A Modal Proof Theory for Polynomial Coalgebras

by

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Abstract

The abstract mathematical structures known as coalgebras are of increasing interest in computer science for their use in modelling certain types of data structures and programs. Traditional algebraic methods describe objects in terms of their construction, whilst coalgebraic methods describe objects in terms of their decomposition, or observational behaviour. The latter techniques are particularly useful for modelling infinite data structures and providing semantics for object-oriented programming languages, such as Java.

There have been many different logics developed for reasoning about coalgebras of particular functors, most involving modal logic. We define a modal logic for coalgebras of polynomial functors, extending Rößiger's logic [33], whose proof theory was limited to using finite constant sets, by adding an operator from Goldblatt [11]. From the semantics we define a canonical coalgebra that provides a natural construction of a final coalgebra for the relevant functor. We then give an infinitary axiomatization and syntactic proof relation that is sound and complete for functors constructed from countable constant sets.

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Chapter 1

Introduction

Traditionally formal methods in computer science have used algebraic methods for describing data structures and programs. However there has recently been increasing interest in the use of coalgebraic methods, as they are able to be used in cases where algebraic methods are not. Algebra is a well-established area of mathematics and can be studied abstractly within universal algebra, and at an even more abstract level within category theory (see e.g. Mac Lane [26]). The notion of a coalgebra comes from category theory, as the formal dual of an algebra. As such, many algebraic concepts and results have dual coalgebraic counterparts — e.g. congruence and bisimulation equivalence, initial algebra and final coalgebra, induction and coinduction. These will be explored in further detail below.

1.1 From algebras to systems

The main conceptual difference between algebraic and coalgebraic methods is that the former represent objects in terms of their *construction* operations. In contrast coalgebraic methods represent objects in terms of the way in which they are *decomposed*, or *observed* to behave and change — using their destructor operations (also called observers, mutators, accessors

or transition maps).

To illustrate these concepts consider the case of (finite) lists, A^* , from a set A . A list is either the empty list, ε , or the concatenation of an element, $a \in A$, and a list l , written $a \cdot l$. Using the singleton set $1 = \{\star\}$ we can combine the construction functions

$$\begin{aligned} \text{empty} : 1 &\longrightarrow A^* : \star \longmapsto \varepsilon \\ \text{cons} : A \times A^* &\longrightarrow A^* : \langle a, l \rangle \longmapsto a \cdot l \end{aligned}$$

into a single entity that we call an algebra:

$$\langle A^*, \alpha_l : (1 + (A \times A^*)) \longrightarrow A^* \rangle$$

where the function α_l constructs a list object from the singleton or an element and list.

To compare, consider the case of streams (or infinite lists), A^ω , from A . Streams have no construction functions, but can be observed through their decomposition functions:

$$\begin{aligned} \text{head} : A^\omega &\longrightarrow A : a \cdot l \longrightarrow a \\ \text{tail} : A^\omega &\longrightarrow A^\omega : a \cdot l \longrightarrow l \end{aligned}$$

We can combine these in a similar fashion to form a structure we call a coalgebra:

$$\langle A^\omega, \alpha_s : A^\omega \longrightarrow (A \times A^\omega) \rangle$$

where the function α_s decomposes a stream in to an element and another stream.

Similarly, the set of finite and infinite lists from A , $A^\infty = A^* \cup A^\omega$, has no construction functions. Its decomposition functions include the same head and tail as before, with the addition of the empty list observation function:

$$\begin{aligned} \text{null} : A^\infty &\longrightarrow 1 : \varepsilon \longmapsto \star \\ \text{head} : A^\infty &\longrightarrow A : a \cdot l \longmapsto a \\ \text{tail} : A^\infty &\longrightarrow A^\infty : a \cdot l \longmapsto l \end{aligned}$$

These functions combine to form a coalgebra

$$\langle A^\infty, \alpha_i : A^\infty \longrightarrow (1 + (A \times A^\infty)) \rangle$$

where the function α_i decomposes a list as the singleton or an element and another (possibly infinite) list.

We have noted that the latter two examples have decomposition functions and no construction functions, and that the first example does have construction functions. Using the same reasoning as above, we can see that the case of lists A^* has decomposition functions too — in fact they are the same as for A^∞ : null, head and tail. Hence we can also form a coalgebra for A^* :

$$\langle A^*, \alpha'_i : A^* \longrightarrow (1 + (A \times A^*)) \rangle$$

It is useful to view coalgebras generically as state based systems, with a *state space*, A , and a transition structure, $\alpha : A \longrightarrow TA$, where T is some operation on the state space. T is known as a *functor* and is a functorial operation, in that it acts on functions between sets as well as on sets (i.e. if $f : B \longrightarrow C$ is a function then so is $Tf : TB \longrightarrow TC$). Formal definitions will be given in Chapter 2, but for now consider this informal definition from Pattinson [30] that a state based system meets the following conditions.

- (1) The behaviour of the system depends on an internal state, which is invisible to the user of the system.
- (2) The system is reactive, i.e. not necessarily terminating, and interacts with its environment.
- (3) The system comes with a set of operations, through which this interaction takes place.

The systems that we consider use polynomial functors, which correspond to deterministic systems with various combinations of input and output, and cover most applications in computer science. They are of the form

Name	Symbol	System actions
Identity	Id	state transition
Constant	$\overline{D}$	output from set D
Power	T^D	input from set D before T
Product	$T_1 \times T_2$	T_1 “and” T_2
Coproduct	$T_1 + T_2$	T_1 “or” T_2

In computer science, common situations where coalgebraic methods are necessary include infinite data structures (e.g. streams and non-well-founded sets [2]) and dynamic black-box systems, where the state internals are hidden from the user (e.g. the object-oriented programming language paradigm). An important and currently very active use of coalgebras in computer science at present is in the formulation of semantics for object-oriented programs, especially in Java [32, 17, 31, 19, 20, 25].

1.2 Relations between systems

In universal coalgebra there are two main ways of describing the relationship between coalgebras. The first is via (homo)morphisms, which transform one coalgebra to another coalgebra of the same functor. As with morphisms of algebras, morphisms of coalgebras are intuitively structure preserving functions.

The second relationship between coalgebras is bisimulation, or behavioural equivalence. Since we are dealing with black-box systems the closest measure we can have of equivalence between systems is of indistinguishable observable behaviour. A bisimulation between coalgebras is intuitively a structure preserving relation. It is dual to the algebraic notion of a substitution relation and relates states that are observationally indistinguishable, or *bisimilar*. Similarly, a bisimulation equivalence — a bisimulation that is also an equivalence relation — is dual to the algebraic

notion of congruence. To illustrate, consider the following systems.

In this system the two states labelled B are behaviourally indistinguishable — they both move to an A state — whilst the states labelled A are distinguishable — one moves to an A state and the other to a B state. The next two pairs of systems are indistinguishable from one another, in the sense that for each state there is a state of the other that can simulate its observable behaviour.

Bisimulation equivalence is an important notion for the following two reasons:

- (1) if two states are bisimilar then one may replace the other without affecting the behaviour of the system; and
- (2) when making assertions about a system we want to refer only to its observable behaviour and not constrain its internal representation.

The notion of bisimulation comes via concurrency theory. It first appeared in [29] as a relation of mutual simulation between states of two automata; it was shown that deterministic automata related by bisimulation

accept the same input. The categorial definition we use (Definition 2.12) is due to Aczel and Mendler [1]. The concept of bisimilar states (or concurrent processes) satisfying the same formulas of a logical language (Theorem 3.6) is due to Hennessy and Milner [13, 14].

1.3 Definitions and proofs: (co)induction

In universal algebra the principle of induction is used for definitions and proofs of algebraic objects. It uses initial algebras, which have a unique morphism *to* any other algebra. There is a dual notion of a final coalgebra, which has a unique morphism *from* any other coalgebra:

$$\left(\begin{array}{c} \text{initial} \\ \text{algebra} \end{array} \right) \xrightarrow[\text{morphism}]{\text{unique}} \left(\begin{array}{c} \text{arbitrary} \\ \text{algebra} \end{array} \right)$$

$$\left(\begin{array}{c} \text{arbitrary} \\ \text{coalgebra} \end{array} \right) \xrightarrow[\text{morphism}]{\text{unique}} \left(\begin{array}{c} \text{final} \\ \text{coalgebra} \end{array} \right)$$

Final coalgebras are used in definitions and proofs by coinduction. In rough terms, for both induction and coinduction, a definition involves establishing the existence of the morphism above, and a proof involves establishing its uniqueness.

For an inductive definition of a function f , we *define the value of f on all constructor functions*. Recall the example of lists A^* , with constructors `empty` and `cons`. We would define the functions `len`, which gives the length of lists, and `double`, which replaces each element in the list with two copies, like so¹:

$$\begin{aligned} \text{len}(\text{empty}(\star)) &= 0 \\ \text{len}(\text{cons}(a, l)) &= 1 + \text{len}(l) \end{aligned}$$

¹The following examples of definition and proof by (co)induction are taken from Jacobs and Rutten [21].

$$\begin{aligned}\text{double}(\text{empty}(\star)) &= \varepsilon \\ \text{double}(\text{cons}(a, l)) &= \text{cons}(a, \text{cons}(a, \text{double}(l)))\end{aligned}$$

Now suppose we want to prove that the length of $\text{double}(l)$ is twice that of l , i.e. $\text{len}(\text{double}(l)) = 2 \cdot \text{len}(l)$. The ordinary way we would prove this is as follows:

$$\begin{aligned}\text{len}(\text{double}(\text{empty}(\star))) &= \text{len}(\text{empty}(\star)) \\ &= 0 \\ &= 2 \cdot 0 \\ &= 2 \cdot \text{len}(\text{empty}(\star)) \\ \text{len}(\text{double}(\text{cons}(a, l))) &= \text{len}(\text{cons}(a, \text{cons}(a, \text{double}(l)))) \\ &= 1 + 1 + \text{len}(\text{double}(l)) \\ &\stackrel{\text{IH}}{=} 2 + 2 \cdot \text{len}(l) \\ &= 2 \cdot (1 + \text{len}(l)) \\ &= 2 \cdot \text{len}(\text{cons}(a, l))\end{aligned}$$

However it is possible to prove this in a more abstract way using the initial algebra, which we shall outline. It can be shown that the list algebra $\langle A^*, \alpha_l \rangle$ is an initial algebra for the functor $T(X) = 1 + A \times X$. This fact will allow us to prove the equality of the functions $\text{len} \circ \text{double}$ and $2 \cdot (-) \circ \text{len}$. First, let us rearrange our definitions in a diagrammatic form:

$$\begin{array}{ccc} 1 + (A \times A^*) & \xrightarrow{\text{id}_1 + \text{id}_A \times \text{len}} & 1 + (A \times \mathbb{N}) \\ \alpha_l \downarrow & & \downarrow \alpha_c \\ A^* & \xrightarrow{\text{len}} & \mathbb{N} \end{array}$$

$$\begin{array}{ccc}
 1 + (A \times A^*) & \xrightarrow{\text{id}_1 + \text{id}_A \times \text{double}} & 1 + (A \times A^*) \\
 \alpha_l \downarrow & & \downarrow \alpha_d \\
 A^* & \xrightarrow{\text{double}} & A^*
 \end{array}$$

where

$$\begin{aligned}
 \alpha_l(x) &= \begin{cases} \text{empty}(x) & \text{if } x = \star \\ \text{cons}(a, l) & \text{if } x = \langle a, l \rangle \end{cases} \\
 \alpha_c(x) &= \begin{cases} 0 & \text{if } x = \star \\ 1 + n & \text{if } x = \langle a, n \rangle \end{cases} \\
 \alpha_d(x) &= \begin{cases} (x) & \text{if } x = \star \\ \text{cons}(a, \text{cons}(a, l)) & \text{if } x = \langle a, l \rangle \end{cases}
 \end{aligned}$$

We see that len and double are structure preserving functions and are intuitively defined as the unique morphisms from the initial algebra $\langle A^*, \alpha_l \rangle$ to the algebras $\langle \mathbb{N}, \alpha_c \rangle$ and $\langle A^*, \alpha_d \rangle$, respectively. A similar approach shows that both of the functions $\text{len} \circ \text{double}$ and $2 \cdot (-) \circ \text{len}$ are morphisms from $\langle A^*, \alpha_l \rangle$ to the algebra $\langle \mathbb{N}, \alpha_p : 1 + A \times \mathbb{N} \rightarrow \mathbb{N} \rangle$, where

$$\alpha_p(x) = \begin{cases} 0 & \text{if } x = \star \\ 2 + n & \text{if } x = \langle a, n \rangle \end{cases}$$

$$\begin{array}{ccccc}
 & & \text{id}_1 + \text{id}_A \times \text{len} \circ \text{double} & & \\
 & \text{id}_1 + \text{id}_A \times \text{double} & \xrightarrow{\quad} & \text{id}_1 + \text{id}_A \times \text{len} & \\
 1 + (A \times A^*) & \xrightarrow{\quad} & 1 + (A \times A^*) & \xrightarrow{\quad} & 1 + (A \times \mathbb{N}) \\
 \alpha_l \downarrow & & \downarrow \alpha_d & & \downarrow \alpha_p \\
 A^* & \xrightarrow{\text{double}} & A^* & \xrightarrow{\text{len}} & \mathbb{N} \\
 & \text{len} \circ \text{double} & & &
 \end{array}$$

But $\langle A^*, \alpha_l \rangle$ is an initial algebra, so there is a unique morphism $f = \text{len} \circ \text{double} = 2 \cdot (-) \circ \text{len}$ from $\langle A^*, \alpha_l \rangle$ to $\langle \mathbb{N}, \alpha_p \rangle$.

Dually, for a coinductive definition of a function g , we *define the values of all decomposition functions on each outcome* $g(x)$. Recall the example of streams A^ω , with destructors *head* and *tail*. We would define the functions *odd*, which removes all elements from a stream in the even numbered positions, *even*, which removes all elements from a stream in the odd numbered positions, and *merge*, which combines two streams one element at a time to form a new stream, like so:

$$\begin{aligned} \text{head}(\text{odd}(l)) &= \text{head}(l) \\ \text{tail}(\text{odd}(l)) &= \text{odd}(\text{tail}(\text{tail}(l))) \end{aligned}$$

$$\text{even} = \text{odd} \circ \text{tail}$$

$$\begin{aligned} \text{head}(\text{merge}(l_1, l_2)) &= \text{head}(l_1) \\ \text{tail}(\text{merge}(l_1, l_2)) &= \text{merge}(l_2, \text{tail}(l_1)) \end{aligned}$$

Now suppose we want to prove that $\text{merge}(\text{odd}(l), \text{even}(l)) = l$. It can be shown that the stream coalgebra $\langle A^\omega, \alpha_s \rangle$ is a final coalgebra for the functor $T(X) = A \times X$. This fact will allow us to prove the equality of the function $\text{merge} \circ \langle \text{odd}, \text{even} \rangle$ and the identity function. Let us rearrange our coin-

ductive definitions in a diagrammatic form:

where $\alpha_s(l) = \langle \text{head}(l), \text{tail}(l) \rangle$, $\alpha_o(l) = \langle \text{head}(l), \text{tail}(\text{tail}(l)) \rangle$ and $\alpha_m(l_1, l_2) = \langle \text{head}(l_1), \langle l_2, \text{tail}(l_1) \rangle \rangle$. Similarly, but dually, to the inductive definitions, we see that odd and merge are structure preserving functions and are coinductively defined as the unique morphisms from the coalgebras $\langle A^\omega, \alpha_o \rangle$ and $\langle A^\omega \times A^\omega, \alpha_m \rangle$, respectively, to the final coalgebra $\langle A^\omega, \alpha_s \rangle$. To prove that $\text{merge}(\text{odd}(l), \text{even}(l)) = l$ we can show that $\text{merge} \circ \langle \text{odd}, \text{even} \rangle$ is a morphism from $\langle A^\omega, \alpha_s \rangle$ to $\langle A^\omega, \alpha_s \rangle$. Thus, since $\langle A^\omega, \alpha_s \rangle$ is a final coalgebra and the identity function $\text{id} : A^\omega \rightarrow A^\omega$ is trivially the unique morphism from $\langle A^\omega, \alpha_s \rangle$ to $\langle A^\omega, \alpha_s \rangle$, it follows that $\text{merge} \circ \langle \text{odd}, \text{even} \rangle = \text{id}$. To show that it is a morphism the following diagram must commute.

