backslash d) - F') \cup f_5$ is a rank-3 cocircuit, and it follows, by Lemma 2.12, that $\mathrm{co}(M\backslash d\backslash q)$ is 3-connected. On the other hand, if $r(E(M\backslash d) - F') \geq 3$, then $(F', \{q\}, E(M\backslash d) - (F' \cup q))$ is a vertical 3-separation of $M\backslash d$, so $\mathrm{si}(M\backslash d/q)$ is not 3-connected, and hence $\mathrm{co}(M\backslash d\backslash q)$ is 3-connected by Bixby's Lemma. So $\{d, q\}$ is an N-detachable pair unless q is in a triad T^* of $M\backslash d$. Note also that, by the foregoing, $q \in \mathrm{cl}(E(M\backslash d) - (F' \cup q))$.

As $q \notin \mathrm{cl}^*_{M\backslash d}(F')$, the triad T^* contains at most one element of F'. By orthogonality, T^* meets $\{f_i, f_3, f_5\}$. Now if $f_5 \notin T^*$, then, as $\{f_1, f_2, f_3, f_4\}$ is a circuit, T^* intersects $\{f_1, f_2, f_3, f_4\}$ in two elements; a contradiction. It follows that $T^* = \{f_5, q, q'\}$, for $q' \in E(M\backslash d) - F'$. Now, as $f_6 \in \mathrm{cl}(F')$ but $f_6 \notin \mathrm{cl}(F' - f_5)$, there is a circuit containing $\{f_5, f_6\}$ that is contained in F. Again by orthogonality, we deduce that $f_6 \in T^*$, so let $q = f_6$. Now $F \cup q'$ is a flan, contradicting that F is a maximal 6-element flan.

It remains to consider the case where M has circuits $\{d, f_1, f_5, g_1\}$ and $\{d, f_2, f_5, g_2\}$ for some $g_1 \in E(M) - \{d, f_1, f_5\}$ and $g_2 \in E(M) - \{d, f_2, f_5\}$. First, suppose that $g_1 = g_2 = f_3$. Then $\{d, f_1, f_3, f_5\}$ and $\{d, f_2, f_3, f_5\}$ are circuits, so $\{f_1, f_2, f_3, f_5\}$ contains a circuit by circuit elimination. But $\{f_1, f_2, f_3, f_5\}$ does not contain a triangle, and $f_5 \notin \mathrm{cl}(\{f_1, f_2, f_3\})$, since $f_5 \in \mathrm{cl}^*_{M\backslash d}(\{f_1, f_2, f_3, f_4\})$, so this is contradictory.

Let $i \in \{1, 2\}$ such that $g_i \neq f_3$. We will show that either $\{f_3, g_i\}$ is an N-detachable pair, or $g_i \in E(M\backslash d) - F$ and there is a 4-element cocircuit C^*_i with $\{f_3, g_i\} \subseteq C^*_i \subseteq F \cup g_i$.

Using a similar argument as in the proof of 6.2.1, it follows from Lemma 2.17 that $M\backslash f_3/f_i/f_5$ has an N-minor for $i \in \{1, 2\}$. Since $\{d, g_i\}$

is a parallel pair in $M/f_i/f_5$, the pair $\{f_3, g_i\}$ is N-labelled for deletion, up to swapping the N-labels on d and g_i.

Suppose that $\mathrm{co}(M\backslash f_3\backslash g_i)$ is not 3-connected. Then $M\backslash f_3\backslash g_i$ has a 2-separation (X, Y) for which $(\mathrm{fcl}(X), Y - \mathrm{fcl}(X))$ and $(X - \mathrm{fcl}(Y), \mathrm{fcl}(Y))$ are also 2-separations. Thus, we may assume that $\{f_1, f_2, d\} \subseteq X$ and X is fully closed. Pick j such that $\{i, j\} = \{1, 2\}$. If $f_5 \in X$, then $g_j \in X$ due to the circuit $\{f_j, f_5, d, g_j\}$, and $f_4 \in X$ due to the cocircuit $\{f_4, f_5, d\}$ of $M\backslash f_3\backslash g_i$. Similarly, if $f_4 \in X$, then $\{f_5, g_j\} \subseteq X$. But then $\{f_3, g_i\} \subseteq \mathrm{cl}(X)$, so $(X \cup \{f_3, g_i\}, Y)$ is a 2-separation of M; a contradiction. So $\{f_4, f_5\} \subseteq Y$. Now, if $g_j \in X$, then $f_5 \in \mathrm{cl}(X)$, so X is not fully closed; a contradiction. So $g_j \in Y$. Consider $\mathrm{fcl}(Y)$. As $d \in \mathrm{fcl}(Y)$, we have $f_j \in \mathrm{fcl}(Y)$, so $f_i \in \mathrm{fcl}(Y)$, and $(X - \mathrm{fcl}(Y), \mathrm{fcl}(Y) \cup \{f_3, g_i\})$ is a 2-separation of M; a contradiction. So $\mathrm{co}(M\backslash f_3\backslash g_i)$ is 3-connected.

So we may assume that M has a 4-element cocircuit C_i^* containing $\{g_i, f_3\}$, otherwise M has an N-detachable pair. By orthogonality, C_i^* meets $\{f_1, f_2, f_4\}$ and $\{d, f_i, f_5\}$. Suppose $d \in C_i^*$. If $f_4 \in C_i^*$, then $\{f_3, f_4, f_5, g_i\}$ is a cosegment of $M\backslash d$. If $g_i \in \{f_1, f_2\}$, then $r^*_{M\backslash d}(F - f_6) = 2$, so $\lambda_{M\backslash d}(F - f_6) = 4 + 2 - 5 = 1$; a contradiction. But on the other hand if $g_i \notin \{f_1, f_2\}$, then this contradicts orthogonality with the circuit $\{f_1, f_2, f_3, f_4\}$. So $\{f_1, f_2\}$ meets C_i^*, and thus $\{f_1, f_2, f_3, g_i\}$ is a cosegment of $M\backslash d$. As before, $g_i \notin \{f_4, f_5\}$, otherwise $\lambda_{M\backslash d}(F) = 1$; and $g_i \neq f_6$, since $f_6 \notin \mathrm{cl}^*_{M\backslash d}(F - f_6)$. Observe that there is a circuit contained in $\{f_1, f_3, f_4, f_5, f_6\}$ with at least four elements. But this contradicts orthogonality. So $d \notin C_i^*$.

Suppose $|C_i^* \cap F| \leq 2$. Then $C_i^* \cap F = \{f_i, f_3\}$. Again, pick j such that $\{i, j\} = \{1, 2\}$, and observe that there is a circuit contained in $\{f_j, f_3, f_4, f_5, f_6\}$ that contains f_3. But this contradicts orthogonality, so $C_i^* \subseteq F \cup g_i$. Now suppose $C_i^* \subseteq F$. Since $F - f_6$ is exactly 3-separating in $M\backslash d$ and $f_6 \in \mathrm{cl}(F - f_6)$, it follows that $f_6 \notin \mathrm{cl}^*_{M\backslash d}(F - f_6)$. So $C_i^* \subseteq F - f_6$. But, as $f_3 \in C_i^*$, and $r^*_{M\backslash d}(\{f_1, f_2, f_3\}) = r^*_{M\backslash d}(\{f_3, f_4, f_5\}) = 2$, the set C_i^* is a 4-element cosegment in $M\backslash d$. Thus $r^*_{M\backslash d}(F - f_6) = 2$, and $\lambda_{M\backslash d}(F - f_6) = 1$; a contradiction. We deduce that $g_i \notin F$.

Finally, we recall that $\{d, f_1, f_5, g_1\}$ and $\{d, f_2, f_5, g_2\}$ are circuits, so, by circuit elimination, there are circuits contained in the sets $\{d, f_1, f_2, g_1, g_2\}$ and $\{f_1, f_2, f_5, g_1, g_2\}$. By the foregoing, if $f_3 \notin \{g_1, g_2\}$, then $\{g_1, g_2\} \cap F = \emptyset$, and $\{g_1, g_2\} \subseteq \mathrm{cl}^*_{M\backslash d}(F)$. Since $\{d, f_3, f_4, f_5\}$ is a cocircuit, we deduce, by orthogonality, that $\{f_1, f_2, g_1, g_2\}$ is a circuit. Thus $g_1 \neq g_2$. Since $\{g_1, g_2\} \subseteq \mathrm{cl}^*_{M\backslash d}(F)$, we have

$$\begin{aligned}\lambda_{M\backslash d}(F \cup \{g_1, g_2\}) &= r_{M\backslash d}(F \cup \{g_1, g_2\}) + r^*_{M\backslash d}(F) - 8\\ &\leq (r(F) + 1) + 4 - 8 = 1;\end{aligned}$$

a contradiction. On the other hand, if $g_2 = f_3$, say, then $g_1 \neq f_3$, and there is a cocircuit $C_1^* \subseteq F \cup g_1$ of $M\backslash d$ with $g_1 \in E(M\backslash d) - F$. Since $\{f_1, f_2, f_5, g_1, g_2\} = \{f_1, f_2, f_3, f_5, g_1\}$ contains a circuit and $f_5 \notin$

$\mathrm{cl}(\{f_1, f_2, f_3\})$, this circuit must contain g_1. But then $g_1 \in \mathrm{cl}_{M\backslash d}(F) \cap \mathrm{cl}^*_{M\backslash d}(F)$; a contradiction. $\triangleleft$

The proposition now follows from 6.2.2 and 6.2.3. $\square$

Next we address the case where $M\backslash d$ has a maximal 5-element flan F. We can break this into two cases depending on whether or not d fully blocks F. The next proposition primarily deals with the case where d fully blocks F. However, we use a more general hypothesis than this, as the same argument also applies in one situation that arises when d does not fully block F. More specifically, suppose F has the ordering $(f_1, f_2, f_3, f_4, f_5)$. The following proposition applies when any 4-element circuit containing $\{f_i, f_5, d\}$ for $i \in \{1, 2\}$ is not contained in $F \cup d$. In particular, observe that this is the case when d fully blocks F.