In other words, we must show that

$$\langle \text{head}, \text{tail} \rangle \circ (\text{merge} \circ \langle \text{odd}, \text{even} \rangle) = (\text{id} \times (\text{merge} \circ \langle \text{odd}, \text{even} \rangle)) \circ \langle \text{head}, \text{tail} \rangle$$

which follows from the following computations.

$$\begin{aligned} & \text{head}(\text{merge}(\text{odd}(l), \text{even}(l))) \\ = & \text{head}(\text{odd}(l)) \\ = & \text{head}(l) \end{aligned}$$

$$\begin{aligned} & \text{tail}(\text{merge}(\text{odd}(l), \text{even}(l))) \\ = & \text{merge}(\text{even}(l), \text{tail}(\text{odd}(l))) \\ = & \text{merge}(\text{even}(l), \text{odd}(\text{tail}(\text{tail}(l)))) \\ = & \text{merge}(\text{odd}(\text{tail}(l)), \text{even}(\text{tail}(l))) \\ = & (\text{merge} \circ \langle \text{odd}, \text{even} \rangle)(\text{tail}(l)) \end{aligned}$$

1.4 Specification language: coalgebraic logics

If a coalgebra is a system then its logic is the system's specification language. Equational logics for algebra have been extensively studied over many decades, but the study of coalgebras is much younger and there remains no definitive approach to logics for coalgebras. The appropriate logic for coalgebras is modal logic, and Kurz [22] has shown that it can be thought of as 'dual' to equational logic, when interpreted in a categorical context. Moss [27] was the first to use modal logic, introducing an infinitary modal logic for general functors, though it lacked a complete axiomatization. Corradini [5, 6], Cîrstea [3, 4] and Goldblatt [10] have published equational logics for monomial functors (polynomial without coproduct). Kurz [23] defined a finitary modal logic for polynomial functors of the form $\prod_{i=1}^n (\overline{B_i} + \overline{C_i} \times \text{Id})^{A_i}$, and Rößiger [33] defined another finitary modal logic for all polynomial functors. The 'syntax trees' of functors used in the latter restrict the proof theory to functors constructed from finite constant sets. Goldblatt [11] defined an equational logic for polynomial functors

that is implicitly modal and is shown to subsume Rößiger's logic; it has no proof theory developed as yet.

We introduce a Hennessy-Milner style modal logic, the models of which are polynomial coalgebras of a fixed functor T . It extends the logic of Rößiger — instead of syntax trees we use the notion of paths, introduced in [18], to characterize the structure of functors and to examine the observational behaviour (output) of systems. Using a characterization of bisimulation through paths we show that logical equivalence of states coincides with bisimilarity. We define an axiomatization and proof theory, which is sound and complete if T has constant sets of denumerable cardinality. Completeness is shown by a Henkin proof [7, 8], which involves the construction of a canonical model (coalgebra), where states are sets of formulas that satisfy all, and only, the formulas they contain. This canonical coalgebra, which can be constructed independent of the proof theory, is shown to be a final coalgebra for T .

Chapter 2

Universal coalgebra and paths for polynomial functors

In this section we provide an exposition of universal coalgebra for polynomial functors over the category of sets, and paths between functors. We closely follow Rutten [34] for universal coalgebra and Goldblatt [11] for paths.

2.1 Set Notation

We briefly review some notions from set theory and the notation used here.

A function f from set A to set B is denoted $f : A \longrightarrow B$, a partial function g from A to B is denoted $g : A \multimap B$ and a function h mapping an element $a \in A$ to an element $b \in B$ is denoted $h : a \longmapsto b$.

The identity function on a set A is denoted $\text{id}_A : A \longrightarrow A$.

For sets A and D the D -th power of A is the set of functions $A^D = \{f \mid D \longrightarrow A\}$. For every $d \in D$ there is an evaluation function $\text{ev}_d : A^D \longrightarrow A$ where $\text{ev}_d(f) = f(d)$.

The product of sets is $A_1 \times A_2 = \{\langle a_1, a_2 \rangle \mid a_1 \in A_1, a_2 \in A_2\}$. For $j \in \{1, 2\}$ there are the projection functions $\pi_j : A_1 \times A_2 \longrightarrow A_j$.

The coproduct (or disjoint union) of sets is $A_1 + A_2 = \{\langle 1, a_1 \rangle \mid a_1 \in A_1\} \cup \{\langle 2, a_2 \rangle \mid a_2 \in A_2\}$. There are the injection functions $\iota_1 : A_1 \longrightarrow A_1 + A_2 : a_1 \mapsto \langle 1, a_1 \rangle$ and $\iota_2 : A_2 \longrightarrow A_1 + A_2 : a_2 \mapsto \langle 2, a_2 \rangle$ and their (partial) inverses, the extraction functions $\varepsilon_j : A_1 + A_2 \multimap A_j : \langle j, a_j \rangle \mapsto a_j$.

The powerset of A , the set of all subsets is $2^A = \{X \mid X \subseteq A\}$.

A relation R between sets A and B is a subset of the product, i.e. $R \subseteq A \times B$. When $\langle a, b \rangle \in R$ we often write $a R b$.

We say that a set $X \subseteq A$ is *closed under* a relation of the form $R \subseteq 2^A \times A$ if, for every $\langle Y, a \rangle \in R$, $Y \subseteq X$ implies $a \in X$.

We adopt the convention that whenever we write “ $f(x) R g(y)$ ” for any relation R and partial functions f and g we mean $x \in \text{Dom } f$ iff $y \in \text{Dom } g$ and if both $f(x)$ and $g(y)$ are defined then $\langle f(x), g(y) \rangle \in R$. In particular, note that we use this convention for the relation “ $=$ ”.

2.2 Polynomial Functors

DEFINITION 2.1 A functor on sets, $T : \mathbf{Set} \longrightarrow \mathbf{Set}$, is a function that maps

- (1) every set A to another set TA ; and
- (2) every function $f : A \longrightarrow B$ to another function $Tf : TA \longrightarrow TB$

such that

- (1) $T\text{id}_A = \text{id}_{TA}$; and
- (2) $T(g \circ f) = (Tg) \circ (Tf)$.

DEFINITION 2.2 A functor T is *polynomial* if it is finitely constructed from the following functors:

- the *identity functor* Id , with $\text{Id}A = A$ on sets A and $\text{Id}f = f$ on functions $f : A \longrightarrow B$

- for a fixed set $D \neq \emptyset$, the *constant functor* $\overline{D}$, with $\overline{D}A = D$ and $\overline{D}f = \text{id}_D$
- the D -th power functor T^D of a functor T , with

$$T^D A = (TA)^D$$

and

$$T^D(f) : (TA)^D \longrightarrow (TB)^D : g \longmapsto Tf \circ g$$

- the product of two functors $T_1 \times T_2$, with

$$(T_1 \times T_2)(A) = T_1(A) \times T_2(A)$$

and

$$\begin{aligned} (T_1 \times T_2)(f) = T_1(f) \times T_2(f) & : T_1 A \times T_2 A \longrightarrow T_1 B \times T_2 B \\ & : \langle a_1, a_2 \rangle \longmapsto \langle T_1(f)(a_1), T_2(f)(a_2) \rangle \end{aligned}$$

- the coproduct of two functors $T_1 + T_2$, with

$$(T_1 + T_2)(A) = T_1(A) + T_2(A)$$

and

$$\begin{aligned} (T_1 + T_2)(f) = T_1(f) + T_2(f) & : T_1 A + T_2 A \longrightarrow T_1 B + T_2 B \\ & : \iota_j(a) \longmapsto \iota_j(T_j(f)(a)) \end{aligned}$$

A functor T_1 used in the construction of a functor T_2 is known as a *component* of T_2 .

From this point forward we shall assume that all functors are polynomial. We shall only use parentheses for clarification where necessary and will assume that product binds more tightly than coproduct and power more tightly than both, i.e. $T_1 + (T_2 \times T_3) = T_1 + T_2 \times T_3 \neq (T_1 + T_2) \times T_3$ and $T_1 \times (T_2^D) = T_1 \times T_2^D \neq (T_1 \times T_2)^D$. Note that this convention includes dropping parentheses on application, i.e. $T_1 + T_2 f = (T_1 + T_2)f$.

EXAMPLE 2.3 Some examples of polynomial functors:

- As seen in the introduction a finite list is either the empty list or the pairing of an element of the constant set A and another list. The construction and decomposition of lists is $T(X) = 1 + A \times X$, so the functor is $T = \bar{1} + \bar{A} \times \text{Id}$.
- Streams decompose as an element and another stream, so the functor is $T = \bar{A} \times \text{Id}$.
- Mealy machines are finite automata that require user input for state transitions and give output at each state. At each state in X input from the constant set I invokes a transition to another state in X and output from the constant set O . The transition can be written as $T(X) = (O \times X)^I$ so the functor is $T = (\bar{O} \times \text{Id})^I$.

2.3 Universal Coalgebra

DEFINITION 2.4 Let T be a functor. A T -coalgebra is a pair $\langle A, \alpha \rangle$, where A is a set and α is a function $\alpha : A \rightarrow TA$. A is called the *carrier set* or *state set* and α is called the *transition structure*.

Since $A = \text{Dom } \alpha$ we often refer to $\langle A, \alpha \rangle$ simply as α .

EXAMPLE 2.5 Some examples of coalgebras of polynomial functors:

- The three examples given above (Example 2.3) have coalgebras of the form $\langle X, \chi : X \rightarrow TX \rangle$.
- The list $\langle 5, 18, 6, 9 \rangle$ can be represented by the $(T = \bar{1} + \bar{\mathbb{N}} \times \text{Id})$ -coalgebra

$\langle X, \chi \rangle$, where $X = \{x_0, \dots, x_4\}$ and

$$\chi(x_0) = \langle 2, \langle 5, x_1 \rangle \rangle$$

$$\chi(x_1) = \langle 2, \langle 18, x_2 \rangle \rangle$$

$$\chi(x_2) = \langle 2, \langle 6, x_3 \rangle \rangle$$

$$\chi(x_3) = \langle 2, \langle 9, x_4 \rangle \rangle$$

$$\chi(x_4) = \langle 1, \star \rangle$$

Recall that the top-level pair is from the disjoint union (coproduct).

- Consider the Mealy machine

where each state outputs its label. It can be represented by the $(T = (\overline{O} \times \text{Id})^I)$ -coalgebra $\langle X, \chi \rangle$, where $I = \{a, b, c\}$, $O = \{A, B, C\}$, $X = \{x_A, x_B, x_C\}$ and

$$\chi(x_A)(a) = \langle A, x_A \rangle$$

$$\chi(x_A)(b) = \langle B, x_B \rangle$$

$$\chi(x_A)(c) = \langle C, x_C \rangle$$

$$\chi(x_B)(a) = \langle A, x_A \rangle \quad \dots$$

DEFINITION 2.6 Let $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ be T -coalgebras. A function $f : A \rightarrow B$ is a (T) -morphism if $\beta \circ f = T f \circ \alpha$, i.e. the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \alpha \downarrow & & \downarrow \beta \\ TA & \xrightarrow{Tf} & TB \end{array}$$

An *isomorphism* is a morphism which has an inverse that is also a morphism. The pair $\langle f, g \rangle$, of T -morphisms $f : A \longrightarrow B$ and $g : A \longrightarrow C$, is called a *span*.

EXAMPLE 2.7 Let $T = \overline{\mathbb{N}} \times \text{Id}$ and $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ be T -coalgebras (i.e. stream automata). We would intuitively say a function $f : A \longrightarrow B$ is a structure preserving transformation if $\text{head}_A(a) = \text{head}_B(f(a))$ and $f(\text{tail}_A(a)) = \text{tail}_B(f(a))$, and is hence a morphism as the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \langle \text{head}_A, \text{tail}_A \rangle = \alpha \downarrow & & \downarrow \beta = \langle \text{head}_B, \text{tail}_B \rangle \\ TA & \xrightarrow{Tf} & TB \end{array}$$

Suppose α represents the stream automaton for $\langle 1, 2, 3, 4, \dots \rangle$ and β represents the stream automaton for $\langle 2, 4, 6, 8, \dots \rangle$, with states $a_0 = \langle 1, 2, 3, \dots \rangle$, $a_1 = \langle 2, 3, 4, \dots \rangle$, $b_1 = \langle 4, 6, 8, \dots \rangle$ and $b_2 = \langle 6, 8, 10, \dots \rangle$ etc. Then the function $f : \text{map list}(\lambda x. 2x)$, that doubles every element, is a morphism from α to β as, for example

$$\begin{array}{ll} Tf(\alpha(a_0)) & Tf(\alpha(a_5)) \\ = Tf(\langle 1, a_1 \rangle) & = Tf(\langle 6, a_6 \rangle) \\ = \langle 2, b_1 \rangle & = \langle 12, b_6 \rangle \quad \dots \\ = \beta(b_0) & = \beta(b_5) \\ = \beta(f(a_0)) & = \beta(f(a_5)) \end{array}$$

PROPOSITION 2.8 Let $\langle A, \alpha \rangle$, $\langle B, \beta \rangle$ and $\langle C, \gamma \rangle$ be T -coalgebras and $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be morphisms from α to β and β to γ respectively.

- (1) id_A is a morphism from α to α .
- (2) $g \circ f : A \longrightarrow C$ is a morphism from α to γ .
- (3) if f is a bijection then it is also an isomorphism.

Proof

(1) The diagram

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \alpha \downarrow & & \downarrow \alpha \\ TA & \xrightarrow{T\text{id}_A} & TA \end{array}$$

commutes as $T\text{id}_A = \text{id}_{TA}$ by definition (2.1).

(2) The diagram

$$\begin{array}{ccccc} & & g \circ f & & \\ & \nearrow & & \searrow & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \alpha \downarrow & & \beta \downarrow & & \downarrow \gamma \\ TA & \xrightarrow{Tf} & TB & \xrightarrow{Tg} & TC \\ & \nwarrow & & \nearrow & \\ & & T(g \circ f) & & \end{array}$$

commutes as $T(g \circ f) = Tg \circ Tf$ by definition (2.1).

(3) Let f be a bijection and $h : B \rightarrow A$ be the inverse of f . Now

$$\begin{aligned} & \alpha \circ h \\ = & Th \circ Tf \circ \alpha \circ h && \text{as } Th \circ Tf = \text{id}_{TA} \\ = & Th \circ \beta \circ f \circ h && \text{as } f \text{ is a morphism} \\ = & Th \circ \beta && \text{as } f \circ h = \text{id}_B \end{aligned}$$

so h is a morphism, and thus f is an isomorphism. ■

Now we define pullbacks, which are useful notions we will use in proving results about bisimulations.

DEFINITION 2.9 Let A, B, C and P be sets and $f : A \rightarrow C$, $g : B \rightarrow C$, $k : P \rightarrow A$ and $l : P \rightarrow B$ be functions with $f \circ k = g \circ l$. Then $\langle P, k, l \rangle$ is a (strong) *pullback* of f and g if, for any set X and functions $i : X \rightarrow A$ and $j : X \rightarrow B$ with $f \circ i = g \circ j$, there exists a unique (called a *mediating*)

function $h : X \longrightarrow P$ with $k \circ h = i$ and $l \circ h = j$:

A *weak* pullback is defined as for a strong pullback, except that the mediating function need not be unique.

A functor T is said to *preserve* pullbacks if it transforms pullbacks to pullbacks, i.e. if $\langle P, \pi_1, \pi_2 \rangle$ is a pullback of f and g then $\langle TP, T\pi_1, T\pi_2 \rangle$ is a pullback of Tf and Tg . Similarly, T preserves weak pullbacks if it transforms weak pullbacks to weak pullbacks.

This object from category theory is always defined for sets:

REMARK 2.10 The pullback of $f : A \longrightarrow C$ and $g : B \longrightarrow C$ is

$$\langle P, \pi_1 : P \longrightarrow A, \pi_2 : P \longrightarrow B \rangle$$

where

$$P = \{ \langle a, b \rangle \in A \times B \mid f(a) = g(b) \}$$

■

THEOREM 2.11 *Polynomial functors preserve both weak and strong pullbacks.*

Proof By induction on the construction of the functor. Let T be a polynomial functor and $\langle P, \rho_1, \rho_2 \rangle$ be a pullback of $f : A \longrightarrow C$ and $g : B \longrightarrow C$.

CASE $T = \text{Id}$: $TP = P$ is a pullback of $Tf = f$ and $Tg = g$.

CASE $T = \overline{D}$: $TP = D$ is obviously a pullback of $Tf = \text{id}_D$ and $Tg = \text{id}_D$:

Now assume functors T_1 and T_2 preserve pullbacks.

CASE $T = T_1^D$: Suppose we have functions k and l from a set X such that

commutes. For every $x \in X$

$$\begin{aligned}
 & T_1 f \circ (k(x)) \\
 = & (T_1^D f \circ k)(x) && \text{by definition} \\
 = & (T_1^D g \circ l)(x) && \text{as the diagram commutes} \\
 = & T_1 g \circ (l(x)) && \text{by definition}
 \end{aligned}$$

so we have

which is a pullback by the induction hypothesis with a unique $h_x : D \rightarrow T_1 P$ as pictured. Define $h : X \rightarrow T_1^D P$ by $h(x) = h_x$. Now, for every $x \in X$, $T_1^D \rho_1 \circ h(x) = T_1^D \rho_1 \circ h_x = k(x)$ and $T_1^D \rho_2 \circ h(x) = T_1^D \rho_2 \circ h_x = l(x)$ so

commutes. If there were some other $h' : X \rightarrow T_1^D P$ with this property then it must disagree with h for at least one $x \in X$. But this would violate the uniqueness of h_x . Hence h is unique and $T_1^D P$ is a pullback of $T_1^D f$ and $T_1^D g$.