Proposition 6.3. *Let M be a 3-connected matroid with a 3-connected matroid N as a minor, where $|E(N)| \geq 4$ and every triangle or triad of M is N-grounded. Suppose that $M\backslash d$ is 3-connected and has a maximal 5-element flan F with ordering $(f_1, f_2, f_3, f_4, f_5)$, where any 4-element circuit containing $\{f_i, f_5, d\}$ for $i \in \{1, 2\}$ is not contained in $F \cup d$, and $M\backslash d\backslash f_3$ has an N-minor. Then M has an N-detachable pair.*

Proof. We start by showing that either there is an N-detachable pair in F, or there are certain 4-element circuits in M containing d and intersecting F.

6.3.1. *For each $i \in \{1, 2\}$, the matroids $M\backslash d/f_i/f_5$ and $M\backslash f_3/f_i/f_5$ have N-minors.*

Subproof. The matroid $M\backslash d\backslash f_3$ has an N-minor, and $\{f_1, f_2\}$ and $\{f_4, f_5\}$ are series pairs in this matroid, so $M\backslash d\backslash f_3/f_i/f_5$ has an N-minor for $i \in \{1, 2\}$. $\triangleleft$

Now it follows from 6.3.1 that none of f_1, f_2, and f_5 is contained in an N-grounded triangle. Hence, we can apply Lemma 6.1 to deduce that $M\backslash d/f_i/f_5$ is 3-connected for each $i \in \{1, 2\}$. Then $\{f_i, f_5\}$ is an N-detachable pair for $i \in \{1, 2\}$ unless d is in a parallel pair in $M/f_i/f_5$. Since neither f_i nor f_5 is contained in an N-grounded triangle, we have that $\{d, f_i, f_5, g_i\}$ is a circuit in M for some $g_i \in E(M) - \{d, f_i, f_5\}$. By hypothesis, $g_i \notin F$ for $i \in \{1, 2\}$. Moreover, if $g_1 = g_2$, then there is a circuit contained in $\{d, f_1, f_2, f_5\}$; a contradiction. So g_1 and g_2 are distinct elements of $E(M\backslash d) - F$.

6.3.2. *$\{f_1, f_2, g_1, g_2\}$ is a circuit of M.*

Subproof. As $r(\{d, f_5, f_1, f_2, g_1, g_2\}) \leq 4$ and $d \notin \mathrm{cl}(E(M) - \{f_3, f_4, f_5\})$, we have $r(\{f_1, f_2, g_1, g_2\}) \leq 3$. Since neither f_1 nor f_2 is in an N-grounded triangle, we deduce that $\{f_1, f_2, g_1, g_2\}$ is a circuit. $\triangleleft$

We now work towards showing that, for each $i \in \{1,2\}$, either $\{f_3, g_i\}$ is an N-detachable pair, or $\{f_3, g_i\}$ is contained in a 4-element cocircuit. To this end, we start by showing that $M\backslash f_3\backslash g_i$ has an N-minor for $i \in \{1,2\}$. Let $i \in \{1,2\}$. By 6.3.1, $M\backslash f_3/f_i/f_5$ has an N-minor. Since $\{d, g_i\}$ is a parallel pair in this matroid and $|E(N)| \geq 4$, $M\backslash f_3\backslash g_i$ has an N-minor.

6.3.3. *Let $i \in \{1,2\}$. If $\{f_3, g_i\}$ is not in a 4-element cocircuit of M, then $\{f_3, g_i\}$ is an N-detachable pair.*

Subproof. We give the proof for $i = 2$; the argument is almost identical when $i = 1$. Let (P, Q) be a 2-separation of $M\backslash f_3\backslash g_2$. Then $(\mathrm{fcl}(P), Q - \mathrm{fcl}(P))$ is also a 2-separation in this matroid, so we may assume that P is fully closed. Without loss of generality, let $\{f_1, f_2, d\} \subseteq P$. If $\{f_5, g_1\}$ meets P, then $\{f_5, g_1\} \subseteq P$, due to the circuit $\{f_1, f_5, d, g_1\}$, and $f_4 \in P$ as well, due to the triad $\{f_4, f_5, d\}$. But then $(P \cup \{f_3, g_2\}, Q)$ is a 2-separation of M; a contradiction. So $\{f_5, g_1\} \subseteq Q$, and, similarly, $f_4 \in Q$. Now consider $\mathrm{fcl}(Q)$. Due to the triad $\{f_4, f_5, d\}$, we have $d \in \mathrm{fcl}(Q)$, and thus $\{f_1, f_2\} \subseteq \mathrm{fcl}(Q)$ due to the circuits $\{f_1, f_5, d, g_1\}$ and $\{f_2, f_5, d, g_2\}$. Thus $(P - \mathrm{fcl}(Q), \mathrm{fcl}(Q) \cup \{f_3, g_2\})$ is a 2-separation of M; a contradiction. So $M\backslash f_3\backslash g_2$ is 3-connected. Since $\{f_3, g_2\}$ is N-deletable, this completes the proof of 6.3.3. $\lhd$

It remains to consider the case where $\{f_3, g_i\}$ is contained in a 4-element cocircuit for each $i \in \{1,2\}$. We next prove two claims regarding the elements that appear in such a cocircuit.

6.3.4. *Let $i \in \{1,2\}$. If $\{f_3, g_i\}$ is in a 4-element cocircuit of M, then either this cocircuit contains f_i, or $g_i \in \mathrm{cl}^*(F \cup d)$.*

Subproof. Let C^* be a 4-element cocircuit of M containing $\{f_3, g_i\}$. Pick i' such that $\{i, i'\} = \{1,2\}$. By orthogonality, C^* meets $\{f_1, f_2, f_4\}$ and $\{f_i, f_5, d\}$. Thus, either $f_i \in C^*$, or C^* meets $\{f_{i'}, f_4\}$ and $\{f_5, d\}$ in which case $g_i \in \mathrm{cl}^*(F \cup d)$. $\lhd$

6.3.5. *Suppose $\{f_i, f_3, g_i, h_i\}$ is a cocircuit of M, for some $i \in \{1,2\}$ and $h_i \in E(M) - (F \cup \{d, g_i\})$. Then, either M has an N-detachable pair, or $h_i \in \mathrm{cl}(F \cup \{d, g_i\}) - \{g_1, g_2\}$.*

Subproof. First, we will show that if $\{f_5, h_i\}$ is not contained in a 4-element circuit, then $\{f_5, h_i\}$ is an N-detachable pair. Pick i' such that $\{i, i'\} = \{1,2\}$. Observe that $\{f_i, f_3, g_1, g_2\}$ is not a cocircuit, by orthogonality with the circuit $\{f_{i'}, f_5, d, g_{i'}\}$. So $h_i \neq g_{i'}$.

Let (P, Q) be a 2-separation in $M/f_5/h_i$, where neither P nor Q is contained in a parallel class. So $(\mathrm{fcl}(P), Q - \mathrm{fcl}(P))$ is also a 2-separation. Without loss of generality, $\{f_i, d, g_i\} \subseteq P$. If P meets $\{f_{i'}, f_3\}$, then $\{f_{i'}, f_3\} \subseteq P$ due to the cocircuit $\{f_1, f_2, f_3, d\}$, and $f_4 \in P$ due to the circuit $\{f_1, f_2, f_3, f_4\}$. But then $(P \cup \{f_5, h_i\}, Q)$ is a 2-separation of M; a contradiction. So $\{f_{i'}, f_3\} \subseteq Q$. Since $\{f_1, f_2, g_1, g_2\}$ is a circuit, by

6.3.2, we have $g_{i'} \in Q$, otherwise $f_{i'} \in \mathrm{fcl}(P) = P$. Now consider the 2-separation $(P', Q') = (P - \mathrm{fcl}(Q), \mathrm{fcl}(Q))$. We have $d \in Q'$, due to the triangle $\{f_{i'}, d, g_{i'}\}$, and it follows that $f_i \in Q'$, due to the cocircuit $\{f_1, f_2, f_3, d\}$; $f_4 \in Q'$, due to the circuit $\{f_1, f_2, f_3, f_4\}$; and $g_i \in Q'$, due to the triangle $\{f_i, d, g_i\}$. So $(P', Q' \cup \{f_5, h_i\})$ is a 2-separation of M; a contradiction. So $M/f_5/h_i$ is 3-connected up to parallel pairs.