CASE $T = T_1 \times T_2$: Suppose we have functions k and l from a set X such that

$$\begin{array}{ccc}
 X & \xrightarrow{\quad l \quad} & T_1 P \times T_2 P \\
 \downarrow k & & \downarrow T_1 \rho_1 \times T_2 \rho_1 \\
 & & T_1 A \times T_2 A \\
 & \xrightarrow{\quad T_1 f \times T_2 f \quad} & T_1 C \times T_2 C \\
 & & \downarrow T_1 g \times T_2 g \\
 & & T_1 B \times T_2 B
 \end{array}$$

commutes. For each $j \in \{1, 2\}$, $T_j P$ is a pullback with a unique h_j :

$$\begin{array}{ccc}
 X & \xrightarrow{\quad \pi_j \circ l \quad} & T_j P \\
 \downarrow \pi_j \circ k & \nearrow \exists! h_j & \downarrow T_j \rho_1 \\
 & & T_j A \\
 & \xrightarrow{\quad T_j f \quad} & T_j C \\
 & & \downarrow T_j g \\
 & & T_j B
 \end{array}$$

Define $h : X \rightarrow T_1 \times T_2 : x \mapsto \langle h_1(x), h_2(x) \rangle$. Now

$$\begin{array}{ll}
 T_1 \times T_2 \rho_1 \circ h(x) & T_1 \times T_2 \rho_2 \circ h(x) \\
 = \langle T_1 \rho_1 \circ h_1(x), T_2 \rho_1 \circ h_2(x) \rangle & \text{and} \quad = \langle T_1 \rho_2 \circ h_1(x), T_2 \rho_2 \circ h_2(x) \rangle \\
 = \langle \pi_1 \circ k(x), \pi_2 \circ k(x) \rangle & = \langle \pi_1 \circ l(x), \pi_2 \circ l(x) \rangle \\
 = k(x) & = l(x)
 \end{array}$$

and h is unique due to the uniqueness of h_1 and h_2 , so $T_1 \times T_2 P$ is a pullback of $T_1 \times T_2 f$ and $T_1 \times T_2 g$.

CASE $T = T_1 + T_2$: Suppose we have functions k and l from a set X such that

commutes. Then, for $j \in \{1, 2\}$, $k(x) \in \text{Dom } \varepsilon_j$ iff $l(x) \in \text{Dom } \varepsilon_j$:

$$\begin{aligned}
 & k(x) \in \text{Dom } \varepsilon_j \\
 \text{iff } & (T_1f + T_2f)(k(x)) \in \text{Dom } \varepsilon_j \quad \text{by the definition of } T_1f + T_2f \\
 \text{iff } & (T_1g + T_2g)(l(x)) \in \text{Dom } \varepsilon_j \quad \text{as the diagram commutes} \\
 \text{iff } & l(x) \in \text{Dom } \varepsilon_j \quad \text{by the definition of } T_1g + T_2g
 \end{aligned}$$

Define $X_j = \{x \in X \mid k(x) \in \text{Dom } \varepsilon_j\}$ for each $j \in \{1, 2\}$. (Note that $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \emptyset$.) Define $k_1 : X_1 \rightarrow T_1A$ by $k_1(x) = \varepsilon_1(k(x))$ and similarly define $k_2 : X_2 \rightarrow T_2B$, $l_1 : X_1 \rightarrow T_1A$ and $l_2 : X_2 \rightarrow T_2B$. For each $j \in \{1, 2\}$, T_jP is a pullback with a unique h_j :

Now define $h : X \rightarrow T_1 + T_2P$ by $h(x) = \begin{cases} \iota_1(h_1(x)) & \text{if } x \in X_1 \\ \iota_2(h_2(x)) & \text{if } x \in X_2 \end{cases}$. For each

$j \in \{1, 2\}$ and $x \in X_j$

$$\begin{array}{ll}
 T_1 + T_2\rho_1 \circ h(x) & T_1 + T_2\rho_2 \circ h(x) \\
 = T_1 + T_2\rho_1 \circ \iota_j \circ h_j(x) & = T_1 + T_2\rho_2 \circ \iota_j \circ h_j(x) \\
 = T_j\rho_1 \circ h_j(x) & \text{and} \quad = T_j\rho_2 \circ h_j(x) \\
 = k_j(x) & = l_j(x) \\
 = k(x) & = l(x)
 \end{array}$$

h is unique by the uniqueness of h_1 and h_2 so $T_1 + T_2P$ is a pullback of $T_1 + T_2f$ and $T_1 + T_2g$.

Hence all polynomial functors preserve (strong) pullbacks. For weak pullbacks repeat the proof without the uniqueness properties. ■

2.3.1 Bisimulation

DEFINITION 2.12 (ACZEL AND MENDLER [1]) For T -coalgebras $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ a relation $R \subseteq A \times B$ is a (T -)bisimulation from α to β if there exists a transition structure $\rho : R \rightarrow TR$ on R such that the projections are morphisms, i.e. the following diagram commutes:

$$\begin{array}{ccccc}
 A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & B \\
 \alpha \downarrow & & \downarrow \rho & & \downarrow \beta \\
 TA & \xleftarrow{T\pi_1} & TR & \xrightarrow{T\pi_2} & TB
 \end{array}$$

A *bisimulation equivalence* is a bisimulation that is also an equivalence relation. Two states $a \in A$ and $b \in B$ are *bisimilar* if there exists a bisimulation $R \subseteq A \times B$ with $\langle a, b \rangle \in R$.

EXAMPLE 2.13 Consider the systems below of the functor $T = \bar{1} + \text{Id}^I$, where $I = \{c, d\}$, which only give output on termination after a (possibly infinite) sequence of input.

$$\begin{array}{ccc}
 a_0 & \xrightarrow{c} & a_1 \xrightarrow{c} \dots \\
 \downarrow d & & \downarrow d \\
 a_0' & & a_1'
 \end{array}
 \qquad
 \begin{array}{c}
 \begin{array}{c} \curvearrowright^c \\ b \end{array} \\
 \downarrow d \\
 b'
 \end{array}$$

The system on the right is the coalgebra $\langle B, \beta \rangle$, where $B = \{b, b'\}$ and $\beta(b)(c) = b$, $\beta(b)(d) = b'$ and $\beta(b') = \star$. The system on the left is the coalgebra $\langle A, \alpha \rangle$, where $A = \{a_0, a_0', a_1, a_1', a_2, \dots\}$ and, for all i , $\alpha(a_i)(c) = a_{i+1}$, $\alpha(a_i)(d) = a_i'$ and $\alpha(a_i') = \star$.

The relation $R = \{\langle a_i, b \rangle\} \cup \{\langle a_i', b' \rangle\}$ is a bisimulation from α to β as the function $\rho : R \longrightarrow TR : r \longmapsto \langle \alpha(\pi_1(r)), \beta(\pi_2(r)) \rangle$ is a transition structure for R and the projections π_1 and π_2 are obviously morphisms from ρ to α and ρ to β respectively.

THEOREM 2.14 *Let α and β be T -coalgebras. A function $f : A \longrightarrow B$ is a morphism from α to β iff its graph $G(f) = \{\langle a, f(a) \rangle \mid a \in A\}$ is a bisimulation from α to β .*

Proof Consider the diagram

($\Leftarrow$) Let $G(f)$ be a bisimulation from α to β and $\gamma : G(f) \longrightarrow T(G(f))$ be a transition structure for $G(f)$. The projections $\pi_1 : G(f) \longrightarrow A$ and $\pi_2 : G(f) \longrightarrow B$ are morphisms and so is π_1^{-1} , by Proposition 2.8(3), as π_1 is bijective. $f = \pi_2 \circ \pi_1^{-1}$ thus, by Proposition 2.8(2), f is a morphism.

($\Rightarrow$) Conversely, suppose f is a morphism. Take $(T\pi_1)^{-1} \circ \alpha \circ \pi_1$ as a

transition structure, γ , on $G(f)$. π_1 and π_2 are both morphisms:

$$\begin{array}{lcl}
 T\pi_1 \circ \gamma & & T\pi_2 \circ \gamma \\
 = T\pi_1 \circ ((T\pi_1)^{-1} \circ \alpha \circ \pi_1) & & = T\pi_2 \circ ((T\pi_1)^{-1} \circ \alpha \circ \pi_1) \\
 = \alpha \circ \pi_1 & & = T(\pi_2 \circ \pi_1^{-1}) \circ \alpha \circ \pi_1 \\
 & & = Tf \circ \alpha \circ \pi_1 \\
 & & = \beta \circ f \circ \pi_1 \\
 & & = \beta \circ \pi_2
 \end{array} \quad \blacksquare$$

Thus morphisms are sometimes called *functional bisimulations*.

Bisimulations have a characterization in terms of liftings of relations, due to [15, 16], which we use later to define another characterization with path functions.

DEFINITION 2.15 For a relation $R \subseteq A \times B$ the *lifting* under a functor T is a relation $R^T \subseteq TA \times TB$ defined inductively as:

$$\begin{aligned}
 R^{\bar{D}} &= \Delta_D = \{\langle d, d \rangle \mid d \in D\} \\
 R^{\text{Id}} &= R \\
 R^{T_1 \times T_2} &= \{\langle x, y \rangle \mid \pi_1 x R^{T_1} \pi_1 y \text{ and } \pi_2 x R^{T_2} \pi_2 y\} \\
 R^{T_1 + T_2} &= \{\langle \iota_1 x, \iota_1 y \rangle \mid x R^{T_1} y\} \cup \{\langle \iota_2 x, \iota_2 y \rangle \mid x R^{T_2} y\} \\
 R^{T_1^D} &= \{\langle f, g \rangle \mid \forall d \in D, f(d) R^{T_1} g(d)\}
 \end{aligned}$$

THEOREM 2.16 $R \subseteq A \times B$ is a bisimulation from α to β iff for every $a \in A$ and $b \in B$

$$a R b \text{ implies } \alpha(a) R^T \beta(b) \quad \blacksquare$$

Now we prove some properties of bisimulations.

THEOREM 2.17 Let $f : A \rightarrow C$ and $g : B \rightarrow C$ be morphisms of T -coalgebras α to γ and β to γ , respectively. A weak pullback $\langle P \subseteq A \times B, \pi_1, \pi_2 \rangle$ of f and g is a bisimulation from α to β .

Proof Let $P \subseteq A \times B$ and

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & B \\ \pi_1 \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

be a weak pullback. T preserves weak pullbacks by Theorem 2.11, so

$$\begin{array}{ccc} TP & \xrightarrow{T\pi_2} & TB \\ T\pi_1 \downarrow & & \downarrow Tg \\ TA & \xrightarrow{Tf} & TC \end{array}$$

is also a weak pullback. But since

$$\begin{aligned} & Tf \circ (\alpha \circ \pi_1) \\ = & \gamma \circ f \circ \pi_1 && \text{as } f \text{ is a morphism} \\ = & \gamma \circ g \circ \pi_2 && \text{as } P \text{ is a pullback} \\ = & Tg \circ (\beta \circ \pi_2) && \text{as } g \text{ is a morphism} \end{aligned}$$

there exists a function $\rho : P \longrightarrow TP$ with $\alpha \circ \pi_1 = T\pi_1 \circ \rho$ and $\beta \circ \pi_2 = T\pi_2 \circ \rho$:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow \pi_1 & \vdots \exists \rho & \searrow \pi_2 & \\ A & & TP & & B \\ \alpha \downarrow & \swarrow f & \swarrow T\pi_1 & \searrow T\pi_2 & \downarrow g \beta \\ TA & & C & & TB \\ & \swarrow Tf & \downarrow \gamma & \searrow Tg & \\ & & TC & & \end{array}$$

Thus $\langle P, \rho \rangle$ is a T -coalgebra and π_1 and π_2 are morphisms. Hence P is a bisimulation from α to β . ■

PROPOSITION 2.18 *The diagonal $\Delta_A = \{\langle a, a \rangle \mid a \in A\}$ of a T -coalgebra $\langle A, \alpha \rangle$ is a bisimulation.*

Proof Follows from Theorem 2.14 and the fact that Δ_α is the graph of the identity $\text{id}_A : A \longrightarrow A$. ■

THEOREM 2.19 *If $\langle R, \rho \rangle$ is a bisimulation from α to β then so is its inverse R^{-1} .*

Proof Let i be the isomorphism $R \longrightarrow R^{-1} : \langle a, b \rangle \longrightarrow \langle b, a \rangle$. Then

$$\langle R^{-1}, T(i) \circ \rho \circ i^{-1} \rangle$$

is a bisimulation from β to α :

The next lemma says that the image of a span is a bisimulation, which we use to prove that the composition and union of bisimulations are bisimulations.

LEMMA 2.20 *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be morphisms from T -coalgebras α to β and α to γ respectively. The image of the span of f and g , $\langle f, g \rangle(A) = \{\langle f(a), g(a) \rangle \mid a \in A\}$, is a bisimulation from β to γ .*

Proof Consider the following diagram:

where the function j is defined by $j(a) = \langle f(a), g(a) \rangle$ and the function i is defined by $i(\langle f(a), g(a) \rangle) = a$. $j \circ i = \text{id}_{\langle f, g \rangle(A)}$ and π_1 and π_2 are projections. Note that everything in the diagram commutes. The function $\rho = Tj \circ \alpha \circ i$ is a transition structure for $\langle f, g \rangle(A)$. It follows that ρ is a bisimulation from

β to γ because

$$\begin{aligned}
 & T\pi_1 \circ \rho \\
 = & T\pi_1 \circ T) \circ \alpha \circ i && \text{by the definition of } \rho \\
 = & T(\pi_1 \circ j) \circ \alpha \circ i && \text{by the definition of functors} \\
 = & Tf \circ \alpha \circ i && \text{as the diagram commutes} \\
 = & \beta \circ f \circ i && \text{as } f \text{ is a functor} \\
 = & \beta \circ \pi_1 && \text{as the diagram commutes}
 \end{aligned}$$

and similarly for π_2 . ■

THEOREM 2.21 *The composition $R \circ S$ of T -bisimulations $R \subseteq A \times C$ and $S \subseteq C \times B$ is a bisimulation from α to β .*

Proof $R \circ S$ is equal to the image $\langle r_1 \circ x_1, s_2 \circ x_2 \rangle(X)$ of the pullback:

where x_i , r_i and s_i are projections. As T preserves weak pullbacks, the pullback X can be supplied with a transition structure, by Theorem 2.17, such that the projections x_1 and x_2 are morphisms. Consequently both $r_1 \circ x_1$ and $s_2 \circ x_2$ are morphisms. By Lemma 2.20, $R \circ S$ is a bisimulation from α to β . ■

THEOREM 2.22 *The union $\bigcup_k R_k$ of a family $\{R_k\}_k$ of bisimulations from coalgebras α to β is again a bisimulation.*

Proof $\bigcup_k R_k$ is the image of

$$A \xleftarrow{k} \sum_i Ri \xrightarrow{l} B$$

where k and l are the componentwise projections. The coproduct of a family of coalgebras is a coalgebra, so it follows from Lemma 2.20 that the union is a bisimulation. ■

COROLLARY 2.23 *The set of all bisimulations from α to β is a complete lattice, with least upper bounds and greatest lower bounds given by*

$$\bigvee_k R_k = \bigcup_k R_k$$

$$\bigwedge_k R_k = \bigcup \left\{ R \mid R \text{ is a bisimulation from } \alpha \text{ to } \beta \text{ with } R \subseteq \bigcap_k R_k \right\}$$

In particular, the greatest bisimulation from α to β exists, and is denoted by $\sim_{\langle \alpha, \beta \rangle}$. It is the union of all bisimulations:

$$\sim_{\langle \alpha, \beta \rangle} = \bigcup \{ R \mid R \text{ is a bisimulation from } \alpha \text{ to } \beta \}$$

Consequently the greatest bisimulation on a coalgebra α , written $\sim_\alpha$, is a bisimulation equivalence. ■

We write $\sim$ for the greatest bisimulation relation when the coalgebras are clear from context, and $\sim_T$ when explicit reference to the functor is needed.

2.3.2 Final coalgebra

DEFINITION 2.24 A T -coalgebra $\langle A, \alpha \rangle$ is *final* (or *terminal*¹) if for any T -coalgebra $\langle B, \beta \rangle$ there exists a unique morphism $!_\beta : B \rightarrow A$

An important property of final coalgebras is that they are unique up to isomorphism.

PROPOSITION 2.25 *If $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ are final T -coalgebras then there exists an isomorphism $f : A \rightarrow B$.*

Proof By finality there are unique morphisms $f : A \rightarrow B$ and $g : B \rightarrow A$, which are mutual inverses and hence bijections. Thus, by Proposition 2.8(3), f is an isomorphism. ■

¹“We prefer *final* to *terminal*, which we associate with *malady*.” [34, p. 41]

THEOREM 2.26 (LAMBKE'S LEMMA [24]) *If $\langle A, \alpha \rangle$ is a final T -coalgebra then α is an isomorphism.*

Proof Since $\langle TA, T\alpha \rangle$ is a T -coalgebra there exists by the finality of α a unique morphism $f : TA \rightarrow A$:

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & TA & \xrightarrow{f} & A \\ \alpha \downarrow & & \downarrow T\alpha & & \downarrow \alpha \\ TA & \xrightarrow{T\alpha} & T^2A & \xrightarrow{Tf} & TA \end{array}$$

Again by the finality of α , $f \circ \alpha = \text{id}_A$. Now

$$\begin{aligned} & \alpha \circ f \\ = & Tf \circ T\alpha && \text{as } f \text{ is a morphism} \\ = & T(f \circ \alpha) && \text{by the definition of functors} \\ = & T\text{id}_A && \text{from above} \\ = & \text{id}_{TA} && \text{by the definition of functors} \end{aligned}$$

Therefore f is an inverse of α , and α is an isomorphism. ■

THEOREM 2.27 (RUTTEN AND TURI [35]) *Let $\langle B, \beta \rangle$ be a T -coalgebra, $\langle A_T, \alpha_T \rangle$ be a final T -coalgebra and $!_\beta : B \rightarrow A_T$ the unique morphism from β to α_T . For every $b_1, b_2 \in B$, $b_1 \sim_\beta b_2$ iff $!_\beta(b_1) = !_\beta(b_2)$*

Proof $(\Rightarrow)$ Let R be a bisimulation on β with projections π_1 and π_2 , and with $\langle b_1, b_2 \rangle \in R$. The compositions $!_\beta \circ \pi_1$ and $!_\beta \circ \pi_2$ are morphisms from R to A_T , by Proposition 2.8(2), and are equal by finality. In particular, $!_\beta(b_1) = (!_\beta \circ \pi_1) \langle b_1, b_2 \rangle = (!_\beta \circ \pi_2) \langle b_1, b_2 \rangle = !_\beta(b_2)$.

$(\Leftarrow)$ Because Δ_{A_T} is a bisimulation on A_T , $!_\beta^{-1}(\Delta_{A_T})$ is a bisimulation on β . If $!_\beta(b_1) = !_\beta(b_2)$ then $\langle b_1, b_2 \rangle \in !_\beta^{-1}(\Delta_{A_T})$, hence $b_1 \sim_\beta b_2$ ■

THEOREM 2.28 *All polynomial functors have a final coalgebra.* ■

This is a well-known result and there are many different methods for constructing a final coalgebra for an arbitrary polynomial functor. Rut-

ten and Turi [35] use canonical solutions of domain equations; other authors [1, 2] exploit an anti foundation axiom in non-wellfounded set theory. In chapter 5 we shall give a model-theoretic construction similar to the canonical model construction in modal logic. This approach is also used by Kurz [23] and Rößiger [33], though for limited subsets of polynomial functors. Rößiger also constructs final coalgebras for all polynomial functors [33, §6] by using states that are sets, not of logical formulas, but of partial mappings from strings of ‘positions’ (c.f. basic paths, Definition 2.29) to observable output.

2.4 Paths

We use the notion of a path, from Jacobs [18] and Goldblatt [11], to make assertions about the structure and contents of transitions. A path specifies the location of one functor inside another and the induced path function allows us to label and make assertions about the result of a transition, which gives us another characterization of bisimulation.

DEFINITION 2.29 A *path* is a finite list of symbols of the kind $\pi_j, \varepsilon_j, ev_d$. We write $p.q$ for the concatenation of lists p and q . We use the notation $T \xrightarrow{p} S$ to mean that p is a path from functor T to functor S , defined as:

- $T \xrightarrow{\langle \rangle} T$, where $\langle \rangle$ is the empty path
- $T_1 \times T_2 \xrightarrow{\pi_j.q} S$, where $j \in \{1, 2\}$ and $T_j \xrightarrow{q} S$
- $T_1 + T_2 \xrightarrow{\varepsilon_j.q} S$, where $j \in \{1, 2\}$ and $T_j \xrightarrow{q} S$
- $T_1 \xrightarrow{D^{ev_d.q}} S$ for every $d \in D$, where $T_1 \xrightarrow{q} S$

A path $T \xrightarrow{p} S$ is a *state path* if $S = \text{Id}$, an *observation path* if $S = \overline{D}$ and a *basic path* if it is either.

REMARK 2.30 There is a path $T \rightsquigarrow S$ iff S is a component (is used in the construction) of T . ■

EXAMPLE 2.31 The following are examples of paths.

$$\begin{array}{ll}
 \bar{1} + \text{Id}^{\mathbb{N}} \xrightarrow{\varepsilon_1} \bar{1} & \bar{\mathbb{N}} \times (\bar{1} + \text{Id}) \xrightarrow{\pi_1} \bar{\mathbb{N}} \\
 \bar{1} + \text{Id}^{\mathbb{N}} \xrightarrow{\varepsilon_2} \text{Id}^{\mathbb{N}} & \bar{\mathbb{N}} \times (\bar{1} + \text{Id}) \xrightarrow{\pi_2} \bar{1} + \text{Id} \\
 \bar{1} + \text{Id}^{\mathbb{N}} \xrightarrow{\varepsilon_2 \cdot \text{ev}_1} \text{Id} & \bar{\mathbb{N}} \times (\bar{1} + \text{Id}) \xrightarrow{\pi_2 \cdot \varepsilon_1} \bar{1} \\
 \bar{1} + \text{Id}^{\mathbb{N}} \xrightarrow{\varepsilon_2 \cdot \text{ev}_{243}} \text{Id} & \bar{\mathbb{N}} \times (\bar{1} + \text{Id}) \xrightarrow{\pi_2 \cdot \varepsilon_2} \text{Id}
 \end{array}$$

LEMMA 2.32 If T contains no components of the form S^E with E uncountable, then the number of paths from T is countable.

Proof Since T is finitely constructed all paths are of length at most n , for some n . For each $0 \leq i \leq n$, let X_i be the set of all paths of length i . This means that X_0 contains only the empty path $\langle \rangle$ and is finite, and $\bigcup_{i \leq n} X_i$ contains all paths.

For each path $q \in X_{i+1}$ there is a path $p \in X_i$ such that

- $q = p \cdot \pi_j, T \xrightarrow{p} S_1 \times S_2$ and $j \in \{1, 2\}$; or
- $q = p \cdot \varepsilon_j, T \xrightarrow{p} S_1 + S_2$ and $j \in \{1, 2\}$; or
- $q = p \cdot \text{ev}_e, T \xrightarrow{p} S^E$ and $e \in E$.

If all such sets E are countable then X_{i+1} can only be countably larger than X_i .

Thus $\bigcup_{i \leq n} X_i$ is countable, as it is the finite union of countable sets. ■

DEFINITION 2.33 A path $T \xrightarrow{p} S$ induces a partial function $p_A : TA \multimap SA$ for every set A defined inductively as:

- $\langle \rangle_A : TA \longrightarrow TA$ is the identity function id_{TA} and is total.
- $(\pi_j \cdot p)_A = p_A \circ \pi_j$, the composition $T_1A \times T_2A \xrightarrow{\pi_j} T_jA \xrightarrow{p_A} SA$. Hence $x \in \text{Dom}(\pi_j \cdot p)_A$ iff $\pi_j(x) \in \text{Dom} p_A$.

- $(\varepsilon_j.p)_A = p_A \circ \varepsilon_j$, the composition $T_1A + T_2A \xrightarrow{\varepsilon_j} T_jA \xrightarrow{p_A} SA$. Hence $x \in \text{Dom}(\varepsilon_j.p)_A$ iff $x \in \text{Dom} \varepsilon_j$ and $\varepsilon_j(x) \in \text{Dom} p_A$.
- $(ev_d.p)_A = p_A \circ ev_d$, the composition $(TA)^D \xrightarrow{ev_d} TA \xrightarrow{p_A} SA$. Hence $f \in \text{Dom}(ev_d.p)_A$ iff $f(d) \in \text{Dom} p_A$.

EXAMPLE 2.34 Consider the $\mathbb{N}$ stream functor $T = \overline{\mathbb{N}} \times \text{Id}$, with paths $T \xrightarrow{\pi_1} \overline{\mathbb{N}}$ and $T \xrightarrow{\pi_2} \text{Id}$. For every T -coalgebra $\langle A, \alpha \rangle$ and state $a \in A$ the transition $\alpha(a)$ is a pair $\langle n, a' \rangle$, where $n \in \mathbb{N}$ and $a' \in A$. Thus the two path functions are

$$(\pi_1)_A : TA \longrightarrow \overline{\mathbb{N}}A = TA \longrightarrow \mathbb{N} : \langle n, a' \rangle \mapsto n$$

$$(\pi_2)_A : TA \longrightarrow \text{Id}A = TA \longrightarrow A : \langle n, a' \rangle \mapsto a'$$

Now consider the Mealy machine $\langle B, \beta \rangle$, a $((\overline{\mathbb{N}} \times \text{Id})^{\mathbb{N}})$ -coalgebra that represents an automaton that cumulatively multiplies user input. At state b_n the transition $\beta(b_n)$ gives a function $f_n : \mathbb{N} \longrightarrow \mathbb{N} \times B : m \mapsto \langle n \times m, b_{n \times m} \rangle$. The path functions for this coalgebra are of the form

$$\begin{aligned} (ev_6)_B(\beta(b_7)) &= (ev_6)_B(f_7) = \langle 42, b_{42} \rangle \\ (ev_6.\pi_1)_B(\beta(b_7)) &= 42 \\ (ev_6.\pi_2)_B(\beta(b_7)) &= b_{42} \\ (ev_{29})_B(\beta(b_2)) &= (ev_{29})_B(f_2) = \langle 58, b_{58} \rangle \end{aligned}$$

LEMMA 2.35 For every polynomial functor T , set A and $x \in TA$ there exists a basic path $T \xrightarrow{p} S$ with $x \in \text{Dom} p_A$.

Proof By induction on the construction of T .

CASE $T = \text{Id}$ or $T = \overline{D}$: The empty path $T \xrightarrow{\langle \rangle} T$ has this property, with $\text{Dom} p_A = TA$.

Now assume the lemma holds for T_1 and T_2 .

CASE $T = T_1 \times T_2$: If $x = \langle x_1, x_2 \rangle \in T_1 \times T_2A$ with $x_1 \in T_1A$ and $x_2 \in T_2A$ then, by the induction hypothesis, for both $j \in \{1, 2\}$, there exists a basic

path $T_j \xrightarrow{p_j} S_j$ with $x_j \in \text{Dom}(p_j)_A$. Therefore $T_1 \times T_2 \xrightarrow{\pi_j \cdot p_j} S_j$ is a basic path with $x_j \in \text{Dom}(\pi_j \cdot (p)_j)_A$.

CASE $T = T_1 + T_2$: If $x \in T_1 + T_2 A$ then $x = \iota_j T_j A$ and $\varepsilon_j(x) \in T_j A$. By the induction hypothesis there exists a basic path $T_j \xrightarrow{p} S$ with $\varepsilon_j(x) \in \text{Dom } p_A$. So $T_1 + T_2 \xrightarrow{\varepsilon_j \cdot p} S$ is a basic path with $x \in \text{Dom}(\varepsilon_j \cdot p)_A$.

CASE $T = T_1^D$: If $x \in T_1^D A$ then, for all $d \in D$, $ev_d(x) \in T_1 A$. By the induction hypothesis there exists a basic path $T_1 \xrightarrow{p} S$ with $ev_d(x) \in \text{Dom } p_A$. Then $T_1^D \xrightarrow{ev_d \cdot p} S$ is a basic path with $x \in \text{Dom}(ev_d \cdot p)_A$. ■

Now we combine the use of path functions with liftings of relations in order to obtain a characterization of bisimulation.

THEOREM 2.36 (GOLDBLATT [11]) *For a polynomial functor T let $R \subseteq A \times B$, $x \in TA$ and $y \in TB$. The following are equivalent:*

- (1) $x R^T y$
- (2) $p_A(x) R^S p_B(y)$, for all paths $T \xrightarrow{p} S$
- (3) $p_A(x) R p_B(y)$, for all state paths $T \xrightarrow{p} \text{Id}$, and
 $p_A(x) = p_B(y)$, for all observation paths $T \xrightarrow{p} \overline{D}$

Proof (1) $\Rightarrow$ (2) By induction on T .

CASE $T = \text{Id}$ or $T = \overline{D}$: The only path from T is the empty path $T \xrightarrow{\langle \rangle} T$, with $p_A = \text{id}_{TA}$ and $p_B = \text{id}_{TB}$. So $x R^T y$ then $p_A(x) R^T p_B(y)$.

Now assume this holds for T_1 and T_2 .

CASE $T = T_1 \times T_2$: If $x R^T y$ then $\pi_1(x) R^{T_1} \pi_1(y)$ and $\pi_2(x) R^{T_2} \pi_2(y)$. A path $T \xrightarrow{p} S$ must have the form $T_1 \times T_2 \xrightarrow{\pi_j \cdot q} S$, for $j \in \{1, 2\}$, with $T_j \xrightarrow{q} S$. By the induction hypothesis $q_A(\pi_j(x)) R^S q_B(\pi_j(y))$ so, since $p_A(x) = q_A(\pi_j(x))$ and $p_B(y) = q_B(\pi_j(y))$, it follows that $p_A(x) R^S p_B(y)$.

CASE $T = T_1 + T_2$: If $x R^T y$ then for some $j \in \{1, 2\}$ $x \in \iota_j T_j A$, $y \in \iota_j T_j B$ and $\varepsilon_j(x) R^{T_j} \varepsilon_j(y)$. A path $T \xrightarrow{p} S$ must have the form $T_1 + T_2 \xrightarrow{ev_k \cdot q} S$, for some $k \in \{1, 2\}$, with $T_k \xrightarrow{q} S$. If $k \neq j$ then neither $\varepsilon_j(x) \in \text{Dom } q_A$ nor $\varepsilon_j(y) \in \text{Dom } q_B$ so neither $p_A(x)$ nor $p_B(y)$ are defined.

If $k = j$ then by the induction hypothesis, since $\varepsilon_j(x) R^{T_j} \varepsilon_j(y)$, we have $q_A(\varepsilon_j(x))$ defined iff $q_B(\varepsilon_j(y))$ defined, and when both are defined $p_A(x) = q_A(\varepsilon_j(x)) R^S q_B(\varepsilon_j(y)) = p_B(y)$.

CASE $T = T_1^D$: If $x R^{T_1^D} y$ then for all $d \in D$ $ev_d(x) R^{T_1} ev_d(y)$. A path $T \xrightarrow{p} S$ must have the form $T_1^D \xrightarrow{ev_d \cdot q} S$, for some $d \in D$, with $T_1 \xrightarrow{q} S$. By the induction hypothesis $q_A(ev_d(x)) R^S q_B(ev_d(y))$ so, since $p_A(x) = q_A(ev_d(x))$ and $p_B(y) = q_B(ev_d(y))$, it follows that $p_A(x) R^S p_B(y)$.

(2) $\Rightarrow$ (3) Immediate from definition.

(3) $\Rightarrow$ (1) By induction on T .

CASE $T = \text{Id}$ or $T = \bar{D}$: Let $p = \langle \rangle$. If (3) holds then we have $x = p_A(x) R^T p_B(y) = y$, giving (1).

Now assume this holds for T_1 and T_2 .

CASE $T = T_1 \times T_2$: If $x \in TA$ and $y \in TB$ then, for each $j \in \{1, 2\}$, $\pi_j(x) \in T_j A$ and $\pi_j(y) \in T_j B$. By Lemma 2.35 there exists a basic path $T_j \xrightarrow{q} S$ with $\pi_j(x) \in \text{Dom } q_A$ so, for $p = \pi_j \cdot q$, $p_A(x) = (\pi_j \cdot q)_A(x) = q_A(\pi_j(x))$ is defined. If (3) holds for T , i.e. $p_A(x) R^S p_B(y)$, then $p_B(y)$ is defined and $\pi_j(y) \in \text{Dom } q_B$.

In fact, for *any* basic path $T_j \xrightarrow{q'} S'$, the above reasoning shows that $q'_A(\pi_j(x))$ is defined iff $q'_B(\pi_j(y))$ is defined, and furthermore that when they are both defined $(\pi_j \cdot q')_A(x) R^{S'} (\pi_j \cdot q')_B(y)$ for T ensures that $q'_A(x) R^{S'} q'_B(y)$.

Thus (3) holds for T_j so by induction hypothesis $\pi_j(x) R^{T_j} \pi_j(y)$. This is true for both $j \in \{1, 2\}$ so we can conclude $x R^T y$.

CASE $T = T_1 + T_2$: If $x \in TA$ and $y \in TB$ then, for exactly one $j \in \{1, 2\}$, $x \in \iota_j T_j A$ and so $\varepsilon_j(x) \in T_j A$. By Lemma 2.35 there exists a basic path $T_j \xrightarrow{q} S$ with $\varepsilon_j(x) \in \text{Dom } q_A$ so, for $p = \varepsilon_j.q$, $p_A(x) = \varepsilon_j.q_A(x) = q_A(\varepsilon_j(x))$ is defined. If (3) holds for T , i.e. $p_A(x) R^S p_B(y)$, then $p_B(y) = (\varepsilon_j.q)_B(y)$ is defined, so $y \in \iota_j T_j B$ and $\varepsilon_j(y) \in \text{Dom } q_B$.