We claim that $M/f_5/h_i$ has an N-minor. Since $M/f_i/f_5\backslash f_3$ has an N-minor, there is an N-labelling (C, D) with $\{f_i, f_5\} \subseteq C$ and $f_3 \in D$. As $\{g_i, d\}$ is a parallel pair in $M/f_i/f_5$, we may assume, up to switching the N-labels on g_i and d, that $g_i \in D$. Now, in $M\backslash f_3\backslash g_i$, we have that $\{f_i, h_i\}$ is a series pair, so, after switching the N-labels on f_i and h_i, we obtain an N-labelling (C', D') with $\{h_i, f_5\} \subseteq C'$. So $M/h_i/f_5$ has an N-minor, as claimed.

We may now assume that $\{f_5, h_i\}$ is contained in a 4-element circuit, otherwise M has an N-detachable pair. By orthogonality, this circuit meets $\{f_3, f_4, d\}$ and $\{f_i, f_3, g_i\}$. If it does not contain f_3, then $h_i \in \mathrm{cl}(\{f_i, f_4, f_5, d, g_i\})$, as required. So suppose it contains f_3. Then, again by orthogonality, it also meets $\{f_1, f_2, d\}$, in which case $h_i \in \mathrm{cl}(F \cup d)$. $\lhd$

Observe that $\{g_1, g_2\} \nsubseteq \mathrm{cl}^*(F \cup d)$. Indeed, if $\{g_1, g_2\} \subseteq \mathrm{cl}^*(F \cup d)$, then

$$\begin{aligned}\lambda(F \cup \{d, g_1, g_2\}) &= r(F \cup d) + r^*(F \cup d) - 8 \\ &\leq 5 + 4 - 8 = 1;\end{aligned}$$

a contradiction. So there exists some $\ell \in \{1, 2\}$ such that $g_\ell \notin \mathrm{cl}^*(F \cup d)$. Then, by 6.3.4, M has a cocircuit $\{f_\ell, f_3, g_\ell, h_\ell\}$ for some $h_\ell \in E(M) - \{f_\ell, f_3, g_\ell\}$. In fact, $h_\ell \notin F \cup d$, since $g_\ell \notin \mathrm{cl}^*(F \cup d)$. Thus, by 6.3.5, we may assume that $h_\ell \in \mathrm{cl}(F \cup \{d, g_\ell\})$ and $h_\ell \notin \{g_1, g_2\}$.

6.3.6. *For each $i \in \{1, 2\}$, we have $g_i \notin \mathrm{cl}^*((F \cup \{d, g_1, g_2\}) - g_i)$. Moreover, there are distinct elements $h_1, h_2 \in E(M) - (F \cup \{d, g_1, g_2\})$ such that $\{f_1, f_3, g_1, h_1\}$ and $\{f_2, f_3, g_2, h_2\}$ are cocircuits of M.*

Subproof. Consider $\lambda(F \cup \{d, g_1, g_2, h_\ell\})$. Observe that $\{g_1, g_2, h_\ell\} \subseteq \mathrm{cl}(F \cup d)$ and $h_\ell \in \mathrm{cl}^*(F \cup g_\ell)$. Thus,

$$\begin{aligned}\lambda(F \cup \{d, g_1, g_2, h_\ell\}) &= r(F \cup d) + r^*(F \cup \{d, g_1, g_2\}) - 9 \\ &\leq r^*(F \cup \{d, g_1, g_2\}) - 4.\end{aligned}$$

Now if either $g_1 \in \mathrm{cl}^*(F \cup \{d, g_2\})$ or $g_2 \in \mathrm{cl}^*(F \cup \{d, g_1\})$, then

$$\lambda(F \cup \{d, g_1, g_2, h_\ell\}) \leq (r^*(F \cup d) + 1) - 4 = 1;$$

a contradiction.

By 6.3.4, $\{f_1, f_3, g_1\}$ and $\{f_2, f_3, g_2\}$ are each contained in a 4-element cocircuit of M. Let these cocircuits be $\{f_1, f_3, g_1, h_1\}$ and $\{f_2, f_3, g_2, h_2\}$ respectively. Observe that, for each $i \in \{1, 2\}$, we have $h_i \in E(M) - (F \cup \{d, g_1, g_2\})$, since $g_i \notin \mathrm{cl}^*((F \cup \{d, g_1, g_2\}) - g_i)$.

Suppose that $h_1 = h_2$. Then $\{f_1, f_2, f_3, g_1, g_2\}$ contains a cocircuit, by cocircuit elimination. Since $g_1 \notin \mathrm{cl}^*(F \cup g_2)$ and $g_2 \notin \mathrm{cl}^*(F \cup g_1)$, it follows that $\{f_1, f_2, f_3\}$ is a cocircuit of M; a contradiction. $\lhd$

Now, by 6.3.6, there are distinct elements $h_1, h_2 \in E(M) - (F \cup \{d, g_1, g_2\})$ such that $\{h_1, h_2\} \subseteq \mathrm{cl}(F \cup \{d, g_1, g_2\}) = \mathrm{cl}(F \cup d)$. Note that $\{h_1, h_2\} \subseteq \mathrm{cl}^*(F \cup \{g_1, g_2\})$. Thus,

$$\begin{aligned}\lambda(F \cup \{d, g_1, g_2, h_1, h_2\}) &= r(F \cup d) + r^*(F \cup \{d, g_1, g_2\}) - 10 \\ &\leq 5 + (r^*(F \cup d) + 2) - 10 \\ &= 1;\end{aligned}$$

a contradiction. This completes the proof. $\square$

Next we handle the case where $M\backslash d$ has a maximal 5-element flan and d does not fully block F. Since d blocks the triads of $M\backslash d$ contained in F, we have that $d \in \mathrm{cl}_M(F)$.

Proposition 6.4. *Let M be a 3-connected matroid with a 3-connected matroid N as a minor, where $|E(N)| \geq 4$, and every triangle or triad of M is N-grounded. Let d be an element of M such that $M\backslash d$ is 3-connected and has a maximal 5-element flan F with ordering $(f_1, f_2, f_3, f_4, f_5)$, where $d \in \mathrm{cl}_M(F)$. Suppose that either*

(a) *$M\backslash d\backslash f_5$ has an N-minor with $|\{f_1, \dots, f_4\} \cap E(N)| \leq 1$, or*
(b) *$M/f_i/f_{i'}$ has an N-minor for all distinct $i, i' \in [3]$.*

Then one of the following holds:

(i) *M has an N-detachable pair,*
(ii) *$F \cup d$ is a skew-whiff 3-separator of M,*
(iii) *$F \cup d$ is an elongated-quad 3-separator of M, or*
(iv) *$F \cup d$ is a twisted cube-like 3-separator of M^*.*

Proof. First, we observe that each element in $F - f_5$ is N-deletable in $M\backslash d$. Indeed, if (a) holds, then since $(F - f_5, \{f_5\}, E(M\backslash d) - F)$ is a cyclic 3-separation of $M\backslash d$, and $F - f_5$ is a circuit, this follows from Lemma 2.18(i). On the other hand, if (b) holds, then since $M/f_i/f_i'$ has an N-minor for all distinct $i, i' \in [3]$, and $\{f_1, f_2, f_3, f_4\}$ is a circuit, it follows that each element in $F - f_5$ is N-deletable up to an N-label switch.

Now each triad of $M\backslash d$ contained in F is not an N-grounded triad, so $\{f_1, f_2, f_3, d\}$ and $\{f_3, f_4, f_5, d\}$ are cocircuits of M. Moreover, as $M\backslash d\backslash f_3$ has an N-minor, and $\{f_1, f_2\}$ and $\{f_4, f_5\}$ are parallel pairs in this matroid, $M\backslash d/f_i/f_5$ has an N-minor for $i \in \{1, 2\}$. By Lemma 6.1(iii), $M\backslash d/f_i/f_5$ is 3-connected, for $i \in \{1, 2\}$. Thus, assuming (i) does not hold, we deduce the existence of 4-element circuits $\{f_1, f_5, d, g_1\}$ and $\{f_2, f_5, d, g_2\}$.

We claim that $\{g_1, g_2\} \subseteq F$ or $\{g_1, g_2\} \subseteq \mathrm{cl}(F \cup d) - (F \cup d)$. Suppose $g_1 \notin F$. Since F is a maximal flan, $g_1 \notin \mathrm{cl}(F)$. By circuit elimination, $\{f_1, f_2, f_5, g_1, g_2\}$ contains a circuit. If this circuit contains g_1, then $g_1 \in \mathrm{cl}(F \cup g_2)$, so $g_2 \notin F$, and $\{g_1, g_2\} \subseteq \mathrm{cl}(F \cup d) - F$ as required. So suppose

$\{f_1, f_2, f_5, g_2\}$ is a circuit. Then $g_2 \in F$, since F is a maximal flan, so $g_2 \in \{f_3, f_4\}$. It follows that $F \subseteq \mathrm{cl}(\{f_1, f_2, g_2\})$; a contradiction.