In fact, for any basic path $T_j \xrightarrow{q'} S'$, the above reasoning shows that $q'_A(\varepsilon_j(x))$ is defined iff $q'_B(\varepsilon_j(y))$ is defined, and when they are both defined $(\varepsilon_j.q')_A(x) R^{S'} (\varepsilon_j.q')_B(y)$ for T ensures $q'_A(\varepsilon_j(x)) R^{S'} q'_B(\varepsilon_j(y))$.

Thus (3) holds for T_j so by induction hypothesis $\varepsilon_j(x) R^{T_j} \varepsilon_j(y)$, which implies $x R^T y$.

CASE $T = T_1^D$: If $x \in TA$ and $y \in TB$ then, for all $d \in D$, $ev_d \in T_1 A$ and $ev_d \in T_1 B$. By Lemma 2.35 there exists a basic path $T_1 \xrightarrow{q} S$ with $ev_d(y) \in \text{Dom } q_B$ so, for $p = ev_d.q$, $p_B(y) = (ev_d.q)_B(y) = q_B(ev_d(y))$ is defined. If (3) holds for T , i.e. $p_A(x) R^S p_B(y)$, then $p_A(x)$ is defined and $ev_d(x) \in \text{Dom } q_A$.

In fact, for any basic path $T_1 \xrightarrow{q'} S'$, the above reasoning shows that $q'_B(ev_d(y))$ is defined iff $q'_A(ev_d(x))$, and when they are both defined (3) for T ensures that $q'_A(ev_d(x)) R^{S'} q'_B(ev_d(y))$.

Thus (3) holds for T_1 so by induction hypothesis $ev_d(x) R^{T_1} ev_d(y)$. This is true for all $d \in D$ so we can conclude $x R^T y$. ■

Now we have the desired “dynamic” characterization of bisimulations and, as a consequence, morphisms.

THEOREM 2.37 (GOLDBLATT [11]) *Let $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ be T -coalgebras. $R \subseteq A \times B$ is a T -bisimulation iff $a R b$ implies*

$$p_A(\alpha(a)) = p_B(\beta(b))$$

for every observation path $T \xrightarrow{p} \bar{D}$ and

$$p_A(\alpha(a)) R p_B(\beta(b))$$

for every state path $T \xrightarrow{p} \text{Id}$.

Proof By Theorem 2.36 the latter half of the biconditional is equivalent to Theorem 2.16. ■

COROLLARY 2.38 (GOLDBLATT [11]) *Let $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ be T -coalgebras. $f : A \rightarrow B$ is a morphism from α to β iff*

$$p_A(\alpha(a)) = p_B(\beta(f(a)))$$

for every observation path p and

$$f(q_A(\alpha(a))) = q_B(\beta(f(a)))$$

for every state path q .

Proof By Theorem 2.14 f is a morphism from α to β iff its graph, $G(f)$, is a bisimulation from α to β . Now apply Theorem 2.37 to $G(f)$. ■

In the preceding theorems we saw that bisimulation is indeed a structure preserving relation. We will explicitly clarify this in a form that is needed in the next chapter.

COROLLARY 2.39 *If $R \subseteq A \times B$ is a T -bisimulation from α to β and $a R b$ then for all paths $T \xrightarrow{p} S$*

$$\alpha(a) \in \text{Dom } p_A \quad \text{iff} \quad \beta(b) \in \text{Dom } p_B$$

Proof By Theorems 2.16 and 2.36, $a R b$ implies $p_A(\alpha(a)) R^S p_B(\beta(b))$ for all paths $T \xrightarrow{p} S$, which has the implicit assumption that $\alpha(a) \in \text{Dom } p_A$ iff $\beta(b) \in \text{Dom } p_B$. ■

Chapter 3

Syntax and Semantics of Formulas

We now define a Hennessy-Milner style modal logic for a fixed polynomial functor, T , which we use throughout. We place no restrictions on T other than that it be non-trivial. The models of the logic are T -coalgebras, and it is expressive enough to distinguish states up to bisimilarity.

DEFINITION 3.1 The set of *well-formed formulas* (wff) Φ_T consists of the following formulas:

- $\perp$
- $(p)\downarrow$, for every path $T \xrightarrow{p} S$
- $(p)c$, for every observation path $T \xrightarrow{p} \overline{D}$ and $c \in D$
- $\varphi \rightarrow \psi$, for every $\varphi, \psi \in \Phi_T$
- $[p]\varphi$, for every state path $T \xrightarrow{p} \text{Id}$ and $\varphi \in \Phi_T$

We also use the following useful abbreviations:

$$\begin{aligned}\neg\varphi &:= \varphi \rightarrow \perp \\ \top &:= \neg\perp \\ (p)\uparrow &:= \neg(p)\downarrow \\ \langle p \rangle\varphi &:= \neg[p]\neg\varphi\end{aligned}$$

$$\varphi \vee \psi := \neg\varphi \rightarrow \psi$$

$$\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$$

$$\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$$

LEMMA 3.2 *If all constant sets used in the construction of T are countable then Φ_T is countable.*

Proof By induction on the construction of Φ_T . Define $\mathbb{D} = \left\{ D \mid T \xrightarrow{p} \overline{D} \text{ for some } p \right\}$ and $\mathbb{E} = \left\{ E \mid T \xrightarrow{p} S^E \text{ for some } p \text{ and } S \right\}$. Let every $C \in \mathbb{D} \cup \mathbb{E}$ be countable and fix an enumeration $c_1, c_2, c_3, \dots$ of each C .

For $n \geq 0$, let Φ_n be a subset of Φ_T restricted such that $\varphi \in \Phi_n$ contains

- no paths (i.e. subformulas $(p)\downarrow, (p)d, [p]\psi$) with a symbol ev_{c_i} where $i > n$ and $c_i \in C \in \mathbb{E}$
- no subformulas of the form $(p)c_j$ where $j > n$ and $c_j \in C \in \mathbb{D}$
- at most n arrows $(\rightarrow)$
- at most n boxes $([p]\psi$ subformulas)

It follows that

- $\Phi_0 = \{\perp\}$
- $\Phi_n \subset \Phi_{n+1}$, for every $n \geq 0$
- each Φ_n is finite
- $\bigcup_{i \geq 0} \Phi_i = \Phi_T$

Therefore Φ_T is countable. ■

DEFINITION 3.3 We define the truth relation $\models$ inductively as follows for all T -coalgebras $\langle A, \alpha \rangle$, with $x \in A$:

- $\alpha, x \not\models \perp$
- $\alpha, x \models (p)\downarrow$ iff $\alpha(x) \in \text{Dom } p_A$
- $\alpha, x \models (p)c$ iff $\alpha, x \models (p)\downarrow$ and $p_A(\alpha(x)) = c$
- $\alpha, x \models [p]\varphi$ iff $\alpha, x \not\models (p)\downarrow$, or $\alpha, x \models (p)\downarrow$ and $\alpha, p_A(\alpha(x)) \models \varphi$
- $\alpha, x \models \varphi \rightarrow \psi$ iff $\alpha, x \models \varphi$ implies $\alpha, x \models \psi$

We say that φ is *true* at x in α , or x *satisfies* φ , if $\alpha, x \models \varphi$, and that φ is *valid* in α , $\alpha \models \varphi$, if it is true in all states in A . We call the set $\{\psi \mid \alpha, x \models \psi\}$ the *truth set* for x .

The following definitions are useful also:

$$\begin{aligned}
 \alpha \models \varphi & \text{ iff } (\forall x \in A) \alpha, x \models \varphi \\
 \alpha, x \models \Gamma & \text{ iff } (\forall \psi \in \Gamma) \alpha, x \models \psi \\
 \Gamma \models^\alpha \varphi & \text{ iff } (\forall x \in A) \alpha, x \models \Gamma \text{ implies } \alpha, x \models \varphi \\
 \Gamma \models_T \varphi & \text{ iff } (\forall \langle B, \beta \rangle) \Gamma \models^\beta \varphi
 \end{aligned}$$

REMARK 3.4 $\alpha, x \not\models \varphi$ iff $\alpha, x \models \neg\varphi$

Proof

$$\begin{aligned}
 & \alpha, x \models \neg\varphi \\
 \text{iff } & \alpha, x \models \varphi \rightarrow \perp & \text{by the definition of } \neg \\
 \text{iff } & \alpha, x \models \varphi \text{ implies } \alpha, x \models \perp & \text{by the semantics of } \rightarrow \\
 \text{iff } & \alpha, x \not\models \varphi & \text{as } \alpha, x \not\models \perp
 \end{aligned}$$

■

Satisfaction of formulas is invariant under the action of morphisms. Since morphisms are functional bisimulations it follows that logical equivalence and bisimilarity coincide.

LEMMA 3.5 Let $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ be T -coalgebras and $f : A \longrightarrow B$ be a morphism from α to β . Then $\alpha, a \models \varphi$ iff $\beta, f(a) \models \varphi$ for every $\varphi \in \Phi_T$.

Proof By induction on the construction of φ :

CASE $\varphi = \perp$: $\alpha, a \not\models \perp$ and $\beta, f(a) \not\models \perp$

CASE $\varphi = (p)\downarrow$: The graph $G(f)$ is a bisimulation by Theorem 2.14, so
 $\alpha(a) \in \text{Dom } p_A$ iff $\beta(f(a)) \in \text{Dom } p_B$, for all paths p , by Corollary 2.39.

So $\alpha, a \models (p)\downarrow$ iff $\alpha(a) \in \text{Dom } p_A$ iff $\beta(f(a)) \in \text{Dom } p_B$ iff $\beta, f(a) \models (p)\downarrow$.

CASE $\varphi = (p)c$:

$$\begin{aligned} & \alpha, a \models (p)c \\ \text{iff } & \alpha, a \models (p)\downarrow \text{ and } p_A(\alpha(a)) = c && \text{by the semantics of } (p)c \\ \text{iff } & \beta, f(a) \models (p)\downarrow \text{ and } p_B(\beta(f(a))) = c && \text{by the previous CASE and} \\ & && \text{Corollary 2.38} \\ \text{iff } & \beta, f(a) \models (p)c && \text{by the semantics of } \rightarrow \end{aligned}$$

Now assume the lemma holds for θ, ψ .

CASE $\varphi = \theta \rightarrow \psi$:

$$\begin{aligned} & \alpha, a \models \theta \rightarrow \psi \\ \text{iff } & \alpha, a \models \theta \text{ implies } \alpha, a \models \psi && \text{by the semantics of } \rightarrow \\ \text{iff } & \beta, f(a) \models \theta \text{ implies } \beta, f(a) \models \psi && \text{by the induction hypothesis} \\ \text{iff } & \beta, f(a) \models \theta \rightarrow \psi && \text{by the semantics of } \rightarrow \end{aligned}$$

CASE $\varphi = [p]\psi$:

$$\begin{aligned} & \alpha, a \models [p]\psi \\ \text{iff } & \alpha, a \models (p)\downarrow \text{ implies } \alpha, p_A(\alpha(a)) \models \psi && \text{by semantics of } [p]\psi \\ \text{iff } & \beta, f(a) \models (p)\downarrow \text{ implies } \beta, f(p_A(\alpha(a))) \models \psi && \text{by induction hyp.} \\ \text{iff } & \beta, f(a) \models (p)\downarrow \text{ implies } \beta, p_B(\beta(f(a))) \models \psi && \text{by Corollary 2.38} \\ \text{iff } & \beta, f(a) \models [p]\psi && \text{by semantics of } [p]\psi \end{aligned}$$

■

Now we are able to show that logical equivalence coincides with bisimilarity.

THEOREM 3.6 *Let $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ be T -coalgebras with $a \in A$ and $b \in B$. The following are equivalent:*

$$(1) \ a \sim b$$

$$(2) \ \alpha, a \models \varphi \text{ iff } \beta, b \models \varphi \text{ for every } \varphi \in \Phi_T$$

Proof (1) $\Rightarrow$ (2) Let $\langle R, \rho \rangle$ be a bisimulation from α to β and $a R b$. The projections $\pi_1 : R \rightarrow A$ and $\pi_2 : R \rightarrow B$ are morphisms so, by Lemma 3.5, $\alpha, a \models \varphi$ iff $\rho, \langle a, b \rangle \models \varphi$ iff $\beta, b \models \varphi$.

(1) $\Leftarrow$ (2) Let $R = \{ \langle a, b \rangle \in A \times B \mid a \text{ and } b \text{ are logically equivalent} \}$. To show that R is a bisimulation relation, by Theorem 2.37, we need to show that

$$(a) \ p_A(\alpha(a)) = p_B(\beta(b)), \text{ for every observation path } \forall T \xrightarrow{p} \overline{D}$$

$$(b) \ p_A(\alpha(a)) R p_B(\beta(b)), \text{ for every state path } \forall T \xrightarrow{p} \text{Id}$$

Firstly, for all basic paths, $\alpha(a) \in \text{Dom } p_A$ iff $\beta(b) \in \text{Dom } p_B$ by Corollary 2.39. For (a),

$$\begin{aligned} & p_A(\alpha(a)) = d \\ \text{iff} \quad & \alpha, a \models (p)d && \text{by the semantics of } (p)d \\ \text{iff} \quad & \beta, b \models (p)d && \text{by the definition of } R \\ \text{iff} \quad & p_B(\beta(b)) = d && \text{by the semantics of } (p)d \end{aligned}$$

For (b), if $\alpha(a) \in \text{Dom } p_A$ then

$$\begin{aligned} & \alpha, p_A(\alpha(a)) \models \varphi \\ \text{iff} \quad & \alpha, a \models [p]\varphi && \text{by the semantics of } [p]\varphi \\ \text{iff} \quad & \beta, b \models [p]\varphi && \text{by the definition of } R \\ \text{iff} \quad & \beta, p_B(\beta(b)) \models \varphi && \text{by the semantics of } [p]\varphi \end{aligned} \quad \blacksquare$$

Chapter 4

Axiomatic Construction

In this section we give the axioms and relation that define the logic in a purely syntactic manner.

DEFINITION 4.1 If φ is a wff and $p_1, \dots, p_n$ are state paths then we say that the wff $[p_1] \dots [p_n]\varphi$ is a *generalization* of φ .

DEFINITION 4.2 Let the set of axioms, Σ_T , consist of the following wffs and their generalizations.

1. all propositional tautologies
2. $(\langle \rangle)\downarrow$

For each path $T \xrightarrow{p} S_1 \times S_2$:

3. $(p)\downarrow \rightarrow ((p.\pi_1)\downarrow \wedge (p.\pi_2)\downarrow)$

For each path $T \xrightarrow{p} S_1 + S_2$:

4. $(p)\downarrow \rightarrow ((p.\varepsilon_1)\downarrow \leftrightarrow \neg(p.\varepsilon_2)\downarrow)$

For each path $T \xrightarrow{p} S^D$ and all $d \in D$:

5. $(p)\downarrow \rightarrow (p.ev_d)\downarrow$

For each observation path $T \xrightarrow{p} \overline{D}$ and all $c, d \in D$ such that $c \neq d$:

$$6. (p)c \rightarrow \neg(p)d$$

$$7. (p)d \rightarrow (p)\downarrow$$

For each state path $T \xrightarrow{p} \text{Id}$:

$$8. \neg[p]\varphi \rightarrow [p]\neg\varphi$$

$$9. (p)\downarrow \rightarrow \neg[p]\perp$$

$$10. (p)\uparrow \rightarrow [p]\perp$$

$$11. [p](\varphi \rightarrow \psi) \rightarrow ([p]\varphi \rightarrow [p]\psi)$$

The axioms 9 and 10 characterize the relationship between formulas of the form $(p)\downarrow$ and $[p]\varphi$, namely that $\perp$ is true at all states accessible via p iff p is “undefined.” However, to characterize the relationship between formulas of the form $(p)\downarrow$ and $(p)d$ we need axiom 7 and its potentially infinite converse $\neg(p)d_1 \wedge \neg(p)d_2 \wedge \dots \wedge (p)d_n \wedge \dots \rightarrow (p)\uparrow$ (for $T \xrightarrow{p} \overline{D}$ and all $d_i \in D$). We do not allow infinite formulas but would like to allow countably infinite constant sets so we use a relation to capture this property.

DEFINITION 4.3 The richness relation $\mathcal{R}_T \subseteq 2^{\Phi_T} \times \Phi_T$ is the set of all pairs of the form

$$\langle \{\neg(p)d \mid d \in D\}, (p)\uparrow \rangle$$

for all observation paths $T \xrightarrow{p} \overline{D}$.

The relation $\mathcal{R}_T^+$ is defined as the smallest extension of $\mathcal{R}_T$ such that, for all state paths q , if $\langle \Sigma, \varphi \rangle \in \mathcal{R}_T^+$ then $\langle [q]\Sigma, [q]\varphi \rangle \in \mathcal{R}_T^+$, where $[q]\Sigma = \{[q]\psi \mid \psi \in \Sigma\}$.

A set $\Gamma \subseteq \Phi_T$ is *rich* if it is closed under $\mathcal{R}_T^+$.

LEMMA 4.4 If T does not contain any components of the form S^E with E uncountable, then $\mathcal{R}_T$ is countable.

Proof $\mathcal{R}_T$ contains exactly one pair for each observation path from T . If T has no power components with uncountable constant sets then it has at most countably many paths, by Lemma 2.32. If there are countably many paths then there are countably many observation paths, so $\mathcal{R}_T$ is countable too. ■

THEOREM 4.5 *All axioms are valid in all T -coalgebras and $\Sigma \mathcal{R}_T^+ \varphi$ implies $\Sigma \models_T \varphi$.*

Proof

1. Propositional tautologies are valid from the semantics of $\perp$ and $\rightarrow$.

2. $\langle \rangle_A$ is always defined, so $\alpha, x \models (\langle \rangle)\downarrow$.

3–5. These three axioms follow from the definition of p_A :

$$\chi \in \text{Dom } p.\pi_{j_A} \text{ iff } \chi \in \text{Dom } p_A$$

$$\chi \in \text{Dom } p.\text{ev}_{d_A} \text{ iff } \chi \in \text{Dom } p_A$$

$$\chi \in \text{Dom } p.\varepsilon_{j_A} \text{ iff } \chi \in \text{Dom } p_A \text{ and } p_A(\chi) \in \text{Dom } \varepsilon_j$$

6. $\alpha, x \models (p)c \rightarrow \neg(p)d$:

If $\alpha, x \models (p)c$ then $p_A(\alpha(x)) = c$, so $p_A(\alpha(x)) \neq d$. Hence $\alpha, x \not\models (p)d$.

7. $\alpha, x \models (p)d \rightarrow (p)\downarrow$:

$\alpha, x \models (p)d$ implies $\alpha, x \models (p)\downarrow$ by definition.

8. $\alpha, x \models \langle p \rangle \varphi \rightarrow [p] \varphi$:

If $\alpha, x \models \langle p \rangle \varphi$ then $\alpha, x \models \neg[p] \neg \varphi$ and $\alpha, x \not\models [p] \neg \varphi$ by definition.

Therefore $\alpha, x \models (p)\downarrow$ and $\alpha, p_A(\alpha(x)) \not\models \neg \varphi$. Hence, $\alpha, p_A(\alpha(x)) \models \varphi$ and $\alpha, x \models [p] \varphi$.

9. $\alpha, x \models (p)\downarrow \rightarrow \langle p \rangle \top$:

If $\alpha, x \models (p)\downarrow$ then $\alpha(x) \in \text{Dom } p_A$. Therefore $\alpha, p_A(\alpha(x)) \not\models \perp$ and $\alpha, x \not\models [p] \perp$. From the definitions it follows that $\alpha, x \models \langle p \rangle \top$.

10. $\alpha, x \models \neg(p)\downarrow \rightarrow [p]\varphi$:

If $\alpha, x \models \neg(p)\downarrow$ then $\alpha, x \not\models (p)\downarrow$ so $\alpha, x \not\models [p]\varphi$.

11. $\alpha, x \models [p](\varphi \rightarrow \psi) \rightarrow ([p]\varphi \rightarrow [p]\psi)$:

If $\alpha, x \models (p)\uparrow$ then $\alpha, x \models [p](\varphi \rightarrow \psi)$ and $\alpha, x \models [p]\varphi$ and $\alpha, x \models [p]\psi$ so let's assume $\alpha, x \models (p)\downarrow$. If $\alpha, x \models [p](\varphi \rightarrow \psi)$ then $\alpha, p_A(\alpha(x)) \models \varphi \rightarrow \psi$. Therefore $\alpha, p_A(\alpha(x)) \models \varphi$ implies $\alpha, p_A(\alpha(x)) \models \psi$ and $\alpha, x \models [p]\varphi$ implies $\alpha, x \models [p]\psi$.

Generalizations: inductively, let φ be an axiom, or a generalization of an axiom, and p a state path. If $\alpha, x \models (p)\uparrow$ then $\alpha, x \models [p]\varphi$ trivially. Otherwise it has been shown that $\alpha, p_A(\alpha(x)) \models \varphi$ so $\alpha, x \models [p]\varphi$ by definition.

Richness:

- Suppose $\langle \{\neg(p)d \mid d \in D\}, (p)\uparrow \rangle \in \mathcal{R}_T$ and, for all $d \in D$, $\alpha, x \models \neg(p)d$. Then $p_A(\alpha(x)) \neq d$ for each d , i.e. $\alpha(x) \notin \text{Dom } p_A$, so $\alpha, x \not\models (p)\downarrow$ and $\alpha, x \models (p)\uparrow$ by Remark 3.4. Hence $\Sigma \mathcal{R}_T \varphi$ implies $\Sigma \models \varphi$.
- Suppose $\Gamma \models \varphi$. Then for some state path q , T -coalgebra α and state $x \in \text{Dom } \alpha$, $\alpha, x \not\models (q)\downarrow$ implies $\alpha, x \models [q]\varphi$, and if $\alpha, x \models (q)\downarrow$

$$\begin{array}{lll}
 & \alpha, x \models [q]\Gamma & \\
 \text{implies} & \alpha, x \models [q]\psi & \text{for each } \psi \in \Gamma \\
 \text{implies} & \alpha, q_A(\alpha(x)) \models \psi & \text{by the semantics of } [q]\psi \\
 \text{implies} & \alpha, q_A(\alpha(x)) \models \Gamma & \\
 \text{implies} & \alpha, q_A(\alpha(x)) \models \varphi & \text{as } \Gamma \models \varphi \\
 \text{implies} & \alpha, x \models [q]\varphi & \text{by the semantics of } [q]\varphi
 \end{array}$$

Thus $\mathcal{R}_T \subseteq \models$ and $\langle \Sigma, \varphi \rangle \in \models$ implies $\langle [p]\Sigma, [p]\varphi \rangle \in \models$. $\mathcal{R}_T^+$ is the smallest relation satisfying these properties, hence $\Sigma \mathcal{R}_T^+ \varphi$ implies $\Sigma \models \varphi$. ■

Chapter 5

Canonical coalgebras

Here we define the canonical T -coalgebra, whose states consist of “maximal theories” of wffs. The construction is similar as for a Henkin proof of completeness in modal logic, as described in [7, 8], and this turns out to be a natural construction of a final T -coalgebra.

DEFINITION 5.1 For $R \subseteq 2^{\Phi_T} \times \Phi_T$, $\Delta \subseteq \Phi_T$ is a TR -theory if

- it contains the set Σ_T of axioms
- it is closed under Detachment, i.e. $\varphi, \varphi \rightarrow \psi \in \Delta$ implies $\psi \in \Delta$
- it is closed under R , i.e. $\langle \Gamma, \varphi \rangle \in R$ and $\Gamma \subseteq \Delta$ implies $\varphi \in \Delta$.

DEFINITION 5.2 A set of wff, Γ , is *negation complete* if either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ for every $\varphi \in \Phi_T$. Γ is a *maximally $\perp$ -free $T\mathcal{R}_T^+$ -theory* if it is a negation complete $T\mathcal{R}_T^+$ -theory and $\perp \notin \Gamma$.

The use of the word ‘maximal’ in the previous definition is justified by the following lemma.

LEMMA 5.3 Let Γ and Δ be $T\mathcal{R}_T^+$ -theories not containing $\perp$. If Γ is negation complete and $\Gamma \subseteq \Delta$ then $\Gamma = \Delta$.

Proof Let Γ and Δ be $T\mathcal{R}_T^+$ -theories, Γ be negation complete, $\perp \notin \Gamma$, $\perp \notin \Delta$ and $\Gamma \subseteq \Delta$.

Assume there is a formula $\varphi \in \Delta$ and $\varphi \notin \Gamma$. By negation completeness $\neg\varphi \in \Gamma$ so $\neg\varphi \in \Delta$. The tautology $\varphi \rightarrow (\neg\varphi \rightarrow \perp) \in \Delta$, so by Detachment $\perp \in \Delta$ (a contradiction). Hence $\Gamma = \Delta$. ■

LEMMA 5.4 *Let $\langle A, \alpha \rangle$ be a T -coalgebra. The truth set $\{\varphi \mid \alpha, x \models \varphi\}$ for $x \in A$ is a maximally $\perp$ -free $T\mathcal{R}_T^+$ -theory.*

Proof $\{\varphi \mid \alpha, x \models \varphi\}$ is a $T\mathcal{R}_T^+$ -theory as it contains the axioms and is rich by Theorem 4.5 and is closed under detachment by the semantics of $\rightarrow$.

It is negation complete by Remark 3.4, and it is $\perp$ -free by the semantics of $\perp$. ■

The following two lemmas are important properties of maximally $\perp$ -free $T\mathcal{R}_T^+$ -theories that are crucial in the construction of the canonical T -coalgebra.

LEMMA 5.5 *Let Γ be a maximally $\perp$ -free $T\mathcal{R}_T^+$ -theory. For each observation path $T \xrightarrow{p} \overline{D}$ if $(p)\downarrow \in \Gamma$ then $(p)d \in \Gamma$ for a unique $d \in D$.*

Proof If $(p)\downarrow \in \Gamma$ then $(p)\uparrow \notin \Gamma$, as Γ is closed under Detachment and contains the tautology $(p)\downarrow \rightarrow ((p)\uparrow \rightarrow \perp)$ but not $\perp$. Γ is also rich, so is closed under the pair $\langle \{\neg(p)d \mid d \in D\}, (p)\uparrow \rangle \in \mathcal{R}_T^+$. But $(p)\uparrow \notin \Gamma$ so $\neg(p)d \notin \Gamma$, for at least one d , and $(p)d \in \Gamma$ by negation completeness. In fact d is unique, as axiom 6, $(p)d \rightarrow \neg(p)d'$, and Detachment ensure that $\neg(p)d' \in \Gamma$ for every $d' \in D$ where $d' \neq d$. ■

LEMMA 5.6 *For all state paths p , if Δ is a $T\mathcal{R}_T^+$ -theory and $(p)\downarrow \in \Delta$ then $\Delta_p = \{\varphi \mid [p]\varphi \in \Delta\}$ is also a $T\mathcal{R}_T^+$ -theory. Furthermore, if Δ is a maximally $\perp$ -free $T\mathcal{R}_T^+$ -theory then so is Δ_p .*

Proof Let Δ be a $T\mathcal{R}_T^+$ -theory and $(p)\downarrow \in \Delta$.

If $\perp \in \Delta$ then $\Delta = \Phi_T$ by the tautology $\perp \rightarrow \varphi \in \Delta$ and Detachment. Therefore $\Delta_p = \Delta$ and is a $T\mathcal{R}_T^+$ -theory. Neither Δ nor Δ_p are maximally $\perp$ -free $T\mathcal{R}_T^+$ -theories.

Suppose $\perp \notin \Delta$. Δ_p contains all axioms by the generalization property and is closed under Detachment by axiom 11. The $[p]$ closure property that extends $\mathcal{R}_T$ to $\mathcal{R}_T^+$ ensures that Δ_p is also rich. Hence it too is a $T\mathcal{R}_T^+$ -theory.

Suppose Δ is negation complete also. Therefore for every φ either $[p]\varphi \in \Delta$ or $\neg[p]\varphi \in \Delta$. But $\neg[p]\varphi \rightarrow [p]\neg\varphi \in \Delta$, by axiom 8, so either $[p]\varphi \in \Delta$ or $[p]\neg\varphi \in \Delta$, by Detachment. Thus either $\varphi \in \Delta_p$ or $\neg\varphi \in \Delta_p$, so Δ_p is negation complete.

From axiom 9 $(p)\downarrow \rightarrow \neg[p]\perp \in x$, so $\neg[p]\perp \in x$ by Detachment. If $\perp \in \Delta_p$ then $[p]\perp \in \Delta$, so $[p]\perp, [p]\perp \rightarrow \perp \in \Delta$, yielding $\perp \in \Delta$, by Detachment, which is a contradiction. Hence $\perp \notin \Delta_p$. ■

Now we can construct the canonical T -coalgebra (A_T, α_T) , with the state set consisting of maximally $\perp$ -free $T\mathcal{R}_T^+$ -theories and the transition structure defined inductively from the basic paths up.

DEFINITION 5.7 $A_T = \{x \mid x \text{ is a maximally } \perp\text{-free } T\mathcal{R}_T^+\text{-theory}\}$

LEMMA 5.8 For every $T \xrightarrow{p} S$ there exists a partial function $\alpha_p : A_T \multimap SA_T$ with $\text{Dom } \alpha_p = \{x \mid (p)\downarrow \in x\}$ such that, for every $S \xrightarrow{q} U$,

$$\text{Dom } \alpha_{p,q} = \{x \in A_T \mid x \in \text{Dom } \alpha_p \text{ and } \alpha_p(x) \in \text{Dom } q_{A_T}\}$$

and $\alpha_{p,q} = q_{A_T} \circ \alpha_p$:

$$\begin{array}{ccc} & & TA_T \\ & & \downarrow p_{A_T} \\ A_T & \xrightarrow{\alpha_p} & SA_T \\ & \searrow \alpha_{p,q} & \downarrow q_{A_T} \\ & & UA_T \end{array}$$

Proof We define α_p by induction on S .

CASE $T \xrightarrow{p} \overline{D}$: By Lemma 5.5 there is a unique $(p)d \in x$. Let $\alpha_p(x) = d$.

CASE $T \xrightarrow{p} \text{Id}$: Let $\alpha_p(x) = x_p$ where $x_p = \{\varphi \mid [p]\varphi \in x\}$ and $x_p \in A_T$ by Lemma 5.6.

CASE $T \xrightarrow{p} S_1 \times S_2$: Let $\alpha_p(x) = \langle \alpha_{p.\pi_1}(x), \alpha_{p.\pi_2}(x) \rangle$.

CASE $T \xrightarrow{p} S_1 + S_2$: If $(p.\varepsilon_j)\downarrow \in x$ then let $\alpha_p(x) = \iota_j \alpha_{p.\varepsilon_j}(x)$. This is the case for exactly one $j \in \{1, 2\}$.

CASE $T \xrightarrow{p} S^D$: for all $d \in D$, let $\alpha_p(x)(d) = \alpha_{p.ev_d}(x)$.

We prove the second part by induction on S again.

CASE $S = \text{Id}$ or $\overline{D}$, $q = \langle \rangle$: In this case $p.q = p$ and $q_{A_T} = \text{id}_{S_{A_T}}$, so $\alpha_{p.q} = \alpha_p = \text{id}_{S_{A_T}} \circ \alpha_p = q_{A_T} \circ \alpha_p$

Now assume that $\alpha_{s.r} = r_{A_T} \circ \alpha_s$ where $s = p.\sigma$ and $q = \sigma.r$ for some symbol σ .

CASE $S = S_1 \times S_2$, $q = \pi_j.r$: From the induction hypothesis we assume $\text{Dom } \alpha_{p.\pi_j.r} = \text{Dom } r_{A_T} \circ \alpha_{p.\pi_j}$ and $\alpha_{p.\pi_j.r} = r_{A_T} \circ \alpha_{p.\pi_j}$. We know that π_j is total and $(p)\downarrow \in x$ implies $(p.\pi_j)\downarrow \in x$ so $\text{Dom } \alpha_{p.\pi_j} = \text{Dom } (\pi_j)_{A_T} \circ \alpha_p$. From the definition of α_p we see $\alpha_{p.\pi_j} = \pi_j \circ \alpha_p$. Hence, since $(\pi_j.r)_{A_T} = r_{A_T} \circ \pi_j$, $\text{Dom } \alpha_{p.\pi_j.r} = \text{Dom } \pi_j.r_{A_T} \circ \alpha_p$ and $\alpha_{p.\pi_j.r} = \pi_j.r_{A_T} \circ \alpha_p$.

CASE $S = S_1^D$, $q = ev_d.r$: From the induction hypothesis we assume for every $d \in D$ $\text{Dom } \alpha_{p.ev_d.r} = \text{Dom } r_{A_T} \circ \alpha_{p.ev_d}$ and $\alpha_{p.ev_d.r} = r_{A_T} \circ \alpha_{p.ev_d}$. We know ev_d is total and $(p)\downarrow \in x$ implies $(p.ev_d)\downarrow \in x$, so $\text{Dom } \alpha_{p.ev_d} = \text{Dom } (ev_d)_{A_T} \circ \alpha_p$. From the definition of α_p we see $\alpha_{p.ev_d} = ev_d \circ \alpha_p$. Hence, since $(ev_d.r)_{A_T} = r_{A_T} \circ ev_d$, $\text{Dom } \alpha_{p.ev_d.r} = \text{Dom } ev_d.r_{A_T} \circ \alpha_p$ and $\alpha_{p.ev_d.r} = ev_d.r_{A_T} \circ \alpha_p$.