If $g_1, g_2 \notin F$, then we can apply Proposition 6.3, so (i) holds. So we may assume that $\{g_1, g_2\} \subseteq F$. Observe that since $\{f_1, f_2, f_3, f_4\}$ and $\{f_1, f_5, d, g_1\}$ are circuits, every element of $F \cup d$ is in a circuit contained in $F \cup d$.

6.4.1. *If C_1 and C_2 are distinct circuits of M contained in $F \cup d$, then $F \cup d = C_1 \cup C_2$. Similarly, if C_1^* and C_2^* are distinct cocircuits of M contained in $F \cup d$, then $F \cup d = C_1^* \cup C_2^*$.*

Subproof. The set $F \cup d$ is exactly 3-separating in M, and $r(F \cup d) = 4$, so $r(E(M) - (F \cup d)) = r(M) - 2$. Suppose that $C_1^* \subseteq F \cup d$ and $C_2^* \subseteq F \cup d$ are distinct cocircuits of M. Then $E(M) - (C_1^* \cup C_2^*)$ is a flat of rank $r(M) - 2$. Thus, if $x \in (F \cup d) - (C_1^* \cup C_2^*)$, then $x \in \mathrm{cl}(E(M) - (F \cup d))$. But this contradicts the fact that every element of $F \cup d$ is contained in some cocircuit that is itself contained in $F \cup d$. The proof of the dual result follows in the same manner due to the fact that $r_M^*(F \cup d) = 4$ and every element of $F \cup d$ is contained in a circuit that is itself contained in $F \cup d$. $\lhd$

Now we assume that (i) does not hold, and show that either (ii), (iii), or (iv) holds. As $\{f_1, f_5, d, g_1\}$ and $\{f_2, f_5, d, g_2\}$ are circuits of M contained in $F \cup d$, either $g_1 = f_2$ and $g_2 = f_1$ so that these circuits coincide, or, by 6.4.1, $\{g_1, g_2\} = \{f_3, f_4\}$. We will show that in the former case (iii) or (iv) holds, whereas in the latter case (ii) holds.

6.4.2. *$M/f_1\backslash f_2\backslash f_5$ and $M/f_2\backslash f_1\backslash f_5$ have N-minors.*

Subproof. First suppose that (a) holds. Since $M\backslash d\backslash f_5$ has an N-minor and $M\backslash d\backslash f_5/f_1/f_3$ is connected, Lemma 2.17 implies that $M\backslash f_5/f_1/f_3$ has an N-minor. Since $\{f_2, f_4\}$ is a parallel pair in this matroid, $M/f_1\backslash f_2\backslash f_5$ has an N-minor, up to an N-label switch. Similarly, $M\backslash f_5/f_2/f_3$ has an N-minor, and, up to an N-label switch, $M/f_2\backslash f_1\backslash f_5$ has an N-minor.

Now suppose (b) holds. Recall that either $\{f_i, f_4, f_5, d\}$ and $\{f_{i'}, f_3, f_5, d\}$ are circuits for some $\{i, i'\} = \{1, 2\}$, or $\{f_1, f_2, f_5, d\}$ is a circuit. Assume the former. Since $M/f_{i'}/f_3$ has an N-minor, and $\{f_i, f_4\}$ and $\{f_5, d\}$ are parallel pairs in this matroid, $M/f_{i'}\backslash f_i\backslash f_5$ has an N-minor. Moreover, $M\backslash d\backslash f_3$ has an N-minor, where $\{f_i, f_{i'}\}$ and $\{f_4, f_5\}$ are series pairs in this matroid, so $M\backslash f_3/f_i/f_4$ has an N-minor. But $\{f_5, d\}$ is a parallel pair in this matroid, so $M\backslash f_5/f_i/f_4$ has an N-minor. Now $\{f_{i'}, f_3\}$ is a parallel pair in this matroid, so $M/f_i\backslash f_{i'}\backslash f_5$ has an N-minor as required.

Now we assume that $\{f_1, f_2, f_5, d\}$ is a circuit. Since, for any $\{i, i'\} = \{1, 2\}$, the matroid $M/f_i/f_{i'}$ has an N-minor, and $\{f_3, f_4\}$ and $\{f_5, d\}$ are parallel pairs in this matroid, $M/f_{i'}\backslash f_4\backslash f_5$ has an N-minor. Since $\{f_3, d\}$ is a series pair in this matroid, $M/f_{i'}/f_3\backslash f_5$ has an N-minor. Now $\{f_i, f_4\}$ is a parallel pair in this matroid, so $M/f_{i'}\backslash f_i\backslash f_5$ has an N-minor as required. $\lhd$

6.4.3. *Either $M/f_1\backslash f_2\backslash f_5$ or $M/f_2\backslash f_1\backslash f_5$ is 3-connected.*

Subproof. Let $\{i,j\} = \{1,2\}$. We start by showing that either $M/f_i\backslash f_j\backslash f_5$ is 3-connected, or there is a 4-element cocircuit $\{f_j, f_j', f_5, h_j\}$ where $f_j' \in \{f_3, f_4\}$ and $h_j \in E(M) - (F \cup d)$. Consider the 3-connected matroid M/f_i. Observe that $r_{M/f_i}((F - f_i) \cup d) = 3$. Since $r_M(F \cup d) = 4$, it follows that $\{f_i, f_3, f_4, d\}$ is independent in M. So $\{f_3, f_4, f_5, d\}$ is a rank-3 cocircuit in M/f_i, with $f_5 \in \mathrm{cl}_{M/f_i}(\{f_3, f_4, d\})$. Thus, by Lemma 2.12, $\mathrm{co}(M/f_i\backslash f_5)$, and indeed $M/f_i\backslash f_5$, is 3-connected. Now $(\{f_3, f_4, d\}, \{f_j\}, E(M) - (F \cup d))$ is a vertical 3-separation in $M/f_i\backslash f_5$. By Bixby's Lemma, $\mathrm{co}(M/f_i\backslash f_j\backslash f_5)$ is 3-connected. So $M/f_i\backslash f_j\backslash f_5$ is 3-connected unless f_j is in a triad of $M/f_i\backslash f_5$ that meets both $\{f_3, f_4, d\}$ and $E(M) - (F \cup d)$. By orthogonality, this triad does not contain d. So $\{f_j, f_j', f_5, h_j\}$ is a cocircuit of M where $f_j' \in \{f_3, f_4\}$ and $h_j \in E(M) - (F \cup d)$, as claimed.

Suppose neither $M/f_2\backslash f_1\backslash f_5$ nor $M/f_1\backslash f_2\backslash f_5$ is 3-connected. Then $\{f_1, f_1', f_5, h_1\}$ and $\{f_2, f_2', f_5, h_2\}$ are cocircuits, where $f_1', f_2' \in \{f_3, f_4\}$.

Recall that $M/f_i\backslash f_j\backslash f_5$ has an N-minor when $\{i,j\} = \{1,2\}$. Since $\{f_j', h_j\}$ is a series pair in this matroid, it follows that $M/f_i/h_j$ has an N-minor.

Next, we claim that either $M\backslash d/f_1/h_2$ or $M\backslash d/f_2/h_1$ is 3-connected. Suppose not, so $M\backslash d/f_i/h_j$ is not 3-connected for $\{i,j\} = \{1,2\}$. Observe that $(F - f_i, \{h_j\}, E(M) - (F \cup \{d, h_j\}))$ is a cyclic 3-separation of $M\backslash d/f_i$, so $\mathrm{si}(M\backslash d/f_i/h_j)$ is 3-connected, by Bixby's Lemma. Thus $M\backslash d/f_i/h_j$ contains a parallel pair, implying that $\{f_i, h_j\}$ is contained in a 4-element circuit in $M\backslash d$ that, by orthogonality, intersects $\{f_1, f_2, f_3\}$ in two elements. But if f_3 is in this circuit, then it also meets $\{f_4, f_5\}$, by orthogonality, in which case $h_j \in \mathrm{cl}(F)$; a contradiction. We deduce that $\{f_1, f_2, h_j, q_j\}$ is a circuit for some $q_j \in E(M) - (F \cup d)$.

Now $\{f_1, f_2, h_1, q_1\}$ and $\{f_1, f_2, h_2, q_2\}$ are both circuits, so $r(\{f_1, h_1, h_2, q_1, q_2\}) \leq 4$. Since $f_1 \in \mathrm{cl}^*(\{f_2, f_3, d\})$, the set $\{h_1, h_2, q_1, q_2\}$ contains a circuit. But such a circuit intersects one of the cocircuits $\{f_1, f_5, f_1', h_1\}$ or $\{f_2, f_5, f_2', h_2\}$ in a single element, contradicting orthogonality.

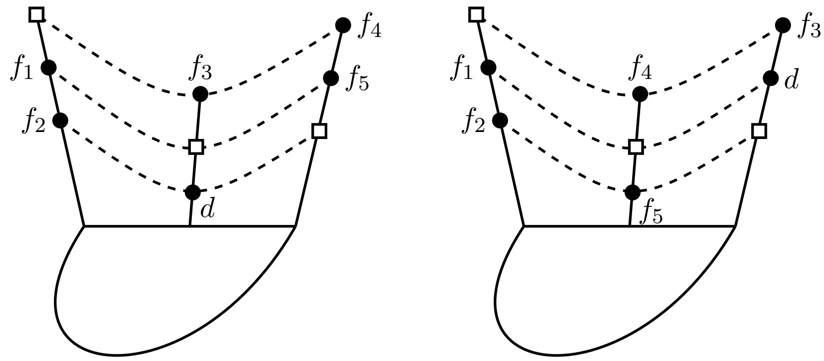
Up to labels, we may now assume that $M\backslash d/f_1/h_2$ is 3-connected. So either $\{f_1, h_2\}$ is an N-detachable pair, contradictory to the assumption that (i) does not hold, or there is a 4-element circuit of M containing $\{d, f_1, h_2\}$. By orthogonality, such a circuit meets $\{f_3, f_4, f_5\}$. So $\{d, f_1, f_\ell, h_2\}$ is a circuit, for $\ell \in \{3,4,5\}$. But then $h_2 \in \mathrm{cl}(F \cup d) \cap \mathrm{cl}^*(F \cup d)$ where $F \cup d$ is exactly 3-separating; a contradiction.

Thus $M/f_1\backslash f_2\backslash f_5$ or $M/f_2\backslash f_1\backslash f_5$ is 3-connected as required. $\triangleleft$

Now 6.4.2 and 6.4.3, together with the assumption that M has no N-detachable pairs, implies that M has a 4-element cocircuit $\{f_1, f_2, f_5, z\}$.

6.4.4. *If $z \notin F$, then $\{f_3, z\}$ is an N-detachable pair.*

Subproof. First we show that $M/f_3/z$ has an N-minor. Suppose (a) holds. Since $M\backslash d\backslash f_5$ has an N-minor and $M\backslash d\backslash f_5/f_2/f_3$ is connected, Lemma 2.17



(a) When $g_1 = f_3$. (b) When $g_1 = f_4$.

FIGURE 5. The two possible labellings of the skew-whiff 3-separator when Proposition 6.4(ii) holds.

implies that $M\backslash f_5/f_2/f_3$ has an N-minor. Since $\{f_1, f_4\}$ is a parallel pair in this matroid, $M/f_3\backslash f_1\backslash f_5$ has an N-minor, up to an N-label switch. Now suppose (b) holds. Since $M/f_1/f_2$ has an N-minor, and $\{f_3, f_4\}$ and $\{f_5, d\}$ are parallel pairs in this matroid, $M/f_2\backslash f_4\backslash f_5$ has an N-minor. Since $\{f_3, d\}$ is a series pair in this matroid, $M/f_2/f_3\backslash f_5$ has an N-minor. Now $\{f_1, f_4\}$ is a parallel pair in this matroid, so $M/f_3\backslash f_1\backslash f_5$ has an N-minor. So in either case $M/f_3\backslash f_1\backslash f_5$ has an N-minor. Now $\{f_1, f_2, f_5, z\}$ is a cocircuit of M, so $\{f_2, z\}$ is a series pair in $M/f_3\backslash f_1\backslash f_5$. It follows that $M/f_3/z$ has an N-minor as required.

Next we show that $\mathrm{si}(M/f_3/z)$ is 3-connected. Evidently, $z \in \mathrm{cl}^*(F \cup d)$, where $F \cup d$ is exactly 3-separating, so $z \notin \mathrm{cl}(F \cup d)$, by Lemma 2.6, implying $z \in \mathrm{cl}^*(E(M) - (F \cup \{d, z\}))$, by Lemma 2.2. Note that M/f_3 is 3-connected by Lemma 6.1(i). Now $((F - f_3) \cup d, \{z\}, E(M) - (F \cup \{d, z\}))$ is a cyclic 3-separation in M/f_3. It follows that $\mathrm{co}(M/f_3\backslash z)$ is not 3-connected, so $\mathrm{si}(M/f_3/z)$ is 3-connected by Bixby's Lemma, as required.

Now, if $M/f_3/z$ is not 3-connected, then M has a 4-element circuit containing $\{f_3, z\}$. By orthogonality, such a circuit C intersects the cocircuits $\{f_1, f_2, f_5, z\}$, $\{f_1, f_2, f_3, d\}$, and $\{f_3, f_4, f_5, d\}$ in at least two elements. So $C \subseteq F \cup \{d, z\}$. But then $z \in \mathrm{cl}(F \cup d)$; a contradiction. We deduce that $M/f_3/z$ is 3-connected. $\lhd$

By 6.4.4, we may now assume that $z \in \{f_3, f_4\}$. Since $\{f_1, f_2, f_3, d\}$ is a cocircuit of M, it follows from 6.4.1 that $z = f_4$ so that $\{f_1, f_2, f_4, f_5\}$ is a cocircuit. Now we examine the potential configurations of the 4-element circuits $\{f_1, f_5, d, g_1\}$ and $\{f_2, f_5, d, g_2\}$, each of which is contained in $F \cup d$. If $g_1 = f_3$, then $g_2 = f_4$ due to 6.4.1. In this situation, it is easily checked that $F \cup d$ is a skew-whiff 3-separator of M, so that (ii) holds, as illustrated in Figure 5(a). Similarly, if $g_1 = f_4$, we obtain a skew-whiff 3-separator as shown in Figure 5(b).

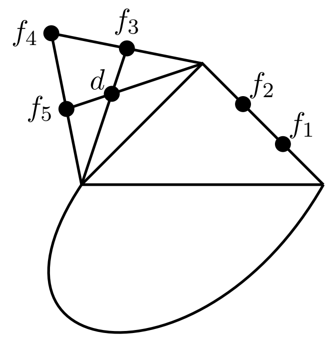


FIGURE 6. The labelling of the elongated-quad 3-separator when Proposition 6.4(iii) holds.

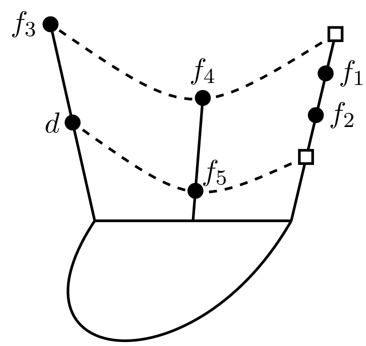


FIGURE 7. The labelling of the twisted cube-like 3-separator of M^* when Proposition 6.4(iv) holds.

The final possibilities arise when $g_1 = f_2$. In this case, 6.4.1 forces $g_2 = f_1$. First, suppose that $\{f_3, f_4, f_5, d\}$ is a circuit. Then $F \cup d$ is an elongated-quad 3-separator with associated partition $(\{f_3, f_4, f_5, d\}, \{f_1, f_2\})$, as illustrated in Figure 6; so (iii) holds. We may now assume $\{f_3, f_4, f_5, d\}$ is independent. Then, since $r(F \cup d) = 4$, the element f_1 (respectively, f_2) is in a circuit contained in $\{f_1, f_3, f_4, f_5, d\}$ (respectively, $\{f_2, f_3, f_4, f_5, d\}$). Since $\{f_1, f_2, f_5, d\}$ is also a circuit contained in $F \cup d$, 6.4.1 implies that these circuits contain $\{f_3, f_4\}$. Similarly, due to the circuit $\{f_1, f_2, f_3, f_4\}$, 6.4.1 implies that these circuits contain $\{f_5, d\}$. So $\{f_1, f_3, f_4, f_5, d\}$ and $\{f_2, f_3, f_4, f_5, d\}$ are circuits. It follows that $F \cup d$ is a twisted cube-like 3-separator in M^*, so (iv) holds. The labelling of the twisted cube-like 3-separator in the dual is illustrated in Figure 7. □

By combining Propositions 6.2 to 6.4, we obtain the following:

Corollary 6.5. *Let M be a 3-connected matroid with a 3-connected matroid N as a minor, where $|E(N)| \geq 4$, and every triangle or triad of M is*

N-grounded. Let d be an element of M such that $M\backslash d$ is 3-connected and has a flan F with ordering $(f_1, f_2, \ldots, f_t)$ where $t \geq 5$. Suppose that $M\backslash d\backslash f_5$ has an N-minor with $|\{f_1, \ldots, f_4\} \cap E(N)| \leq 1$. Then either

(i) *M has an N-detachable pair, or*
(ii) *$F \cup d$ is either a skew-whiff 3-separator of M, an elongated-quad 3-separator of M, or a twisted cube-like 3-separator of M^*.*

Proof. If $t \geq 6$, then (i) holds by Proposition 6.2. So suppose $t = 5$ and F is a maximal flan. First, suppose that d fully blocks F. Towards an application of Proposition 6.3, we claim that f_3 is N-deletable in $M\backslash d$. Observe that $(F - f_5, \{f_5\}, E(M\backslash d) - F)$ is a cyclic 3-separation of $M\backslash d$. Since $F - f_5$ is a circuit, Lemma 2.18(i) implies that $M\backslash d\backslash f_3$ has an N-minor, as claimed. Now, by Proposition 6.3, we may assume that d does not fully block F. Then $d \in \mathrm{cl}(F)$, and so, by Proposition 6.4, the corollary follows. □

7. Unveiling the 3-separating set X

In this section, we prove our main result, Theorem 7.4. For the entirety of the section, we work under the following setting. Let M be a 3-connected matroid and let N be a 3-connected minor of M where $|E(N)| \geq 4$, and every triangle or triad of M is N-grounded. Suppose, for some $d \in E(M)$, that $M\backslash d$ is 3-connected and has a cyclic 3-separation $(Y, \{d'\}, Z)$ with $|Y| \geq 4$, where $M\backslash d\backslash d'$ has an N-minor with $|Y \cap E(N)| \leq 1$.