CASE $S = S_1 + S_2$, $q = \varepsilon_j.r$: From the induction hypothesis we assume $\text{Dom } \alpha_{p.\varepsilon_j.r} = \text{Dom } r_{A_T} \circ \alpha_{p.\varepsilon_j}$ and $\alpha_{p.\varepsilon_j.r} = r_{A_T} \circ \alpha_{p.\varepsilon_j}$. We know $x \in \text{Dom } \alpha_p$ iff $(p)\downarrow \in x$ and $x \in \text{Dom } \alpha_{p.\varepsilon_j}$ iff $(p.\varepsilon_j)\downarrow \in x$. Also, $(p)\downarrow \in x$ implies either $(p.\varepsilon_1)\downarrow \in x$ or $(p.\varepsilon_2)\downarrow \in x$ and $(p.\varepsilon_j)\downarrow \in x$ implies $\alpha_p(x) = \iota_j \alpha_{p.\varepsilon_j}(x)$. Therefore $\text{Dom } \alpha_{p.\varepsilon_j} = \text{Dom } \varepsilon_j \circ \alpha_p$ and $\alpha_{p.\varepsilon_j} = \varepsilon_j \circ \alpha_p$. Hence, since $(\varepsilon_j.r)_{A_T} = r_{A_T} \circ \varepsilon_j$, $\text{Dom } \alpha_{p.\varepsilon_j.r} = \text{Dom } \varepsilon_j.r_{A_T} \circ \alpha_p$ and $\alpha_{p.\varepsilon_j.r} = \varepsilon_j.r_{A_T} \circ \alpha_p$. ■

DEFINITION 5.9 Let $\alpha_T = \alpha_{\langle \rangle}$, i.e. α_p for $T \xrightarrow{\langle \rangle} T$.

COROLLARY 5.10

- (1) $\text{Dom } \alpha_T = A_T$
- (2) For every path $T \xrightarrow{q} U$, $\text{Dom } \alpha_q = \{x \mid \alpha_T(x) \in \text{Dom } q_{A_T}\}$ and $\alpha_q = q_{A_T} \circ \alpha_T$.

Proof

- (1) Every maximally $\perp$ -free $T\mathcal{R}_T^+$ -theory contains axiom 2, $(\langle \rangle)\downarrow$, so $A_T = \text{Dom } \alpha_T$.
- (2) Let $p = \langle \rangle$ in Lemma 5.8. ■

An important property of the canonical coalgebra is that each state x satisfies all and only the wffs that it contains.

LEMMA 5.11 (TRUTH LEMMA) $\alpha_T, x \models \varphi$ iff $\varphi \in x$.

Proof By induction on the structure of φ :

CASE $\varphi = \perp$: $\alpha_T, x \not\models \perp$ and $\perp \notin x$ by definition.

CASE $\varphi = (p)\downarrow$: $\alpha_T, x \models (p)\downarrow$
 iff $\alpha_T(x) \in \text{Dom } p_{A_T}$ by the semantics of $(p)\downarrow$
 iff $x \in \text{Dom } \alpha_p$ by Corollary 5.10
 iff $(p)\downarrow \in x$ by the definition of α_p

CASE $\varphi = (p)d$: ($\Rightarrow$) If $\alpha_T, x \models (p)d$ then $\alpha_T, x \models (p)\downarrow$ and $p_{A_T}(\alpha_T(x)) = d$. Therefore $(p)\downarrow \in x$, by the previous CASE, and $\alpha_p(x) = d$, by Corollary 5.10. Hence, by Lemma 5.5 and the definition of α_p (Lemma 5.8), $(p)d \in x$.

($\Leftarrow$) If $(p)d \in x$ then $(p)\downarrow \in x$ by axiom 7. By its definition (Lemma 5.8) $\alpha_p(x) = d$ and thus $p_{A_T}(\alpha_T(x)) = d$, by Corollary 5.10. Since $\alpha_T, x \models (p)\downarrow$ by the previous CASE it follows that $\alpha_T, x \models (p)d$.

Now assume the lemma holds for θ and ψ .

CASE $\varphi = \theta \rightarrow \psi$:

$$\begin{aligned}
& \alpha_T, x \models \theta \rightarrow \psi \\
\text{iff } & \alpha_T, x \models \theta \text{ implies } \alpha_T, x \models \psi && \text{by the semantics of } \rightarrow \\
\text{iff } & \theta \in x \text{ implies } \psi \in x && \text{by the induction hypothesis} \\
\text{iff } & \theta \rightarrow \psi \in x && \text{by Detachment and the} \\
& && \text{tautology } \theta \rightarrow (\psi \rightarrow (\theta \rightarrow \psi)) \in x
\end{aligned}$$

CASE $\varphi = [p]\psi$: ($\Rightarrow$) Suppose $\alpha_T, x \models [p]\psi$. If $\alpha_T, x \not\models (p)\downarrow$ then $(p)\uparrow \in x$. From axioms 10, 1 and 11, $(p)\uparrow \rightarrow [p]\perp$, $[p](\perp \rightarrow \psi)$, $[p](\perp \rightarrow \psi) \rightarrow ([p]\perp \rightarrow [p]\psi) \in x$, and so by Detachment it follows that $[p]\psi \in x$.

Otherwise, $\alpha_T, x \models (p)\downarrow$ and $\alpha_T, p_{A_T}(\alpha_T(x)) \models \psi$. Therefore $(p)\downarrow \in x$ and $\alpha_T, x_p \models \psi$, as $p_{A_T}(\alpha_T(x)) = \alpha_p(x) = x_p$ by Lemma 5.8 and Corollary 5.10. Hence $\psi \in x_p$ by the induction hypothesis and $[p]\psi \in x$.

($\Leftarrow$) Suppose $[p]\psi \in x$. By negation completeness either $(p)\uparrow \in x$ or $(p)\downarrow \in x$. For the former it immediately follows that $\alpha_T, x \not\models (p)\downarrow$ and therefore $\alpha_T, x \models [p]\psi$. For the latter $\alpha_T, x \models (p)\downarrow$ and $\psi \in x_p \in A_T$ so, by the induction hypothesis, $\alpha_T, x_p \models \psi$. But $x_p = p_{A_T}(\alpha_T(x))$; hence $\alpha_T, x \models [p]\psi$. ■

Now we are able to show that the canonical T -coalgebra is a final T -coalgebra.

THEOREM 5.12 (A_T, α_T) is a final T -coalgebra.

Proof Let $\langle B, \beta \rangle$ be any T -coalgebra. Define

$$!_\beta : B \longrightarrow A_T : b \longmapsto \{\varphi \mid \beta, b \models \varphi\}$$

Now we show that $!_\beta$ is the unique morphism from β to α_T .

- $!_\beta(b) \in A_T$: $!_\beta(b)$ is a truth set, so is in A_T by Lemma 5.4.
- $!_\beta$ is a morphism: by Corollary 2.38 we need to show $p_B(\beta(b)) = p_{A_T}(\alpha_T(!_\beta(b)))$ for observation paths and $!_\beta(p_B(\beta(b))) = p_{A_T}(\alpha_T(!_\beta(b)))$ for state paths.

Firstly, $\beta(b) \in \text{Dom } p_B$
iff $\beta, b \models (p)\downarrow$ by the semantics of $(p)\downarrow$
iff $(p)\downarrow \in !_\beta(b)$ by the definition of $!_\beta$
iff $\alpha_T, !_\beta(b) \models (p)\downarrow$ by the Truth Lemma (5.11)
iff $\alpha_T(!_\beta(b)) \in \text{Dom } p_{A_T}$ by the semantics of $(p)\downarrow$

For observation paths, when $\beta, b \models (p)\downarrow$:

$p_B(\beta(b)) = c$
iff $\beta, b \models (p)c$ by the semantics of $(p)c$
iff $(p)c \in !_\beta(b)$ by the definition of $!_\beta$
iff $\alpha_T, !_\beta(b) \models (p)c$ by the Truth Lemma (5.11)
iff $p_{A_T}(\alpha_T(!_\beta(b))) = c$ by the semantics of $(p)c$

For state paths, when $\beta, b \models (p)\downarrow$:

$\varphi \in !_\beta(p_B(\beta(b)))$
iff $\beta, p_B(\beta(b)) \models \varphi$ by the definition of $!_\beta$
iff $\beta, b \models [p]\varphi$ by the semantics of $[p]\varphi$
iff $[p]\varphi \in !_\beta(b)$ by the definition of $!_\beta$
iff $\varphi \in p_{A_T}(\alpha_T(!_\beta(b)))$ by the definition of α_T

i.e. $!_\beta(p_B(\beta(b))) = p_{A_T}(\alpha_T(!_\beta(b)))$.

Hence, $!_\beta$ is a T -morphism.

- uniqueness of $!_\beta$: Let $f : B \rightarrow A_T$ be a morphism from β to α_T . Now we need to show $\varphi \in f(b)$ iff $\varphi \in !_\beta(b)$.

$\varphi \in !_\beta(b)$
iff $\beta, b \models \varphi$ by the definition of $!_\beta$
iff $\alpha_T, f(b) \models \varphi$ by Lemma 3.5
iff $\varphi \in f(b)$ by the Truth Lemma (5.11) ■

As can be seen, $\langle A_T, \alpha_T \rangle$ is a natural construction of a final coalgebra for any polynomial functor T , providing a proof of Theorem 2.28.

Chapter 6

Proof Theory

Now we will develop a proof theoretical relation $\vdash_T$. It will be seen that this relation is both sound (Theorem 6.4) and complete (Theorem 6.14) with respect to the semantic model. For the latter we require Φ_T to be countable, so we restrict T to those functors constructed using countable constant sets (c.f. Lemma 3.2). The approach used to show completeness is called a “Henkin proof”, which uses a canonical model (coalgebra), as constructed in the previous chapter. This technique was first introduced by Henkin [12] for first order logic and has been described abstractly for modal logics by Goldblatt [7, 8].

First, we need to extend the relation $\mathcal{R}_T$ further, as the relation $\mathcal{R}_T^+$ used in the previous chapters is not strong enough to define $\vdash_T$. The new relation, which we call $\mathcal{R}_T^{++}$, could have been defined and used in Chapter 5 to construct the canonical T -coalgebra instead of $\mathcal{R}_T^+$. Doing so would slightly simplify the thesis as a whole, but would have introduced redundancies in the chapter as a self-contained final T -coalgebra construction and proof of Theorem 2.28.

DEFINITION 6.1 The relation $\mathcal{R}_T^{++} \subseteq 2^{\Phi_T} \times \Phi_T$ extends $\mathcal{R}_T^+$ with the addition of an implication property. It is defined as the smallest relation extending

$$\mathcal{R}_T = \left\{ \langle \{ \neg(p)d : d \in D \}, (p)\uparrow \rangle \mid p \text{ is an observation path } T \xrightarrow{p} \overline{D} \right\}$$

such that

- if $\langle \Sigma, \varphi \rangle \in \mathcal{R}_T^+$ then $\langle [q]\Sigma, [q]\varphi \rangle \in \mathcal{R}_T^+$ for every state path q
- if $\langle \Sigma, \varphi \rangle \in \mathcal{R}_T^+$ then $\langle \psi \rightarrow \Sigma, \psi \rightarrow \varphi \rangle \in \mathcal{R}_T^+$ for every $\psi \in \Phi_T$

where $[q]\Sigma = \{[q]\psi \mid \psi \in \Sigma\}$ and $\psi \rightarrow \Gamma = \{\psi \rightarrow \theta \mid \theta \in \Gamma\}$.

A set $\Gamma \subseteq \Phi_T$ is *implication rich* if it is closed under $\mathcal{R}_T^+$.

The properties shown for $\mathcal{R}_T^+$ and $T\mathcal{R}_T^+$ -theories can be extended to apply to $\mathcal{R}_T^{++}$ and $T\mathcal{R}_T^{++}$ -theories as well. In particular we want to extend Lemma 4.4, Theorem 4.5 and Lemma 5.6.

LEMMA 6.2

- (1) If all of the constant sets used in the construction of T are countable then $\mathcal{R}_T^{++}$ is countable.
- (2) If $\Sigma \mathcal{R}_T^{++} \varphi$ then $\Sigma \models_T \varphi$.
- (3) For every state path p , if Δ is a $T\mathcal{R}_T^{++}$ -theory and $(p)\downarrow \in \Delta$ then $\Delta_p = \{\varphi \mid [p]\varphi \in \Delta\}$ is also a $T\mathcal{R}_T^{++}$ -theory.

Proof

- (1) We define a sequence of relations $R_i \subseteq 2^{\Phi_T} \times \Phi_T$, where $R_0 = \mathcal{R}_T$ and R_{n+1} is the result of adding to R_n all pairs of the form $\langle [q]\Sigma, [q]\varphi \rangle$ and $\langle \psi \rightarrow \Sigma, \psi \rightarrow \varphi \rangle$, where $\langle \Sigma, \varphi \rangle \in R_n$, $\psi \in \Phi_T$ and q is a state path from T . $\mathcal{R}_T^{++} = \bigcup_{i \geq 0} R_i$ as it is the smallest relation satisfying the properties in Definition 6.1.

If R_n is countable then the set of pairs of the form $\langle [q]\Sigma, [q]\varphi \rangle$ is countable, by Lemma 2.32, and the set of pairs of the form $\langle \psi \rightarrow \Sigma, \psi \rightarrow \varphi \rangle$ is countable, by Lemma 3.2. Thus R_{n+1} is countable too.

But $\mathcal{R}_T$ is countable, by Lemma 4.4, so $\mathcal{R}_T^{++}$ is a countable union of countable sets, and hence is countable.

- (2) Recall, from the proof of Theorem 4.5, that $\mathcal{R}_T \subseteq \models_T$ and that $\langle \Gamma, \varphi \rangle \in \models_T$ implies $\langle [p]\Gamma, [p]\varphi \rangle \in \models_T$. Furthermore, suppose that $\Gamma \models_T \varphi$ and that $\alpha, x \models_T \psi \rightarrow \Gamma$, for some $\psi \in \Phi_T$, T -coalgebra α and state $x \in \text{Dom } \alpha$. Then

$$\begin{array}{lll}
 & \alpha, x \models_T \psi & \\
 \text{implies} & \alpha, x \models_T \Gamma & \text{by the semantics of } \rightarrow \\
 \text{implies} & \alpha, x \models_T \varphi & \text{as } \Gamma \models_T \varphi \\
 \text{hence} & \alpha, x \models_T \psi \rightarrow \varphi & \text{by the semantics of } \rightarrow
 \end{array}$$

Thus $\langle \Gamma, \varphi \rangle \in \models_T$ implies $\langle \psi \rightarrow \Gamma, \psi \rightarrow \varphi \rangle \in \models_T$. $\mathcal{R}_T^{++}$ is the smallest relation to satisfy these three properties, hence $\Sigma \mathcal{R}_T^{++} \varphi$ implies $\Sigma \models_T \varphi$.

- (3) As for Lemma 5.6. ■

We define the proof relation $\vdash_T$ with respect to $T\mathcal{R}_T^{++}$ -theories, which are purely syntactic constructs.

DEFINITION 6.3 $\Gamma \vdash_T \varphi$ iff $\varphi \in \bigcap \{ \Delta \mid \Gamma \subseteq \Delta \text{ and } \Delta \text{ is a } T\mathcal{R}_T^{++}\text{-theory} \}$ i.e. φ belongs to every $T\mathcal{R}_T^{++}$ -theory extending Γ .

THEOREM 6.4 (SOUNDNESS) *If $\Gamma \vdash_T \varphi$ then $\Gamma \models_T \varphi$.*

Proof Suppose $\Gamma \vdash_T \varphi$ and $\alpha, x \models \Gamma$ for some T -coalgebra $\langle A, \alpha \rangle$ and $x \in A$. We need to show $\alpha, x \models \varphi$, so let $\Delta = \{ \psi \mid \alpha, x \models \psi \}$.

Δ is a $T\mathcal{R}_T^{++}$ -theory — it contains all axioms and is closed under $\mathcal{R}_T^{++}$ by Theorem 4.5 and Lemma 6.2 and it is closed under Detachment by the semantic definition of $\rightarrow$. Therefore, since $\Gamma \subseteq \Delta$ and $\Gamma \vdash_T \varphi$, $\varphi \in \Delta$. Hence $\alpha, x \models \varphi$. ■

There are a number of properties that we would like $\vdash_T$ to have.