First, we handle the cases where Y contains either a 4-element cosegment, or a particular configuration of two triads.

Lemma 7.1. *If Y contains a 4-element cosegment of $M\backslash d$, then M has an N-detachable pair.*

Proof. Suppose X is a 4-element cosegment of $M\backslash d$ contained in Y. If $X \subseteq \mathrm{cl}^*(Z)$, then $X \cup d'$ is a cosegment in $M\backslash d$. Since $M\backslash d\backslash d'$ has an N-minor, neither d nor d' is in a triad of M, and any pair of elements in $\mathrm{cl}^*_{M\backslash d\backslash d'}(X)$ is N-contractible. In particular, there are no triads of M contained in $\mathrm{cl}^*(X \cup \{d, d'\})$, and so $M^*|(X \cup \{d, d'\}) \cong U_{3,6}$. Now, by Lemma 4.4, M has an N-detachable pair.

So we may assume that $|X \cap \mathrm{cl}^*(Z)| \leq 1$. Let $x \in X$, where $x \in \mathrm{cl}^*(Z)$ if such an element exists. Since $x' \in \mathrm{cl}^*(X - \{x, x'\})$ for each $x' \in X - x$, we have $x' \notin \mathrm{cl}(\mathrm{cl}^*(Z))$. Thus, by Lemma 2.18, each $x' \in X - x$ is doubly N-labelled in $M\backslash d$. As d blocks every triad of $M\backslash d$ contained in X, the set $X \cup d$ is a 5-element coplane in M. If d does not fully block X, then $d \in \mathrm{cl}(X)$, in which case $X \cup d$ is 3-separating in M, and M has an N-detachable pair by the dual of Proposition 4.6. So we may assume that d fully blocks X.

Let $p' \in X - x$. Towards an application of Proposition 4.7, we claim that for distinct elements $u, v \in \mathrm{cl}(X) - X$, either $M/p'/u$ or $M/p'/v$ has an N-minor. Recall that $M\backslash d/p'$ has an N-minor. By the dual of Lemma 2.8, $M\backslash d/p'$ is 3-connected. Now $(Y - p', \{d'\}, Z)$ is a path of 3-separations in

$M\backslash d/p'$. Let $Z' = \mathrm{cl}^*_{M\backslash d}(Z) - d'$ and $Y' = Y - Z'$. Then $(Y' - p', \{d'\}, Z')$ is a path of 3-separations of $M\backslash d/p'$ where $Z' \cup d'$ is coclosed, and $X - x \subseteq Y'$. Note that $Y' - p'$ contains a circuit in $M\backslash d/p'$, since Y' contains a circuit in $M\backslash d$. In order to show that $(Y' - p', \{d'\}, Z')$ is a cyclic 3-separation of $M\backslash d/p'$, it remains only to observe that $d' \in \mathrm{cl}^*_{M\backslash d}(Y' - p')$, which follows from the fact that $p' \in \mathrm{cl}^*_{M\backslash d}(X - \{x, p'\})$.

Suppose there are distinct elements $u, v \in \mathrm{cl}(X) - X$. Then $\{u, v\} \subseteq \mathrm{cl}(Y)$. If $\{u, v\} \subseteq Y - p'$, then either $M/p'/u$ or $M/p'/v$ is 3-connected by Lemma 2.18(ii). Moreover, if $\{u, v\} \cap Z \neq \emptyset$, then, since $d' \in \mathrm{cl}^*_{M\backslash d}(Y) \cap Z$, Lemma 2.13 implies that $\{u, v\} - Z \neq \emptyset$. So suppose, without loss of generality, that $v \in Z$ and $u \in Y$. By Lemma 2.18(ii) again, the claim holds unless $u \in \mathrm{cl}_{M/p'}(Z')$. But then it follows that $Y - \{p', u\}$ is exactly 3-separating in $M\backslash d/p'$, with $\{u, v\} \subseteq \mathrm{cl}_{M/p'}(Y - \{p', u\}) \cap Z$ and $d' \in \mathrm{cl}^*(Y - \{p', u\}) \cap Z$, contradicting Lemma 2.13.

Now M has an N-detachable pair by the dual of Proposition 4.7. □

Lemma 7.2. *Suppose that Y contains a set $X = \{s_1, s_2, t_1, t_2, u\}$ such that the following hold:*

(a) *$\{s_1, s_2, u\}$ and $\{t_1, t_2, u\}$ are triads of $M\backslash d$;*
(b) *M has a $U_{3,6}$ restriction contained in $\mathrm{cl}(X \cup d) - (X \cup d)$, containing distinct elements v_3 and w_3; and*
(c) *$\{s_1, u, d, v_3\}$ and $\{s_2, u, d, w_3\}$ are circuits of M.*

Then M contains an N-detachable pair.

Proof. We show that $\{d, v_3\}$ is an N-detachable pair. Since $\{s_1, t_3, d, v_3\}$ and $\{s_2, t_3, d, w_3\}$ are circuits, $\{s_1, s_2, t_3, v_3, w_3\}$ contains a circuit, by circuit elimination. But since $t_3 \in \mathrm{cl}^*(\{d, t_1, t_2\})$, Lemma 2.2 implies that $\{s_1, s_2, v_3, w_3\}$ is a circuit. Note that if $d' \in \mathrm{cl}^*_{M\backslash d}(\{t_1, t_2, t_3\})$, then $\{d', t_1, t_2, t_3\}$ is a cosegment in $M\backslash d$ whose triads are blocked by d, so $\{d, d', t_1, t_2, t_3\}$ is a 5-element plane in M^*. But then, by the dual of Proposition 4.6, M has an N-detachable pair. So we may assume that $d' \notin \mathrm{cl}^*_{M\backslash d}(\{t_1, t_2, t_3\})$. Now $M\backslash d/s_1$ is 3-connected by the dual of Lemma 2.12, and $M\backslash d\backslash d'/s_1$ has an N-minor by Lemma 2.17. Applying Lemma 2.18(ii), we deduce that $M\backslash d/s_1/s_2$ has an N-minor, since $d' \notin \mathrm{cl}^*_{M\backslash d/s_1}(X - \{s_1, s_2\})$. As $\{v_3, w_3\}$ is a parallel pair in $M\backslash d/s_1/s_2$, the pair $\{d, v_3\}$ is N-deletable, up to switching N-labels. By Lemma 4.1, $M\backslash d\backslash v_3$ is 3-connected, so $\{d, v_3\}$ is an N-detachable pair. □

Lemma 7.3. *Suppose that Y contains a set $X = \{s_1, s_2, t_1, t_2, u\}$ such that the following hold:*

(a) *$\{s_1, s_2, u\}$ and $\{t_1, t_2, u\}$ are triads of $M\backslash d$,*
(b) *X is closed in $M\backslash d$,*
(c) *X is 3-separating in $M\backslash d$,*
(d) *X is not a cosegment in $M\backslash d$, and*
(e) *there are no 4-element circuits contained in X.*

Then M contains an N-detachable pair.

Proof. Since X is the union of two triads that meet at u, but X is not a cosegment, $r^*_{M\backslash d}(X) = 3$. As X is a 5-element 3-separating set, $r_{M\backslash d}(X) = 4$. It follows that $E(M\backslash d) - X$ is coclosed, due to (e). Since $x \in \mathrm{cl}^*_{M\backslash d}(X - x)$ for each $x \in X$, we also have that $E(M\backslash d) - X$ is closed. Hence each element in X is doubly N-labelled in $M\backslash d$ by Lemma 2.18. It follows that each $x \in X$ is not contained in an N-grounded triangle or triad.

Assume that M does not contain an N-detachable pair.

7.3.1. *For distinct $s \in \{s_1, s_2, u\}$ and $t \in \{t_1, t_2, u\}$, the matroid $M\backslash d/s/t$ is 3-connected and has an N-minor.*

Subproof. Let $s \in \{s_1, s_2\}$ and $t \in \{t_1, t_2, u\}$. Since X is a corank-3 circuit, and s is not contained in a triangle, the dual of Lemma 2.12 implies that $M\backslash d/s$ is 3-connected. Moreover, $X - s$ is a corank-3 circuit in $M\backslash d/s$, so $M\backslash d/s/t$ is 3-connected unless $\{s, t\}$ is contained in a 4-element circuit of $M\backslash d$. But, by orthogonality, such a circuit contains another element of X, and so, as X is closed in $M\backslash d$, the circuit is contained in X; a contradiction. It follows by symmetry that $M\backslash d/s/t$ is 3-connected.

It remains to show that $M\backslash d/s/t$ has an N-minor. By swapping the labels on $\{s_1, s_2\}$ and $\{t_1, t_2\}$, if necessary, we may assume that $s \neq u$. Recall that $M\backslash d/s$ has an N-minor. Now $M\backslash d/s$ is 3-connected by the dual of Lemma 2.12, and $M\backslash d\backslash d'/s$ has an N-minor by Lemma 2.17. Applying Lemma 2.18(ii), we deduce that $M\backslash d/s/t$ has an N-minor, since $t \in \mathrm{cl}^*_{M\backslash d}(\{t_1, t_2, u\} - t)$, so $t \notin \mathrm{cl}(E(M) - \{t_1, t_2, u\})$. $\lhd$

Now, as M has no N-detachable pairs, 7.3.1 implies that for each distinct pair s, t with $s \in \{s_1, s_2, u\}$ and $t \in \{t_1, t_2, u\}$, there is a circuit of M containing $\{d, s, t\}$.

7.3.2. *There are no 4-element circuits of M contained in $X \cup d$.*

Subproof. Suppose $X \cup d$ contains a 4-element circuit C. Then $d \in C$, by (e). Let $S = \{s, s'\} \in \{\{s_1, s_2\}, \{t_1, t_2\}\}$, and $T = \{t, t', t''\} = X - S$. We may assume, without loss of generality, that $C = \{d, s, t, x\}$, where $x \in X - \{s, t, t'\}$. Now $\{d, s, t'\}$ is also contained in a 4-element circuit, $\{d, s, t', y\}$ say. By circuit elimination, $\{s, t, t', x, y\}$ contains a circuit. By (e), $\{s, t, t', x\}$ is independent, so (b) implies that $y \in X$, and $y \notin \{s, t, t', x\}$, and thus $\{x, y\} = \{s', t''\}$.

If $x = t''$, then $\{d, s, t, t''\}$ and $\{d, s, t', s'\}$ are circuits, but there is also a 4-element circuit containing $\{d, s', t\}$. So let $\{d, s', t, z\}$ be a circuit, for some z. Now $\{s, t, t'', s', z\}$ contains a circuit, by circuit elimination, and it follows, by (b) and (e), that $z \in X - \{s, s', t, t''\}$, so $z = t'$. But then circuit elimination on the circuits $\{d, s, t', s'\}$ and $\{d, s', t, t'\}$ implies that $\{s, s', t, t'\}$ contains a circuit; a contradiction. The argument is similar when $x = s'$. $\lhd$

Let $W = E(M\backslash d) - X$. Now, letting $t_3 = u$, for each $i \in [3]$ there are elements $v_i, w_i \in W \cap \mathrm{cl}(X \cup d)$ such that $\{s_1, t_i, d, v_i\}$ and $\{s_2, t_i, d, w_i\}$ are circuits. Observe also that $d \notin \mathrm{cl}(X)$, since $v_i, w_i \notin \mathrm{cl}(X)$.

7.3.3. *M has a $U_{3,6}$ restriction contained in $W \cap \mathrm{cl}(X \cup d)$.*

Subproof. If $v_i = v_{i'}$ for distinct $i, i' \in [3]$, then $\{s_1, t_i, t_{i'}, d\}$ contains a circuit, by the circuit elimination axiom, contradicting 7.3.2. Similarly, the w_i are pairwise distinct for $i \in [3]$. Say $v_i = w_j$ for some $i, j \in [3]$. Then, again by circuit elimination, there is a circuit $\{s_1, s_2, t_i, t_j, v_i\}$. But X is closed in $M\backslash d$, so $v_i \notin \mathrm{cl}(\{s_1, s_2, t_i, t_j\})$. Hence $\{s_1, s_2, t_i, t_j\}$ is a circuit of M, contradicting (e). Hence the elements $v_1, v_2, v_3, w_1, w_2, w_3$ are pairwise distinct. Now $W \cap \mathrm{cl}(X \cup d)$ has rank at most 3. If $r(\{v_1, v_2, v_3\}) \leq 2$, then $\{s_1, d, v_1, v_2, v_3\}$ has rank at most four, but spans the rank-5 set $X \cup d$; a contradiction. We deduce that $M|\{v_1, v_2, v_3, w_1, w_2, w_3\} \cong U_{3,6}$. $\triangleleft$

Now M has an N-detachable pair by Lemma 7.2, thus completing the proof. $\square$

Finally, we come to the main result of this paper.

Theorem 7.4. *Suppose M has no N-detachable pairs. Then there is a subset X of Y such that*

(i) $|X| \geq 4$ *and X is 3-separating in $M\backslash d$, and*

(ii) *either*

 (a) *$X \cup \{c, d\}$ is an elongated-quad 3-separator of M, a skew-whiff 3-separator of M, or a twisted cube-like 3-separator of M^*, for some $c \in \mathrm{cl}^*_{M\backslash d}(X) - X$; or*

 (b) *for every $x \in X$,*

  (I) $\mathrm{co}(M\backslash d\backslash x)$ *is 3-connected,*

  (II) $M\backslash d/x$ *is 3-connected, and*

  (III) *x is doubly N-labelled in $M\backslash d$.*

Proof. Choose $X \subseteq Y$ that is minimal with respect to (i). Let $W = E(M\backslash d) - X$. Suppose that (a) does not hold; then, it remains to show that (b) holds. By Lemma 7.1, we may assume that X is not a cosegment of $M\backslash d$.

7.4.1. *Every element in $Y \cup d'$ is N-deletable in $M\backslash d$, and every element in X is doubly N-labelled in $M\backslash d$.*

Subproof. If there is some element $x \in X \cap \mathrm{cl}^*_{M\backslash d}(Z)$, then $X - x$ is 3-separating, by Lemma 2.3. If $|X| > 4$, this contradicts the minimality of X. On the other hand, if $|X| = 4$, then $X - x$ is a triad, since $X - x$ cannot be an N-grounded triangle by Lemma 2.18(ii). But then X is a 4-element cosegment, contradicting Lemma 7.1.

Now we may assume that $Z \cup d'$ is coclosed in $M\backslash d$. By Lemma 2.18(i), every element in Y is N-deletable, while d' is N-deletable by hypothesis. If there is some element $x \in X$ that is not N-contractible, then $x \in \mathrm{cl}(Z)$ by Lemma 2.18(ii). Then, the minimality of X implies that $|X| = 4$.

Since $X - x$ is not an N-grounded triangle, $X - x$ is a triad, and X is a circuit. Moreover, $(X - x, \{x\}, W)$ is a vertical 3-separation, so $\mathrm{co}(M\backslash d\backslash x)$ is 3-connected by Bixby's Lemma. Since M has no N-detachable pairs, x is in a triad of $M\backslash d$ that meets $X - x$ and W. Let this triad be $\{x', x, w\}$ where $x' \in X - x$. Since $M\backslash d\backslash x'$ has an N-minor, and $\{x, w\}$ is a series pair in this matroid, up to an N-label switch the matroid $M\backslash d/x$ has an N-minor after all, thus completing the proof of 7.4.1. $\lhd$

Note, in particular, that no triangle meets X.

7.4.2. *If $|X| = 4$ and $X \cup f_5$ is a flan for some $f_5 \in W$, with flan ordering $(f_1, f_2, f_3, f_4, f_5)$ for some labelling $\{f_1, f_2, f_3, f_4\}$ of X, then either M has an N-detachable pair, or* (a) *holds.*

Subproof. Suppose f_5 is N-deletable in $M\backslash d$. Then, by Corollary 6.5, either M has an N-detachable pair, or (a) holds. So we may assume that f_5 is not N-deletable in $M\backslash d$. By 7.4.1, $f_5 \in Z$. Now $(Y \cup f_5, \{d'\}, Z - f_5)$ is a path of 3-separations in $M\backslash d$, by Lemma 2.3. By Lemmas 2.2 and 2.4, $d' \in \mathrm{cl}^*_{M\backslash d}(Z - f_5)$. Moreover, $Z - f_5$ contains a circuit, since Z contains a circuit and $f_5 \notin \mathrm{cl}(Z - f_5)$, so this path of 3-separations is a cyclic 3-separation, and $|(Y \cup f_5) \cap E(N)| \le 1$, by Lemma 2.17 and since $|Y \cap E(N)| \le 1$ and $|E(N)| \ge 4$.

Suppose there is some $f_6 \in \mathrm{cl}(X \cup f_5) \cap (W - f_5)$ so that $X \cup \{f_5, f_6\}$ is a flan. Now $(Y \cup \{f_5, f_6\}, \{d'\}, Z - \{f_5, f_6\})$ is a path of 3-separations where d' is a coguts element, using a similar argument as in the previous paragraph. To show this is a cyclic 3-separation, we now require only that $r^*_{M\backslash d}(Z - \{f_5, f_6\}) \ge 3$. Suppose not. Since $M\backslash d\backslash d'$ has an N-minor with $|(Y \cup f_5) \cap E(N)| \le 1$, Lemma 2.17 implies that $|(Y \cup \{f_5, f_6\}) \cap E(N)| \le 1$. But now $r^*_{M\backslash d\backslash d'}(Z - \{f_5, f_6\}) \le 1$; a contradiction. By Lemma 2.18(ii), since f_5 is not N-deletable we have $f_5 \in \mathrm{cl}^*(Z - \{f_5, f_6\})$. But $f_6 \in \mathrm{cl}(Z - \{f_5, f_6\})$ and $d' \in \mathrm{cl}^*(Z - \{f_5, f_6\})$, contradicting Lemma 2.13. So $X \cup f_5$ is a maximal flan.