LEMMA 6.5 $\vdash_T$ *satisfies:*

- (1) Closure Theorem (CT): *If $\Gamma \vdash_T \psi$ for all $\psi \in \Delta$ and $\Delta \vdash_T \varphi$ then $\Gamma \vdash_T \varphi$*
- (2) Deduction Theorem (DT): *$\Gamma \cup \{ \varphi \} \vdash_T \psi$ implies $\Gamma \vdash_T \varphi \rightarrow \psi$*

- (3) Monotonicity: If $\Gamma \vdash_T \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_T \varphi$
- (4) Detachment: If $\Gamma \vdash_T \varphi$ and $\Gamma \vdash_T \varphi \rightarrow \psi$ then $\Gamma \vdash_T \psi$
- (5) If $\Gamma \vdash_T \varphi$ and $\Gamma \cup \{\varphi\} \vdash_T \perp$ then $\Gamma \vdash_T \perp$
- (6) If $\Gamma \cup \{\neg\varphi\} \vdash_T \perp$ then $\Gamma \vdash_T \varphi$
- (7) Box Rule (BR): If $\Gamma \vdash_T \varphi$ then $[p]\Gamma \vdash_T [p]\varphi$ for all state paths p
- (8) Implication Rule (IR): If $\Gamma \vdash_T \varphi$ then $\psi \rightarrow \Gamma \vdash_T \psi \rightarrow \varphi$

Proof

- (1) If $\Gamma \vdash_T \psi$ for all $\psi \in \Delta$ and $\Delta \vdash_T \varphi$ then any $T\mathcal{R}_T^+$ -theory containing Γ will also contain Δ and therefore φ . Hence $\Gamma \vdash_T \varphi$
- (2) Suppose $\Gamma \cup \{\varphi\} \vdash_T \psi$. Let $\Delta = \{\chi \mid \Gamma \vdash_T \varphi \rightarrow \chi\}$. We want to show $\psi \in \Delta$, and can do so by showing that Δ is a $T\mathcal{R}_T^+$ -theory containing $\Gamma \cup \{\varphi\}$.
 $\theta \rightarrow (\varphi \rightarrow \theta)$ is a tautology (axiom 1) so if $\theta \in \Gamma$ or $\theta \in \Sigma_T$ then $\Gamma \vdash_T \varphi \rightarrow \theta$. $\varphi \rightarrow \varphi$ is also a tautology so $\Gamma \vdash_T \varphi \rightarrow \varphi$. Hence $\Sigma_T \cup \Gamma \cup \{\varphi\} \subseteq \Delta$.
 To show Δ closed under Detachment, suppose $\theta, \theta \rightarrow \chi \in \Delta$. Then from the tautology $(\varphi \rightarrow \theta) \rightarrow ((\varphi \rightarrow (\theta \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi))$ it follows that $\Gamma \vdash_T \varphi \rightarrow \chi$.
 To show Δ is implication rich, let $\langle \Sigma, \chi \rangle \in \mathcal{R}_T^+$ and suppose $\Sigma \subseteq \Delta$. Then $\Gamma \vdash_T \varphi \rightarrow \theta$ for every $\theta \in \Sigma$. So, any $T\mathcal{R}_T^+$ -theory that contains Γ will also contain $\varphi \rightarrow \Sigma$. But $\langle \varphi \rightarrow \Sigma, \varphi \rightarrow \chi \rangle \in \mathcal{R}_T^+$ so every $T\mathcal{R}_T^+$ -theory that contains Γ will also contain $\varphi \rightarrow \chi$. Therefore $\chi \in \Delta$.
 Since Δ is a $T\mathcal{R}_T^+$ -theory containing $\Gamma \cup \{\varphi\}$ it follows that $\Gamma \cup \{\varphi\} \vdash_T \psi$ implies $\Gamma \vdash_T \varphi \rightarrow \psi$.
- (3) If $\Gamma \vdash_T \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_T \psi$ for all $\psi \in \Gamma$. Therefore, by CT, $\Delta \vdash_T \varphi$.
- (4) $\{\varphi, \varphi \rightarrow \psi\} \vdash_T \psi$ so if $\Gamma \vdash_T \varphi$ and $\Gamma \vdash_T \varphi \rightarrow \psi$ then $\Gamma \vdash_T \psi$ follows from CT.
- (5) If $\Gamma \vdash_T \varphi$ and $\Gamma \cup \{\varphi\} \vdash_T \perp$ then $\Gamma \vdash_T \varphi \rightarrow \perp$ by DT. So $\Gamma \vdash_T \perp$ by (4).

- (6) If $\Gamma \cup \{\neg\varphi\} \vdash_T \perp$ then $\Gamma \vdash_T \neg\varphi \rightarrow \perp$ by DT. By tautology $\Gamma \vdash_T (\neg\varphi \rightarrow \perp) \rightarrow \varphi$. Hence, by $\{\neg\varphi \rightarrow \perp, (\neg\varphi \rightarrow \perp) \rightarrow \varphi\} \vdash_T \varphi$ and CT, $\Gamma \vdash_T \varphi$.
- (7) Suppose $\Gamma \vdash_T \varphi$. To show $[p]\Gamma \vdash_T [p]\varphi$ let $\Delta \supseteq [p]\Gamma$ be a $T\mathcal{R}_T^+$ -theory and define $\Delta_p = \{\psi \mid [p]\psi \in \Delta\}$. We want to show $\varphi \in \Delta_p$ — it is sufficient to show that Δ_p is a $T\mathcal{R}_T^+$ -theory containing Γ . But $\Gamma \subseteq \Delta_p$, by definition, and Δ_p is a $T\mathcal{R}_T^+$ -theory, by Lemma 5.6.
- (8) This is a general proof that IR follows from CT, DT and propositional inference.

Suppose $\Gamma \vdash_T \varphi$. Since $(\psi \rightarrow \Gamma) \cup \{\psi\} \vdash_T \theta$ for all $\theta \in \Gamma$ it follows that $(\psi \rightarrow \Gamma) \cup \{\psi\} \vdash_T \varphi$ by CT. Therefore $\psi \rightarrow \Gamma \vdash_T \psi \rightarrow \varphi$, by DT. ■

One of the most important concepts associated with $\vdash_T$ is the notion of consistency. It is important to recognise when a set of wff can be used to prove something that is not true (e.g. $\perp$) and is thus inconsistent.

DEFINITION 6.6 Γ is $\vdash_T$ -inconsistent if $\Gamma \vdash_T \perp$ and $\vdash_T$ -consistent otherwise. Γ is *finitely* $\vdash_T$ -consistent if all finite subsets of Γ are $\vdash_T$ -consistent. Γ is a *maximally* $\vdash_T$ -consistent $T\mathcal{R}_T^+$ -theory if it is a negation complete and $\vdash_T$ -consistent $T\mathcal{R}_T^+$ -theory.

The use of the word ‘maximal’ in the previous definition is justified by the following lemma.

LEMMA 6.7 Suppose Γ and Δ are $\vdash_T$ -consistent sets, Γ is negation complete and $\Gamma \subseteq \Delta$. Then $\Gamma = \Delta$.

Proof As for Lemma 5.3, with Lemma 6.8(3). ■

LEMMA 6.8

- (1) $T\mathcal{R}_T^+$ -theories are $\vdash_T$ -closed, i.e. if Δ is a $T\mathcal{R}_T^+$ -theory and $\Delta \vdash_T \varphi$ then $\varphi \in \Delta$
- (2) If $\perp \in \Gamma$ then Γ is $\vdash_T$ -inconsistent.

(3) If Δ is a $T\mathcal{R}_T^+$ -theory then it is $\vdash_T$ -consistent iff $\perp \notin \Delta$.

Proof

- (1) Suppose Δ is a $T\mathcal{R}_T^+$ -theory, $\Gamma \vdash_T \varphi$ and $\Gamma \subseteq \Delta$. By definition φ is a member of every $T\mathcal{R}_T^+$ -theory that contains Δ , hence $\varphi \in \Delta$.
- (2) If $\perp \in \Gamma$ then $\perp$ will be a member of every $T\mathcal{R}_T^+$ -theory that contains Γ . Therefore $\Gamma \vdash_T \perp$.
- (3) The $(\Rightarrow)$ direction follows from the contraposition of (2) above. For $(\Leftarrow)$ assume that Δ is a $T\mathcal{R}_T^+$ -theory and $\perp \notin \Delta$. It follows that $\perp \notin \{\Delta' \mid \Delta' \text{ is a } T\mathcal{R}_T^+ \text{-theory and } \Delta \subseteq \Delta'\}$, hence by definition $\Delta \not\vdash_T \perp$. ■

LEMMA 6.9 *All truth sets are maximally $\vdash_T$ -consistent $T\mathcal{R}_T^+$ -theories.*

Proof As for Lemma 5.4: $\{\varphi \mid \alpha, x \models \varphi\}$ is a $T\mathcal{R}_T^+$ -theory by Theorem 4.5, Lemma 6.2(1) and the semantics of $\rightarrow$. It is negation complete by Remark 3.4 and is $\vdash_T$ -consistent by Lemma 6.8(3). ■

At this point in a Henkin proof we would normally construct a canonical model (coalgebra) using the proof relation — in this case using maximally $\vdash_T$ -consistent $T\mathcal{R}_T^+$ -theories. However we have already constructed a canonical coalgebra, using maximally $\perp$ -free $T\mathcal{R}_T^+$ -theories. These two notions are actually the same and we can use the existing canonical coalgebra with the alternate definition $A_T = \{x \mid x \text{ is a maximally } \vdash_T\text{-consistent } T\mathcal{R}_T^+ \text{-theory}\}$.

LEMMA 6.10 *A set of wffs is a maximally $\perp$ -free $T\mathcal{R}_T^+$ -theory iff it is a maximally $\vdash_T$ -consistent $T\mathcal{R}_T^+$ -theory.*

Proof $(\Leftarrow)$ Every $T\mathcal{R}_T^+$ -theory is a $T\mathcal{R}_T^+$ -theory as $\mathcal{R}_T^+ \supseteq \mathcal{R}_T^+$. So Lemma 6.8(3) gives that every maximally $\vdash_T$ -consistent $T\mathcal{R}_T^+$ -theory is a maximally $\perp$ -free $T\mathcal{R}_T^+$ -theory.

$(\Rightarrow)$ If Δ is a maximally $\perp$ -free $T\mathcal{R}_T^+$ -theory then it is a member of A_T and, by the Truth Lemma (5.11), $\alpha_T, \Delta \models \varphi$ iff $\varphi \in \Delta$. Thus Δ is a truth set and hence, by Lemma 6.9, is a maximally $\vdash_T$ -consistent $T\mathcal{R}_T^+$ -theory. ■

LEMMA 6.11

- (1) *If Γ is finitely $\vdash_T$ -consistent then so is one of $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ for all φ .*
- (2) *If Γ is negation complete and finitely $\vdash_T$ -consistent then Γ is closed under Detachment and contains all axioms.*
- (3) *If Γ is finitely $\vdash_T$ -consistent then Γ is maximally finitely $\vdash_T$ -consistent iff it is negation complete.*

Proof

- (1) If the conclusion is false then $\Gamma_0 \cup \{\varphi\} \vdash_T \perp$ and $\Gamma_1 \cup \{\neg\varphi\} \vdash_T \perp$ for some φ and some $\Gamma_0, \Gamma_1 \subseteq \Gamma$. Then, by DT and monotonicity, $\Gamma_0 \cup \Gamma_1 \vdash_T \varphi \rightarrow \perp$ and $\Gamma_0 \cup \Gamma_1 \vdash_T \neg\varphi \rightarrow \perp$.

But $\{\varphi \rightarrow \perp, \neg\varphi \rightarrow \perp\} \vdash_T \perp$ as every $T\mathcal{R}_T^+$ -theory that contains $\varphi \rightarrow \perp$ and $\neg\varphi \rightarrow \perp = (\varphi \rightarrow \perp) \rightarrow \perp$ also contains $\perp$, by Detachment. Therefore, by CT, $\Gamma_0 \cup \Gamma_1 \vdash_T \perp$.

Therefore Γ is not finitely $\vdash_T$ -consistent.

- (2) Let Γ be negation complete and finitely $\vdash_T$ -consistent.

Γ is closed under Detachment as if $\varphi, \varphi \rightarrow \psi \in \Gamma$ and $\psi \notin \Gamma$ then $\neg\psi \in \Gamma$, so Γ would contain the finite $\vdash_T$ -inconsistent set $\{\varphi, \varphi \rightarrow \psi, \neg\psi\}$.

If φ is an axiom then $\{\neg\varphi\}$ is a finite $\vdash_T$ -inconsistent set, so $\varphi \in \Gamma$ by negation completeness.

- (3) Let Γ be maximally finitely $\vdash_T$ -consistent. Then for all φ by (1) either $\Gamma \cup \{\varphi\} = \Gamma$ or $\Gamma \cup \{\neg\varphi\} = \Gamma$. Therefore Γ is negation complete.

Conversely, if Γ is finitely $\vdash_T$ -consistent and negation complete let $\Gamma \subset \Delta$. Then there exists $\varphi \in \Delta$ with $\neg\varphi \in \Gamma$, so Δ contains the finitely $\vdash_T$ -inconsistent set $\{\varphi, \neg\varphi\}$. Therefore Γ has no finitely $\vdash_T$ -consistent extension and is maximally finitely $\vdash_T$ -consistent. ■

COROLLARY 6.12 *If Γ is maximally finitely $\vdash_T$ -consistent and implication rich then Γ is a maximally $\vdash_T$ -consistent $T\mathcal{R}_T^+$ -theory.*

Proof If Γ is maximally finitely $\vdash_T$ -consistent and implication rich then by (2) and (3) of Lemma 6.11 it is also negation complete, closed under Detachment and contains all axioms, and therefore a $T\mathcal{R}_T^+$ -theory.

If Γ was $\vdash_T$ -inconsistent, i.e. $\Gamma \vdash_T \perp$, then $\{\perp\} \subseteq \Gamma$, by Lemma 6.8(3), contrary to the fact that it is finitely $\vdash_T$ -consistent. Hence it is a maximally $\vdash_T$ -consistent $T\mathcal{R}_T^+$ -theory. ■

LEMMA 6.13 (EXTENSION LEMMA) *Every $\vdash_T$ -consistent set can be extended to a maximally $\vdash_T$ -consistent $T\mathcal{R}_T^+$ -theory.*

Proof Suppose Γ is $\vdash_T$ -consistent. Recall that Φ_T and $\mathcal{R}_T^+$ are countable, by Lemma 3.2 and Lemma 6.2(1), using the assumption for this chapter that no uncountable constant sets are used in the construction of T . Fix an enumeration $\varphi_0, \varphi_1, \dots, \varphi_n, \dots$ of Φ_T and an enumeration $\langle \Sigma_0, \psi_0 \rangle, \langle \Sigma_1, \psi_1 \rangle, \dots, \langle \Sigma_n, \psi_n \rangle, \dots$ of $\mathcal{R}_T^+$.

Let $\Delta_0 = \Gamma$. Assume Δ_n is defined and $\vdash_T$ -consistent. If $\Delta_n \vdash_T \varphi_n$ then let

$$\Delta_{n+1} = \Delta_n \cup \{\varphi_n\}$$

which is $\vdash_T$ -consistent by Lemma 6.5(5). Alternatively, $\Delta_n \not\vdash_T \varphi_n$. If $\varphi_n \neq \psi_m$, for every m , then let

$$\Delta_{n+1} = \Delta_n \cup \{\neg\varphi_n\}$$

which is $\vdash_T$ -consistent by Lemma 6.5(6). Otherwise there is an m such that $\psi_m = \varphi_n$. If $\Delta_n \cup \{\neg\varphi_n\} \vdash_T \psi_m$ then $\Delta_n \vdash_T \varphi_n$ — a contradiction — so $\Delta_n \cup \{\neg\varphi_n\} \not\vdash_T \psi_m$. Therefore, since $\Sigma_m \vdash_T \psi_m$, by CT there exists a $\theta \in \Sigma_m$ such that $\Delta_n \cup \{\neg\varphi_n\} \not\vdash_T \theta$. Let

$$\Delta_{n+1} = \Delta_n \cup \{\neg\varphi_n, \neg\theta\}$$

which is $\vdash_T$ -consistent by Lemma 6.5(6).

Now let $\Delta = \bigcup_{i \geq 0} \Delta_i$.

Δ is finitely $\vdash_T$ -consistent — all finite subsets of Δ are subsets of some Δ_i , so if a finite subset was not $\vdash_T$ -consistent then Δ_i 's $\vdash_T$ -consistency would be violated.

Δ is negation complete by construction, so is maximally finitely $\vdash_T$ -consistent, by Lemma 6.11(2). But it is also implication rich by construction. Hence, by Corollary 6.12, Δ is a maximally $\vdash_T$ -consistent $T\mathcal{R}_T^+$ -theory. ■

This now allows us to show that $\vdash_T$ is complete.

THEOREM 6.14 (COMPLETENESS) *The following are equivalent:*

- (1) $\Gamma \vdash_T \varphi$
- (2) $\Gamma \models_{\overline{T}} \varphi$
- (3) $\Gamma \models^{\alpha_T} \varphi$

Proof

(1) $\Rightarrow$ (2) Theorem 6.4

(2) $\Rightarrow$ (3) Follows as α_T is a T -coalgebra.

(3) $\Rightarrow$ (1) If $\Gamma \not\vdash_T \varphi$ then $\Gamma \cup \{\neg\varphi\}$ is $\vdash_T$ -consistent, by Lemma 6.5(6). Therefore, by the Extension Lemma (6.13), there exists a maximally $\vdash_T$ -consistent $T\mathcal{R}_T^+$ -theory $x \supseteq \Gamma \cup \{\neg\varphi\}$. By the Truth Lemma (5.11), $\alpha_T, x \models \Gamma$ and $\alpha_T, x \not\models \varphi$, hence $\Gamma \not\models^{\alpha_T} \varphi$. ■

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