Note that $M\backslash d\backslash f_3$ has an N-minor, by 7.4.1. If d fully blocks $X \cup f_5$, then, by Proposition 6.3, M has an N-detachable pair. Towards an application of Proposition 6.4, we show that $M/f_i/f_{i'}$ has an N-minor for all distinct $i, i' \in [3]$. Let $i \in \{1, 2\}$. By Lemma 6.1 and 7.4.1, $M\backslash d/f_i$ is 3-connected and has an N-minor. Now $((Y - f_i) \cup f_5, \{d'\}, Z - f_5)$ is a cyclic 3-separation in $M\backslash d/f_i$. Since $\{f_3, f_4, f_5\}$ is a triad in $M\backslash d$, we have $f_3 \notin \mathrm{cl}(Z - f_5)$, so $M\backslash d/f_i/f_3$ has an N-minor by Lemma 2.18(ii). Now, $\{f_5, d'\} \subseteq \mathrm{cl}^*_{M\backslash d/f_i}(Z - f_5)$, so no element in $(Y - f_i) \cup f_5$ is also in $\mathrm{cl}_{M\backslash d/f_i}(Z - f_5)$ by Lemma 2.13. Hence $M\backslash d/f_1/f_2$ also has an N-minor by Lemma 2.18(ii). Now, by Proposition 6.4, either M has an N-detachable pair or (a) holds, thus completing the proof. $\lhd$

Next we prove that (I) holds for each $x \in X$. Towards a contradiction, let x be an element of X such that $\mathrm{co}(M \backslash d \backslash x)$ is not 3-connected, and let $(P, \{x\}, Q)$ be a cyclic 3-separation of $M \backslash d$.

7.4.3. *$W \cap P \neq \emptyset$ and $W \cap Q \neq \emptyset$.*

Subproof. Suppose that $W \cap Q = \emptyset$. Then $Q \cup x \subseteq X$ and $|Q| \geq 3$. But Q and $Q \cup x$ are 3-separating, so the minimality of X implies that $X = Q \cup x$ and $|Q| = 3$. Since Q contains a circuit, Q is a triangle of $M \backslash d$, and hence of M. But, by 7.4.1, Q is not N-grounded; a contradiction. So $W \cap Q$ and, by symmetry, $W \cap P$ are non-empty. $\triangleleft$

7.4.4. *Up to swapping P and Q, $|X \cap Q| = 2$ and $|W \cap P| \geq 2$.*

Subproof. Since $|W| \geq 3$, we may assume that $|W \cap P| \geq 2$. By uncrossing, $X \cap Q$ and $(X \cap Q) \cup x$ are 3-separating in $M \backslash d$. If $|X \cap Q| \leq 1$, then $|W \cap Q| \geq 2$, in which case $X \cap P$ and $(X \cap P) \cup x$ are also 3-separating in $M \backslash d$, by uncrossing. By the minimality of X, it follows that $|X| = 4$, so either $X \cap Q = \emptyset$ and $|X \cap P| = 3$, or $|X \cap Q| = 1$ and $|X \cap P| = 2$. In the first case, $X - x$ is a triad, since it cannot be an N-grounded triangle, so X is a 4-element cosegment, contradicting Lemma 7.1. In the latter case, 7.4.4 holds after swapping P and Q. On the other hand, if $|X \cap Q| > 2$, then the minimality of X implies that $X \cap P = \emptyset$. But then $X - x$ is a triad, so X is a 4-element cosegment, contradicting Lemma 7.1. $\triangleleft$

Now, note that if $|W \cap Q| = 1$, then Q is a triangle in $M \backslash d$, but Q is not an N-grounded triangle since, by 7.4.1, it contains an N-contractible element; a contradiction. So $|W \cap Q| \geq 2$.

7.4.5. *$|X \cap P| = 2$.*

Subproof. By uncrossing, $X \cap P$ and $(X \cap P) \cup x$ are 3-separating. If $|X \cap P| > 2$, then this contradicts the minimality of X. So assume that $X \cap P = \{t\}$, say. Now $X - t$ is a triad, and $t \in \mathrm{cl}^{(*)}(X - t)$. If $t \in \mathrm{cl}^*(X - t)$, then X is a 4-element cosegment, contradicting Lemma 7.1. So $t \in \mathrm{cl}(X - t)$. By the dual of Lemma 2.11, $\mathrm{co}(M \backslash d \backslash t)$ is 3-connected, so, as M has no N-detachable pairs, t is in a triad that, by orthogonality, meets $X - t$. If this triad does not contain x, then, by the dual of Lemma 2.11 again, $\mathrm{co}(M \backslash d \backslash x)$ is 3-connected; a contradiction. Let f_5 be the element of the triad in W, and let $X \cap Q = \{f_1, f_2\}$. Now X is contained in a 5-element flan with ordering (f_1, f_2, x, t, f_5). Thus, by 7.4.2, either M has an N-detachable pair or (a) holds; a contradiction. $\triangleleft$

7.4.6. *X is closed in $M \backslash d$.*

Subproof. Suppose $c \in \mathrm{cl}(X) - X$. We may assume that $c \in P$. Since $|W \cap Q| \geq 2$, both $X \cap P$ and $(X \cap P) \cup c$ are 3-separating, by uncrossing. So $c \in \mathrm{cl}(X \cap P)$, and $(X \cap P) \cup c$ is a triangle. Since this triangle contains an N-contractible element, it is not N-grounded, which is contradictory. $\triangleleft$

7.4.7. *X contains no 4-element circuits.*

Subproof. Let $X \cap P = \{p_1, p_2\}$ and $X \cap Q = \{q_1, q_2\}$. Suppose X has a 4-element circuit. Either this circuit contains x or it does not. Suppose that it does: without loss of generality, let $\{p_1, p_2, x, q_1\}$, be this circuit. Since $\{p_1, p_2, x\}$ is a triad, $\{p_1, p_2, x, q_1\}$ is 3-separating, contradicting the minimality of X. Now we may assume there is no 4-element circuit in X containing x. Thus $r(\{p_1, p_2, x, q_1, q_2\}) = 4$, and it follows, by Lemma 2.2, that $x \in \mathrm{cl}^*(W)$, so $X - x$ is 3-separating by Lemma 2.3, again contradicting the minimality of X. $\lhd$

Now, since 7.4.4–7.4.7 hold, we can apply Lemma 7.3 to deduce that M has an N-detachable pair; a contradiction. This proves that each $x \in X$ satisfies (I). Recall that each $x \in X$ satisfies (III) by 7.4.1.

It remains to consider (II). Suppose $M\backslash d/x$ is not 3-connected for some $x \in X$. Since x is not in a triangle, $\mathrm{si}(M\backslash d/x)$ is not 3-connected, so $M\backslash d$ has a vertical 3-separation $(P, \{x\}, Q)$. We may assume, without loss of generality, that $|W \cap P| \geq 2$. Thus, by uncrossing, both $X \cap Q$ and $(X \cap Q) \cup x$ are 3-separating. By the minimality of X, we have $|X \cap Q| \leq 3$, and if $|X \cap Q| = 3$, then $X \cap P = \emptyset$. If $|X \cap Q| = 2$, then $(X \cap Q) \cup x$ is a triangle or a triad, but as $x \in \mathrm{cl}(Q)$ and X contains no triangles, this leads to a contradiction.

Suppose $|X \cap Q| \leq 1$. Then $|W \cap Q| \geq 2$, in which case $X \cap P$ and $(X \cap P) \cup x$ are 3-separating, by uncrossing. Now $|X \cap P| \geq 2$, but, by the minimality of X, $|X \cap P| \leq 3$ and if $|X \cap P| = 3$ then $X \cap Q = \emptyset$. Moreover, $|X \cap P| \neq 2$ since $x \in \mathrm{cl}(X \cap P)$ and X does not contain any triangles. It follows that $X \cap Q = \emptyset$ and $|X \cap P| = 3$.

Now $\{|X \cap P|, |X \cap Q|\} = \{0, 3\}$, $X - x$ is a triad, and X is a circuit. Since $\mathrm{co}(M\backslash d\backslash x)$ is 3-connected, but $M\backslash d\backslash x$ is not, x is in a triad T^* of $M\backslash d$ that meets both $X - x$ and W, by orthogonality. Let $T^* \cap W = \{f_5\}$, and observe that $X \cup f_5$ is a 5-element flan of $M\backslash d$. By 7.4.2, either M has an N-detachable pair or (a) holds; a contradiction. So each $x \in X$ also satisfies (II), as required. $\square$

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